Deterministic Modelling of Water Waves in the Frequency Domain

Henrik Bredmose
PhD thesis
November 2002
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PhD Thesis
Technical University of Denmark, Maritime Engineering.
November, 2002
ISBN 87-89502-62-0
Keywords: Wave models, deterministic evolution equations, efficiency, FFT, breaking, roller model, amplitude dispersion, fully dispersive theory, cubic nonlinearity, wave-wave interaction
Preface

This thesis is submitted as one of the requirements for obtaining the Danish Ph.D. degree at the Technical University of Denmark (DTU). The work has been carried out over the period October 1, 1999 till November 1, 2002.

The study was financed by DTU. This support is gratefully acknowledged.

The first 27 months were carried out at Department of Mathematical Modelling (IMM), which from January 1, 2001 became a part of the new department Informatics and Mathematical Modelling (IMM). Due to a reorganization of the University’s departments, the last 10 months of the project were carried out at the Section of Maritime Engineering (MT), in the Department of Mechanical Engineering (MEK).

The study was supervised by Professor Per A. Madsen (MEK) and Dr. Hemming A. Schäffer (DHI Water & Environment). I would like to thank both of my supervisors for encouraging discussions and competent feedback during the period of study. Their willingness to discuss the various issues involved has been an invaluable source of support for this work.

During the period February 4, till June 15, 2001, I had the opportunity to visiting Dr. Yehuda Agnon, Faculty of Civil Engineering, Israel Institute of Technology (Technion), Haifa, Israel. A large part of the last two chapters of the thesis is an outcome of the discussions during this visit. Special thanks are therefore directed to Dr. Yehuda Agnon for his encouraging and inspiring attitude during and after the stay.

I would like to thank my colleagues at IMM and MEK for a nice working atmosphere and inspiring interaction. A big word of thanks also goes to my family and friends for their support during these three years.

Lyngby November 1st 2002
Henrik Bredmose

In this official edition of the thesis, some improvements of wording have been made.

Bristol June 20th 2003
Henrik Bredmose
Executive Summary

Deterministic modelling of water waves in the frequency domain

The purpose of this study is to extend the applicability of deterministic evolution equations for wave-wave interaction. Various aspects of evolution equations are treated. In the following, the scope of each thesis chapter is outlined. Afterwards, on page vi, the most important findings of the study are listed. This may serve as a guidance for reading the thesis.

Chapter 1: Introduction

In Chapter 1, the basic concept and methods of deterministic evolution equations are accounted for. A method for deriving evolution equations is presented and a literature review is given. The review covers evolution equations for triad interactions as well as evolution equations for four-wave interaction at deep water.

Chapter 2: Boussinesq based models

During the last 10–15 years, a number of increasingly accurate time domain Boussinesq formulations have appeared in the literature. The aim of Chapter 2 is to investigate if these improvements in accuracy can be retained in the corresponding evolution equations.

The Boussinesq equations of Nwogu (1993) are transformed to the frequency domain and the resulting model is implemented as a boundary value problem. This model serves as a benchmark for examining the influence of the various approximations involved when deriving evolution equations. It turns out that the higher-order linear derivatives cannot be retained if the model is to be solved in a space-marching manner. It is possible, though, to derive a set of evolution equations formulated in the amplitudes of the free surface elevation and the amplitudes of the velocity variable. In this way some of the approximations of a conventional derivation of evolution equations can be avoided, but as the model suffers from so-called spurious reflections on varying depth, this model is not recommended for general use.

The overall conclusion of Chapter 2 is that evolution equations should be used due to their computational efficiency rather than their accuracy.
**Executive Summary**

Chapter 3: Efficiency of evolution equations  Following the conclusion of Chapter 2, the aspect of computational efficiency of evolution equations is treated in Chapter 3. Even though evolution equations appear to be simple and quick to solve, the convolution sums result in a computational effort of $O(N^2)$ for taking a spatial step. Here $N$ is the number of frequencies.

In spectral methods for partial differential equations, however, it is known that convolution sums can be treated with a computational effort of $O(N \log N)$ utilizing the FFT-algorithm. We go into this in Chapter 3 and show how this speed-up can be applied to the evolution equations of Madsen and Sørensen (1993).

To examine the practical efficiency of evolution equations using this trick, we present comparisons of CPU-times of a time domain Boussinesq model and evolution equations for weakly nonlinear waves passing a submerged bar. For each model, a curve relating the accuracy of a model run to the CPU-time used is constructed. By comparing these curves, it is found that the evolution equations are around 1000 times faster for a test of regular waves and around 100 times faster for a test of irregular waves. The reason that the time domain model is less efficient than the evolution equations is the need for resolving the total phase variation for all waves and the need of taking much smaller time steps than needed for describing a given maximum frequency at the Nyquist frequency.

Chapter 4: Inclusion of roller breaking  Breaking is usually included in evolution equations through an empirical damping term $-d_p \alpha_p$, added to the right hand side. Such breaking formulations do not take the phases of the wave field into account and do therefore not provide a deterministic description of the wave breaking. As the damping coefficient is only real, no phase changes due to breaking are implied by such formulations. In time domain Boussinesq models, several deterministic breaking schemes exist. Tao (1983) and others have developed an eddy-viscosity breaking formulation and Schäffer et al. (1993) incorporated the surface roller breaking model into a Boussinesq model. This breaking model was later used by Madsen et al. (1997a,b) to simulate regular as well as irregular wave motion in the surf zone, including swash zone dynamics and surf beat.

Hence, in Chapter 4, the corresponding evolution equations are extended with a roller breaking formulation. For doing this, the roller breaking formulation is adapted to work on time series in a fixed location. An equation for the mean water level variation is derived as well. Results for the spilling breaker test of Ting and Kirby (1994) are presented, and the new breaking formulation is compared to a conventional breaking formulation as well as results of a time domain Boussinesq formulation.

It turns out that two problems occur within the simulations of the breaking waves: 1) the phase speed of the breaking waves of the evolution equations is too large when comparing to the time domain model, and 2) the shape of the breaking waves does not resemble the characteristic saw-tooth shape of the time domain model and the experimental results.
Chapter 5: Amplitude dispersion in evolution equations  The problems related to the phase speed of the waves in Chapter 4 motivated an investigation of the amplitude dispersion within the evolution equations of Madsen and Sørensen (1993). The amplitude dispersion of the evolution equations is not identical to the amplitude dispersion of the corresponding time domain Boussinesq formulation. The different approximations applied in the transformation to the frequency domain influence the amplitude dispersion. The influence of the various approximations applied is therefore investigated. Both a weakly nonlinear approach and a fully nonlinear approach are presented. It is found that the amplitude dispersion within the evolution equations of Madsen and Sørensen (1993) is much stronger than for the corresponding time domain model. The main reason for this deviation is the neglect of spatial derivatives of the wave amplitudes in the nonlinear terms.

Chapter 6: Fully dispersive evolution equations, 2nd order  Even though it appears to be possible to avoid the over-prediction of the amplitude dispersion in the evolution equations of Madsen and Sørensen (1993), by retaining the derivatives of the wave amplitudes in the nonlinear terms, it was decided to shift the focus towards fully dispersive evolution equations. The reason for this is the improved accuracy of this formulation, which allows for exact linear dispersion and linear shoaling characteristics. Hence a more accurate description of the wave field, including amplitude dispersion was expectable. Thus in Chapter 6, a new method of deriving these evolution equations is presented, eliminating the need for specifying the vertical profile of the horizontal velocity. The resulting evolution equations therefore have exact second-order transfer functions when comparing to Stokes wave theory. Utilizing the so-called resonance assumption, the models of Agnon et al. (1993), Kaihatu and Kirby (1995) and Eldeberky and Madsen (1999) are recovered.

While these ‘resonant’ models allow for calculating the nonlinear terms using FFT, the models with exact transfer functions do not. This problem can be overcome by approximating the kernel function (the interaction coefficients) of the new model with a kernel function that allows for the use of FFT. A first suggestion of such an approximation is given.

The amplitude dispersion of fully dispersive evolution equations is analyzed using the weakly nonlinear approach of Chapter 5. It is found that the amplitude dispersion is over-predicted to a large extent for the models formulated in $\eta$, when comparing to third-order Stokes wave theory. Models formulated in $\phi$ over-predict the amplitude dispersion in shallow and intermediate water, while the amplitude dispersion decays towards zero in deep water.

The surface roller breaking formulation of Chapter 4 is incorporated into fully dispersive evolution equations, and an equation for the mean water level variation is derived. Model results are compared to a time domain Boussinesq formulation as well as a conventional breaking formulation. It turns out that the shape of the breaking waves is rather poorly described by the fully dispersive equations.

Chapter 7: Fully dispersive evolution equations, 3rd order  The reason for retaining only quadratic nonlinearity in evolution equations is the heavy computational cost of the
Executive Summary

associated convolution sums. While quadratic terms gives a computational work of $O(N^2)$ in each spatial step, cubic terms gives a work of $O(N^3)$. Using the FFT procedure, the quadratic terms can be calculated at $O(N \log N)$. This gives hope, that also cubic terms can be evaluated at this low cost. However, as just mentioned, for quadratic fully dispersive models, the exact kernel cannot be speeded up using FFT, and therefore an extension to cubic nonlinearity with fast calculation times is not possible. Nevertheless, a cubic extension to the fully dispersive quadratic model is derived in Chapter 7. The depth is assumed to vary slowly in space, at a scale similar to the wave nonlinearity, $h = h(\varepsilon x)$. The resulting equation can be regarded as a spatial Zakharov model for varying depth. At third order, the wave induced mean current and set-down interacts with the wave field. These interactions are included in the model. For one single periodic wave component the model is shown to coincide with the Nonlinear Schrödinger Equation. A numerical test of the spatial evolution of wave groups in a wave flume is presented. The test follows a test of Shemer et al. (2001). While qualitatively reproducing the results of Shemer et al., it appears that the phases of the results are erroneous. This failure is believed to be due to an error in the numerical implementation of the model.

Highlights of the thesis

The following five items are regarded to be the most important results of the thesis:

**Speeding up the calculation of the nonlinear terms using FFT** The application of FFT to speed up the calculation of the nonlinear terms from an $O(N^2)$-process to an $O(N \log N)$-process is regarded as a large step forward for evolution equations. This improvement certainly widens the application area of evolution equations, since spectra with a large number of frequencies can now be treated efficiently. The technique of using FFT is described in Section 3.2.

**Adaption of the surface roller breaking formulation** Although the results are not as promising as was hoped initially, the incorporation of the roller breaking formulation into evolution equations is an important conceptual step towards improving the description of breaking waves in evolution equations. With the roller formulation, the breaking process is described in a deterministic way in contrast to the conventional breaking formulations which simply damps the whole spectrum. The inclusion of variations of the mean water level is also an important extension in the surf zone. These two developments are described in Sections 4.3–4.4.

**Analysis of amplitude dispersion** The weakly nonlinear method of analyzing the amplitude dispersion in evolution equations is new and brings insight into the ‘mechanics’ of evolution equations. The finding that the amplitude dispersion is larger for the Boussinesq evolution equations of Madsen and Sørensen (1993) than for the corresponding time domain model is new. It is also new that fully dispersive evolution equations overpredict the amplitude dispersion severely. The method of weakly nonlinear analysis is
presented in Section 5.2 along with results for the evolution equations of Madsen and Sørensen (1993), while the analysis of fully dispersive evolution equations is given in Section 6.5.

**Fully dispersive evolution equations with exact second-order transfer functions**

The new derivation of fully dispersive evolution equations with no assumption of the vertical variation of the velocity field is an improvement of the models of Agnon et al. (1993), Kaihatu and Kirby (1995) and Eldeberky and Madsen (1999). For the first time, evolution equations having exact second-order transfer functions are presented. The derivation is given in Section 6.2.

**A cubic model on mildly sloping sea bed**

The cubic model, being valid on a mildly sloping bottom, can be seen as a spatial version of the Zakharov equation. Thus this model is an interesting theoretical development, since the Zakharov model is only valid on constant depth. The derivation of the model is given in Sections 7.1–7.4.
Executive Summary
Synopsis

Deterministisk modellering af vandbølger i frekvensdomænet

Formålet med dette arbejde er at udvide anvendelsesområdet for deterministiske udviklingsligninger for interaktion mellem vandbølger. En række aspekter angående udviklingsligninger behandles.


Det vises hvorledes beregningen af de ikke-lineære led kan effektiviseres ved brug af Fast Fourier Transformation (FFT). Herved reduceres beregningsarbejdet fra $O(N^2)$ til $O(N \log N)$, hvor $N$ er antallet af frekvenser. Den praktiske beregnings effektivitet af udviklingsligninger med denne forbedring undersøges ved at sammenligne CPU-tider mellem en tidsdomæne Boussinesq model og de tilhørende udviklingsligninger. For hver model konstrueres en kurve, der afbilder nøjagtighed mod CPU-tid. Ved sammenligning af disse kurver findes det, at udviklingsligninger er omtrent 1000 gange hurtigere for en test af regelmæssige bølger, og omtrent 100 gange hurtigere for en test af uregelmæssige bølger. Grunden til den mindre effektivitet af tidsdomenemodellen er at denne er nødt til at oplose den fulde fasevariation af alle bølger. Ydermere er det nødvendigt at bruge et meget mindre tidsskridt end hvad der ville svare til at beskrive den højeste frekvens af interesse ved Nyquist frekvensen. For den regelmæssige bølgetest bidrager opvarmningstiden for det numeriske domæne også i væsentlig grad til beregningstiden.

Med det formål at forbedre beskrivelsen af brydning i udviklingsligninger, tilpasses den såkaldte roller model, kendt fra tidsdomæne Boussinesq formuleringer til udviklingsligninger. Til dette formål modificeres roller formuleringen, så den baserer sig på lokale tidsserier. Der udledes også en ligning for variationen af middelvandspejlet. Resultater for ‘spilling breaker’


Det viser sig at amplitudedispersionen for de fuldt dispersive udviklingsligninger er overestimeret i stor grad for modellerne formuleret i $\eta$. Modellerne formuleret i $\phi$ overestimerer amplitudedispersionen på lavt og mellemdybt vand og underestimerer amplitudedispersionen på dybt vand. Roller brydningsmodellen inkluderer de fuldt dispersive udviklingsligninger. Modelresultater sammenlignes med en tidsdomæne Boussinesqformulering og en konventionel brydningsformulering. Det viser sig at formen af brydende bølger er relativt ringe beskrevet.


Indholdet af de enkelte kapitler er opremset i det engelske resumé, side iii, og de vigtigste resultater af arbejdet er udpeget på side vi. Sidstnævnte afsnit kan tjene som en hjælp til læsningen af afhandlingen.
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<td>time average of variable $q$ over one wave period</td>
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Chapter 1

Introduction

This dissertation is about deterministic evolution equations for water waves.

Starting from a time domain formulation of the governing equations, evolution equations can be obtained by assuming time periodicity. Hereby, the flow variables are expanded into Fourier series in time with spatially varying coefficients. The time coordinate is thus eliminated from the problem and a system of partial differential equations in the spatial variation of the Fourier coefficients is obtained. The removal of the time coordinate changes the structure of the mathematical problem significantly. With the time coordinate preserved, the problem is hyperbolic in time, allowing a time-stepping solution procedure. When time is removed, the problem is divided into a set of coupled problems, elliptic in space. Usually, the depth as well as the Fourier coefficients are assumed to be slowly varying in space. Hence higher-order derivatives of these quantities can be neglected and the problem can be reduced to a set of parabolic equations in space. This allows for the application of a space-marching integration procedure to solve the problem.

In this introductory section, different aspects of evolution equations are described. The motivation for studying evolution equations is given, and through an example, the method of deriving evolution equations is described. This provides an initial insight into the field of evolution equations. In the following sections, some of the early literature on evolution equations is outlined. Further, different extensions of evolution equations are discussed (breaking, currents, 2D-wave fields) and an account for evolution equations at deep water is given.

1.1 Why evolution equations?

For more than 30 years, a number of numerical methods for calculation of nonlinear water waves have been available. Stream function theory for regular waves on constant depth
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(Chappelear, 1961; Dean, 1965; Rienecker and Fenton, 1981; Fenton, 1988) provides a fully nonlinear solution to the water wave problem at all depths. Boundary integral element methods provide exact time stepping for any irrotational wave field taking full nonlinearity into account (Svendsen, 1971; Longuet-Higgins and Cokelet, 1976; Dold and Peregrine, 1986; Grilli et al., 2001). Volume of fluid methods make it possible to solve free surface flows with a Navier Stokes solver, including drop ejection, vorticity and turbulence (Muzaferija et al., 1998; Christensen, 1998; Lin and Liu, 1998; Mayer and Madsen, 2000). Hence, accurate methods for the fully nonlinear wave problem are available describing the flow in great detail.

However, these accurate methods are still computationally expensive. Usually only a few wave lengths are resolved, and the calculation times are rather large. For engineering purposes, faster tools are needed. Also, within engineering applications, the full details of the flow are not the objective of the simulations. A less accurate solution is acceptable, if it can be obtained fast.

In the last 10–15 years, the field of Boussinesq methods has extended rapidly. Boussinesq formulations are time domain models, derived by expanding the governing equations from the shallow water limit. In this process, the vertical coordinate is eliminated, such that a three dimensional problem can be solved on a 2D horizontal grid. Usually, the models have been confined to small depths and weak nonlinearity (see e.g., Peregrine, 1967), but now fully nonlinear models are available, being accurate to dimensionless wave numbers of $kh = 30$ both linearly and nonlinearly (see Madsen et al., 2002b). These models are faster than the accurate models mentioned above, but still quite expensive as an engineering tool for large areas. In larger model systems, coupling the coastal wave field with a sediment transport module and morphological evolution, time domain Boussinesq models have been claimed to be too slow to provide an alternative to the linear wave models usually used.

In cases like this, evolution equations may be a relevant alternative. Due to their parabolic nature, they are solved rather fast, and only the upwave and lateral boundary conditions need to be considered. The built-in assumption of periodic flow and one-way transfer of the wave field from the off-shore boundary eliminates the warm-up time associated with the time domain models. Further, the boundary condition needed: a wave spectrum with phases, is almost always available — at least with random phases. The disadvantages of the method is that (usually) only quadratic nonlinearity is retained in the models and that reflections cannot be described. Thus, if the bathymetry does not give rise to (too strong) reflections and the wave field can be considered to be only moderately nonlinear, evolution equations can be used. In Chapter 3 two examples of the computational efficiency are given.

1.2 How to derive evolution equations

As an introduction to the methods and ideas within evolution equations, we here derive evolution equations for the Korteveg-de Vries equation on constant depth. This equation
reads
\[ \eta_t + \sqrt{gh} \eta_x + \frac{3}{2} \sqrt{gh} \eta_x^2 + \frac{1}{6} h^2 \sqrt{gh} \eta_{xxx} = 0, \]
(1.1)
where \((x, t)\) are the spatial and temporal coordinates, \(\eta\) the free surface elevation measured upwards from the still water level, \(g\) the acceleration of gravity and \(h\) the still water depth.

**Expanding the flow variables**  We expand \(\eta\) in the following Fourier series
\[
\eta(x, t) = \sum_{p=-N}^{N} \hat{\eta}_p(x) e^{i \omega_p t} = \sum_{p=-N}^{N} a_p(x) e^{i(\omega_p t - k_p x)},
\]
(1.2)
Here \(\omega_p = p \omega\), \(\omega\) being the angular frequency of the lowest frequency in the series and \(k_p\) is the free wave number of the model for frequency \(p\). That is, \((\omega_p, k_p)\) satisfy the linear dispersion relation of (1.1). The wave amplitudes \(\hat{\eta}_p\) are complex numbers, holding information on both amplitude and phase of each frequency component. For \(\eta(x, t)\) to be real we require
\[
\hat{\eta}_{-p} = \hat{\eta}_p^*, \quad \hat{\eta}_0 \in \mathbb{R}, \quad k_{-p} = -k_p, \quad \omega_{-p} = \omega_p \quad (1.3)
\]
and similarly \(a_{-p} = a_p^*, \ a_0 \in \mathbb{R}\).

The first form of the expansion in (1.2) is a pure Fourier series in time, the Fourier coefficients being functions of space. The number of frequencies is \(N\) plus the zeroth frequency. In the second summation, the linear spatial variation of \(\hat{\eta}_p\) has been written explicitly in the argument of the exponential function. In this way, the linear spatial variation of the wave field is taken out of the solution variable \(\hat{\eta}_p(x)\). If the nonlinearity is weak, \(a_p(x)\) can therefore be expected to vary slowly in space. Note that the last sum of (1.2) is not a Fourier series in space, since the free wave numbers \(k_p\) do not have fixed increments.

**Inserting the expansion**  For inserting (1.2) into the linear terms of (1.1), the spatial and temporal derivatives of \(\eta\) must be expressed in terms of the expansion. As examples we get
\[
\eta_t = \sum_{p=-N}^{N} [i \omega_p a_p] e^{i(\omega_p t - k_p x)},
\]
(1.4)
\[
\eta_{xxx} = \sum_{p=-N}^{N} [a_{p,xxx} - 3i k_p a_{p,xx} - 3k^2_p a_{p,x} + i k^3_p a_p] e^{i(\omega_p t - k_p x)}.
\]
(1.5)
The linear part of the KdV-equation can therefore be written
\[
\sum_{p=-N}^{N} \left[ \frac{1}{6} h^2 \sqrt{gh} a_{p,xxx} - \frac{1}{2} i k_p h^2 \sqrt{gh} a_{p,xx} + \sqrt{gh} (1 - \frac{1}{2} k^2_p h^2) a_{p,x} + i (\omega_p - k_p \sqrt{gh} + \frac{1}{6} \sqrt{gh} k^3_p h^2) a_p \right] e^{i(\omega_p t - k_p x)} = 0.
\]
(1.6)
For the nonlinear term, we use the results of Appendix A. We neglect the spatial derivative of \( a_p(x) \) in the nonlinear term, thus only expressing \( \eta_x \) in terms of the fast variation of the exponential function. We hereby get

\[
\frac{3}{2} \sqrt{g/h} \eta_x = \frac{3}{2} \sqrt{g/h} \sum_{p=-2N}^{2N} e^{i p \omega t} \sum_{s=-p-N}^{N} (-i k_s) a_s a_{p-s} e^{-i (k_s + k_{p-s}) x}.
\] (1.7)

Substitution of \( s = p - u \) in the inner sum and later substituting \( u = s \), yields an expression similar to the above, except for the change of \( k_s \) to \( k_{p-s} \) in the inner sum. We can therefore make the sum symmetric by taking half of each form of the sum. All together, the KdV equation then takes the form

\[
\sum_{p=-N}^{N} \left[ \frac{1}{6} h^2 \sqrt{g h a_{p,xxx}} - \frac{1}{2} i k_p h^2 \sqrt{g h a_{p,xx}} + \sqrt{g h} (1 - \frac{1}{2} k_p h^2) a_{p,x} \right] e^{i (p \omega t - k_p x)} \\
+ i (\omega_p - k_p \sqrt{g h} + \frac{1}{6} \sqrt{g h k_p^3 h^2}) a_p \\
+ \frac{3}{4} \sqrt{g/h} \sum_{p=-2N}^{2N} e^{i (p \omega t - k_p x)} \sum_{s=-p-N}^{N} (-i (k_s + k_{p-s})) a_s a_{p-s} e^{-i (k_s + k_{p-s} - k_p) x} = 0.
\] (1.8)

**Separating the frequencies** Equation (1.8) is a polynomial in \( e^{i \omega t} \). Hence to be valid for all times, we require that the coefficient to each power of \( e^{i \omega t} \) is zero. For the powers \( p = \pm \{N + 1, \ldots, 2N\} \), we assume that the number of frequencies is high enough to capture all contributions from the nonlinear terms. Hence these powers are automatically zero. For the other frequencies we get

\[
\frac{1}{6} h^2 \sqrt{g h a_{p,xxx}} - \frac{1}{2} i k_p h^2 \sqrt{g h a_{p,xx}} + \sqrt{g h} (1 - \frac{1}{2} k_p h^2) a_{p,x} \\
+ i (\omega_p - k_p \sqrt{g h} + \frac{1}{6} \sqrt{g h k_p^3 h^2}) a_p = -\frac{3}{4} \sqrt{g/h} \sum_{s=-N}^{N} (-i (k_s + k_{p-s})) a_s a_{p-s} e^{-i (k_s + k_{p-s} - k_p) x} \\
p = 1, \ldots, N
\] (1.9)

**The parabolic approximation** The system (1.9) provides a set of elliptic problems, coupled through the nonlinear term. By the assumption of weak nonlinearity, the wave amplitudes \( a(x) \) are assumed to vary slowly in space, that is \( a(x) = a(\delta x) \). Here \( \delta \) is a small quantity, and it follows that \( (\partial^p/\partial x^p) a_p = O(\delta^n) \). Neglecting all but the first power of \( \delta \) corresponds to neglecting all but first-order derivatives of \( a_p(x) \). This leaves us with a set of first-order coupled differential equations for \( a_p(x) \), parabolic in space. Defining

\[
\beta_{0,p} = \omega_p - k_p \sqrt{g h} + \frac{1}{6} \sqrt{g h k_p^3 h^2}
\]

\[
\beta_{1,p} = \sqrt{g h} (1 - \frac{1}{2} k_p^2 h^2)
\] (1.10) (1.11)
the evolution equations therefore read

\[
\beta_{1,p} a_{p,x} + i \beta_{0,p} a_{p} = + i \frac{3}{4} \sqrt{g/h} \sum_{s=p-N}^{N} (k_s + k_{p-s}) a_{s} a_{p-s} e^{-i(k_s+k_{p-s}-k_p)x} \\
p = 1, \ldots, N. \tag{1.12}
\]

The equation defined by \( \beta_{0,p} = 0 \) is the dispersion relation for frequency \( p \) and is satisfied when \( k_p \) is a free wave number. The coefficient \( \beta_1 \) is the group velocity of the KdV-equation at frequency \( p \). Choosing \( k_p \) as free wave numbers, we can write up the final result for the evolution equations as

\[
a_{p,x} = \frac{i}{\beta_{1,p}} \frac{3}{4} \sqrt{g/h} \sum_{s=p-N}^{N} (k_s + k_{p-s}) a_{s} a_{p-s} e^{-i(k_s+k_{p-s}-k_p)x} \\
p = 1, \ldots, N. \tag{1.13}
\]

Given a set of initial wave amplitudes, this set of equations is easily integrated in space, thus predicting the spatial evolution of the wave spectrum.

This example illustrates the ideas and concepts involved in the theory of evolution equations. Derivation of evolution equations involves 1) choice of governing time domain formulation, 2) insertion of Fourier series, 3) assumption of weak nonlinearity (otherwise more than quadratic terms should be included), and 4) assumption of slowly varying wave amplitudes (allowing for the parabolic approximation).

It should be noted, that the above model is not suited for any practical calculations beyond \( kh = \sqrt{2} \). The reason is the linear dispersion relation of the KdV-equation, which approximates the exact dispersion relation \( \omega \sqrt{h/g} = (kh \tanh kh)^{1/2} \) with the polynomial \( \omega \sqrt{h/g} = kh(1 - \frac{1}{6}(kh^2)) \). At \( kh = \sqrt{2} \), \( \partial \omega / \partial k = \beta_1 \) is zero, thus giving a singularity in the right hand side of (1.13).

In the next section, we outline some of the pioneering literature on evolution equations.

### 1.3 Early work in the field of evolution equations

The first appearances of evolution equations were associated with investigations of the physical mechanism of near resonant interaction of waves at shallow water. The theory of nonlinear energy exchange among deep-water waves had been developed by Phillips and Longuet-Higgins among others in the late fifties and sixties (see Phillips, 1977). Now these ideas were extended to shallow water, to explain phenomena like second-harmonic generation. In this context, the principle of near-resonant interaction was developed. In the following we shall focus on two papers dealing with this phenomenon, Mei and Ünlüata (1972) and Bryant (1973).
1.3.1 Mei and Ünlüata (1972)

Mei and Ünlüata (1972) took basis in observations of (among others) Goda (1967) on second-harmonic generation in a shallow water wave flume. This paper reported the appearance of secondary wave crests at down-wave stations in the flume, when the amplitude of the wave maker was sufficiently large. The secondary crests moved with a smaller phase speed than the primary wave, and their strength varied in an oscillatory pattern along the tank length.

Mei and Ünlüata analysed this phenomenon using the Boussinesq equations of Peregrine (1967)

\begin{align}
\eta_t + (\eta U)_x + h U_x &= 0 \quad \text{(1.14a)} \\
U_t + g \eta_x + \frac{1}{2}(U^2)_x - \frac{1}{3}h^2 U_{xxt} &= 0. \quad \text{(1.14b)}
\end{align}

Here, \( U \) is the depth averaged velocity, i.e., \( U = \int_{-h}^{\eta} u(z) \, dz / (h + \eta) \), defined in terms of the horizontal particle velocity \( u \). The Boussinesq equations (1.14) were combined into one equation

\begin{equation}
U_{xx} - \frac{1}{gh} U_{tt} + \frac{1}{3} h U_{xxtt} = \frac{1}{gh} \left( \frac{1}{2} (U^2)_{xt} - g(\eta U)_{xx} \right) \quad \text{(1.15)}
\end{equation}

and two analyses were presented. We here repeat the results of Mei and Ünlüata in dimensional form, rather than the nondimensional form of the original paper.

**Regular perturbation analysis**  Assuming small flap motion and thus weakly nonlinear waves, the depth averaged velocity and the free surface elevation were expanded as

\begin{align}
\eta(x, t) &= \varepsilon \eta_0(x, t) + \varepsilon^2 \eta_1(x, t) \\
U(x, t) &= \varepsilon U_0(x, t) + \varepsilon^2 U_1(x, t). \quad \text{(1.16)}
\end{align}

Here \( \varepsilon \) is a small parameter measuring the nonlinearity. To analyze the observations from the laboratory, a wave flap with the motion

\begin{equation}
x_b(t) = \varepsilon b e^{i \omega t} + \text{c.c.} \quad \text{(1.17)}
\end{equation}

was introduced. Expanding around \( x = 0 \) gave the boundary condition

\begin{equation}
U(x_b, t) = U(0, t) + x_b U_x(0, t) = \frac{dx_b}{dt}. \quad \text{(1.18)}
\end{equation}

The first-order solution is

\begin{align}
u_0 &= i b \omega e^{i(\omega t - kx)} + \text{c.c.} \\
\eta_0 &= a e^{i(\omega t - kx)} + \text{c.c.} \quad \text{(1.19)}
\end{align}

where \( a \) is related to the flap motion by the relation \( a = i b k h \). Here \( k \) is the linear wave number for the frequency \( \omega \). For later use, we define the dispersion operator \( D(\omega, k) \) as

\begin{equation}
D(\omega, k) \equiv -k^2 + \frac{1}{gh} \omega^2 (1 + \frac{1}{3} k^2 h^2) \quad \text{(1.20)}
\end{equation}
1.3 Early work in the field of evolution equations

and \( k \) is then the positive solution of \( D(\omega, k) = 0 \).

At second order, (1.18) reads

\[
    u_1(0, t) = -x_b u_{0,x} = -\frac{2\omega}{kh^2}|a|^2 + \left[ \frac{\omega}{kh^2} a^2 e^{2i\omega t} + c.c. \right].
\]

(1.21)

This should be satisfied by the sum of the inhomogeneous solution for \( u_1 \) forced through the nonlinear terms in (1.15), the homogeneous solution \( u_1 = A e^{i(2\omega t - k_2 x)} \) and a constant independent of space and time. Matching with (1.21) gives

\[
    u_1(x, t) = -2 \frac{\omega}{kh^2}|a|^2 + \left[ \frac{2\omega^3/(gk) + 4\omega k h}{h^3 D(2\omega, 2k)} a^2 e^{2i(\omega t - kx)} \right.
    + \left( \frac{\omega}{kh^2} - \frac{2\omega^3/(gk) + 4\omega k}{h^3 D(2\omega, 2k)} \right) a^2 e^{i(2\omega t - k_2 x)} + c.c. \right] .
\]

(1.22)

The first term in each line arises from the flap motion, imposed through (1.21). The exponential term in the first line is the inhomogeneous solution, forced by the linear solution through the nonlinear terms of (1.15), while the second term in the second line makes this vanish at the wave maker (in \( x = 0 \)), since the flap motion (1.17) does not impose any second-order wave motion. If we disregard the details of the flap motion, the second-order part of \( u_1 \) can be written as

\[
    \hat{u}_1(x, t) = \frac{2\omega^3/(gk) + 4\omega k}{h^3 D(2\omega, 2k)} a^2 e^{i(2\omega t - k_2 x)} \left( e^{i(k_2 - 2k)x} - 1 \right) + c.c.
\]

(1.23)

This describes a free wave with periodically varying amplitude. The period of the oscillation is \( 2\pi/(k_2 - 2k_1) \). The quantity \( k_2 - 2k_1 \) is denoted the phase mismatch and refers to the difference in wave number between a bound second-harmonic wave and a free wave with the same period. The result (1.23) therefore explains the laboratory observations of secondary crests with periodically varying amplitudes away from the wave maker.

The regular perturbation solution provides insight into the mechanism of second-harmonic generation. The second-order bound solution emerges from the nonlinearity in the water wave problem, and to obtain the linear motion at the wave flap, a free wave with the same frequency is added. The difference in wave length between these two contributions results in an oscillating amplitude of the second-order wave field. One thing (among others) is however not explained by this theory: from where does the energy flux associated with the second-order wave field come? This answer is provided by the second analysis of Mei and Ünlüata (1972), utilizing the technique of multiple scales. This analysis leads to a set of evolution equations.

Multiple scales analysis In the multiple scale analysis, the modulation of the free wave mode amplitudes due to the nonlinear terms is studied. Mei and Ünlüata introduced a slowly varying spatial variable, \( X = \varepsilon x \), and the expansions

\[
    U(x, t) = \varepsilon u_0(x, X, t) + \varepsilon^2 u_1(x, X, t) \quad , \quad \eta(x, t) = \varepsilon \eta_0(x, X, t) + \varepsilon^2 \eta_1(x, X, t).
\]

(1.24)
The slow and fast x-variables are now treated independently. To zeroth order, one gets the linearized form of (1.15) for \( u_0 \). To study the interaction of two harmonics, Mei and Ünlüata picked the solution

\[
\eta_0 = a_1(X)e^{i(\omega_1 t - k_1 x)} + a_2(X)e^{i(\omega_2 t - k_2 x)} \quad \text{(1.25a)}
\]

\[
u_0 = \frac{\omega_1}{k_1 h} a_1 e^{i(\omega_1 t - k_1 x)} + \frac{\omega_2}{k_2 h} a_2 e^{i(\omega_2 t - k_2 x)} \quad \text{(1.25b)}
\]

where \( \omega_2 = 2\omega_1 \) and \((k_1, k_2)\) are the free wave numbers of \((\omega_1, \omega_2)\). Note here that contrary to the regular perturbation analysis, no assumptions of the relative magnitude of the harmonics is made. The second harmonic can be just as large, or even larger than the first harmonic.

To second order, (1.15) reads

\[
u_{1,xx} - \frac{1}{gh} \nu_{1,tt} + \frac{1}{3} \nu_{1,xtt} = -2(1 - \frac{1}{3} h \partial_t) u_{0,xx} + \frac{1}{gh} \left( \frac{1}{2} (u_0^2)_{xt} - g (\eta_0 u_0)_{xt} \right). \quad \text{(1.26)}
\]

The right-hand side can be calculated explicitly from the linear solution (1.25). This gives

\[
\text{RHS} = 2 \frac{\omega_1}{h} \left( 1 - \frac{1}{3} \frac{h}{\omega_1} \right) \frac{da_1}{dx} e^{i(\omega_1 t - k_1 x)} + 2 \frac{\omega_2}{h} \left( 1 - \frac{1}{3} \frac{h}{\omega_2} \right) \frac{da_2}{dx} e^{i(\omega_2 t - k_2 x)}
\]

\[
+ \frac{1}{gh} \left[ \left( \frac{\omega_1^2}{k_1 h^2} \omega_1 + \frac{4k_1 \omega_1}{h} \right) a_1^2 e^{2i(\omega_1 t - k_1 x)} \right. \\
+ \left. \left( \frac{\omega_1 \omega_2}{k_1 k_2 h^2} \omega_1 (k_2 - k_1) + g \left( \frac{\omega_1}{k_1 h} + \frac{\omega_2}{k_2 h} \right) (k_2 - k_1) \right)^2 \right] a_2 a_1^* e^{i(\omega_1 t - (k_2 - k_1) x)} + \text{c.c.} \quad \text{(1.27)}
\]

Here, contributions to other frequencies than \( \pm \{\omega_1, \omega_2\} \) from the nonlinear terms have been left out. We now observe that the linear operator on the left hand side of (1.26) working on \( u_1 \) is the same as the linear operator governing \( u_0 \). Further, the right hand side contains terms proportional to the homogeneous solutions to this operator. Before eliminating these terms, we discuss the similarity with the homogeneous solution in more detail. The proportionality is clearly seen for the first two terms involving derivatives of the wave amplitudes \((a_1, a_2)\). The nonlinear terms are not directly proportional to the homogeneous solution, but nearly resonant with the homogeneous solution.

The latter point can be realized by inspection of the linear dispersion relation, (1.20). Shallow water corresponds to small wave numbers. Here, there is almost proportionality between \( \omega \) and \( k \), and therefore, the difference between the bound wave number \( 2k_1 \) and the free wave number \( k_2 \) of the same frequency is small. This is depicted in Figure 1.1, where both the dispersion relation (1.20) and the exact dispersion relation \( \omega^2 = gh \tanh kh \) are plotted. Writing the phasemismatch as \( \delta = k_2 - 2k_1 \), we see that the exponential functions at the right hand side (1.27) can be rewritten in the following way

\[
e^{2i(\omega_1 t - k_1 x)} = e^{i(\omega_2 t - k_2 x)} e^{i\delta x} \quad \text{(1.28)}
\]

\[
e^{i(\omega_1 t - (k_2 - k_1) x)} = e^{i(\omega_1 t - k_1 x)} e^{-i\delta x}. \quad \text{(1.29)}
\]
1.3 Early work in the field of evolution equations

Figure 1.1: The principle of near resonance for small wave numbers. The solid line is the dispersion relation (1.20), while the dashed line is the exact dispersion relation \( \omega^2 = gk \tanh kh \).

Since \( \delta \) is small, these forcing terms are nearly resonant with the natural wave modes of the linearized equation.

To eliminate this resonance, we require that the resonant terms cancel out. We collect terms having the same time variation in the exponential functions, and hereby obtain two differential equations for the wave amplitudes \((a_1, a_2)\):

\[
\frac{da_1}{dX} = iS_1 a_2 a_1^* e^{i(2k_1 - k_2)x} \tag{1.30a}
\]
\[
\frac{da_2}{dX} = iS_2 a_1^2 e^{-i(2k_1 - k_2)x} \tag{1.30b}
\]

with

\[
S_1 = \frac{1}{gh} \left[ \frac{\omega_1 \omega_2}{k_1 k_2 h^2} \omega_1 (k_2 - k_1) + g \left( \frac{\omega_1}{k_1 h} + \frac{\omega_2}{k_2 h} \right) (k_2 - k_1)^2 \right] \left[ \frac{2 \omega_1}{h} \left( 1 - \frac{1}{3} \frac{h}{g} \omega_1^2 \right) \right]^{-1} \tag{1.30c}
\]
\[
S_2 = \frac{1}{gh} \left[ \frac{\omega_1^2}{k_1 h^2} \omega_1 + g \frac{4k_1 \omega_1}{h} \right] \left[ \frac{2 \omega_2}{h} \left( 1 - \frac{1}{3} \frac{h}{g} \omega_2^2 \right) \right]^{-1} \tag{1.30d}
\]

This result differs from the result of Mei and Ünliiata in two points. First, Mei and Ünliiata has an extra factor of 2 in the second term of \( S_1 \) and the first term of \( S_2 \). Second, the sign of the right-hand sides are opposite to the results of the paper, because we use the phase function \( \theta = \omega t - kx \), having the opposite sign of the phase function of Mei and Ünliiata. The first discrepancy has been pursued by comparing the intermediate result for the right hand
side (1.27) to the result of Dingemans (1997, equation (7.24)). Besides the dimensional form of the present results, the expressions agree perfectly. The mismatch is therefore regarded as a misprint in Mei and Unluata (1972).

The system (1.30) governs the slow modulation of the free wave amplitudes \( (a_1, a_2) \) due to their nonlinear interaction with each other. Contrary to the regular perturbation analysis, both harmonics are now affected by the interaction. This is due to the removal of the assumption \( a_2 \ll a_1 \) of the regular perturbation analysis.

Further properties of the two-harmonic system  

The coupled system of two harmonics can be solved analytically in terms of Jacobian elliptic functions. This is summarized briefly in Mei and Unluata (1972), referencing the work of Armstrong et al. (1962) within nonlinear optics. Similar calculations can be looked up in Mei (1983); Bryant (1973) and Dingemans (1997). With a bit of algebra, further insight into the nature of the interaction system can be gained. Upon multiplication of the complex conjugate of (1.30a) with \( S_2 a_1 \) and addition of \( S_1 a_2^* \) times (1.30b), one obtains

\[
S_2 a_1 a_1^{*x} + S_1 a_2 a_2^{*x} = 0. 
\]

Adding the complex conjugate of this equation and integrating once in space yields

\[
S_2 |a_1|^2 + S_1 |a_2|^2 = \text{const}. 
\]

Within nonlinear optics this equation is called the Manley-Rowe equation. The pair of amplitudes \( (|a_1|, |a_2|) \) forms an ellipse in the plane. Clearly, the gain in amplitude of one harmonic results in decaying amplitude of the other harmonic, the rate of change being governed by \( S_1 \) and \( S_2 \).

The analysis of Mei and Unluata gives insight in the physics of harmonic generation as well as the mathematical treatment of the wave interactions. Near-resonant wave interaction in shallow water is also denoted ‘tripad interaction’. The two-harmonic system is an example of a set of evolution equations for free wave amplitudes. The derivation is easily extended to include more than two harmonics. The method can be viewed as a variant of the more direct derivation method demonstrated in Section 1.2. The two methods give identical results.

1.3.2 Bryant (1973)

Another early paper in the field of evolution equations is the paper of Bryant (1973). Here, three different wave models and their evolution equations were examined. The first model is the KdV-equation, as given by (1.1). The second is the BBM-equation

\[
\eta_t + \sqrt{gh} \eta_x + \frac{3}{2} \sqrt{gh} \eta \eta_x - \frac{1}{6} h^2 \eta_{xxx} = 0, 
\]
1.3 Early work in the field of evolution equations

(Benjamin et al., 1972), a variant of the KdV-equation. It can be derived from the KdV-equation by utilizing the nondispersive, linear relation implied by the two first terms, to manipulate the dispersive term. This is consistent with the accuracy of the KdV-equation. The BBM-equation turns out to have much better dispersion properties than the KdV-equation and the problem of zero group velocity at \( kh = \sqrt{2} \) does not occur for the BBM-equation. The third model is the fully dispersive water wave problem, i.e., the Laplace equation for the velocity potential with the bottom boundary condition and the kinematic and dynamic boundary conditions at the free surface.

Bryant (1973) derived evolution equations for the time dependence of the spatial Fourier coefficients. In the dimensionless variables used, the expansion used is

\[
\eta(x, t) = \frac{1}{2} \sum_{k=1}^{\infty} B_k(t)e^{i(kx-\omega_k t)} + \text{c.c.} \equiv \frac{1}{2} \sum_{k=1}^{\infty} A_k(t)e^{ik(x-t)} + \text{c.c.} \tag{1.34}
\]

This differs from the expansion (1.2) used in Section 1.2. The Fourier expansion is now performed in the spatial domain, thus assuming spatial periodicity. The frequencies \( \omega_k \) are the solutions of the dispersion relation for the evenly spaced wave numbers \( k \). The Fourier coefficients are constant in space, but allowed to vary in time. Hence, the evolution equations derived, describes the temporal evolution of each linear wave mode in the wave field.

Utilizing the above expansions, each wave model was recast into a set of evolution equations of the form

\[
\frac{dA_k}{dt} = iP_k A_k - i \sum_{l=1}^{\frac{1}{2}(k-1)} \frac{1}{2^k} S_{kl} A_l A_{k-l} - i \sum_{l=1}^{\infty} R_{kl} A_l^* A_{k+l}. \tag{1.35}
\]

The coefficients \( (P_k, S_{kl}, R_{kl}) \) are functions of the wave numbers and frequencies, and vary from model to model. Bryant compared the different set of coefficients, and presented analytical solutions for the interaction of two and three harmonics. The solution for two harmonics correspond to the regular perturbation approach of Mei and Ünlüata (1972). That is, the presence of a constant first harmonic forces an explicit solution of the second harmonic. For three harmonics, the solution consists of an explicit solution of the two-harmonic interaction problem in terms of Jacobian elliptic functions, while the third harmonic was determined as a regular perturbation solution forced by the explicit solution of the first and second harmonic. Bryant also considered the system of four harmonics interacting with each other. Using explicit solutions for the third and fourth harmonics, the governing equations for the first and second harmonics were expressed as a closed system in the first and second harmonic amplitudes. However, no analytical solution was found for this case.

Bryant also presented numerical results for solving the evolution equations. Focussing on harmonic generation, the parameter space of \( (\varepsilon, \mu) \) i.e., (nonlinearity,dispersion) was investigated, varying \( \mu \) for a fixed value of \( \varepsilon \). The results clearly show how the second-harmonic wave component is forced to oscillate in time between the initial value of zero and a finite amplitude. The smaller the dispersion, the larger the amplitude of the higher harmonics.
This is due to the approach of perfect resonance in the shallow water limit for decreasing values of \( \mu \). In the most resonant case considered, the second-harmonic wave obtained an amplitude of similar size as that of the first harmonic. Still a periodic recurrence in time was found.

### 1.4 Boussinesq evolution equations for triad interactions

Within the pioneering work of shallow water evolution equations, the scope was primarily to explain the phenomenon of harmonic generation and provide models that could predict the interaction of a primary wave and its higher harmonics. The practical use of evolution equations as a tool for calculating real wave fields was initiated by Freilich and Guza (1984), who took basis in the Boussinesq equations and applied the resulting evolution equations to model field data of wind waves. Here, 240 components in the spectrum were modelled, thereby moving from the controlled laboratory frame to a more engineering fashioned application.

After the work of Freilich and Guza, a number of authors have presented evolution equation models based on Boussinesq equations (Liu et al., 1985; Yoon and Liu, 1989; Madsen and Sørensen, 1993; Kaihatu and Kirby, 1998). Usually, the Boussinesq models suffered from poor linear dispersion properties for short waves. For example, the Boussinesq model (1.14) formulated in the depth-averaged velocity underpredicts the linear phase speed by 10% at \( kh = 2.59 \). However, inspired by the work of Witting (1984), Madsen et al. (1991) presented a Boussinesq model with improved linear dispersion. The new model incorporated Padé[2,2] dispersion, thus moving the 10% error limit to \( kh = 4.98 \). This motivated a number of new papers on evolution equations based on Boussinesq models with improved linear dispersion properties.

In this section we shall focus on a few of the papers mentioned, namely the papers of Freilich and Guza (1984), Madsen and Sørensen (1993), Chen and Liu (1995) and Kaihatu and Kirby (1998).

#### 1.4.1 Freilich and Guza (1984)

Freilich and Guza (1984) started from the Boussinesq equations formulated in the depth-averaged velocity potential. Elimination of \( \eta \) gave the wave equation

\[
\ddot{\phi}_{xtt} - h\ddot{\phi}_{xxx} - \frac{1}{2}\rho\alpha h^2\ddot{\phi}_{xxxxxt} + \frac{1}{2}\alpha(\ddot{\phi}_x^2)_{xt} - 2\alpha h_x\ddot{\phi}_{xx} - \alpha(\eta\ddot{\phi_x})_{xx} = 0, \tag{1.36}
\]
1.4 Boussinesq evolution equations for triad interactions

which by the definition $U \equiv \bar{\phi}_x$ is identical to (1.15). In the notation of Freilich and Guza, the variables have been normalized by the scaling

$$x' = l_0 x \quad t' = (l_0/\sqrt{g h_0}) t \quad \eta' = a_0 \eta \quad \bar{\phi}' = \frac{a_0}{h_0} l_0 \sqrt{g h_0} \bar{\phi}.$$  

Here the prime denotes physical variables, while the unprimed variables are dimensionless. $(a_0, l_0, h_0)$ are reference wave amplitudes, wave lengths and depths, respectively. The parameters $\alpha$ and $\rho$ are defined by

$$\alpha = \frac{a_0}{h_0} \quad \rho = \left[ \frac{a_0 l_0^2}{h_0^3} \right]^{-1}$$  

such that $\rho \alpha = (h_0/l_0)^2$ is a measure of the dispersion, while $\alpha$ is a measure of the nonlinearity. The potential and free surface elevation were expanded in the Fourier series

$$\bar{\phi}(x, t) = \sum_{n=1}^{\infty} Q_n(\alpha x) \sin(\Psi_n(x) - \omega_n t) \quad \eta(x, t) = \sum_{n=1}^{\infty} a_n(\alpha x) \cos(\Psi_n(x) - \omega_n t)$$  

with the phase function $\Psi$ defined by

$$\Psi_{n,x} = k_n + \alpha T_n(\alpha x)$$  

i.e., the local linear wave number plus a phase shift varying slowly in space, accounting for the phase changes not captured by the linear wave number. Also the depth was assumed to vary slowly in space, i.e., $h = h(\alpha x)$.

With basis in this expansion, two sets of evolution equations were derived. The so-called ‘consistent shoaling model’ took basis in (1.36), linearized by setting $\alpha = 0$. The resulting equation is a linear shallow water equation, with the nondispersive dispersion relation

$$k_n = \frac{\omega_n}{\sqrt{h}}.$$  

With this definition of $k_n$, the expansions (1.38) were inserted in the wave equation (1.36). At $O(\alpha)$ a set of first-order differential equations for $Q_n$ and a set of explicit expressions for $T_n$ emerged. The expression for $T_n$ corrected the nondispersive choice of $k_n$ in the phase function through (1.38b).

The so-called ‘dispersive shoaling model’ did not assume that the dispersive term was small compared to the lowest-order terms in the equations. The zeroth-order equations were therefore the linearized version of (1.36), i.e., with the dispersive term retained. This gives the dispersion relation

$$k^2 = \frac{\omega^2/h}{1 - \frac{1}{3} h \omega^2}.$$  

for determining the linear wave numbers. At second order, insertion of the expansion (1.38) gave a system of ODE’s for $Q_n$ and a set of explicit expressions for $T_n$, just as for the consistent model. In the dispersive shoaling model, the expressions for $T_n$ include only nonlinear terms, since all linear phase variation is incorporated directly in $k_n$ through the linear dispersion relation (1.40).

The two models were tested on data measured at the Torrey Pines Beach, California. At three different dates, time series at different depths were sampled. For each day, these were broken up into individual time series of 1024 s duration, and the corresponding Fourier coefficients calculated. The offshore values were used as initial conditions for the models, while the Fourier coefficients from stations towards the shore were used for comparison with the model results. 240 frequencies were used in the calculations, and the results of all the separated time series averaged, to yield the power spectrum in each station of the wave field. Also comparisons with linear wave theory applied to each single harmonic were made.

In general, Freilich and Guza (1984) found that the two nonlinear shoaling models were successfull in predicting the spectral evolution of the waves. For higher frequencies and low-energetic conditions, linear wave theory was somewhat better to predict the phases of the wave field, due to the limited dispersion properties of the underlying Boussinesq model. For low frequencies all models were good at predicting the phases. For the nonlinear spectral evolution, the models were able to predict the cross-spectral exchange of energy, which is caused by triad interactions. In the simulations, the spectral growth at high frequencies was usually overpredicted by the consistent shoaling model and underpredicted by the dispersive shoaling model. This deviation is due to the different choice of wave numbers altering the interaction coefficients in the nonlinear terms. For the consistent shoaling model, the interaction coefficients are based on nondispersive wave numbers, therefore effectively showing no depth-dependence of the nonlinear interactions.

### 1.4.2 Madsen and Sørensen (1993)

Madsen and Sørensen (1993) were the first to derive evolution equations incorporating Padé$[2,2]$-dispersion. In their paper, the constant depth Boussinesq model of Madsen et al. (1991) was extended to a mildly sloping bed. The dispersion relation of their model can be written as

$$\frac{\omega^2}{k^2 gh} = \frac{1 + B k^2 h^2}{1 + (B + \frac{1}{3})k^2 h^2}$$

which for $B = 1/15$ is the Padé$[2,2]$ expansion of the similar expression of linear Stokes waves, $\omega^2/(k^2 gh) = (\tanh kh)/kh$. The Boussinesq model was formulated with the free surface elevation $\eta$ and the depth averaged flux $P$ as dependent variables. To derive evolution equations, the expansions

$$\eta(x, t) = \sum_{p=-N}^{N} a_p(x) e^{i(\omega_p t - \int k_p dx)} \quad \quad P = \sum_{p=-N}^{N} \frac{\omega_p}{k_p} a_p(x) e^{i(\omega_p t - \int k_p dx)}$$


were inserted into the Boussinesq equations. In contrast to the approach of Freilich and Guza, the wave amplitudes $a_p(x)$ are here complex variables, thus holding information on both wave amplitude and phase. This eliminates the need for adding an extra phase correcting function $T_p(x)$ in the argument to the exponential function. Further, the choice of phase function satisfies the local condition $\eta_{p,x} = -ik_p\eta_p$ and thereby contains the total linear phase variation on a mildly sloping bed.

When the Fourier series (1.42) were inserted into the governing Boussinesq equations, a system of fourth-order ODE’s for $a_p(x)$ was obtained. Instead of just neglecting all but first derivatives of $a_p(x)$, Madsen and Sørensen chose to include also the second derivative of $a_p(x)$. The system was subsequently rewritten to a first-order system using an approximative integration procedure, also used by Bryant (1973). This procedure, however, was only used at constant depth. For varying depth, only the first derivative of $a_p(x)$ was included. We shall go more into detail with the model of Madsen and Sørensen (1993) in Chapter 2.

1.4.3 Chen and Liu (1995)

Chen and Liu (1995) derived evolution equations for two horizontal dimensions and varying depth based on the Boussinesq model of Nwogu (1993). Nwogu’s (1993) equations were rederived in terms of the velocity potential $\phi_\alpha$ at an arbitrary vertical level $z_\alpha$. For $z_\alpha/h = -0.53$, the model has Padé[2,2]-dispersion. Fourier series with complex wave amplitudes for the free surface elevation and $\phi_\alpha$ were now inserted into the governing equations, resulting in a set of fourth-order differential equations (involving both $x$-derivatives and $y$-derivatives) for $\phi_\alpha$. This system was reduced by assuming weak $y$-dependence of the wave amplitudes, thus neglecting $y$-derivatives higher than second order in $\phi_\alpha$. Further, each equation was splitted into two second-order equations in the $x$-direction, one for the propagating wave modes and one for the evanescent wave modes. Neglection of the evanescent wave modes rendered the problem as a set of second-order differential equations in the $x$-direction. Application of the splitting technique once again — now splitting between the forward and backward propagating waves — and further neglect of the backward propagating wave field lead to a set of first-order differential equations for the $x$-variation of $\phi_\alpha$.

Having reduced the initially fourth-order differential equations to a set of first-order differential equations, Chen and Liu (1995) modified the interaction coefficients of the nonlinear terms to be correct for both free and bound waves. The model was tested on the laboratory measurements of Whalin (1971) of weakly two-dimensional wave propagation over a semi-circular shoal.

The above model is only suited for wave propagation in primarily one direction. Chen and Liu wrote that for incident wave angles exceeding $30^\circ$, the approximation involved in neglecting the higher-order $y$-derivatives is invalid. Hence, for waves of arbitrary angles of incidence, an angular spectrum model was derived, valid for wave directions of $\pm90^\circ$ from the dominant wave direction. We describe this extension in more detail in Section 1.6.3.
A note on the splitting technique  The splitting technique applied by Chen and Liu (1995) was introduced in the water wave literature by Radder (1979). It has been used in optics (Corones, 1975) and acoustics (McDaniel, 1975) on similar problems. In this paragraph we will demonstrate that the splitting technique is indeed an approximate technique, even if the wave field consists of only forward propagating waves.

Radder (1979) applied the splitting technique to the mild slope equation (here in one dimension)

\[ \phi_{xx} + k^2 \phi = 0 \]  (1.43)

rewriting it to

\[ \frac{\partial}{\partial x} \left( \phi^+ - \phi^- \right) = \left[ ik - \frac{1}{2k} k_x \right] \left( \phi^+ - \phi^- \right) \]  (1.44)

where \( \phi^+ \) is denoted the potential of the forward propagating wavefield and \( \phi^- \) the potential of the backward propagating wavefield. In the splitting procedure, no approximations are made, and (1.44) can be brought back to (1.43) by differentiating once and utilizing \( \phi = \phi^+ + \phi^- \). Neglect of the backward propagating wavefield \( \phi^- \) leads to a first-order differential equation for \( \phi^+ \)

\[ \frac{\partial \phi^+}{\partial x} = \left( ik - \frac{1}{2k} k_x \right) \phi^+ . \]  (1.45)

Hence, if only forward propagating waves are present, (1.45) is expected to be exact.

To investigate this, we eliminate \( \phi^- \) from (1.44) to obtain

\[ \phi_{xx}^+ + 2i k_x \phi_x^+ - 2i k_x \phi^- - k_{xx} \phi_x^+ + ik \frac{k_{xx}}{k_x} \phi^- + k^2 \phi^+ = 0. \]  (1.46)

We now have three equations, describing the forward propagating wave field 1) the mild slope equation (1.43), 2) the splitted equation for the forward propagating wave field (1.45) and 3) a non-splitted equation for \( \phi^+ \) (1.46). We insert the solution

\[ \phi = \phi^+ = A(x) e^{i \int k(x) dx} \]  (1.47)

to obtain

from (1.43): \[ A_{xx} + 2i k A_x + i k_x A = 0 \]  (1.48)
from (1.45): \[ 2i k A_x^+ + i k_x A^+ = 0 \]  (1.49)
from (1.46): \[ A_{xx}^+ + 2i k A_x^+ + i k_x A^+ + \frac{k_{xx}}{k} A_x^+ - \frac{k_{xx}}{k_x} A_x^+ = 0 \]  (1.50)

If \((A, k)\) are assumed to be functions of a slow variable, i.e. \((A, k) = (A, k)(\varepsilon x)\), we see that the three equations for \( A \) are only identical in the \( O(1) \) and \( O(\varepsilon) \)-terms. The mild slope
equation gives the equation (1.43), which is regarded as the correct result. We use this as reference solution. The result obtained via the splitting method, (1.45), can be obtained by neglecting $A_{xx}$ in reference solution. Hence, the splitting method does not yield an exact result for the forward propagating wave field. Further, we see that the exact equation for $A^+$ (1.46) contains additional terms which are not present in the reference solution.

The reason for these discrepancies is that $\phi^+$ is not the true forward propagating wave mode. Similarly, $\phi^-$ is not the true backward propagating wave mode. Further details are given in Bredmose (1999).

It is hereby clear that the splitting method does not leave an exact equation for the forward propagating wave field. Further, the results are the same as can be obtained by inserting the WKB-expansion (1.47) into the mild slope equation and neglect higher-order derivatives of the envelope function $A$. Since this latter method is much easier to use, we recommend using this approach rather than the splitting technique.

1.4.4 Kaihatu and Kirby (1998)

Kaihatu and Kirby (1998) has presented a model of evolution equations for Nwogu’s equations as well. They manipulated Nwogu’s equations in the time domain to obtain a single equation having only $\eta$ as dependent variable in the linear terms. This wave equation retained the dispersion characteristics of Nwogu’s equations. Afterwards the equation was modified by an enhancement technique in order to improve the shoaling characteristics.

After insertion of the solution ansatz

$$\eta(x,t) = \frac{1}{2} \sum_{p=1}^{N} \tilde{\eta}_p(x,y)e^{-ip\omega t} + c.c. \quad (1.51)$$

$$u(x,t) = \frac{1}{2} \sum_{p=0}^{N} \tilde{u}_p(x,y)e^{-ip\omega t} + c.c. \quad (1.52)$$

$\tilde{u}$ was eliminated from the nonlinear terms, resulting in $N$ coupled differential equations for $\tilde{\eta}_p(x,y)$. The fast spatial variation of the wave field was taken out by substitution of

$$\tilde{\eta}_p(x,y) = a_p(x,y)e^{i \int k_p dx} \quad (1.53)$$

(in our nomenclature). The resulting equations contained fourth-order derivatives of $\hat{\eta}_p$ in both horizontal directions. These higher-order derivatives were neglected, assuming a slow variation of $\hat{\eta}_p$ in space. In the resulting equations the highest derivatives retained were $\hat{\eta}_{p,x}$ and $\hat{\eta}_{p,yy}$. This corresponds to the small-angle model of Chen and Liu (1995).

The linear terms of the model were compared against results of the mild slope equation for data of Berkhoff et al. (1982) over a sloping bed with a spherical hump focusing the waves.
A good match between the two models were found. Next, the model was tested on the data of Whalin (1971) and compared to results of the small angle model of Chen and Liu (1995) and the Kadomtsev-Petviashivilli model, which is a weakly two-dimensional KdV-equation. The deepest part of the basin had a depth of 0.45m, and experiments of $T=(1,2,3)$ s were carried out. To stay within the dispersion range of the model, only results for $T=(2,3)$ s were compared and relatively few harmonics were included (5 for $T=3$ s and 3 for $T=2$ s). For these comparisons, the new model as well as the small angle model of Chen and Liu (1995) improved the predictions of the Kadomtsev-Petviashivilli model. As this model retains only the lowest-order terms in an expansion from the shallow water limit, this is not a surprise.

### 1.5 Fully dispersive evolution equations for triad interactions

Derivation of evolution equations requires a wave model in which Fourier series for the flow variables can be inserted. Lower-order Boussinesq models are suited for this, since they are relatively simple and include a limited number of higher-order spatial derivatives. However, it is also possible to work directly from the Laplace equation with its boundary conditions at the bottom and free surface. On constant depth, temporal evolution equations are relatively easily derived, as we have already seen in Bryant (1973). The reason is that the resulting equations only include a second-order derivative in time, which is easily reduced to a system of first-order ODE’s in the time dependence of the spatial Fourier amplitudes. However, for varying depth, the interaction coefficients are not constant in space, and therefore one has to treat all spatial points separately at each frequency. This prevents the elegant formulation and solution technique of the constant depth case.

Therefore, for treating varying depth, temporal periodicity rather than spatial periodicity must be assumed, and one must solve for the spatial variation of the Fourier coefficients. Agnon et al. (1993) derived fully dispersive evolution equations from the Laplace equation. The model was formulated in the still water velocity potential $\Phi$. Kaihatu and Kirby (1995) have presented a similar model.

Both models were corrected by Eldeberky and Madsen (1999) with respect to the second-order terms in the transformation from $\Phi$ to the free surface elevation $\eta$. Kaihatu (2001) describes the improvement of this correction within the model of Kaihatu and Kirby (1995) and further compares wave profiles and phase speeds to stream function theory. However, it turns out that even after the correction, the second-order transfer functions do not match Stokes second-order theory exactly. We look further into fully dispersive models and their second-order transfer functions in Chapter 6.
1.6 Extensions of evolution equations

The evolution equation models presented so far are valid for weakly nonlinear waves in one horizontal dimension. They assume irrotational flow and do not allow for any currents or wave breaking. In this section we will mention examples of extension of the models to improve on these shortcomings. More specifically, we will describe the inclusion of currents and wave breaking. Further, the methods for modelling two-dimensional wave fields are described.

1.6.1 Wave breaking

Evolution equations are based on potential theory, and have therefore no chance of modelling any rotational flow, hereunder wave breaking, directly. Hence, breaking must be described in an empirical way. Usually, a simple damping term is added to the equations, giving the schematic representation

\[ a_{p,x} = \text{shoal + nonlinear} - d_p a_p \]  \hspace{1cm} (1.54)

where \(d_p\), the damping coefficient, is determined empirically. The terms ‘shoal’ and ‘nonlinear’ refer to the terms modelling linear shoaling and quadratic interactions. Mase and Kirby (1992) incorporated a breaking term of the above type into a KdV-based evolution equation model. The total amount of energy dissipation in each spatial point was determined using the empirical formula of Thornton and Guza (1983). When the total energy dissipation is known, it must be distributed among the wave components at different frequencies. A constant value of \(d_p\) corresponds to dissipating energy at each frequency proportionally to its energy content. This is seen by multiplying (1.54) with \(a_p^*\) and adding the complex conjugate. One then gets \(\frac{\partial}{\partial x} |a_p|^2 = -2d_p |a_p|^2\). From laboratory experiments, however, Mase and Kirby (1992) found that introducing an \(f_p^2\)-dependence of \(d_p\) gave the best agreement between the model results and the experimental data.

Eldeberky and Battjes (1996) incorporated a similar, but frequency-independent damping term into the evolution equations of Madsen and Sørensen (1993). The need for a frequency dependent damping coefficient was later investigated in detail by Chen et al. (1997), who found that the shape of the power spectrum is rather insensitive to the frequency dependence of \(d_p\), while higher-order measures such as skewness and asymmetry of the wave field are better reproduced by a weighting towards higher frequencies.

We shall look further into the modelling of wave breaking within the framework of evolution equations in Chapter 4.

1.6.2 Currents

The evolution equation models mentioned so far, describe only the pure wave motion. A vertically uniform current can, however, rather easily be incorporated. Yoon and Liu (1989)
extended the Boussinesq equations of Peregrine (1967) to account for wave-current interaction. In terms of the Boussinesq scaling parameters \( \varepsilon = a_0/h_0 \) and \( \mu = k_0 h_0 \), the depth, current and current-induced set-down were assumed to vary slowly in space \( (\partial_x = O(\mu)) \) and to be known apriori. The current velocity was assumed to be nearly uniform in the vertical direction with a maximum departure from uniformness of \( O(\varepsilon \mu, \mu^3) \). Also a small vorticity for the current of order \( O(\varepsilon \mu^2, \mu^4) \) was allowed.

In the model derived, the velocities and free surface elevation were separated into the contributions from the current and wave motion, respectively. This was necessary, because the Boussinesq model used is not fully nonlinear but only retains terms up to \( O(\varepsilon, \mu^2) \). If the \( O(1) \)-current is treated as part of the velocity variable, the quadratic terms are no longer \( O(\varepsilon) \). This may violate the derivations carried out. Further, the current effects will be imposed through the \( O(\varepsilon) \)-terms, and will therefore not affect the dispersion properties correctly, since the \( O(\varepsilon \mu^2) \)-terms are missing.

Hence, in weakly nonlinear models, the variables must be decomposed into current-variables and wave motion variables, and the model derivations be carried out taking into account that the current terms can have a magnitude of order 1. In fully nonlinear models, however, the current can be treated implicitly in the flow variables, see e.g., Madsen and Schäffer (1998, Section 8).

After deriving the new current terms for the Boussinesq model, Yoon and Liu (1989) derived evolution equations for the wave field. As described for other Boussinesq evolution equations, the model was valid for weakly two-dimensional wave fields. The model did non incorporate the feedback of the wave field to the current field. This is consistent with the quadratic accuracy of the model, and Yoon and Liu mentioned the possibility of applying an iterative scheme, first calculating the current field without waves, then the wave field on the current, next the modification of the current due to the waves and so on until convergence.

Kaihatu and Kirby (1995) show in an appendix, how their fully dispersive evolution equation model can be extended to include a slowly varying current and set-down. Here, the equations are formulated at the mean water level implied by the current, which similar to the work of Yoon and Liu (1989) is assumed to be known apriori. The model obtained reproduces the Doppler-shift of linear Stokes waves on a current.

### 1.6.3 Wave motion in two horizontal dimensions

Untill now, we have limited the discussion of evolution equations to one horizontal dimension. Some of the papers referenced, however, treat two-dimensional wave propagation. We will discuss the approaches and the underlying assumptions behind them in the following. Two methods appear in the literature, the so-called small angle approach and the wide angle approach based on Fourier collocation.
Small angle approach  For waves propagating primarily in one dimension, the small-angle models can be used. Here, the wave field is expanded as

\[ \eta(x, t) = \sum_{p=-N}^{n} a_p(x, y) e^{i(\omega_p t - \int \tilde{k}_p dx)} \]  

(1.55)

where \( \tilde{k}_p \) is a reference wave number being constant in the \( y \)-direction. Usually \( \tilde{k}_p \) is chosen as an averaged value in the longshore direction of the free wave numbers for a given \( x \)-value. By choosing a constant reference wave number, differentiations of \( \eta \) in the \( y \)-direction will produce \( y \)-derivatives of \( a_p \), while the phase function will not contribute. The assumption of weak spatial variation in the \( y \)-direction is imposed by

\[ \partial_x a_p = O(\varepsilon^n) \quad \partial_y a_p = O(\varepsilon^{n/2}) \quad k_p(x, y) = \tilde{k}_p(x) + \varepsilon k_p^\delta(x, y) \]  

(1.56)

where the slow \( x \)-variation of \( a_p \) is due to the absorption of the linear wave motion in the exponential function in the expansion (1.55). Here \( \varepsilon \) is the small parameter associated with the nonlinearity of the problem.

For Boussinesq evolution equations built on e.g., Nwogu’s (1993) equations, this scaling results in terms involving \( a_{p,x}, a_{p,y} \) and \( a_{p,yy} \). These terms are moved to the right hand side along with the nonlinear term. The resulting evolution equations then has the schematic form

\[ a_{p,x} = i(\tilde{k}_p - k_p)a_p + \beta_p a_{p,yy} + \text{shoal} + \text{nonlinear} \]  

(1.57)

where ‘shoal’ represents linear shoaling and ‘nonlinear’ represents the convolution sums due to the quadratic nonlinearity. The evolution equations can then be integrated on a grid, as shown in Figure 1.2, determining \( a_{p,yy} \) by a finite difference approximation for each value of \( x \). Hence, the small-angle approach can be viewed as solving several sets of evolution equations on parallel lines, coupled through the \( a_{p,yy} \)-term.

Models using the small-angle approach goes back to the paper of Radder (1979) on the parabolic approximation technique and has been applied within nonlinear evolution equations by among others Liu et al. (1985), Chen and Liu (1995), Kaihatu and Kirby (1995) and Kaihatu and Kirby (1998).
Chapter 1. Introduction

The wide-angle approach  For waves propagating at angles larger than $30^\circ$ from the cross shore direction, the small angle approximation breaks down. The reason is that the fast phase variation is not confined to the $x$-direction for large angles. Hence, the assumption of weak $y$-dependence of $a_p$ is violated.

For a directional wave component, the wave number is a vector in the $(x, y)$-plane having the same direction as the wave propagation. The length of the vector is given by the local dispersion relation. We write this as

$$k = \begin{pmatrix} k_x \\ k_y \end{pmatrix} \quad k^2 = |k|^2 = k_x^2 + k_y^2 \quad \omega^2 = gk \tanh kh. \quad (1.58)$$

Assuming periodicity in the longshore direction of the wave field allows to Fourier expand the flow variables along the $y$-axis. Hence an expansion of the form

$$\eta(x, t) = \sum_{l=1}^{M} \sum_{p=-N}^{N} a_{p,l}(x)e^{i(\omega_{p,l}t - \int k_{x,p}dx - \kappa_{l}y)} \quad (1.59)$$

can be used. Note that the set of wave numbers in the $y$-direction $\kappa_l$ are simply chosen with fixed increments. Hence for each $k^2$, being a solution to the dispersion relation for a given frequency, and a given $\kappa_l$, the wave number in the cross-shore direction $k_x$ is known. Further, since the periodicity assumption ensures that the $y$-variation can be completely described by the Fourier series in the $y$-direction, the amplitudes $a_{p,l}$ is only a function of $x$.

As an example such evolution equations we cite the result of Agnon and Sheremet (1997) for their fully dispersive evolution equation model:

$$\frac{\partial a_{p,l}}{\partial x} = -\frac{a_{p,l}}{2c_{g,p,l}} \frac{\partial c_{g,p,l}}{\partial x} + \sum_{l=1}^{M} \sum_{p=-N}^{N} R_{s,p-s,l-l-t} a_{s,t} a_{p-s,t-l} e^{-i\int (k_s + k_{p-s} - k_p)dx}. \quad (1.60)$$

This describes the evolution of the $l$'th $y$-modes $p$'th Fourier component. This result is derived assuming constant depth in the $y$-direction. The expansion in the $y$-direction results in an extra convolution in the nonlinear sums. Like for the small-angle approach, an $O(\varepsilon)$ deviation from uniform depth in the $y$-direction can be handled by adding extra terms, see Chen and Liu (1995).

Models of this type are also denoted angular-spectrum models, since they are ideal for modelling the evolution of a directionally spread wave spectrum. Expansions of this form have been used by Dalrymple and Kirby (1988), Dalrymple et al. (1989), Suh et al. (1990), Chen and Liu (1995) and Agnon and Sheremet (1997).

1.7 Evolution equations for four-wave interaction

The evolution equations presented so far have all dealt with quadratic interactions, where two interacting waves force a third wave. This is called three-wave interaction or ‘triad
interaction’ and approaches resonance at shallow water. In deep water, four-wave interaction or ‘quartet interactions’ are known to take place. Four-wave interaction requires cubic nonlinearity, since the interaction requires the forcing of one wave mode by three other waves.

Schematically, the governing equation for a wave mode forced by three-wave interaction can be written as

$$\frac{\partial a_3}{\partial t} = \varepsilon g_{12} a_1 a_2$$  \hfill (1.61)$$

and the governing equation for a wave mode forced by four-wave interaction as

$$\frac{\partial a_4}{\partial t} = \varepsilon^2 g_{123} a_1 a_2 a_3.$$  \hfill (1.62)$$

Here $\varepsilon$ is the nonlinearity parameter and $g_{ij}(l)$ some interaction coefficients. This illustrates that the time scale of three-wave interaction is $O(\varepsilon^{-1})$ and the time scale of four-wave interaction is $O(\varepsilon^{-2})$. Hence, four-wave interaction requires more time (or space) to build up. In shallow water where the three-wave interaction is near-resonant, the four-wave interaction is therefore generally not allowed to be an effective interaction mechanism for the water waves, simply because the rapid changes implied by the three-wave interaction or depth-changes ruins the long time evolution needed for the four-wave interaction.

At deep water, however, the three-wave interactions are too far from resonance, and therefore only produces second-order bound waves of the primary waves. Hence, the sufficient time scale for the four-wave interaction is provided here, and the four-wave interaction is the dominating interaction process, due to its resonance.

In the following we shall look a little closer on the mechanism of four-wave interaction and discuss two evolution equations describing four-wave interaction, the Zakharov equation and the Nonlinear Schrödinger equation.

### 1.7.1 Four-wave interaction

We here give a brief account of the mechanism of four-wave interaction. The description is mainly based on Phillips (1977). We consider four wave components of two-dimensional wave propagation

$$\eta(x,t) = a_1 e^{i(\omega_1 t - k_1 \cdot x)} + a_2 e^{i(\omega_2 t - k_2 \cdot x)} + a_3 e^{i(\omega_3 t - k_3 \cdot x)} + a_4 e^{i(\omega_4 t - k_4 \cdot x)} + c.c.$$  \hfill (1.63)$$

where the wave amplitudes $a_i$ are allowed to vary slowly in time and space. The wave numbers $k_i = |k_i|$ and frequencies $\omega_i$ are solutions to the linear dispersion relation. A certain combination of the frequencies $\omega_{1,\ldots,4}$ may produce cubic forcing terms having the same frequency as one of the wave components. We may write this as

$$\omega_a + \omega_b + \omega_c = \omega_i$$  \hfill (1.64)$$
where \(a, b\) and \(c\) represent any of the indices \(\pm 1, \pm 2, \pm 3, \pm 4\). The modulation of the receiving wave amplitude can be expressed schematically by

\[
\frac{\partial a_i}{\partial t} = g_{abc}a_ia_be^{-i(k_a+k_b+k_c-k_i)x}.
\]

(1.65)

This forcing is resonant with the receiving wave mode \(a_i\) if \(k_a + k_b + k_c - k_i = 0\). In other words, for resonance to exist, the conditions

\[
\begin{align*}
\omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 &= 0 \\
k_1 \pm k_2 \pm k_3 \pm k_4 &= 0
\end{align*}
\]

(1.66)

(1.67)

must be fulfilled.

For a lot of the above sign combinations, the linear dispersion relation \(\omega_i^2 = gk_i \tanh k_i h\) does not allow for non-trivial solutions of the resonance conditions. However, the combination

\[
\begin{align*}
\omega_1 + \omega_2 &= \omega_3 + \omega_4 \\
k_1 + k_2 &= k_3 + k_4
\end{align*}
\]

(1.68a)

(1.68b)

is known to have non-trivial solutions (Phillips, 1977). For one-dimensional wave propagation, this is illustrated in Figure 1.3. Here, the bold coordinate system shows the linear dispersion relation in the \((k, \omega)-plane\). Two free waves are marked with black points, having the wave numbers \(k_1\) and \(k_3\) and corresponding frequencies \(\omega_1\) and \(\omega_3\) (not marked on the figure). Another coordinate system \((k', \omega')\) rotated 180° showing the dispersion relation

\[
\text{Figure 1.3: Four resonant free waves.}
\]

once again is plotted on top. It is positioned such that the dispersion relation curve in the \((k', \omega')\)-system passes through the free wave points of the bold coordinate system. The
intersection points defines two waves in the \((k', \omega')\)-system, having the wave numbers \(k_2\) and \(k_4\) and frequencies \(\omega_2\) and \(\omega_4\) (not shown on the figure). By inspection, it can be seen that

\[
\begin{align*}
  k_1 + k_2 &= k_3 + k_4 \quad (1.69) \\
  \omega_1 + \omega_2 &= \omega_3 + \omega_4. \\end{align*}
\]

Hence, the figure shows an example of resonant four-wave interaction of the type \((1.68)\). In the figure, the dispersion curves of the two coordinate systems also intersects at two other points, close to the origin of the bold coordinate system. Totally there are therefore four points of intersection, and any two among these, define a quartet of resonant waves.

For two-dimensional wave propagation, the dispersion relation is not just a curve, but a surface. In figure 1.3, the \(k\)-axis is changed to a \(k_x\)-axis and a third axis perpendicular to the paper must be added, representing \(k_y\). The dispersion relation is then the surface resulting from rotating the dispersion curve around the \(\omega\)-axis. The points of intersection therefore generalizes to deformed circular curves around the origin in the coordinate systems. If the upper coordinate system is moved upwards towards the top of the page, the two inner intersection points merge and disappear. In two dimensions this corresponds to a merging of the two circular curves into one curve. For arbitrary scaling, the projection of these curves onto the \((k, \omega)\)-plane is shown in Figure 1.4. The points A and B correspond to the origins of the \((k, \omega)\) system and \((k', \omega')\) system of Figure 1.3. On any two points \((C, C')\) on a curve, two wave vectors can be drawn from point A. The vectors from these points to point B then defines the other two wave vectors, needed to form a resonant quartet. Here, the closed curves around A and B can be regarded as ‘one curve’ in pairs of two. That is, the innermost curves surrounding A and B belong together, and the two points \((C, C')\) can be chosen as one point on each or as two points on one of them.

**1.7.2 The Zakharov model**

Quartet interactions can be modelled by any fully nonlinear wave model with accurate dispersion properties at deep water. The effect of quartet interactions is however most
directly studied and modelled in terms of evolution equations. A very general set of evolution equations for this purpose is the Zakharov model, which is often denoted ‘the Zakharov equation’.

Zakharov (1968) derived a set of evolution equations for the temporal evolution of the wave amplitudes in deep water due to four-wave interactions. We here briefly cite the derivation and assumptions. We follow the derivation of Stiassnie and Shemer (1984), rather than the work of Zakharov.

Stiassnie and Shemer’s (1984) Zakharov equation is derived for constant, finite depth. The kinematic and dynamic free surface conditions are expressed in terms of the free surface variables \( \eta, \tilde{\phi} \) and \( \tilde{w} \) and Fourier transformed from the spatial domain to the wave number domain. Here, the tilde denotes values at the free surface. For a given wave number \( k \), the Laplace equation and bottom boundary condition has the solution

\[
\hat{\phi}(k, z, t) = \hat{\Phi}(k, t) \cosh |k|(z + h)
\] (1.71)

thus enabling elimination of the vertical dimension of the problem. Using the above result, and further assuming \( \varepsilon = |k|\eta \ll 1 \), \( \tilde{w} \) is eliminated from the problem. Next, introduction of the complex variable

\[
b(k, t) = \sqrt{\frac{g}{2\omega(k)}} \hat{\eta}(k, t) + i \sqrt{\frac{\omega(k)}{2g}} \hat{\phi}(k, t)
\] (1.72)

allows the surface conditions to be merged into one equation in \( b(k, t) \). Next, the expansion

\[
b(k, t) = [\varepsilon \tilde{B}(k, t_2, t_3) + \varepsilon^2 B'(k, t, t_2, t_3) + \cdots]e^{-i\omega(k)t}
\] (1.73)

is introduced. The resulting problem at order (\( \varepsilon \)) is identically satisfied, while the \( O(\varepsilon^2) \) yields an equation for the fast temporal variation of \( B' \):

\[
i \frac{\partial B'}{\partial t} = \int \int \int_{-\infty}^{\infty} V_{0,1,2} \tilde{B}_1 \tilde{B}_2 \delta(k - k_1 - k_2) e^{i(\omega(k) - \omega(k_1) - \omega(k_2))t} dk_1 dk_2.
\] (1.74)

Here, \( V_{0,1,2} \) is an interaction coefficient. This equation\(^1\) can be integrated directly in \( t \), since \( \tilde{B} \) is not a function of fast time. \( \tilde{B} \) represents the second-order bound waves, forced by the free waves. They are not resonant with any free waves.

To next order the following equation for the slow temporal evolution of \( \tilde{B} \) emerges:

\[
i \frac{\partial \tilde{B}}{\partial t_2} = \int \int \int_{-\infty}^{\infty} T_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta(k + k_1 - k_2 - k_3) e^{i(\omega(k) + \omega(k_1) - \omega(k_2) - \omega(k_3))t} dk_1 dk_2 dk_3.
\] (1.75)

\(^1\)the equation of Stiassnie and Shemer (1984) has two extra terms in the integrand, similar to the one shown here.
This is the Zakharov equation for finite depth. $T_{0,1,2,3}$ is an interaction coefficient and $\delta()$ is the Dirac delta function. The equation describes the slow temporal evolution of the free wave amplitudes $\tilde{B}$. Note that this equation can be solved without explicitly calculating the second-order bound wave field, since it is formulated solely in the amplitudes of the free waves. Stiassnie and Shemer (1984) carried out the analysis one order higher, to obtain evolution equations describing the effects of resonant quintet (five-wave) interactions.

The Zakharov equation is a quite general equation for water wave motion. It requires constant depth and small nonlinearity, but no assumptions of the relative size of the harmonics is made. Hence, it can model a spectrum of any width. The equation has often been used for describing instability phenomena and determining appropriate parameter spaces for certain instabilities. For practical computations some work is needed to discretize the continuous integrals into discrete convolution sums of the wave amplitudes. Such a discretization has been presented by Rasmussen and Stiassnie (1999). Further, wave measurements are usually given as time series in discrete locations. This makes it difficult to use practical measurements as initial conditions for the model, since it requires a spectrum in $k$-space, corresponding to knowing the whole spatial variation of the wave field in one instant. Shemer et al. (2001) have transformed the Zakharov equation to a spatial form, thus describing the spatial evolution of the free wave amplitudes rather than the temporal. We shall derive a similar spatial cubic model in Chapter 7 for sloping sea bed.

### 1.7.3 The Nonlinear Schrödinger equation

The Nonlinear Schrödinger governs the slow spatial and temporal evolution of the wave amplitude for a narrow spectrum. It is derived under the same assumptions as the Zakharov equation, and it can be viewed as a particular case of this model.

We here cite the Nonlinear Schrödinger equation for finite depth, as written in Mei (1983). Defining the wave amplitude $A$ by

$$\eta(x, t) = \frac{1}{2} A(x_1, t_1)e^{i(kx-\omega t)} + c.c., \quad x_1 = \varepsilon x, \quad t_1 = \varepsilon t, \quad (1.76)$$

the Nonlinear Schrödinger equation reads for one-dimensional wave propagation

$$\left( \frac{\partial}{\partial t_1} + C_g \frac{\partial}{\partial x_1} \right) A + i\varepsilon \left\{ -\frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \frac{\partial^2 A}{\partial x_1^2} + \frac{\omega k^2(cosh 4k\hbar + 8 - 2tanh^2 k\hbar)}{16 sinh^3 k\hbar} |A|^2 A \\
- \left( \frac{k^2}{2\omega cosh^2 k\hbar} \frac{\partial \phi_{10}}{\partial t_1} - k \frac{\partial \phi_{10}}{\partial x_1} \right) A \right\} = 0, \quad (1.77)$$

governing the slow modulation of the wave train and its interaction with the mean flow. The mean flow potential $\phi_{10}$ is governed by the equation

$$\frac{\partial^2 \phi_{10}}{\partial t_1^2} - gh \frac{\partial^2 \phi_{10}}{\partial x_1^2} = \frac{\omega^3 cosh^3 k\hbar}{2k sinh^2 k\hbar} (A^*)_{x_1} - \frac{\omega^3}{4 sinh^2 k\hbar} (A^*)_{t_1}. \quad (1.78)$$
Chapter 1. Introduction

The Nonlinear Schrödinger equation is also known within nonlinear optics and has a number of particular solutions, including solitons. For constant $A$, the solution is identical to Stokes third-order theory.

1.8 Structure of the thesis

The structure of the thesis is outlined in the executive summary, page iii. At page vi, the most important findings of the study are listed. The reader is referred to these sections for an overview of the thesis contents.
Chapter 2

Boussinesq Based Models

Evolution equations of modern literature are usually based on Boussinesq formulations. One important exception is the fully dispersive models, which we look further into in Chapters 6 and 7. The reason for using Boussinesq formulations as basis for evolution equations is that they are depth-integrated and usually consists of just two equations in two variables, the free surface elevation and a velocity variable. From here the equations can rather easily be simplified to evolution equations.

During the last 10–15 years, a number of increasingly accurate time domain Boussinesq models have emerged. Madsen et al. (1991) and Nwogu (1993) enhanced the accuracy of the linear dispersion relation by incorporating ‘Padé[2,2]-dispersion’ in the models. For these formulations, the squared phase speed is the Padé[2,2] approximant of the result of Stokes wave theory. Later on, Padé[4,4]-dispersion has been incorporated (see e.g., Schäffer and Madsen, 1995; Gobbi et al., 2000), further extending the depth-range of applicability. Along with the improvement in linear dispersion and shoaling properties, also the accuracy of the nonlinear terms has been improved to cover larger and larger relative depths. Recently, Madsen et al. (2002a) have presented a Boussinesq model with linear and nonlinear properties accurate to $kh = 40$.

These developments make it natural to investigate if the improved accuracy of the new Boussinesq formulations can be retained in the corresponding evolution equations. A study of the accuracy of evolution equations and the effects of the approximations usually applied is therefore motivated.

As was shown in Chapter 1, the derivation of Boussinesq evolution equations is traditionally based on the following steps: First, the free surface elevation and the velocity variable are
expressed as Fourier series

\[
\eta(x, t) = \sum_{p=-N}^{N} a_p(x) e^{i\int k_p(x) dx - \omega_p t},
\]
\[
u(x, t) = \sum_{p=-N}^{N} b_p(x) e^{i\int k_p(x) dx - \omega_p t},
\]

and inserted in the time domain equations. Usually \( k_p \), the reference wave numbers, are chosen as solutions of the models linear dispersion relation. Second, all but first-order derivatives of \( a_p, b_p, k_p \) and \( h \) in the linear terms and all derivatives in the nonlinear terms are neglected. This is justified by the assumption of weak nonlinearity and slow spatial variation of the depth, wave numbers and amplitude functions.

However, these assumptions do not always apply. Wave fields with fast varying amplitudes do exist, and it is therefore interesting to investigate to which extent the assumption of slowly varying amplitudes can be avoided.

Such an investigation is the main objective of this chapter. The Boussinesq model of Nwogu (1993) is chosen as basis for the investigations due to its similarity with the more accurate Boussinesq formulations, still having a limited number of terms. Three approximate models resembling usual evolution equation formulations are derived and compared. Further, we derive a frequency domain model retaining all terms, thus being completely equivalent to the time domain formulation. With reference solutions from this model, the influence of 1) derivatives in the nonlinear terms, 2) higher-order linear derivatives, 3) elimination of the free surface elevation amplitude and 4) neglect of \( b_{p,xx} \) is investigated.

One of the approximate models, denoted the two-equation model, is formulated directly in \( a_p(x) \) and \( b_p(x) \). This facilitates a more direct approach to evolution equations, since no elimination of the velocity variable is needed. However, it turns out that this model is vulnerable to fast depth variations. We demonstrate that the choice of reference wave numbers has a significant impact on the solutions of this model and compare its performance on varying depth to time domain results.

### 2.1 Nwogu’s equations in the frequency domain

For one dimensional wave propagation and slowly varying bathymetry, the time domain equations of Nwogu (1993) equations read

\[
\eta_t + h u_x + (\alpha + 1/3) h^3 u_{xxx} + h_x u + c_1 h^2 h_x u_{xx} = -(\eta u)_x
\]
\[
u_t + g \eta_x + \alpha h^2 u_{xxt} + c_2 h h_x u_{xt} = -u u_x,
\]

where \( \alpha \) is defined as \( \alpha = \frac{1}{2}(z_\alpha/h)^2 + z_\alpha/h \) and \( z_\alpha \) is the vertical level of the velocity variable \( u \). The coefficients \( c_1 \) and \( c_2 \) are functions of \( \alpha \) defined by \( c_1 = 3\alpha + 2(1 + 2\alpha)^{1/2} \).
2.2 Conventional approximations and three approximate models

$c_2 = 2(1 + 2\alpha)^{1/2} - 2$, see e.g., Schäffer and Madsen (1995, equation 45–46). The equations (2.2) have been derived assuming $h_x \ll 1$, thus neglecting all higher-order derivatives of $h$, products like $h_x^2$ and nonlinear terms including $h_x$. Further, the parameter $\alpha$ has been assumed constant in the domain, corresponding to choosing the vertical level of the velocity variable $u$ as a constant fraction of depth.

Insertion of the expansions (2.1) yields

$$\alpha_{1,p} a_p + \alpha_{2,p} b_p + \alpha_{12,p} b_{p,x} + \alpha_{22,p} b_{p,xx} + \alpha_{32,p} b_{p,xxx}$$

$$\beta_{1,p} a_p + \beta_{2,p} b_p + \beta_{11,p} a_{p,x} + \beta_{12,p} b_{p,x} + \beta_{22,p} b_{p,xx}$$

for $p = 0, \ldots, N$, where

$$\alpha_{1,p} = -i\omega_p$$
$$\alpha_{2,p} = hik_p - i(\alpha + 1/3)h^3 k_p^3 - c_1 h_x h^2 k_p^2 - 3(\alpha + 1/3)h^3 k_p k_{p,x} + h_x$$
$$\alpha_{12,p} = h - 3(\alpha + 1/3)h^3 k_p^2 + 3i(\alpha + 1/3)h^3 k_{p,x} + 2ci_1 h_x h^2 k_p$$
$$\alpha_{22,p} = 3i(\alpha + 1/3)k_p h^3 + c_1 h^2 h_x$$
$$\alpha_{32,p} = (\alpha + 1/3)h^3$$
$$\beta_{1,p} = gik_p$$
$$\beta_{2,p} = -i\omega_p + i\omega_p h^2 k_p^2 + \alpha \omega_p h^2 k_{p,x} + c_2 \omega_p h_x h k_p$$
$$\beta_{11,p} = g$$
$$\beta_{12,p} = 2\alpha \omega_p h^2 k_p - c_2 i\omega_p h h_x$$
$$\beta_{22,p} = -ia h^2 \omega_p$$

In the above equations, no terms have been neglected when transforming to the frequency domain, except for higher-order derivatives of $h$ and $k$ and products of derivatives of $h$ and $k$. The two equations in (2.3) and (2.4) are therefore equivalent to the time domain equations on constant depth, as long as a sufficiently large number of harmonics are included in the calculations.

2.2 Conventional approximations and three approximate models

Conventionally, the reference numbers $k_p$ in evolution equations are chosen as the free wave numbers of the linearized underlying wave model. For linear wave motion on constant depth,
Chapter 2. Boussinesq Based Models

the series expansions (2.1) are therefore solutions to the equations with constant amplitude functions \( a_p \) and \( b_p \). Hence, for weakly nonlinear wave propagation on slowly varying depth, the amplitude functions can be assumed to vary slowly in space. This justifies to neglect 1) the derivatives of the amplitude functions in the nonlinear terms and 2) the higher-order derivatives in the linear terms. This is illustrated below.

\[
\alpha_1p a_p + \alpha_2p b_p + \alpha_{12,p} b_{p,x} + \alpha_{22,p} b_{p,xx} + \alpha_{32,p} b_{p,xxx} \\
\text{approx 2: neglect}
\]

\[
= - \sum_{s=p-N}^{N} \frac{\partial}{\partial x} [a_s b_{p-s} e^{i \int (k_p + k_s) dx}] e^{-i \int k_p dx} \\
\text{approx 1: neglect } a_{s,x} \text{ and } b_{p-s,x}
\]

\[
\beta_1p a_p + \beta_2p b_p + \beta_{11,p} a_{p,x} + \beta_{12,p} b_{p,x} + \beta_{22,p} b_{p,xx} \\
\text{approx 2: neglect}
\]

\[
= - \sum_{s=p-N}^{N} b_s \frac{\partial}{\partial x} [b_{p-s} e^{i \int k_p dx}] e^{i \int (k_s - k_p) dx} \\
\text{approx 1: neglect } b_{p-s,x}
\]

Using approximation 1 and 2, we can derive an approximate set of evolution equations.

### 2.2.1 A two-equation model

Utilizing approximation 1 and 2 in the system (2.3) we get the model

\[
a_{p,x} = \frac{1}{\beta_{11,p}} \left( \frac{\beta_{12,p}}{\alpha_{12,p}} \alpha_{1,p} - \beta_{1,p} \right) a_p + \frac{1}{\beta_{11,p}} \left( \frac{\beta_{12,p}}{\alpha_{12,p}} \beta_{2,p} - \beta_{2,p} \right) b_p - \frac{1}{\beta_{11,p}} S2_p + \frac{\beta_{12,p}}{\beta_{11,p} \alpha_{12,p}} S1_p \\
b_{p,x} = - \frac{\alpha_{1,p}}{\alpha_{12,p}} a_p - \frac{\alpha_{2,p}}{\alpha_{12,p}} b_p - \frac{1}{\alpha_{12,p}} S1_p \\
p = 1, \ldots, N.
\]

Here, \( S1_p \) and \( S2_p \) denote the nonlinear interaction sums of the mass equation and the momentum equation respectively, which under approximation 1 read

\[
S1_p = \sum_{s=p-N}^{N} i(k_s + k_{p-s}) a_s b_{p-s} e^{i \int (k_s + k_{p-s} - k_p) dx} \\
S2_p = \sum_{s=n-N}^{N} i(k_s + k_{p-s}) b_s b_{p-s} e^{i \int (k_s + k_{p-s} - k_p) dx}.
\]

The model (2.6) is denoted the two-equation model in the following.
2.2 Conventional approximations and three approximate models

2.2.2 A one-equation model

A simpler model can be derived by eliminating \( a_p \) from (2.3) and neglecting derivatives of higher than second order in \( b_p \). We only derive the constant-depth version of this model here.

In the nonlinear interaction sums, \( a_p \) is eliminated using a linear solution to the constant depth version of the time domain equations (2.2):

\[
a_p = \sqrt{\frac{h}{g}} \Gamma_p b_p, \quad \Gamma_p = (1 - \alpha k_p^2 h^2)^{1/2}(1 - (\alpha + 1/3)k_p^2 h^2)^{1/2}.
\]  

(2.9)

We denote this elimination approximation 3. The resulting model is then

\[
\hat{a}_{2,p} b_{p,xx} + \hat{a}_{1,p} b_{p,x} + \hat{a}_{0,p} b_p = \sum_{s=n-N}^{N} \left[ -\omega_p k_{p-s} - \sqrt{gh} \Gamma_s (k_s + k_{p-s})^2 \right] b_s b_{p-s} e^{i(k_s + k_{p-s} - k_p)x},
\]  

(2.10)

with coefficients

\[
\begin{align}
\hat{a}_{2,p} &= -gh + 6(\alpha + 1/3)gh^3 k_p^2 - \alpha \omega_p^2 h^2, \\
\hat{a}_{1,p} &= -2i \omega_p h k_p + 4i(\alpha + 1/3)gh^3 k_p^3 - 2i\alpha \omega_p^2 h^2 k_p, \\
\hat{a}_{0,p} &= -\omega_p^2 + gh k_p^2 - (\alpha + 1/3)gh^3 k_p^4 + \alpha \omega_p^2 h^2 k_p^2.
\end{align}
\]  

(2.11)

We recognize that \( \hat{a}_0 = 0 \) is the linear dispersion relation of the time domain equations, and the term \( \hat{a}_0 b_n \) therefore drops out when free wave numbers are chosen for \( k_p \). The second derivative of \( b_p \) is now eliminated following a technique of Bryant (1973): multiply each side of the equation by \( e^{(\hat{a}_{1,p}/\hat{a}_{2,p})x} \) and integrate once in space, neglecting the spatial variation of \( b_s b_{p-s} \) in the nonlinear terms. This yields the model

\[
b_{p,x} = \sum_{s=n-N}^{N} \frac{-\omega_p k_{p-s} - \sqrt{gh} \Gamma_s (k_s + k_{p-s})^2}{i \hat{a}_{2,p}(k_s + k_{p-s} - k_p)} b_s b_{p-s} e^{i(k_s + k_{p-s} - k_p)x}
\]  

(2.12)

which consists of only one equation in one variable for each frequency component.

2.2.3 A crude one-equation model

An even simpler model can be derived by simply neglecting \( b_{p,xx} \) in (2.10). When free reference wave numbers are used, \( \hat{a}_{0,p} \) again vanishes and we are left with the model

\[
b_{p,x} = \sum_{s=n-N}^{N} \frac{-\omega_p k_{p-s} - \sqrt{gh} \Gamma_s (k_s + k_{p-s})^2}{\hat{a}_{1,p}} b_s b_{p-s} e^{i(k_s + k_{p-s} - k_p)x}
\]  

(2.13)

\[ p = 1, \ldots, N \]

We shall come back to the results of these three models in Section 2.5.
2.3 Retaining higher-order derivatives in linear terms

The reason for applying approximation 2, i.e., neglecting the higher-order \( x \)-derivatives, is that the resulting model forms a first-order ODE system, which can be integrated in a space-marching manner. However, it is well known that a higher-order ODE system can be reduced to a first-order system by introducing extra utility variables. Hence, the neglect of the higher-order derivatives in (2.3) may be avoided, still retaining the advantages of a space marching solution procedure.

To try this out, we define \( b_{p,xx} \) and \( b_{p,xxx} \) as separate variables and rewrite (2.3) to a set of first-order systems

\[
\begin{pmatrix}
  a_p \\
  b_p \\
  b_{p,x} \\
  b_{p,xx}
\end{pmatrix}
= \begin{pmatrix}
  a_p \\
  b_p \\
  b_{p,x} \\
  b_{p,xx}
\end{pmatrix} + \text{nonlinear terms}
\]

for \( p = 0, \ldots, N \), where \( A_p \) are \( 4 \times 4 \) matrices formed from the coefficients of (2.3).

Even though this formulation forms a first-order system in space, it turns out that it cannot be solved by a space-marching integration procedure due to a numerical instability. This arises from the roots of the linear dispersion relation of Nwogu’s (1993) equations, which for constant depth can be written

\[
(\alpha + 1/3)(kh)^4 - (1 + \alpha \omega^2 \frac{h}{g})(kh^2) + \omega^2 \frac{h}{g} = 0.
\]

Given \( \omega \sqrt{h/g} \), this equation has four solutions for \( kh \). For \( \alpha < 1/3 \) there is one positive and one negative solution for \( (kh)^2 \), and we can therefore write the solutions for \( k \) as \( k_{\text{time domain}} = \{ \pm k, \pm ik \} \), \( k, \kappa \in \mathbb{R}^2 \). The eigenvalues for \( A_p \) are given by \( \lambda_p = i(k_{\text{time domain}} - k_{\text{ref}}) \), \( k_{\text{ref}} \) being the reference wave numbers chosen. For \( A_p \) constant, a system like (2.14) is known to have linear eigensolutions of the form \( v e^{\lambda x} \), where \( v \) is the eigenvector of the eigenvalue \( \lambda \). In our case, \( A_p \) varies with \( x \) due to depth variations, but the local form of the solution will still be composed of such eigenfunctions. From this we see that no matter which reference wave numbers are used, one eigenvalue of \( A_p \) will always possess a positive real part, thus representing an exponentially growing eigensolution. When (2.14) is solved by a space-marching integration procedure, rounding errors will soon excite the exponentially increasing wave mode, and this will rapidly dominate the solution completely.

For this reason, higher-order linear derivatives cannot be retained in evolution equations without spoiling the possibility of integrating the wave field in one spatial sweep. This also holds for evolution equations build on higher-order Boussinesq models incorporating even higher derivatives of the flow variables. The reason is that these equations will only have two real wave numbers for a given frequency, corresponding to a forward and a backward progressive wave mode. All other roots to the dispersion relation will be imaginary, resulting in eigenvalues with positive real parts of the matrix \( A_p \).
When higher-order derivatives of the wave amplitudes are taken out, the eigenvalue spectrum of the system is changed. For example, neglecting only $b_{p,xxx}$ in (2.3) results in a $3 \times 3$ first-order system for each frequency. One of the eigenvalues is zero, corresponding to the forward propagating wave mode ($\lambda_p = i(k_p \text{ time domain} - k_{p \text{ ref}})$). The other two eigenvalues are imaginary for small values of $kh$, but turn into a pair of complex numbers at $\omega \sqrt{h/g} \approx 0.82$. More details can be found in Bredmose (1999). Hence, also $b_{p,xx}$ must be neglected. This yields the two-equation model (2.6), which does not have stability problems.

In the next section we present an approach that allows the higher-order linear derivatives to be retained. In this approach, the system (2.3) is solved as a boundary value problem.

### 2.4 Boundary value problem formulation

The linear stability problems mentioned above can be overcome by solving (2.3) as a boundary value problem, taking both up-wave and down-wave boundary conditions into account. We present such a formulation in the following. The boundary value problem approach does not have the same computational efficiency as a space-marching solution procedure, since all points in the domain are treated at once. Hence, the model is not suited for practical application, but is developed as a tool for assessing the importance of the higher-order terms, usually neglected in evolution equation models.

#### 2.4.1 Spatial discretization

The equations (2.3) are solved in a finite difference formulation. The spatial domain is discretized in $M$ points on an equidistant grid and five-point centered finite-difference operators are used to express the derivatives in (2.3). The use of these five-point approximations give truncation errors proportional to the fifth and sixth derivative of the dependent variables, and therefore the truncation errors will not interfere with the highest derivatives ($b_{p,xxx}$) in the equations. The truncation error is of second order for the term $b_{p,xxx}$, while it is of fourth order for the first and second derivatives.

#### 2.4.2 Boundary conditions

At each boundary an additional ghost point is introduced. At the incident-wave boundary the condition $a_{p1} = a_{p3}^*$, $b_{p1} = b_{p3}^*$ is used, motivated by inspection of (2.14) in the case of real initial amplitude coefficients. The subscript numbers are the index of the harmonic, followed by the spatial index in the computational grid, starting with 1. In all calculations
with the boundary value problem model, the amplitude functions were real at the incident boundary. A linear wave train of the form
\[ \eta(x, t) = \sum_{p=1}^{N} A_p \cos(k_p x - \omega_p t) \]
is specified by
\[ a_{p,2} = A_p / 2 \quad b_{p,2} = \frac{gk_p}{\omega_p (1 - \alpha(k_p h)^2)} A_p / 2, \quad p = 1, \ldots, N. \] (2.16)

which is identical to the linear solution (2.9).

In a region close to the down-wave boundary, a sponge layer is used to damp the waves. In the time domain, a sponge layer effect can be obtained by adding terms proportional to \( \eta \) and \( u \) in the mass and momentum equations, (2.2). This corresponds to adding real numbers to the coefficients \( \alpha_{1,p}, \alpha_{2,p}, \beta_{1,p}, \) and \( \beta_{2,p} \) within the sponge layer. Following this idea, we choose to modify the coefficients \( a_2 \) and \( b_1 \) in the following way

\[ \alpha_{2,p,\text{sponge}} = \alpha_{2,p}(1 - i f_s) \] (2.17a)
\[ \beta_{1,p,\text{sponge}} = \beta_{1,p}(1 - i f_s), \] (2.17b)

thus rotating them in the complex plane. Here \( f_s \), the sponge layer function, is given by

\[ f_s = \gamma_1 e^{\gamma_2 s^3} - 1 \quad e^{\gamma_2} - 1, \quad s \in [0; 1]. \] (2.18)

In the calculations the parameter values \( \gamma_1 = 10, \gamma_2 = 1 \) were used.

### 2.4.3 Numerical solution procedure

When the frequency domain equations (2.3) are discretized, the following two algebraic equations are obtained in each spatial point and for each harmonic

\[ \sigma_1 a_{p,j-2} + \sigma_2 b_{p,j-2} + \cdots + \sigma_9 a_{p,j+2} + \sigma_{10} b_{p,j+2} = -S_{1,p} \] (2.19a)
\[ \tau_1 a_{p,j-2} + \tau_2 b_{p,j-2} + \cdots + \tau_9 a_{p,j+2} + \tau_{10} b_{p,j+2} = -S_{2,p} \] (2.19b)

where \( (\sigma_j, \tau_j) \) are coefficients and \( S_{1,p} \) and \( S_{2,p} \) denote the nonlinear sums in (2.3a) and (2.3b), respectively. In these terms, the derivatives of the amplitude functions \( a_p \) and \( b_p \) were included using a five-point finite difference approximation. For each harmonic, the linear terms in these equations establish a linear equation system in \( a_{1,\ldots,M} \) and \( b_{1,\ldots,M} \), with a coefficient matrix of bandwidth 10.

Two strategies for solving the model equations numerically have been examined: Gauss-Seidel iteration and the Newton-Raphson method for systems of nonlinear equations. The first approach was based on solving the linear part of the discretized equations for each harmonic in turn, treating the nonlinear terms and the conjugate boundary condition at the right hand sides of these linear systems. This method, however, resulted in stability
problems when too large wave amplitudes were used, or a too large number of harmonics were included in the calculations.

Alternatively, the discretized equations can be written in the form \( F(a, b) = 0 \) and be treated by the Newton-Raphson method. However, since complex conjugation is a non-differentiable operation, it was necessary to transform the equations to real variables. The system to solve could hereafter be written \( F(X) = 0 \) and in general only a few iterations were needed for convergence. As convergence criterion, it was used that the two-norm of the relative deviation between two successive solution vectors should be smaller than \( 1 \cdot 10^{-10} \). The length of \( X \) is \( 4MN \), thus making the effort needed for each iteration \( O(M^3N^3) \). The method is therefore not suited for practical applications, but only as a test of the importance of the different approximations.

2.4.4 Comparison with time domain calculations

To compare the boundary value problem model with the equivalent time domain formulation, a test of harmonic generation on constant depth was carried out. The test parameters are identical to test ‘A’ of Chapalain et al. (1991), specifying a linear wave of period \( T = 2.5 \) s, and a wave height of \( H = 0.084 \) m on a depth of \( h = 0.4 \) m. We will use this test case throughout the chapter for comparing the different models presented. In the frequency domain six harmonics were included in the calculations, and the spatial discretization was \( \Delta x = 0.2 \) m. The domain was 45 m long and a sponge layer of length 15 m was used in the down-wave part of the domain. In the time domain a grid spacing of \( \Delta x = 0.05 \) m with a time step of \( \Delta t = 0.025 \) s was used. The time domain model uses the numerical scheme of Wei and Kirby (1995), which is of fourth order in space and time. Note that the spatial discretization in the frequency domain can be more coarse than in the time domain, since the amplitude functions vary more slowly than the physical variables themselves. Further, the Courant number requirement \( C_r = \sqrt{gh \frac{\Delta t}{\Delta x}} \leq 1 \) results in a much smaller value of \( \Delta t \) than needed to resolve the sixth frequency. For the present choice of \((\Delta x, \Delta t)\), the free wave length of the sixth harmonic is resolved with 7.5 points in space and 16.6 points in time, the latter due to the Courant number requirement. This temporal discretization corresponds to resolving the 50th harmonic at the Nyquist frequency. In contrast, frequency domain models do not need to resolve any other frequencies, than the frequencies chosen for a given simulation.

The amplitude of the first four harmonics are depicted in Figure 2.1. The amplitude of the incoming wave is too large to preserve a linear form of the propagating wave. Consequently energy from the first harmonic is transferred to the second and higher harmonics in an almost cyclic pattern. The spatial period of this recurrence is denoted the beat length.

In general the results of the two models compare very well. The largest deviations are observed for the third harmonic which seems to build up a phase difference away from the incoming boundary. Also the amplitude of the fourth harmonic is over-predicted slightly by
the boundary value problem model. The deviations may be due to our neglect of the zeroth harmonic in the boundary value problem model, or to the omission of higher harmonics. All in all, however, the agreement is very satisfactory.

### 2.4.5 More on the physics of harmonic generation

Before we move on to study the influence of the different approximations, we pay a little more attention to the physics of harmonic generation. More specifically, we give a linear measure of the beat length and relate the strength of the energy exchange between the harmonics to the Ursell number

\[ U = \frac{a_0 l_0^3}{h_0^3}, \]

where \((a_0, l_0, h_0)\) are the linear wave amplitude, the linear wave length and the depth, respectively. We use the crude one-equation model (2.13) for the analysis, thus taking basis in a weakly dispersive model.

#### Linear estimate of the beat length

We consider the forcing of wave mode \(p\) in equation (2.13). If the forcing amplitudes \((b_s, b_{p-s})\) are considered constant, we have the explicit solution

\[
b_p = b_{p,0} + \frac{\omega_p k_{p-s} + \sqrt{g \Gamma_s (k_s + k_{p-s})^2}}{\alpha_1 (k_s + k_{p-s} - k_p)} b_s b_{p-s} e^{i(k_s+k_{p-s}-k_p)x}. \tag{2.20}
\]

The solution is oscillating with the wave number \(\delta = k_s + k_{p-s} - k_p\) which is the phase mismatch. The recurrence length of the oscillation is

\[
L_{\text{beat}} = 2\pi / \delta = 2\pi / (k_s + k_{p-s} - k_p). \tag{2.21}
\]
This is a linear estimate, since we have regarded the amplitudes of $b_s$ and $b_{p-s}$ as constants in the integration of (2.13).

### The asymptotic behavior of the phase mismatch

The asymptotic behavior of the phase mismatch can be studied using the linear dispersion relation. For Stokes waves this reads $\omega^2 h/g = k h \tanh k h$. Defining $\kappa = k h$ and $\Omega = \omega \sqrt{h/g}$ gives

$$\Omega = (\kappa \tanh \kappa)^{1/2} = \kappa (1 - \frac{1}{6} \kappa^2) + O(\kappa^5).$$

To lowest order we have $\Omega = \kappa + O(\kappa^3)$ which can be substituted in the last term. Hereby we can invert the dispersion relation to

$$\kappa = \Omega (1 + \frac{1}{6} \Omega^2) + O(\Omega^5).$$

For the phase mismatch we now get

$$\kappa_s + \kappa_{p-s} - \kappa_p = \frac{1}{6} (s^3 + (p - s)^3 - p^3) \Omega_s^3 + O(\Omega_s^5) = \frac{1}{2} p s (s - p) \Omega_s^3 + O(\Omega_s^5)$$

where $\Omega_s = \Omega_p/p$. Hence, the phase mismatch varies like $\Omega^3$ and thereby like $\kappa^3$.

### The strength of the energy exchange

With the above results, we can now give a measure of the strength of the energy exchange between the harmonics. We consider the explicit solution to the crude one-equation model (2.20). We introduce the scaling

$$b_p = \frac{a_0}{h_0} \sqrt{gh_0} b_{p,\text{nondim}} = \varepsilon \sqrt{gh_0} b_{p,\text{nondim}}$$

which is the standard normalization for shallow water waves, see e.g. Madsen and Schäffer (1998). Further, we use that $\Gamma_s = 1 + O(\kappa_s^2)$ and $\tilde{\alpha}_{1,p} = -2 i g \kappa_p + O(\kappa_p^3)$, see (2.11b) and (2.9). Hereby, (2.20) can be written

$$b_{p,\text{nondim}} \sim \varepsilon \frac{\kappa_p (\kappa_s + \kappa_{p-s} - \kappa_p)}{\kappa_p (\kappa_s + \kappa_{p-s} - \kappa_p)} b_{k,\text{nondim}} b_{p-s,\text{nondim}} \sim \varepsilon \frac{a_0 l_0^2}{h_0^3} \sim \frac{\varepsilon}{\kappa_p^2}$$

where the second simplified result is obtained using (2.24). The above result shows that the strength of the energy exchange is governed by the Ursell number. The nonlinear energy exchange is a growing function of nonlinearity and a decreasing function of dispersion. The latter is due to the departure from near resonance for increasing dispersion. Further, we see that the strength is governed by $a_0 l_0^2/h_0^3 = \varepsilon/\mu^2$, and thus by the balance between $\varepsilon$ and $\mu^2$. This result has also been derived by Whitham (1973, see page 473), who calculated the strength of the second harmonic and higher harmonics for the Korteweg-deVries equation.

### 2.5 The importance of the approximations

In the following, we present results of the different models, demonstrating the effects of the approximations applied. We concentrate on results for harmonic generation and base the first description on the single test, already depicted in Figure 2.1. Two more tests have been carried out. The results of these are given in Appendix B and summarized in Section 2.5.6.
2.5.1 Approximation 1: Influence of derivatives in nonlinear terms

With the implementation of the boundary value problem formulation, the influence of derivatives in the nonlinear terms is easily investigated. In Figure 2.2, results with and without inclusion of the nonlinear derivatives are compared. The effect of neglecting these terms is clearly seen. The transfer of energy to the higher harmonics is over-predicted and also the beat length is overestimated.

Figure 2.2: Influence of approximation 1. Boundary value problem formulation with and without derivatives in nonlinear terms.

2.5.2 Approximation 2: Influence of the higher-order linear derivatives

To investigate the effect of neglecting the higher-order linear derivatives, we compare results of the boundary value problem formulation without nonlinear derivatives to results of the two-equation formulation. The difference between these two models is exactly the $b_{p,xx}$ and $b_{p,xxx}$ terms appearing in (2.3). The results are shown in Figure 2.3. We see that for the first and second harmonic, approximation 2 does not influence the amplitudes much. The beat length is decreased slightly. For the third and fourth harmonics, the amplitudes are reduced, thus showing that the energy transfer between the harmonics is reduced. These changes are the opposite as those implied by approximation 1. Comparing Figure 2.3 to Figure 2.2, approximation 1 seems to influence the results the most. This is substantiated by the results of Figure 2.4, where the results of the full boundary value problem formulation and the two-equation model are compared. This plot shows the combined influence of approximation
2.5 The importance of the approximations

Figure 2.3: Influence of approximation 2. Comparison of boundary value problem formulation excluding derivatives in nonlinear terms with the two-equation model.

1 and 2. The net effect is to increase the energy transfer between the harmonics and to increase the beat length.

For completeness and later use, we present comparisons of the velocity amplitudes in Figure 2.5. Again, we compare the boundary value problem formulation without nonlinear derivatives to the two-equation model. For the first three harmonics, the qualitative differences between the boundary value formulation and the approximate results are very similar to the differences found for the free surface elevation amplitudes, while for the fourth harmonic, the amplitude is slightly under-predicted for the velocity, while being slightly over-predicted for the free surface elevation. As this discrepancy is rather small, the general deviations are regarded as practically similar for the two flow quantities.

2.5.3 Approximation 3: Elimination of $a_p$ in the nonlinear terms by a linear approximation

The combined effect of approximation 1, 2 and 3 can be assessed by comparing results of the full boundary value problem formulation to results of the one-equation model. Since this model is formulated in the velocity amplitude $b_p$, the comparison is studied for this variable rather than the surface elevation amplitude $a_p$. The comparison is presented in Figure 2.6. The energy exchange to the higher harmonics is overestimated and also the beat length is overestimated. This is in line with the general influence implied by approximation 1. Further, when comparing to the results of Figure 2.5, we see that the deviations to the boundary value problem results are larger for the one-equation model than for the two-equation model.
Figure 2.4: Comparison of two-equation model with boundary value problem formulation.

Figure 2.5: Comparison of two-equation model with full boundary value problem formulation. Velocity amplitudes.
2.5 The importance of the approximations

2.5.4 Results for the crude one-equation model

In Figure 2.7 the accuracy of the crude one-equation model is examined by comparing its results to the results of the full boundary value problem formulation. The results compares better to the boundary value problem results than for the two-equation model and the one-equation model. Comparing the results of Figures 2.6 and 2.7 we observe that the neglect of $b_{p,xx}$ rather than rewriting it using the approximate integration procedure decreases the transfer of energy and reduces the beat length. These reductions in beat length and energy transfer almost eliminates the overestimation of these quantities obtained with the one-equation model. Hence the neglect of the second derivative seems to have a large effect on the results.

2.5.5 Two other tests of harmonic generation

Besides the test just presented, two more tests on harmonic generation with differing parameters were carried out. One had a larger wave period, thus representing a more shallow water situation, while the other had a stronger nonlinearity. The parameters are lined up in Table 2.1 along with the linear wave length of the first harmonic, $l_0$, the nonlinearity parameter $\varepsilon = a_0/h_0$, the dispersion parameter $\mu = h_0/l_0$, the Ursell number $U = \varepsilon/\mu^2$, the linearly estimated beat length $L_{\text{beat, est}}$ and the observed beat length $L_{\text{beat, obs}}$. The various comparisons of the models are presented in Appendix B. In the following the results are outlined.
Figure 2.7: Comparison of crude one-equation model to boundary value problem formulation.

<table>
<thead>
<tr>
<th>Case</th>
<th>$h$</th>
<th>$T$</th>
<th>$a_0$</th>
<th>$l_0$</th>
<th>$\varepsilon$</th>
<th>$\mu$</th>
<th>$U$</th>
<th>$L_{\text{beat, est}}$</th>
<th>$L_{\text{beat, obs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test A</td>
<td>0.4</td>
<td>2.5</td>
<td>0.042</td>
<td>4.74</td>
<td>0.11</td>
<td>0.084</td>
<td>14.8</td>
<td>15.4</td>
<td>14</td>
</tr>
<tr>
<td>Shallow</td>
<td>0.4</td>
<td>4.0</td>
<td>0.021</td>
<td>7.79</td>
<td>0.053</td>
<td>0.051</td>
<td>19.9</td>
<td>72.0</td>
<td>60</td>
</tr>
<tr>
<td>Nonlinear</td>
<td>0.4</td>
<td>2.5</td>
<td>0.084</td>
<td>4.74</td>
<td>0.21</td>
<td>0.084</td>
<td>29.5</td>
<td>15.4</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 2.1: Parameters for tests on harmonic generation.
2.5 The importance of the approximations

The nonlinear test  For calculating the nonlinear test with the boundary value problem formulation, four harmonics were used. Due to convergence problems, it was not possible to increase the number of harmonics. Also, no convergence was obtained when using free wave numbers as reference wave numbers for the full boundary value problem formulation. Therefore, bound wave numbers were used. This, however, is not expected to influence the results, since no terms are neglected in the full boundary value problem formulation. For this model, the choice of wave numbers do therefore not influence the results. This is demonstrated in Section 2.6.1 for ‘test A’. For the calculations without derivatives in the nonlinear terms, no convergence problems with free wave numbers were detected. Hence, free wave number results are used for this test case. For consistency, the same number of harmonics were used in the other models for the comparisons for this test case. A grid spacing of $\Delta x = 0.2$ m was used. In Figure 2.8, a comparison with time domain calculations for the four harmonics is shown. In the time domain calculation, the grid size was $\Delta x = 0.05$ m and the time step $\Delta t = 0.025$ s. The results of the boundary value problem formulation do not match the time domain results well. The energy exchange is rather similar between the two models, but the beat length of the boundary value problem formulation is under-estimated. The reason is probably the low number of harmonics for the boundary value problem formulation. This may weaken the conclusions drawn for this test, since they are based on results of the boundary value problem formulation as reference solution. However, the calculations presented are accurate regarding the interactions between the four harmonics retained. As the other evolution equation models are truncated in the same way, the comparison is consistent, even though it may partly deviate from the physical solution of the physical test.

The shallow water test  The shallow water test was calculated with 5 harmonics in all models. Also this was due to convergence problems for the boundary value problem formulation. The grid spacing was $\Delta x = 0.4$ m and the total length of the domain was 80 m, with a sponge layer of 20 m in the right end. No comparisons with time domain calculations have been made. However, as the different evolution equation models are found to agree rather well for this test, the comparison with a time domain calculation is expected to be of similar quality as for test A.

The plots for judging the effects of approximation 1, 2 and 3 are given in Appendix B. In this appendix the plots of test A are also given for completeness. The exchange of energy between the harmonics, relative to the amplitude of the first harmonic is strongest for the nonlinear test, followed by the shallow test. The smallest energy exchange occurs for test A. This agrees with the values of the Ursell numbers, being largest for the nonlinear test and smallest for test A. For all tests the observed beat lengths are smaller than the estimated beat lengths. The deviation between the observed and the estimated beat length, normalized with the estimated beat length is 9%, 16% and 22% for test A, the shallow test and the nonlinear test, respectively, again matching the increasing value of the Ursell number. Hence the nonlinearity has a reducing effect on the beat length.
Figure 2.8: Comparison of boundary value problem formulation and time domain model for the nonlinear test case of harmonic generation.
2.5.6 Summary of approximation effects

The effects of the various approximations are summarized in Table 2.2 for the three test cases. From the three tests we deduce that the omission of derivatives in the nonlinear terms (approximation 1) increases the energy exchange and beat length. Neglecting the higher-order linear derivatives (approximation 2) reduces the energy exchange, but with a smaller amount than the increase implied by approximation 1. Approximation 2 also implies a reduction of the beat length. This change can be both larger and smaller than for approximation 1.

When the velocity amplitudes $b_p$ are considered rather than the free surface elevation amplitudes $a_p$, the relative magnitude of the higher harmonics are smaller. This is probably due to the fact that the velocity variable is the velocity at $z = -0.553h$, i.e., in the middle of the water column. Here, the faster decay of the velocity profile for the higher harmonics makes the velocity amplitudes appear weaker than the primary wave component. Elimination of the free surface elevation amplitude (i.e. $2\text{EQ} \rightarrow 1\text{EQ}$) implies an increase in beat length, while the energy exchange is unaffected. Neglecting $b_{p,xx}$ rather than approximating it (i.e. $1\text{EQ} \rightarrow \text{crude 1EQ}$) reduces the beat length and the energy exchange.

For the three approximate models, the deviations to the boundary value problem formulation are due to the combined effect of the approximations involved. The two-equation model is obtained using approximation 1 and 2, and we see that it generally overestimates the energy exchange while the beat length can be both under-predicted and over-predicted. The one-equation model has a similar over-prediction of the energy exchange as well as an over-prediction of the beat length. Further, in the nonlinear test, this model shows some strange phase problems. Ironically, the crude one-equation model has a rather good estimation of the beat length. Even though the energy exchange is under predicted, the qualitative variation of the harmonics is well modelled. It is now relevant to ask which model is the best to use.

As we have seen in Section 2.3, the higher-order linear derivatives cannot be retained. Hence, approximation 2 cannot be avoided. There is a chance of avoiding approximation 1 by retaining the derivatives in the nonlinear terms. These could be estimated by a backward finite difference approximation. This may provide a good model, since approximation 1 has been found to affect the results more than approximation 2. Judging from Figures B.2 and B.8, this would lead to model with a slight underestimation of the beat length and a slight under-prediction of the energy exchange. From the observations, the one-equation model cannot be recommended due to its relatively large over-prediction of the beat length. Here, the crude one-equation model gives a fine resemblance for the beat length, even though the energy exchange is under predicted. The results of this model are of similar quality as those of the two-equation model. Hence, these two models are the best candidates of the three models considered. However, as we shall see in the next section, the two-equation model has some problems on varying depth, making it less attractive for sloping beds.
Chapter 2. Boussinesq Based Models

<table>
<thead>
<tr>
<th>test</th>
<th>approx 1</th>
<th>approx 2</th>
<th>approx 1+2</th>
<th>(a_p \rightarrow b_p)</th>
<th>(2\text{EQ} \rightarrow 1\text{EQ})</th>
<th>(1\text{EQ} \rightarrow \text{crude 1EQ})</th>
</tr>
</thead>
<tbody>
<tr>
<td>test A</td>
<td>Exch ↑ beat ↑</td>
<td>Exch ↓ beat ↓</td>
<td>Exch ↑ beat ↑</td>
<td>BVP: harmonics</td>
<td>Exch (↑) beat ↑</td>
<td>Exch ↓ beat ↓</td>
</tr>
<tr>
<td></td>
<td>(not as much as for approx 1)</td>
<td>(4th harmonic very good)</td>
<td></td>
<td>relatively smaller for velocity than for elevation</td>
<td></td>
<td>Phase is good, harmonics under-predicted</td>
</tr>
<tr>
<td>shallow</td>
<td>Exch ↑ beat ↑</td>
<td>Exch → beat →</td>
<td>Exch ↑ beat ↑ (as for approx 1)</td>
<td>similar for velocity as for elevation</td>
<td>Exch → beat →</td>
<td>Exch ↓ beat →</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Phase is good, harmonics slightly under-predicted</td>
</tr>
<tr>
<td>nonlinear</td>
<td>Exch ↑ beat ↑</td>
<td>Exch → beat ↓</td>
<td>Exch ↑ (harm 1,2,3) ↓ (harm 4)</td>
<td>BVP: harmonics</td>
<td>Exch ↓ beat ↑</td>
<td>Exch ↓ beat ↓</td>
</tr>
<tr>
<td></td>
<td>(similar to approx 1)</td>
<td></td>
<td>beat ↓</td>
<td>relatively smaller for velocity than for elevation</td>
<td></td>
<td>Phase is good, harmonics under-predicted</td>
</tr>
<tr>
<td>conclusion</td>
<td>Exch ↑ beat ↑</td>
<td>Exch →/↓ beat ↓</td>
<td>Exch ↑ beat ↑ / ↓</td>
<td>BVP: harmonics</td>
<td>Exch ↑ /↓ beat ↑</td>
<td>Exch ↓ beat ↓</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>relatively smaller for velocity than for elevation</td>
<td></td>
<td>Phase is good, harmonics under-predicted</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of effects of approximations. ↑: increasing, ↓: decreasing, →: unchanged.
2.6 More results of the two-equation model

The above analysis is for constant depth. Among the three approximate models, the two-equation model and the crude one-equation model are the most attractive. The two-equation model has the advantage of being very close to the underlying time domain model. In this section some extra results of the two-equation model are presented. First, it is demonstrated that the choice of reference wave numbers affects the solution, and second, it is shown that the two-equation structure leads to problems on varying depth.

2.6.1 Bound or free reference wave numbers

The reference wave numbers, \(k_p\), appearing in the solution ansatz (2.1) are usually chosen as free wave numbers, i.e., as solutions of the linear dispersion relation at the local depth. For modelling a steady higher-order Stokes wave, however, it would be more natural to choose bound wave numbers as reference wave numbers, i.e., \(k_p = pk_1\) where \(k_1\) is the free wave number of the first harmonic.

For the two approximate one-equation models, we are compelled to using free reference wave numbers, since this was utilized in the derivation of the models. The two-equation model and the boundary value problem formulation, however, leave the choice of the reference wave numbers free. In the boundary value problem formulation, the choice of reference wave numbers does not influence the results, since the model is equivalent to the time domain formulation which is ‘wave number free’. This is demonstrated in Figure 2.9, where the wave amplitudes obtained with bound and free reference wave numbers are plotted. The deviations are very small. The most pronounced deviations are small phase differences for the third and fourth harmonics. These deviations are probably due to truncation errors of the spatial discretization (\(\Delta x = 0.2\) m). The amplitude functions for the two sets of wave numbers are not identical with respect to phase, and therefore the discretization errors are different.

For the two-equation formulation, the results depend on the choice of reference wave numbers, independently of the grid. This is illustrated in Figure 2.10. The results calculated with free reference wave numbers predict a larger energy transfer to the higher harmonics and a larger beat length. From a first glance, this is surprising. Imagine that a given wave component is expressed in terms of a free and a bound wave number amplitude function

\[\eta(x, t) = a_{\text{free}}(x)e^{i(k_{\text{free}}x - \omega t)} = a_{\text{bound}}(x)e^{i(k_{\text{bound}}x - \omega t)} \tag{2.27}\]

From this equation we get

\[a_{\text{bound}}(x) = a_{\text{free}}(x)e^{i(k_{\text{free}} - k_{\text{bound}})x} \tag{2.28}\]

and obviously \(|a_{\text{free}}| = |a_{\text{bound}}|\). We would therefore expect the plots of the modulus of the amplitude functions to be identical. The deviation, however, can be explained by the truncation of derivatives of the amplitude functions in the two-equation model. Observing (2.28),
Figure 2.9: Comparison of results of the boundary value problem formulation using bound and free reference wave numbers.

Figure 2.10: Comparison of results of two-equation model using bound and free reference wave numbers.
we see that \((\partial/\partial x)^n a_{\text{bound}}(x) \neq (\partial/\partial x)^n a_{\text{free}}(x)\) and the influence of neglecting derivatives in the model therefore depends on the choice of reference wave numbers. In general, one should choose reference wave numbers representing the true solution as accurately as possible, since this would result in the slowest variation of the amplitudes and thus in the smallest magnitude of the derivatives neglected.

### 2.6.2 The two-equation model on varying depth

The two-equation model (2.6) is derived for varying depth. In the following we present results for a test of wave propagation over a submerged bar. The parameters of the test are the parameters of test ‘A’ of Luth et al. (1994) with an initial depth of \(h_1 = 0.4\) m and a depth of \(h_2 = 0.1\) m on top of the bar. The upward slope of the bar is 1:20, while the downward slope is 1:10. The incoming wave has a period of \(T = 2.02\) s and a wave height of \(H = 0.02\) m.

Conventionally, the assumption of slowly varying wave amplitudes and slowly varying depth leads to neglect of \(h_x\) and \(k_x\)-terms in the coefficients multiplying on \(a_{p,x}\) and \(b_{p,x}\) in the evolution equations. This would correspond to retaining the depth slope terms in \(\alpha_{2,p}\) and \(\beta_{2,p}\) only, see (2.4). In the calculations shown here, however, the \(h_x\) and \(k_x\) terms of \(\alpha_{12,p}\) and \(\beta_{12,p}\) are retained as well. This is motivated by the initial wish for modelling fast variations of the depth and amplitude functions as good as possible. For a linear test on the bar, the difference between retaining and leaving out these extra terms was smaller than 5\% for the wave amplitudes.

**Spurious reflections in the model.** To check the performance of the linear shoaling terms in the model, a linear test with just one harmonic was carried out. The resulting amplitude variation is shown in Figure 2.11 together with two representations of the bar. Besides the usual bar, a soft bar is shown, having no corners. It is given by

\[
h = \begin{cases} 
0.4 - 0.3 \exp \left[ - (0.21(x - 13\ m))^2 \right] & x \leq 13\ m \\
0.1 & 13 < x \leq 13.5 \\
0.4 - 0.3 \exp \left[ - (0.4(x - 13.5\ m))^2 \right] & x > 13.5\ m 
\end{cases} \tag{2.29}
\]

The results for the physical ‘hard’ bar shows some spurious oscillations in the wave amplitude function on top of and behind the bar. These oscillations are mainly due to the discontinuity of the bed-slope at the corners of the bar and correspond to a reflection in the time domain. For a single harmonic, the linearized model can be written

\[
\begin{pmatrix} a_p \\ b_p \end{pmatrix}_x = A_p \begin{pmatrix} a_p \\ b_p \end{pmatrix}. \tag{2.30}
\]

The linear solution components vary like \(e^{\lambda x}\) where \(\lambda\) is an eigenvalue of \(A_p\). Since \(A_p\) is a \(2 \times 2\)-matrix it has two eigenvalues, and for free reference wave numbers one of these is zero.
This corresponds to the natural linear wave mode of the equations. The other eigenvalue, however, represents a spurious, unphysical wave mode. This wave mode is excited when sudden depth-changes are encountered and since we solve for the wave field in a forward space marching procedure, this ‘reflection’ is not allowed to propagate backwards from the bar. Consequently the excitement of the spurious wave mode is carried along forward in the domain. In Figure 2.11 results for the smooth representation of the bar are shown as well. The smoothing of the bar reduces the oscillations significantly.

**Comparison with time domain results.** In Figure 2.12 results of the two-equation model are compared to the results of a time domain simulation. The parameters are the same as above, the incoming wave being a second-order Stokes wave. For this incident wave the second harmonic is a bound wave component and therefore bound reference wave numbers were used in the two-equation model. The results compare quite satisfactory. In the region behind the bar, the transfer of energy to the second harmonic is over-predicted, while the transfer of energy to the third harmonic is under-predicted. However, the amplitude of the third harmonic behind the bar seems to be rather sensitive to the number of harmonics included in the calculations. This should be investigated further.
2.7 Summary and conclusions

Nwogu’s (1993) equations have been transformed to the frequency domain and implemented as a boundary value problem. This model is completely equivalent to the time domain formulation, and has thus been used to examine the influence of the approximations involved in deriving evolution equations. These approximations are 1) neglect of derivatives in the nonlinear terms, 2) neglect of higher-order linear derivatives, 3) elimination of the free surface amplitude and 4) further neglect of \( b_{p,xx} \), (i.e. the second derivative of the velocity amplitude).

For three tests of harmonic generation, it has been found that approximation 1 increases the energy exchange and beat length, while approximation 2 reduces both of these quantities. The increase in energy exchange implied by approximation 1 is larger than the decrease implied by approximation 2. Elimination of the free surface elevation amplitude (approximation 3) increases the beat length, and can both increase and decrease the energy exchange. Further neglect of \( b_{p,xx} \) (approximation 4) reduces the energy exchange and beat length.

It is not possible to retain the higher-order derivatives in the linear terms, if a spatially marching solution procedure is to be used. This is due to the imaginary wave number solutions of the dispersion relation, which give exponentially growing wave amplitudes and therefore result in instability. Hence approximation 2 cannot be avoided.

Figure 2.12: Results for a smooth representation of the bar.
Three approximate models have been considered: a two-equation model (approximation 1 and 2) a one-equation model (approximation 1, 2 and 3) and the so-called crude one-equation model (approximation 1, 2, 3 and 4). Summarizing, the two-equation model and the crude one-equation model are the best of these approximate models. This shows that the influence of the various approximations can compensate for each other. There is a possibility of extending the two-equation model to include derivatives in the nonlinear terms using a finite difference approximation. The resulting model would then only be affected by approximation 2. However, as the two-equation model has problems with spurious reflections on varying depth, this extension does not seem attractive. It has therefore not been pursued further.

Further, it has been demonstrated for the two-equation model that the choice of reference wave numbers affects the results of the model. This is not a shortcoming of the model, but just a consequence of the possibility of choosing the reference wave numbers freely.

One of the original intentions for this Ph.D. study was to derive evolution equations for higher-order Boussinesq formulations, matching their accuracy. However, from the results of this chapter it seems unlikely that the accuracy of higher-order Boussinesq equations can be retained in the corresponding evolution equations. Linearly, the dispersion relation is well preserved, but the weak nonlinearity and the need for neglecting the higher-order linear derivatives limit the accuracy of the models. Nevertheless, we have seen that a very simple model can reproduce the wave field rather satisfactorily. Hence there is still good reason for using evolution equations, if a less accurate solution can be accepted for the advantage of a fast numerical model. For this reason, the attention was therefore redirected towards increasing the computational efficiency for more simple models. We go further into the aspects of computational efficiency in Chapter 3.
Chapter 3

Efficiency of Evolution Equations

The use of evolution equations is usually motivated by their simple formulation and their computational efficiency. The latter point, however, usually only holds for wave fields consisting of a few frequency components, since the direct calculation of the quadratic interaction sums requires a computational effort of $O(N^2)$, where $N$ is the number of frequencies. The interaction sums are convolution sums in the frequency domain, and within spectral methods for partial differential equations, it has been known since the late sixties, see e.g. Canuto et al. (1987), that such sums can be handled with a computational effort of $O(N \log N)$ using the FFT-algorithm. However, until now, this knowledge has not been used within the field of spatial evolution equations for water waves.

A speed up from $O(N^2)$ to $O(N \log N)$ is practically as good as to $O(N)$, since log $N$ grows very slowly with $N$. Hence, such a speed up has a tremendous impact on calculation times for large values of $N$ and certainly extends the applicability of evolution equations. In this chapter we demonstrate how this speed up technique can be used in evolution equations.

With this speed up, evolution equations can handle a physical wave field with $N$ frequencies and $M$ spatial points (in one horizontal dimension) with a computational effort of $O(MN \log N)$. This is attractive, but standard time domain models are able to do simulations of a similar wave field at an asymptotic work load of $O(MN)$. This is easily explained: If the highest frequency in the spectrum is to be resolved with, say, $n$ time steps, the total number of time steps needed is

$$\frac{T}{\Delta t} = \frac{1}{f_{\text{min}}} \frac{1}{T_{f_{\text{max}}}/n} = \frac{f_{\text{max}}}{f_{\text{min}}} n = nN.$$  

In a finite difference implementation, the discretization of the spatial differential operators results in a banded matrix with side lengths proportional to $M$. Thereby, such models can solve the same physical problem with a computational effort of $O(MN)$.

From this asymptotic consideration, one may argue that time domain models should be preferred to evolution equations. Besides the equivalent (or slightly better) asymptotic work
load, these models are able to describe reflections of the waves and are not confined to weak nonlinearity. Such a statement, however, calls for a closer investigation. One thing is the asymptotic behavior, but if the constants multiplying on this behavior differ significantly between the models, there may be large differences in the computational effort for practical applications.

This chapter deals with these aspects of evolution equations. The evolution equations of Madsen and Sørensen (1993) are used for the practical computations. In Section 3.1, the derivation of this model is therefore given. In Section 3.2 it is shown how the nonlinear terms of this model can be calculated using the FFT-algorithm. The concept of aliasing is also discussed. Two practical tests of the efficiency of evolution equations with embedded FFT transformations are presented in Sections 3.3 and 3.4. The tests cases are chosen as regular and irregular waves passing a submerged bar. For the regular test it is found that the evolution equations can calculate the stationary wave field 1000 times faster than the time domain model, while the gain in speed is a factor 100 for the irregular wave case.

3.1 The evolution equations of Madsen and Sørensen (1993)

The deterministic evolution equations of Madsen and Sørensen (1993) are based on a Boussinesq formulation in the depth-integrated velocity \( P \) and the free surface elevation \( \eta \). The Boussinesq formulation is valid on slowly varying depth and incorporates Padé[2,2] dispersion. The model consists of two conservation equations for mass and momentum

\[
\eta_t + P_x = 0 \tag{3.1a}
\]

\[
P_t + \left( \frac{P^2}{h + \eta} \right)_x + g (h + \eta) \eta_x - Bgh^3 \eta_{xxx} - (B + 1/3)h^2 P_{xxt} - h_x \left( 2Bgh^2 \eta_{xx} + \frac{1}{3} h P_{xt} \right) = 0, \tag{3.1b}
\]

where \( \eta \) is measured upwards from the still water level, \( h \) is the undisturbed depth and \( g \) is the acceleration of gravity. \( B \) is a free parameter governing the linear dispersion properties of the equation. For \( B = 1/15 \), the Padé[2,2]-dispersion relation is achieved.

To derive the evolution equations, the two equations (3.1) are combined into a single wave equation, by eliminating \( P \) in the linear terms. This wave equation is given in equations (2.6a)–(2.6d) of the original paper and reads

\[
\eta_t - gh \eta_{xx} + Bgh^3 \eta_{xxxx} - (B + 1/3)h^2 \eta_{xxtt} - (g \eta_x + (2B + 1)h \eta_{xtt} - 5Bgh^2 \eta_{xxx}) h_x = \left( \frac{1}{2} g \eta^2 + \frac{P^2}{h + \eta} \right)_{xx}, \tag{3.2}
\]
where the terms in the first line are the linear terms, and the terms in the second line are the bottom slope terms and the nonlinear terms, respectively. To derive this equation, only the lowest-order terms in \( h_x \) have been preserved in the linear terms, while no effect of varying depth has been included in the nonlinear terms.

**Transformation to the frequency domain** To transform the model to the frequency domain, we substitute the following series for \( \eta \) and \( P \):

\[
\eta(x, t) = \sum_{p=-N}^{N} a_p(x) e^{i(\omega_p t - \int k_p dx)} \quad \quad P(x, t) = \sum_{p=-N}^{N} b_p(x) e^{i(\omega_p t - \int k_p dx)}.
\]

To help the comparison with Madsen and Sørensen (1993) we multiply the wave equation by \((-1)\) before inserting the sums. By evaluating all the derivatives and collecting terms we get

\[
\sum_{p=-N}^{N} \left\{ \beta_{4,p} a_{p,xxxx} - i\beta_{3,p} a_{p,xxx} - \beta_{2,p} a_{p,xx} + i\beta_{1,p} a_{p,x} 
- i\beta_{k,p} k_p a_p - \beta_{h,p} h_x a_p + \beta_{0,p} a_p \right\} e^{i(\omega_p t - \int k_p dx)} = -\left( \frac{1}{2} g \eta^2 + \frac{P^2}{h + \eta} \right)_{xx}
\]

where the nonlinear terms have not yet been treated. Again, only the leading order bottom-slope terms have been retained in the linear terms, while no effects of varying depth have been retained in the nonlinear terms. The \( \beta \)-coefficients are given by

\[
\begin{align*}
\beta_{4,p} &= -Bgh^3 \\
\beta_{3,p} &= -4Bgh^3 k_p \\
\beta_{2,p} &= -(gh + 6Bgh^3 k_p^2 - (B + 1/3)h^2 \omega_p^2) \\
\beta_{1,p} &= -2(ghk_p + 2Bgh^3 k_p^3 - (B + 1/3)h^2 \omega_p^2 k_p) \\
\beta_{k,p} &= gh + 6Bgh^3 k_p^2 - (B + 1/3)h^2 \omega_p^2 \\
\beta_{h,p} &= gk_p + 5gBh^2 k_p^3 - (2B + 1)h \omega_p^2 k_p \\
\beta_{0,p} &= \omega_p^2 - ghk_p - Bgh^3 k_p^4 + (B + 1/3)h^2 \omega_p^2 k_p^2
\end{align*}
\]

where the expressions for \( \beta_{4,p}, \ldots, \beta_{0,p} \) agree with Madsen and Sørensen (1993). We note that similar to the derivation in Section 1.2, the equation \( \beta_{0,p} = 0 \) establishes the linear dispersion relation of the wave model.

For the nonlinear terms we apply the linear approximation \( b_p = \frac{\omega_p}{h} a_p \) derived from (3.1a) neglecting the spatial derivative of \( b_p \). For this second-order model it is consistent to use this linear approximation in the nonlinear terms. Furthermore, still consistent to second order, we replace \((h + \eta)\) by \( h \) in the denominator of the second nonlinear term. We also neglect spatial derivatives of \( a_p \) in the nonlinear terms, as well as derivatives of \( h \) and \( k \), still being
consistent with the derivation of the original model (3.1). Applying these approximations we get

\[
- \left( \frac{1}{2} g \eta^2 + \frac{P^2}{h + \eta} \right)_{xx}
= - \sum_{p=-2N}^{2N} e^{i \omega_p t} \sum_{s=p-N}^{N} \left( \left( \frac{1}{2} g a_s a_{p-s} + \frac{1}{h k_s} \frac{\omega_p}{k_{p-s}} a_s a_{p-s} \right) e^{-i \int (k_s + k_{p-s}) dx} \right)_{xx}
= g \sum_{p=-2N}^{2N} e^{i \omega_p t} \sum_{s=p-N}^{N} (k_s + k_{p-s})^2 \left( \frac{1}{2} + \frac{1}{gh k_s} \frac{\omega_p}{k_{p-s}} a_s a_{p-s} \right) e^{-i \int (k_s + k_{p-s}) dx}.
\]  

(3.6)

The details for calculating the product of two Fourier series can be found in Appendix A. Now, by demanding that the coefficients to all powers of \( e^{i \omega_p t} \) in equation (3.4) vanish, we get after division by \( e^{-i \int k_p dx} \)

\[
\beta_{4,p} a_{p,xxxx} - i \beta_{3,p} a_{p,xxx} - \beta_{2,p} a_{p,xx} + i \beta_{1,p} a_{p,x} - i \beta_{h,p} h_x a_p + \beta_{0,p} a_p
= g \sum_{s=p-N}^{N} (k_s + k_{p-s})^2 \left( \frac{1}{2} + \frac{1}{gh k_s} \frac{\omega_p}{k_{p-s}} a_s a_{p-s} \right) e^{-i \int (k_s + k_{p-s}) dx},
\]

\[p = 0, \ldots, N. \]  

(3.7)

**Final approximations and manipulations** The model is further simplified by neglecting all but first-order derivatives in \( a_p \). This is justified by the assumption of weak nonlinearity and slowly varying depth, thus resulting in a slow variation of \( a_p \). Furthermore, we rewrite the nonlinear interaction sum, such that summation is formulated only in positive indices for \( a_p, k_p \) and \( \omega_p \). This rewriting is similar to the one applied in Section 1.2 and is described in Appendix A in detail. We get

\[
i \beta_{1,p} a_{p,x} - i \beta_{h,p} h_x a_p + \beta_{0,p} a_p
= g \sum_{s=0}^{p} (k_s + k_{p-s})^2 \left( \frac{1}{2} + \frac{1}{gh k_s} \frac{\omega_s \omega_{p-s}}{k_{p-s}} \right) a_s a_{p-s} e^{-i \int (k_s + k_{p-s} - k_p) dx} + 2g \sum_{s=1}^{N-p} (k_{p+s} - k_s)^2 \left( \frac{1}{2} + \frac{1}{gh k_s k_{p+s}} \right) a_s a_{p+s} e^{-i \int (k_{p+s} - k_s - k_p) dx}.
\]  

(3.8)

Finally, the terms related to \( h_x \) and \( k_x \) are combined into one shoaling term. This is done by differentiating the linear dispersion relation (given by \( \beta_{0,p} = 0 \)), thus giving a relation
3.2 Calculating the nonlinear terms using FFT

between $h_x$ and $k_x$. The model can then be written

$$a_{p,x} = -\frac{i \beta_{s,p}}{\beta_{1,p}} a_p - \frac{h_x}{h} a_p$$

$$- \frac{i}{\beta_{1,p}} g \sum_{s=0}^{p} (k_s + k_{p-s})^2 \left( \frac{1}{2} + \frac{1}{gh} \frac{\omega_s \omega_{p-s}}{k_s k_{p-s}} \right) a_s a_{p-s} e^{-i f(k_s + k_{p-s} - k_p) dx}$$

(3.9)

$$- \frac{2i}{\beta_{1,p}} g \sum_{s=1}^{N-p} (k_{p+s} - k_s)^2 \left( \frac{1}{2} + \frac{1}{gh} \frac{\omega_s \omega_{p+s}}{k_s k_{p+s}} \right) a_s^* a_{p+s} e^{-i f(k_{p+s} - k_s - k_p) dx}$$

(3.9)

with

$$\beta_{s,p} = \left( 1 + (4B - 1)k_p^2 h^2 + \frac{6}{3} B k_p^4 h^4 + \frac{6}{3} B^3 + \frac{4}{3} B^2 + \frac{1}{3} B^6 \right) k_p^6 h^6$$

$$+ (B^4 - \frac{1}{3} B^2) k_p^8 h^8 \right) / 4 \left( 1 + 2Bk_p^2 h^2 + (B^2 + \frac{1}{3} B) k_p^4 h^4 \right)^2$$

(3.10)

agreeing with the result of Madsen and Sørensen (1993), page 384.

3.2 Calculating the nonlinear terms using FFT

We now turn the focus towards speeding up the calculation of the nonlinear terms. Usually, the nonlinear convolution sums in the model are calculated explicitly as they appear in (3.9). The work amount of this is $O(N^2)$. However, within spectral methods for partial differential equations, it is known that such convolution sums can be handled with a computational effort of $O(N \log N)$ using the Fast Fourier Transform algorithm (FFT). This technique was developed independently by Orszag (1969, 1970) and Eliasen et al. (1970) (Canuto et al., 1987). Fornberg and Whitham (1978) used a variant of this method for time stepping the KdV-equation. The equation was formulated in time domain variables, but instead of approximating the $\eta_x$ and $\eta_{xxx}$ terms with finite differences, $\eta$ was Fourier transformed to the $k$-space, and differentiated by multiplying the coefficients with $ik$. The resulting set of Fourier coefficients were then transformed back to the spatial domain using inverse Fourier transform. In this way $\eta_x$ and $\eta_{xxx}$ were obtained with spectral accuracy with a work load of $O(N \log N)$ due to the use of FFT. However, within the field of spatial evolution equations, the speed-up technique of applying FFT to the nonlinear terms does not seem to have been used earlier.

The idea behind the speed-up technique is rather simple: As we have seen in the derivation of the evolution equations of Madsen and Sørensen (1993), the nonlinear summation term, as a whole, is nothing but the $p$'th Fourier component of $-(g\eta^2 + P^2/h)_{xx}$ divided by $i \beta_{1,p}$, see (3.4). We write this in a more concise form in the following. First, we define the Fourier
components $\hat{\alpha}_p$ of the nonlinear term by

$$
(gq^2/2 + P^2/h)_{xx} = \sum_{p=-2N}^{2N} \hat{\alpha}_p e^{ip\omega t}
$$

(3.11)

where the summation is carried out from $-2N$ to $2N$ due to the quadratic form of the nonlinear term. Following the derivation leading to (3.9), we find that the evolution equations can be written

$$
a_{p,x} = \text{linear terms} + i \beta_{1,p} \hat{\alpha}_p e^{i \int k_p dx}.
$$

(3.12)

Hence the nonlinear term can be calculated by toggling to the time domain with a backward Fourier transform, calculate $(gq^2/2 + P^2/h)_{xx}$ and determine $\hat{\alpha}_p$ by a forward Fourier transform.

In the following subsections, we go through the definition of the Discrete Fourier Transform, the symmetry properties of the Fourier coefficients for real time series and the technique of zero-padding to avoid aliasing. Further, it is described how the toggling between time domain and frequency domain is carried out and a precise recipe for calculating the nonlinear term is given.

### 3.2.1 The Discrete Fourier Transform

We consider a time series $x_j, j = 1, \ldots, M$. The Discrete Fourier Transform (DFT) of $x$ is defined as

$$
\text{DFT : } X_k = \sum_{j=1}^{M} x_j e^{2\pi i (j-1)(k-1)/M}, \quad k = 1, \ldots, M
$$

(3.13)

with the inverse

$$
\text{IDFT : } x_j = \frac{1}{M} \sum_{k=1}^{M} X_k e^{2\pi i (j-1)(k-1)/M}, \quad j = 1, \ldots, M.
$$

(3.14)

This definition is not unique. The signs of the arguments in the exponential functions can be interchanged, and the factor $1/M$ can be multiplied on any one of the two transformations. The convention used here is identical to the one used in Matlab.

We now consider a time series $x$, defined in $M$ points ($M$ odd) having the Fourier representation

$$
x = \sum_{k=-(M-1)/2}^{(M-1)/2} b_k e^{ik\omega t}.
$$

(3.15)
3.2 Calculating the nonlinear terms using FFT

This is similar to the usual expansion of the flow variables used in this thesis, see e.g., (1.2). For \( t = (j - 1)\Delta t \) and \( \omega = 2\pi/T \), we write this as

\[
x_j = \sum_{k=-(M-1)/2}^{(M-1)/2} b_k e^{i2\pi(j-1)k}. \tag{3.16}
\]

Here \( T \) is the duration of the time series and \( \omega \) the corresponding angular frequency, which is the smallest angular frequency of the signal. Insertion into (3.13) gives

\[
X_u = \sum_{j=1}^{M} x_j e^{-2\pi i (j-1)(u-1)}
\]

\[
= \sum_{j=1}^{M} \sum_{k=-(M-1)/2}^{(M-1)/2} b_k e^{i2\pi(j-1)(k-u+1)}
\]

\[
= \sum_{k=-(M-1)/2}^{(M-1)/2} M b_k \delta_{k-u+1+rM}
\]

\[
= M b_{u-1+rM}, \quad r \in \mathbb{Z}. \tag{3.17}
\]

where \( \delta_k \) is the discrete Kronecker delta. Here we have used that \( \sum_{j=1}^{M} e^{2\pi i (j-1)k} = M\delta_k \) for \( (M, j, k) \in \mathbb{N} \). Note that adding \( rM \) to the argument factor \( (k - u + 1) \) in the exponential function is equivalent to multiplying each term in the series by \( e^{2\pi i} = 1 \). The above result thus shows that applying the Discrete Fourier Transform to the signal (3.16) gives the vector

\[
X_{1, \ldots, M} = M \begin{bmatrix} b_0 & b_1 & \ldots & b_{(M-1)/2} & b_{-(M-1)/2} & \ldots & b_{-1} \end{bmatrix}^T, \quad M \text{ odd}, \tag{3.18}
\]

such that for the positive frequencies, \( b_k = X_{k+1} \).

For \( M \) even, the expansion (3.15) is not symmetric. An extra frequency is added for example among the positive frequencies, giving the expansion

\[
x = \sum_{k=-(M/2-1)}^{M/2} b_k e^{ik\omega t}. \tag{3.19}
\]

With this expansion, the same calculation as in (3.17) can be made, leading to the same result: the \( b \)'s are positioned in \( X \) starting with \( b_0 \) and wrapped around the highest frequency. That is

\[
X_{1, \ldots, M} = M \begin{bmatrix} b_0 & b_1 & \ldots & b_{M/2-1} & b_{M/2} & b_{-(M/2-1)} & \ldots & b_{-1} \end{bmatrix}^T, \quad M \text{ even} \tag{3.20}
\]

This result would also have been obtained, if the extra frequency was added at the negative side of the expansion (3.16). This frequency is denoted the Nyquist frequency, and is always real for a real time series. We show that in Section 3.2.2. In the following we will take \( M \)
to be even. The reason for this is, that the FFT algorithm in most implementations works fastest for \( M = 2^n \), \( n \) being a positive integer.

Mathematically, there is no difference between the Discrete Fourier Transform and the Fast Fourier Transform. The difference lies in the way, the transformation is computed. While straight-forward use of (3.13) and (3.14) requires a computational effort of \( O(M^2) \), the Fast Fourier Transform algorithm gives the same results with a computational effort of \( O(M \log M) \). Hence, the difference between the Discrete Fourier Transform and Fast Fourier Transform is a numerical issue, and below we concentrate on the Discrete Fourier Transform, since this representation of the algorithm is easy to work with. We note that the speed-up implied by the toggle to the time domain is solely due to the existence of FFT. Direct use of DFT would still require a computational effort of \( O(N^2) \).

### 3.2.2 Symmetry of the Fourier coefficients

For our purposes, the time series \( x \) is always real. This allows for working only in half of the Fourier components, in agreement with the formulation of the evolution equation model (3.9), where only positive \( p \)'s are considered. In the following we explore this in more detail.

First consider the complex conjugate of \( X_k \) according to the definition (3.13). For \( x_j \in \mathbb{R} \) we get

\[
X_k^* = \sum_{j=1}^{M} x_j e^{-2\pi i M (j-1)(-k+1)} = \sum_{j=1}^{M} x_j e^{-2\pi i M (j-1)(M-k+1)} = X_{M-k+2}.
\]

Hence, for \( M \) even, there is symmetry in the Fourier coefficients \( X \) around \( X_{M/2+1} \). Applying (3.21) for \( k = M/2 + 1 \) we get \( X_{M/2+1} = X_{M/2+1}^* \), showing that the highest frequency represented, the Nyquist frequency, will always have a real Fourier coefficient and thus no phase information. In (3.20), this coefficient is \( b_{M/2} \). The lacking phase information makes it necessary to ensure that no significant energy is present at the highest frequency for getting an accurate representation in the frequency domain.

Second, still taking \( M \) to be even, we can split up the sum defining \( x_j \) in equation (3.14) in the following manner

\[
x_j = \frac{1}{M} \left\{ X_1 + X_{M/2+1} (-1)^{(j-1)} + \sum_{k=2}^{M/2} X_k e^{2\pi i M (j-1)(k-1)} + \sum_{k=M/2+2}^{M} X_k e^{2\pi i M (j-1)(k-1)} \right\}.
\]
In the second sum we substitute \( k = M - p + 2 \) to obtain
\[
\sum_{k=M/2+2}^{M} X_k e^{\frac{2\pi i}{M} (j-1)(k-1)} = \sum_{p=2}^{M/2} X_{M-p+2} e^{\frac{2\pi i}{M} (j-1)(M-p+1)}
\]
\[
= \sum_{p=2}^{M/2} X_p^* e^{-\frac{2\pi i}{M} (j-1)(p-1)}
\]
which we recognize as the complex conjugate of the first sum in (3.22). We therefore have
\[
x_j = \frac{1}{M} \left( X_1 + X_{M/2+1}(-1)^{j-1} \right) + \frac{2}{M} \text{Re} \left\{ \sum_{k=2}^{M/2} X_k e^{\frac{2\pi i}{M} (j-1)(k-1)} \right\}.
\]
(3.24)

Thus for \( X_1 = X_{M/2+1} = 0 \) we can construct a vector \( X_{\text{pad}} \) as
\[
X_{\text{pad}} = \begin{bmatrix} X_1 & X_2 & \ldots & X_{M/2} & [0]_{1 \times M/2} \end{bmatrix}^T
\]
and calculate \( x \) as
\[
x_{1\ldots M} = 2 \text{Re} \{ \text{IDFT} \{ X_{\text{pad}} \} \}.
\]
(3.26)

In other words, we can calculate the real time series \( x \) using only the Fourier coefficients of the positive frequencies, inserting zeros at the last \( M/2 \) entries in \( X \). If a given number of time steps of the output time series is needed, the number of added zeros can just be increased, such that the total number of elements in \( X_{\text{pad}} \) is equal to the number of time steps required. This corresponds to adding higher frequencies with zero energy to the signal. If the zeroth-frequency is nonzero, one can use
\[
x_{1\ldots M} = 2 \text{Re} \{ \text{IDFT} \{ X_{\text{pad}} \} \} - X_1.
\]
(3.27)

### 3.2.3 Procedure for toggling

Using the above results valid for \( x \in \mathbb{R} \), the FFT-algorithms of Matlab are easily applied. In the sum defining \( a_p \), equation (3.3), we have
\[
\eta(x, t) = \sum_{p=-N}^{N} a_p(x) e^{i(\omega_p t - \int k_p dx)} = \sum_{p=-N}^{N} \hat{\eta}_p(x) e^{i\omega_p t}
\]
(3.28)

with \( \hat{\eta}_p = a_p e^{-i \int k_p dx} \). The definition of \( \hat{\eta}_p \) corresponds to \( b_k \) in equation (3.15) and we therefore have \( X_m = M \hat{\eta}_p \) where the mapping between \( m \) and \( p \) is given in (3.20). To account for the factor of \( M \) we use the FFT-algorithms in the following way
\[
\eta = 2 M \text{Re} \{ \text{IFFT} \{ \hat{\eta}_{\text{pad}} \} \} - \hat{\eta}_0
\]
(3.29)
\[
\hat{\eta}_{0\ldots N} = \frac{1}{M} [\text{FFT} \{ \eta \}]_{1\ldots N+1}.
\]
(3.30)
Expressed in terms of $a_p$ this is

$$\eta = 2M \text{Re} \left\{ \text{IFFT} \left\{ \left[ a e^{-i \int k dx} \right]_{\text{pad}} \right\} \right\} - a_0$$ (3.31)

$$a_{0,...,N} = \frac{1}{M} \left[ \text{FFT} \{ \eta \} e^{i \int k dx} \right]_{1,...,N+1}.$$ (3.32)

### 3.2.4 Aliasing

When products of time series are converted to Fourier space, the frequency resolution is important. If too few frequencies are used, or the time series is represented on a too coarse grid, the high-frequency part of the signal may be aliased to lower frequencies. As an example, consider the product of two cosine functions represented at the Nyquist frequency, i.e., at the points $x = \{0, \pi, 2\pi, 3\pi, \ldots \}$. Both functions are represented by the sequence $\{1, -1, 1, -1, \ldots \}$. For the continuous signal, we have $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$, and in the points chosen, this signal is really evaluated to the sequence of 1's, obtained. The discrete representation, however, does not resolve the oscillation of $\cos 2x$. If the discrete representation of $\cos^2 x$ is Fourier transformed, we would get that $X_1 = M$, $X_2, ..., M = 0$. The content of the high-frequent component $\cos 2x$ is mapped or aliased to the zeroth-frequency.

#### The mapping of high frequencies due to aliasing

We look a little further into the mapping of the high-frequency content of a signal represented on a too coarse grid. First, we define a time series with $N/2$ positive frequencies ($N$ even), evaluated at $M$ points. We take $M$ to be even and consider $M < N$. We restrict the considerations to a real time series. The time series is

$$x_j = \frac{1}{M} X_1 + \frac{1}{M} \sum_{k=2}^{N/2+1} \left\{ X_k e^{2\pi i (k-1)(j-1)} + \text{c.c.} \right\}$$

$$= \frac{1}{M} X_1 + \frac{1}{M} \sum_{k=2}^{N/2+1} \left\{ X_k e^{2\pi i (k-1)(j-1)} + X_{N-k+2} e^{-2\pi i (k-1)(j-1)} \right\}, \quad j = 1, \ldots, M,$$ (3.33)

where (3.21) has been used to express the complex conjugate of $X_k$. Second, we calculate the Discrete Fourier Transform of this time series using (3.13). We denote the Fourier coefficients
\( \tilde{X}_u \) and obtain

\[
\tilde{X}_u = \sum_{j=1}^{M} x_j e^{-\frac{2\pi i}{M}(j-1)(u-1)}
\]

\[
= \frac{1}{M} \sum_{j=1}^{M} X_1 e^{-\frac{2\pi i}{M}(j-1)(u-1)} + \frac{1}{M} \sum_{k=2}^{N/2+1} \left\{ X_k e^{\frac{2\pi i}{M}(j-1)(k-u)} + X_{N-k+2} e^{-\frac{2\pi i}{M}(j-1)(k+u-2)} \right\}
\]

\[
= X_1 \delta_{1-u+rM} + X_k \delta_{k-u+rM} + X_{N-k+2} \delta_{k+u-2+rM}
\]

\( k = 2, \ldots, N/2 + 1, \quad u = 1, \ldots, M, \quad r \in \mathbb{Z}. \)

This result shows how the frequency content of the signal represented by \( X_k \) is mapped to \( \tilde{X}_u \). Term 1 shows the contribution to the zeroth frequency, while terms 2 and 3 show the contributions to the other frequencies. We look at this mapping graphically. Since the time series is real, we consider just \( u = 1, \ldots, M/2 + 1 \). The contributions from term 1, term 2 and term 3 are depicted in Figure 3.1. The solid line shows the mapping implied by term 1 and term 2. This is simply the identity mapping, thus giving \( \tilde{X}_u = X_k \) for small \( u \). For larger \( u \)'s, the dashed line gives a contribution as well. This line represents term 3. Thus for the higher frequencies, both term 2 and term 3 contribute. We see that the contribution from term 3 can be avoided by taking \( M = N \), i.e., using a grid representing the highest frequency with two points in the time series. The figure illustrates how the high-frequency content of the time series is ‘folded’ over the Nyquist frequency of the coarse grid and back onto the frequencies represented on the coarse grid.

Figure 3.1: Mapping of the frequencies of a time series, represented on a too coarse grid.
Aliasing for a quadratic product  When the quadratic products of the evolution equations are calculated via time series, aliasing may occur. As an example, we consider the quadratic product of the two Fourier series

\begin{equation}
\hat{u}_j = \frac{(M-1)}{2} \sum_{k=-(M-1)/2}^{(M-1)/2} \hat{u}_k e^{2\pi i (j-1)k}, \tag{3.35a}
\end{equation}

\begin{equation}
\hat{v}_j = \frac{(M-1)}{2} \sum_{k=-(M-1)/2}^{(M-1)/2} \hat{v}_k e^{2\pi i (j-1)k}. \tag{3.35b}
\end{equation}

Defining the number of positive frequencies as $N \equiv (M - 1)/2$, we write their product as

\begin{equation}
\hat{u}_j \hat{v}_j = \sum_{k=-N}^{N} \sum_{l=-N}^{N} \hat{u}_k \hat{v}_l e^{2\pi i (j-1)(k+l)} = \sum_{p=-2N}^{2N} \sum_{s=\max\{p-N,N\}}^{\min\{p+N,N\}} \hat{u}_s \hat{v}_{p-s} e^{2\pi i (j-1) p}. \tag{3.36}
\end{equation}

Further details of the derivation of the above expression can be found in Appendix A. The min and max operators are not used in the appendix, but are needed here, since $p$ can take both positive and negative signs. We see that the product time series contains components having the double frequency of the original signals. We transform the product time series to Fourier space using (3.13) to obtain

\begin{equation}
X_k = \sum_{j=1}^{M} \sum_{p=-2N}^{2N} \sum_{s=\max\{p-N,N\}}^{\min\{p+N,N\}} \hat{u}_s \hat{v}_{p-s} e^{2\pi i (j-1)p+k+1}, \quad r \in \mathbb{N}, \quad k = 1, \ldots, M. \tag{3.37}
\end{equation}

For $k = 1$, we see that the delta function takes the value 1 for $p = 0$. Hence, $X_1$ is the zeroth frequency of the product signal as expected. For $k = 2$, the delta function is 1 for $p = \{1, -(M-1)\} = \{1, -2N\}$. Hence, $X_2$ gets contributions from both the first frequency and the highest frequency of the product series. Similarly, for $k = 3$, the delta function is non-zero for $p = \{2, -(M-2)\} = \{2, -2N + 1\}$ and thus $X_3$ contains contributions from the second frequency as well as the next-highest frequency. This pattern continues up to $k = M$, and we thereby see that all the Fourier coefficients are affected by aliasing.

The aliasing can be removed by increasing the number of frequencies, which is the same as using more points in the time series. In this way the high-frequency parts of the time series are resolved and not aliased to lower frequencies. To avoid aliasing among the frequencies $-\frac{3}{2}M, \ldots, \frac{3}{2}M$ frequencies (and points) are needed. In this way, not all the high frequency content of the product time series is resolved, but the aliasing does not affect the frequencies $-\frac{3}{2}M, \ldots, \frac{3}{2}M$. In practice, since the Fast Fourier Transform works fastest if the number of points is a power of 2, the closest power of 2 being larger than $\frac{3}{2}M$ is chosen.
3.2.5 Calculating the nonlinear term

Using the results of this section, the nonlinear term (3.11) is easily calculated. The recipe is as follows. First we expand the spatial derivatives to get

$$\text{NL}(t) \equiv \left( g \eta^2 / 2 + P^2 / h \right)_{xx} = g \left( \eta_{xx} + \eta_x^2 \right) + \frac{2}{h} \left( PP_{xx} + P_x^2 \right). \quad (3.38)$$

The derivatives are calculated spectrally by multiplying the Fourier coefficients by $-ik_p$ and $-k_p^2$, respectively. We hereby ignore the spatial variation of the Fourier coefficients and $k_p$, consistent with the convolution formulation. The flux, $P$ is calculated using the linear approximation $b_p = \omega_p k_p a_p$ for the Fourier coefficients, again consistent with the convolution formulation.

The products in (3.38) are now calculated in the time domain, and the result transformed back to the frequency domain using (3.32). With reference to (3.11), we introduce the following notation for the $p$'th Fourier component of the nonlinear time series NL:

$$\left( g \eta^2 / 2 + P^2 / h \right)_{x}^{[p]} = \hat{\alpha}_p. \quad (3.39)$$

The evolution equations can then be written

$$a_{p,x} = -\frac{\beta_{0,p}}{\beta_{1,p}} a_p - \beta_{s,p} \frac{h_x}{h} a_p + \frac{i}{\beta_{1,p}} g \eta^2 / 2 + P^2 / h \right)_{xx}^{[p]} e^{ik_p dx}. \quad (3.40)$$

3.3 Regular waves — a practical test

Having shown how the calculations of the nonlinear terms can be speeded up with the aid of FFT, we turn the focus towards two practical tests. As mentioned in the introductory part of this chapter, this speed up allows physical problems to be solved with a computational effort of $O(MN \log N)$, where $M$ is the number of spatial points and $N$ the number of frequencies.

However, as time domain Boussinesq models can solve the same problems with an asymptotic work load of $O(MN)$, a practical investigation of the computational performance of evolution equations versus time domain models is necessary to judge, whether evolution equations are an interesting alternative to time domain models.

Such a test is the objective of this and the following section. We compare the performance of the evolution equations of Madsen and Sørensen (1993) to the performance of the corresponding time domain formulation. The time domain equations solved are given in Section 3.1, equations (3.1). Both models incorporate Padé[2,2]-dispersion, and are valid for weakly nonlinear wave propagation on mildly sloping beds.

In this section, the physical test concerns regular waves passing a submerged bar. Before presenting the results obtained with the two models, the overall approach of the investigation is described and the physical processes of the bar test for regular waves discussed.
3.3.1 Overall strategy of test

Since the two models are not identical, it is not appropriate to compare solutions of the models directly to each other. For a given set of physical parameters, each formulation has a unique mathematical solution, which under a given discretization can be approximated numerically. Thus for a given discretization, the quality of the corresponding numerical solution can be assessed by comparing it to an estimate of the unique solution. As the unique solution is not known in general, the solution obtained with the finest discretization is used. In this way, a deviation measure for each choice of discretization can be calculated, and be linked to the CPU-time needed for calculating the solution. This leads to a curve relating the deviation from the unique (converged) solution of the model to CPU-time. The efficiency of the models can then be compared with basis in these curves.

Using this strategy, results of the two models are never compared directly. This is appropriate since the models are not identical. On the other hand, if one of the models is much simpler than the other, it may show a fast convergence with small CPU-times, still producing a solution of low quality in a physical sense. From a practical point of view, however, it is well known that evolution equations are not as accurate as the corresponding time domain formulations, since they are restricted to forward propagating waves. Hence, when considering the use of evolution equations, it has already been accepted that a less accurate solution (in a physical sense) will be obtained. The results of the present investigation are therefore intended to show how fast each model can be solved satisfactorily, once the user has accepted the approximations involved with the model.

3.3.2 About the bar test

The bar test here reported uses the experimental parameters of Luth et al. (1994). The bar is positioned in a wave flume with a depth of 0.4 m. The upward slope is 1:20 while the downward slope is 1:10. The upper flat section of the bar is 2 m long and has a depth of 0.1 m. We here use the parameters of test ‘A’, that is, a wave period of 2.02 s and a wave height of 0.02 m.

In Figure 3.2 the amplitudes of the wave field are plotted against space. The results shown are calculated with the frequency domain model. Below the plot, the bathymetry is shown. The incoming wave condition is a bound wave with 7 harmonics. The method for finding this initial condition is discussed in Chapter 5. The second harmonic amplitude makes out about 4% of the first harmonic amplitude, and the wave is therefore very close to being linear. This can also be seen by comparing the wave number of the nonlinear wave to the wave number of a linear free wave: the linear wave number is \( \frac{kh}{2} = 0.6725 \) (Padé[2,2]-dispersion), while the nonlinear wave has a wave number of \( kh = 0.6716 \).

As the wave propagates along the upward slope, the amplitude increases due to shoaling. A linear run reveals that practically only the first harmonic is affected by linear shoaling. The
increase in the second and higher harmonics is due to the nonlinear steepening of the wave profile, which forces the higher harmonics to gain in amplitude.

At the top of the bar, the wave field exhibits the same behavior as known from harmonic generation: the incoming wave amplitudes at the flat bed do not have the right ratios to produce a perfect bound wave. Energy is therefore transferred in a cyclic recurrence pattern between the harmonics. This can be seen as the sudden decrease in the first harmonic wave amplitude at the top of the bar. However, as the bar is only 2 m long, we do not see the whole pattern. In Figure 3.3, the results for a shelf topography are shown. The shelf corresponds to an infinitely long bar. From this figure, the recurrence of the wave field initiated at the upper first corner of the shelf is clearly seen.

When the wave enters the downward slope, the recurrence pattern of harmonic generation is interrupted. The first harmonic undergoes de-shoaling, while the second harmonic maintains its growth in amplitude, but at a slower rate. The higher harmonics decrease their amplitude corresponding to the decrease in steepness of the wave profile.

At the transition to the flat bed following the bar, the wave amplitudes do not resemble the shape of a perfect bound wave, and the wave field therefore again enters a cyclic process of energy exchange. The strong amplitude of the second harmonic — most of it being a free wave — results in a doubling of the number of wave crests after the bar, when inspecting the free surface (not shown in the figure).
3.3.3 Running the frequency domain model

The frequency domain model was implemented in Fortran 90 on an IBM RS6000 43P model 260 (2 processors at 200 MHz). The FFT library `fftpack` from [www.netlib.org](http://www.netlib.org) was used to compute the FFT’s needed. The integration scheme is of fourth-order accuracy and is due to Scraton (1964). The scheme is described in Appendix C and is very similar to a fourth-order explicit Runge-Kutta method. The reason for not using a standard Runge-Kutta method is the following. When integrating the solution in the interval \( x = [x_j; x_j + \Delta x] \), the standard Runge-Kutta method evaluates the right-hand side of the ODE-system in the points \((x_j, x_j + \frac{\Delta x}{2}, x_j + \Delta x)\). If \(x_j + \Delta x\) is a corner point of the bar, the solution in \(x_j + \Delta x\) will be affected by the new bed slope of the corner point. The effect will be \(O(\Delta x \Delta h_x)\), where \(\Delta h_x\) is the change in \(h_x\) over the corner point. Hereby, the fourth-order accuracy of the integration scheme is violated. Note that assigning the pre-corner value of the bed slope to the corner point does not solve the problem, but only moves it to the next grid point after the corner.

The method of Scraton evaluates the right-hand side in the points \((x_j, x_j + \frac{\Delta x}{9}, x_j + \frac{1}{3} \Delta x, x_j + \frac{3}{4} \Delta x, x_j + \frac{9}{10} \Delta x)\). The right-hand side is thus not evaluated in the end point of the integration interval. This makes it possible to retain the fourth-order accuracy of the method, even though the bed slope is discontinuous in the domain.

Alternatively, a bar with smoothed corners can be used in the calculations. However, the smooth representation of the bar should then be continuous in its derivatives to a similar order as the order of the integration scheme to retain the order of the integration scheme. The approach here used seems easier, since a bathymetry identical to the one of the time
domain can be used. The only restriction is that the corners of the bar must be located in grid points.

To test the efficiency of the model, a number of runs were carried out with varying number of frequencies and step length. For each run, the initial wave amplitudes were calculated with a finite amplitude solver of the model, hence producing a fully nonlinear bound incoming wave. The method of obtaining this initial condition is described in Chapter 5. Note that this wave is different from both a stream function theory wave and a finite amplitude solution to the time domain model, due to the different approximations involved in the derivation of these models.

In the runs 4, 5, 6, 7, 8, 12 and 16 harmonics were used, and the step lengths were set to \((1 \cdot 10^0, 5 \cdot 10^{-1}, 2.5 \cdot 10^{-1}, 1 \cdot 10^{-1}, 1 \cdot 10^{-2}, 1 \cdot 10^{-3}, 1 \cdot 10^{-4})\) m. For each set of parameters, the CPU-time was measured with the UNIX command `time`, which returns the CPU-time in terms of two numbers: the user time and the system time. The user time is the time used within the program called, while the system time is the time used for system calls. Usually the system time is a very small fraction of the user time.

In Table 3.1 the CPU times measured for each run are shown. Each run was measured twice. Inspection of the table shows that the two results of CPU-time for each set of parameters vary very little. Further, the CPU-time is proportional to the number of spatial steps needed for each simulation. Hence halving the step size leads to a doubling in CPU-time. For fixed step length, the CPU-time increases with the number of frequencies, but in a step-like manner. The reason for this is that the number of points used by the FFT-routine in the program was set to the lowest power of 2 exceeding \(3^N\). In this way aliasing is eliminated and the length of the time series worked on is a power of 2, being advantageous for the FFT algorithm. The results of \(N = (4, 5)\) are therefore calculated with 16 points in the time series, the results of \(N = (6, 7, 8)\) with 32 points and the results of \(N = (12, 16)\) with 64

<table>
<thead>
<tr>
<th>dx (m)</th>
<th>N=4</th>
<th>N=5</th>
<th>N=6</th>
<th>N=7</th>
<th>N=8</th>
<th>N=12</th>
<th>N=16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\cdot10^0</td>
<td>0.03u+0.01s</td>
<td>0.03u+0.01s</td>
<td>0.04u+0.02s</td>
<td>0.07u+0.00s</td>
<td>0.06u+0.01s</td>
<td>0.09u+0.01s</td>
<td>0.10u+0.01s</td>
</tr>
<tr>
<td>5\cdot10^{-1}</td>
<td>0.05u+0.02s</td>
<td>0.05u+0.01s</td>
<td>0.10u+0.01s</td>
<td>0.11u+0.02s</td>
<td>0.11u+0.01s</td>
<td>0.18u+0.01s</td>
<td>0.20u+0.01s</td>
</tr>
<tr>
<td>2.5\cdot10^{-1}</td>
<td>0.10u+0.02s</td>
<td>0.11u+0.01s</td>
<td>0.19u+0.01s</td>
<td>0.20u+0.01s</td>
<td>0.21u+0.00s</td>
<td>0.38u+0.01s</td>
<td>0.39u+0.00s</td>
</tr>
<tr>
<td>2\cdot10^{-1}</td>
<td>0.13u+0.01s</td>
<td>0.14u+0.01s</td>
<td>0.22u+0.01s</td>
<td>0.26u+0.01s</td>
<td>0.27u+0.01s</td>
<td>0.45u+0.00s</td>
<td>0.50u+0.00s</td>
</tr>
<tr>
<td>1\cdot10^{-1}</td>
<td>0.26u+0.01s</td>
<td>0.27u+0.01s</td>
<td>0.47u+0.02s</td>
<td>0.52u+0.00s</td>
<td>0.52u+0.03s</td>
<td>0.88u+0.01s</td>
<td>0.97u+0.01s</td>
</tr>
<tr>
<td>1\cdot10^{-2}</td>
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<td>2.69u+0.01s</td>
<td>4.77u+0.01s</td>
<td>4.96u+0.01s</td>
<td>5.17u+0.02s</td>
<td>8.79u+0.03s</td>
<td>9.65u+0.03s</td>
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<tr>
<td>1\cdot10^{-3}</td>
<td>4.35u+0.02s</td>
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<td>4.97u+0.00s</td>
<td>5.20u+0.01s</td>
<td>8.79u+0.01s</td>
<td>9.50u+0.01s</td>
</tr>
</tbody>
</table>

Table 3.1: CPU times for different values of N and dx. All times are in seconds and are given in the format \(t_1u+\ t_2s\) where \(t_1\) is the user time in seconds and \(t_2\) is the system time in seconds.
Chapter 3. Efficiency of Evolution Equations

points. Besides this grouping, the overall trend is that the CPU-time is roughly proportional to \( N \), agreeing with the asymptotic work amount of the FFT-routine of \( O(N \log N) \).

The results of each run were written to a file in terms of the complex wave amplitudes calculated for each meter in physical space. On the basis of this output, time series for each meter of the physical domain were calculated (using FFT) and compared. 64 points were used in each time series — enough to capture the influence of the 16’th harmonic in the time series produced. A reference solution produced with \( N = 16 \) and \( dx = 1 \cdot 10^{-4} \) was used for measuring the accuracy of each run. An error measure defined as

\[
Err_{\text{time series}} = \frac{1}{M} \sum_{j=1}^{M} |\eta_j - \eta_{\text{ref},j}| \over \frac{1}{M} \sum_{j=1}^{M} |\eta_{\text{ref},j}|
\]  

(3.41)

was used to calculate the relative deviation from the reference time series in each point. The error measure can be interpreted as the mean absolute error, normalized by the mean absolute value of the reference signal.

A similar error measure was defined for the skewness of the time series:

\[
Err_{\text{skewness}} = \left| \frac{S - S_{\text{ref}}}{S_{\text{ref}}} \right|
\]  

(3.42)

The skewness is a measure of the nonlinearity of the wave field. Its definition is given in Chapter 4. Finally, to describe the deviation from the reference solution by one single number, the mean error

\[
Err^* = \frac{1}{8} \sum_{x=17}^{24} Err_{\text{time series}}(x)
\]  

(3.43)

was defined, being the mean value of the time series errors on the flat bottom after the bar. When this single number is plotted against CPU-time for each run, a curve illustrating the accuracy as function of CPU-time can be obtained.

Results for varying step size For \( N = 16 \), the effect of varying the step size is shown in Figure 3.4. The upper panel shows \( Err_{\text{time series}} \), the middle panel \( Err_{\text{skewness}} \) and the lower panel a plot of the time series in \( x = 24 \) m. As reference solution, the parameters \( N = 16 \) and \( dx = 1 \cdot 10^{-4} \) m were used. The errors in skewness show the same trend and error levels as the time series errors, although the curves are not as smooth. We here base the discussion on the time series error.

Generally, the error is constant in the flat region in front of the bar and increases on the upward shoal and on top of the bar. This is due to the stronger nonlinearity on the bar top. After the bar, the error maintains a constant level. This indicates that the local error of integration after the bar is smaller than or equal to the error introduced on top of the
bar, which is now propagated along with the integration. For \( x \geq 16 \) m, the solution for \( dx = 1.0 \) m blew up, and thus no results are available for \( x \)-values exceeding this location.

The error in the time series decreases monotonically with the step size. The fourth order accuracy of the integration algorithm can be observed by realizing that the error is reduced by a factor of \( 1 \cdot 10^4 \) each time the step size is reduced by a factor 10. For \( dx = 1 \cdot 10^{-3} \) m the solution seems to have converged, since the error does not decrease by a factor of \( 1 \cdot 10^4 \) in the domain after the bar, when comparing to the solution of \( dx = 1 \cdot 10^{-2} \) m.

In the plot of the time series, only the coarsest calculations have been included, since the results for \( dx = 1 \cdot 10^{-2} \) m and finer falls right on top on the target solution in the plot.

From the results we see that a step size of \( dx = 20 \cdot 10^{-1} \) gives an error in the time series below 1%.

**Results for varying number of frequencies** To check the error as function of the number of frequencies, a series of runs with \( N \) taking values between 4 and 16 was made. For all the runs \( dx \) had the value \( dx = 1 \cdot 10^{-4} \) m. In Figure 3.5, the errors of each run and time series of \( x = 24 \) m are depicted in the same fashion as in Figure 3.4. The dependence of space of the errors is similar to the behavior in Figure 3.4: the error is small within the first flat section of the topography and increases as the waves passes the bar.

We concentrate on the results in the upper panel (\( Err_{\text{time series}} \)), since the results for \( Err_{\text{skewness}} \) are similar, although less smooth. In front of the bar, it makes no improvement of the results to include more than 6 harmonics. As the waves propagate along the upward shoal, the presence of the higher harmonics begins to influence the results. For \( N = 12 \), the time series error decreases in the region after the bar. This indicates that the missing frequencies 13, \ldots, 16 (which are present in the reference solution), have an influence on the results at the bar top, where the wave field is most nonlinear, but does not affect the wave field after the bar much.

To stay within an error of about 1%, \( N = 6 \) is sufficient for the present test.

**Accuracy versus CPU-time** For each run, \( Err^* \) as defined in (3.43) is plotted versus the CPU-time of Table 3.1 in Figure 3.6. Runs produced with the same number of frequencies are connected with lines, and each step size has been given a symbol. For \( N \) fixed, the errors converge towards a fixed value for decreasing step length. This reflects that once this limiting step length is used, the physical variation of the frequencies retained is properly resolved. In Appendix D it is demonstrated that this step length is related to the beat length of the highest harmonic on top of the bar. The converged error level for each value of \( N \) is identical to the levels of the upper plot in Figure 3.5.

The interesting part of the curve is the envelope produced by the lowest CPU-times for each value of the error. The error level \( Err^* < 1\% \) can be obtained in 0.27 s with \( N = 7 \) and \( dx = 2 \cdot 10^{-1} \) m.
Figure 3.4: Results for $N = 16$ and varying step size. Upper panel: time series error, middle panel: skewness error and lower panel: time series in $x = 24$ m. The reference solution is shown as the solid line with dots in the lower panel.
3.3 Regular waves — a practical test

Figure 3.5: Results for \( dx = 10^{-4} \) and varying number of frequencies. Upper panel: time series error, middle panel: skewness error, lower panel: time series in \( x = 24 \) m. Reference solution is shown as the solid line with dotted points.
Figure 3.6: Error measure plotted against CPU-time. The numbers in the box are the values of $dx$ (m).

### 3.3.4 Running the time domain model

To test the efficiency of the time domain model, a number of simulations of the bar test was carried out. The numerical scheme of the time domain model used is the Mike 21 scheme, known from the software package ‘Mike 21’ of DHI Water & Environment. This scheme is staggered in space and time and has second order accuracy in space and time.

#### Spatial domain and boundary conditions

The spatial domain used for the simulations is identical to the domain used in the frequency domain, except for a length of 30 m rather than 24 m. This extension was made to accommodate a sponge layer in each end of the domain of 4 meters length. The sponge layer strength was governed by the function

$$f_{sp}(s) = 1 + a \frac{\exp(bs^3) - 1}{\exp(b) - 1}$$

where $s$ takes the value 0 at the entrance of the sponge layer and 1 at the end of the sponge layer. In each point within the sponge layer, the surface elevation and the flux was divided
with \( f_{sp}(s) \) in each time step. For the calculations here presented, \( a = 10 \) and \( b = 1 \) were used.

The incoming wave conditions were calculated as a third-order bound wave solution of the model for regular waves, see Chapter 5. From the harmonic amplitudes hereby found, time series of the flux and surface slope (\( \eta_x \)) were calculated and used to drive the wave field internally in \( x = 5 \) m.

**Time step and CPU-times** A series of runs were now calculated with varying grid spacing. The spacings \( dx = (1 \cdot 10^{-1}, 5 \cdot 10^{-2}, 2 \cdot 10^{-2}, 1 \cdot 10^{-2}, 5 \cdot 10^{-3}) \) m were used. The time steps were chosen such that the Courant number defined as

\[
C_r = \frac{c}{dx} = \frac{\sqrt{gh} dt}{dx} \tag{3.45}
\]

was close to 0.5. This means that each time the spatial grid spacing was halved, also the time step was halved, thus doubling the number of time steps needed for a run. The wave conditions at ‘deep’ water (the flat sections before and after the bar) determines the time step, since the waves here have the fastest phase speed. The actual time step values are shown in Table 3.2 along with the number of time steps in a wave period, \( M_p \), and the CPU-times of each run. As a reference, the run published in Madsen and Schäffer (1999) of the same physical test was made with a grid spacing of \( dx = 1 \cdot 10^{-2} \) m and a time step of \( dt = 2 \cdot 10^{-3} \) s. This corresponds to the present run with the same grid spacing, but with a 20% smaller time step. Since the system time was usually about 1 s and therefore a small fraction of the total CPU-time, only the sum of the user time and system time is shown for each run.

The CPU-times increases like \( M_p^2 \). This is due to the fixed Courant number. Thus when the grid spacing is halved, both the number of spatial and temporal points are doubled, resulting in a four times increase of the CPU-time.

<table>
<thead>
<tr>
<th>( dx ) [m]</th>
<th>( dt ) [s]</th>
<th>( M_p )</th>
<th>CPU time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\cdot10^{-1}</td>
<td>2.525\cdot10^{-2}</td>
<td>80</td>
<td>6</td>
</tr>
<tr>
<td>5\cdot10^{-2}</td>
<td>1.2625\cdot10^{-2}</td>
<td>160</td>
<td>19</td>
</tr>
<tr>
<td>2\cdot10^{-2}</td>
<td>5.05\cdot10^{-2}</td>
<td>400</td>
<td>116</td>
</tr>
<tr>
<td>1\cdot10^{-2}</td>
<td>2.525\cdot10^{-3}</td>
<td>800</td>
<td>481</td>
</tr>
<tr>
<td>5\cdot10^{-3}</td>
<td>1.2625\cdot10^{-3}</td>
<td>1000</td>
<td>1829</td>
</tr>
</tbody>
</table>

Table 3.2: Grid spacing, time step and CPU-time for the runs. \( M_p \) is the number of time steps per period.

**Stationarity of the wave field** When the time stepping is started, the waves generated in \( x = 5 \) m enters the domain with initially calm water. After a certain time (about 11 s
for this test) the transient wave reaches the down-wave boundary. However, the transient motion of the wave field needs some time to decay. This increases the physical time required for obtaining a stationary wave solution to several wave periods. To judge when the solution was stationary, the amplitude of the first harmonic in $x = 24$ m was plotted against time for all the runs. For $dx = 5 \cdot 10^{-3}$ s, this curve is shown in Figure 3.7. For the other runs, the curves are similar. It was judged that the solutions are stationary after 24 wave periods corresponding to 48.48 s. Hence, all results used for the comparisons between the runs are taken from the time interval $[24T; 25T]$, and the CPU-times reported represents the time needed to time step the wave field through 25 periods.

This cut-off, however, is not a trivial decision. A closer zoom on the plot in Figure 3.7 reveals that a small oscillation is still present after 24 wave periods. On the other hand, one might as well argue that on a given grid, the solution is accurate enough after a shorter time, e.g. after $t = 25$ s. The loss in accuracy may be well balanced by the reduction in CPU-time. A closer investigation of this is recommended. Such an investigation, however, has not been pursued. The largest influence is expected to be a factor of $1/2$ on the CPU-times.

**Wave amplitudes** In Figure 3.8, the wave amplitudes for the solution obtained with $dx = 5 \cdot 10^{-3}$ are depicted. The amplitudes are overlayed by an oscillating variation. This is due to backward propagating waves, reflected from the bar or the down wave sponge layer.
Error measures and time series  For all runs, output was produced for each 0.1 m. The temporal interval of output was 0.05 s, corresponding to 40 points per wave period. To compare the solutions obtained, the error measures defined in (3.41)–(3.43) were calculated for each run with the run of $dx = 5 \cdot 10^{-3}$ as reference solution. In contrast to the approach within the frequency domain, the errors were here calculated for each 0.1 m and $Err^*$, the mean of $Err_{\text{time series}}$ in $x \in [17; 24]$ m, was therefore based on the information for all these points, rather than from points with a spacing of 1 m. As the results of the time domain model are less clean than the results of the frequency domain model, it seemed appropriate to base $Err^*$ on more points. In Figure 3.9 the spatial variation of $Err_{\text{time series}}$ and $Err_{\text{skewness}}$ are depicted. The time series error shows the same behavior as within the frequency domain model: the error increases most rapidly, as the wave passes the bar. Here, however, the level after the bar maintains a slow spatial growth after the bar. Also, when comparing to the frequency domain results, the time series error curve has a more rough variation. The solution of $dx = 1 \cdot 10^{-2}$ m has a time series error level of around 1% in the end of the domain. In $x = 5$ m, the specified position of wave generation, the error is larger than for $x > 5$ m. The reason for this is that the actual point of wave generation is a little shore-ward of the specified point. This shift implies that the point $x = 5$ m is a little behind the point of wave generation. This distance, however is proportional to the grid spacing, thus giving large discrepancies when comparing to the reference solution in this point.

The error measure based on skewness shows a rather messy spatial variation.

In Figure 3.10, time series in $x = 6$ m and $x = 24$ m are plotted. While the results are very similar for $x = 6$ m, the coarse grids results in amplitude errors and phase errors in $x = 24$ m. From a graphical point of view, only the solution of $dx = 0.01$ m has converged.
Chapter 3. Efficiency of Evolution Equations

Figure 3.9: Results for varying grid spacing, $dx$. Left: time series error. Right: skewness error.

Figure 3.10: Time series in $x = 6$ m (left) and $x = 24$ m (right) for different grid spacings.
3.3 Regular waves — a practical test

**Accuracy versus CPU-time** The values of $Err^*$ are plotted versus the CPU-times in Figure 3.11. In the double logarithmic plot, there is an almost linear dependence between $Err^*$ and the CPU-time for the four points established. An error of about 1% can be obtained within approximately 800 s.

![Figure 3.11: CPU-time versus $Err^*$ for time domain runs.](image)

### 3.3.5 Comparison of CPU-times

The main result of the investigations is the plots relating the average time series error after passing the bar $Err^*$ (as defined in (3.43)) to the CPU-time needed. We now compare the results by fitting straight lines through the points obtained. These fits allow for quantifying the ratio between CPU-times for the two numerical models as function of the time series error level.

From the time series plots shown in Figures 3.4, 3.5 and 3.10, we judge that the interval of interest for $Err^*$ is $Err^* = [1 \cdot 10^{-2}; 2 \cdot 10^{-1}]$. Lower values of $Err^*$ are not visible to the eye and are therefore not important for a practical application. In this interval of $Err^*$, the envelope of the points for the frequency domain model in Figure 3.6 is well described by two lines in the double logarithmic plot. In Figure 3.12 such two lines have been fitted through the points. The time domain results of Figure 3.11 are also shown in the figure and represented by a single line fitted to them. The lines are determined as a least squares minimizing fit of the difference in CPU-time between the points and the line. Two horizontal lines in the plot marks the error levels $Err^* = (0.01, 0.5)$. For this interval, the ratio between the CPU-time of the time domain model and the frequency domain model is plotted in Figure
3.13. The ratio between the CPU-times is increasing for smaller error levels. For an error level of $Err = 10\%$, which we regard as an acceptable level of $Err^*$, the ratio is 1059 or 1000 in round numbers.

Figure 3.12: Lines fitted through CPU-time results for regular waves.

Figure 3.13: Ratio of CPU-time between the time domain model and the frequency domain model. Regular waves.
3.4 Irregular waves — a practical test

Similarly to the test of computational efficiency for regular waves, a test for irregular waves has been carried out. For the regular wave case, we saw that around seven frequencies was sufficient to model the wave field including nonlinear effects. Contrary, for irregular waves, a large number of frequencies is needed within the frequency domain model. For the time domain model, the difference between modelling regular and irregular waves, only affects the boundary condition used to drive the wave field. Hence a comparison of performance for irregular waves is a benchmark test in favor of the time domain model.

Before we describe the results of each model, we describe the choice of incoming wave spectrum.

3.4.1 Incoming wave spectrum

The test chosen is very similar to the regular wave test, since the same topography is used. The peak period is chosen as $T = 2.02$ s, similar to the period of the regular waves test.

For a wave field described by

$$\eta(x, t) = \sum_{p=-N}^{N} \hat{\eta}_p e^{ip\omega t}, \quad (3.46)$$

the local, depth integrated mean energy density is

$$E = \sum_{p=1}^{N} \frac{1}{2} \rho g \hat{H}_p^2 = \sum_{p=1}^{N} 2\rho g |\hat{\eta}_p|^2 \quad (3.47)$$

and thereby $E_p = 2\rho g |\hat{\eta}_p|^2$. Often the expression $E_p = \frac{1}{2} |\hat{A}_p|^2$ is seen. This expression is valid for the alternative expansion $\eta(x, t) = \sum_{p=-N}^{N} \frac{1}{2} \hat{A}_p e^{i\omega t}$, apart from the factor of $\rho g$, which has been let out.

A wave spectrum is defined by its spectrum function $S_{\eta}$. This function is defined such that the mean energy (besides the factor $\rho g$) in a given frequency interval $\Delta f$ is equal to $S_{\eta}(f) \Delta f$ in the limit $\Delta f \to 0$. We therefore have $2|\hat{\eta}_p|^2 = \Delta f S_{\eta}(f_p)$ and thus

$$|\hat{\eta}_p| = \sqrt{\frac{1}{2} \Delta f S_{\eta,p}}. \quad (3.48)$$

The wave spectrum chosen is a JONSWAP (JOint North Sea WAve Project) spectrum where $S_{\eta}$ is given as

$$S_{\eta}(f) = \Gamma \frac{1}{f^3} \exp \left( -\frac{5}{4} \left( \frac{f}{f_{\text{peak}}} \right)^{-4} \right) \gamma^a \quad (3.49)$$
with
\[ a = \exp \left( -\frac{(f - f_{\text{peak}})^2}{2\sigma^2 f_{\text{peak}}^2} \right) \] (3.50)
and
\[ \sigma = \begin{cases} 0.07 & f \leq f_{\text{peak}} \\ 0.09 & f > f_{\text{peak}} \end{cases}, \quad \gamma = 3.3, \] (3.51)
see e.g., Sumer and Fredsøe (1997). Here \( f_{\text{peak}} \) denotes the peak frequency i.e., the frequency with the highest energy content in the wave spectrum. \( \Gamma \) is a scaling constant, which we will return to shortly.

For a wave spectrum, the \( p \)'th moment is defined as
\[ m_p = \int_0^\infty f^p S_\eta(f) df, \] (3.52)
and for a discrete spectrum the zero'th moment may be approximated as
\[ m^*_0 = \sum_{p=1}^N \Delta f S_\eta(f_p). \] (3.53)

The significant wave height for a wave spectrum, \( H_s \), is defined as the average wave height of the highest one-third of all waves. It is therefore often denoted \( H_{1/3} \) and is related to the spectrum function by the result
\[ m_0 = \frac{1}{16} H_s^2 \] (3.54)

Using these results, the wave amplitudes of the incoming wave spectrum were determined as follows. First, the cut-off frequency and the significant wave height were chosen. Next, the frequency resolution \( \Delta f = f_1 \) and thereby the number of frequencies was chosen. The relation (3.54) defines \( m_0 \), and through the requirement
\[ m_0 = m^*_0 \] (3.55)
the scaling of the spectrum was found, thus giving the value of \( \Gamma \). Next, the wave amplitudes were obtained using (3.48). The requirement (3.55) corresponds to assigning the energy of the truncated frequencies to the discrete frequencies chosen.

It was decided to simulate 200 peak periods in the time domain. This determines the lowest frequency within the evolution equations to be \( f_{\text{peak}}/200 \). The highest frequency was chosen as \( f_{\text{max}} = 4f_{\text{peak}} \), and therefore the number of frequencies in the runs was 800. The phases were chosen randomly from a uniform distribution in the interval \([0; 2\pi]\).
3.4 Irregular waves — a practical test

Choice of significant wave height

The significant wave height was chosen such that the highest wave during the test was close to breaking. The reason for this is that the linear wave motion is almost completely described in the frequency domain model (besides shoaling) prior to any calculations. Hence some nonlinearity is needed to validate the efficiency of the model. Fenton (1990) gave the following expression for the highest regular wave of permanent form, based on numerical results of Williams (1981)

\[
\frac{H_{\text{max}}}{h} = \frac{0.141063(L/h) + 0.0095721(L/h)^2 + 0.0077829(L/h)^3}{1 + 0.0788340(L/h) + 0.0317567(L/h)^2 + 0.0093407(L/h)^3}.
\] (3.56)

For a wave having a period corresponding to the peak frequency of \(T_p = 2.02\) s, the linear wave number at the top of the bar is found to be \(kh = 0.32\), resulting in \(H_{\text{max}}/h = 0.77\) from the above expression. Further, Sumer and Fredsøe (1997) cites Longuet-Higgins (1952) for the result

\[
\frac{H_{\text{max}}}{H_s} = \sqrt{\frac{1}{2} \ln N},
\] (3.57)

\(N\) being the number of waves in a wave record. For a time series of 200 records this gives \(H_{\text{max}}/H_s = 1.63\). Based on these numbers, the significant wave height on top of the bar can be estimated. A linear run of the model shows that linear shoaling at the peak frequency results in an increase of approximately 30\% when the wave propagates from the deep water region of \(h = 0.4\) m to the top of the bar. Hereby we estimate the significant wave height at \(h = 0.4\) m to

\[
H_s = 0.77 \times 0.1 \text{ m}/(1.63 \times 1.3) = 3.63 \cdot 10^{-2} \text{ m}.
\] (3.58)

With this choice of significant wave height, the highest wave on top of the bar is expected to be in the breaking regime. We note that since the model is only weakly nonlinear, the steepness of the waves is usually underestimated. Hence the above choice of significant wave height will not produce a wave as high as in nature on top of the bar.

In Figure 3.14 a short part of the time series in \(x = 14\) m, the down-wave corner of the upper bar section, is shown. The incident shows one of the highest waves within the series. The wave height is about 0.072 m, which is not far from the 0.077 m chosen above.

Choice of cut-off frequency

For the depths \(h = 0.4\) m and \(h = 0.1\) m, the nondimensionalized wave numbers for the peak frequency and its super harmonics are shown in Table 3.3. For \(kh = 3.33\), the error in linear phase velocity compared to Stokes theory is 3.4\%, while the error for \(kh = 5.14\) is 10.7\%. The values of \(kh\) for a given harmonic are largest for \(h = 0.4\) m. From a physical point of view, the spectrum should therefore be truncated somewhere between 3 and 4 times \(f_{\text{peak}}\). On the other hand, truncating the spectrum already at the third harmonic of the peak frequency seems a little drastic, especially when inspecting the amplitude spectrum in Figure 3.15. Here the energy spectrum function \(S_\eta\) is plotted as well as the length of the complex wave amplitudes \(|\hat{a}_j|\). These
Chapter 3. Efficiency of Evolution Equations

Figure 3.14: Time series in $x = 14$ m.

<table>
<thead>
<tr>
<th>frequency $f_p$</th>
<th>$2f_p$</th>
<th>$3f_p$</th>
<th>$4f_p$</th>
<th>$5f_p$</th>
<th>$6f_p$</th>
<th>$7f_p$</th>
<th>$8f_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kh for $h = 0.4$ m</td>
<td>0.67</td>
<td>1.68</td>
<td>3.33</td>
<td>5.14</td>
<td>6.88</td>
<td>8.55</td>
<td>10.19</td>
</tr>
<tr>
<td>kh for $h = 0.1$ m</td>
<td>0.32</td>
<td>0.67</td>
<td>1.11</td>
<td>1.68</td>
<td>2.45</td>
<td>3.33</td>
<td>4.25</td>
</tr>
</tbody>
</table>

Table 3.3: Non-dimensionalized wave numbers for the peak frequency and its harmonics. The wave numbers are determined by the Padé[2,2]-dispersion relation.

Figure 3.15: Left: Energy and amplitude spectra for random wave test. Right: Time series in $x = 24$ m for different choices of cut-off frequency.
3.4 Irregular waves — a practical test

quantities are related by (3.48) and the square root operation implies a much slower decay of the amplitudes for high frequencies than for the energy spectrum. However, as there is no point in retaining high frequencies which are described wrongly, it was chosen to cut off the spectrum in \( f = 4f_{\text{peak}} \). This gives an error of 23\% for the highest frequency at \( h = 0.4 \text{ m} \). This is a rather large error, and for practical use a smaller cut-off frequency is generally recommended. On the other hand, this error is valid for the deep parts of the domain, while for the shallow part of the domain, the largest error in phase speed is 0.4\%.

To investigate the impact on the wave field of this truncation, a series of runs with different number of frequencies was made. For each run the input spectrum was identical: the JONSWAP spectrum with parameters as specified above. For each run, this spectrum was extended with more frequencies beyond \( f = 4f_p \), initiated with zero energy. Hereby it could be monitored how the truncation of energy generated at higher frequencies affect the results. The deviations are most pronounced after the bar. This can be seen in the right panel of Figure 3.15, where time series for different cut-off frequencies are shown. The truncation results in phase errors as well as amplitude errors. The convergence towards the solution with the frequency axis truncated in \( f = 8f_p \) is slow.

This shows that the calculations are sensitive to the choice of cut-off frequency. However, as described above, there is not much to do about it, since inclusion of the higher frequencies cannot be done accurately. For fully dispersive models, this problem is eliminated, since the linear dispersion relation is exact for all frequencies.

3.4.2 Results of the frequency domain model

In the frequency domain model, the influence of spatial step size was investigated for \( dx = (1, 5 \cdot 10^{-1}, 2.5 \cdot 10^{-1}, 2 \cdot 10^{-1}, 1 \cdot 10^{-1}, 5 \cdot 10^{-2}, 1.10^{-2}) \text{ m} \). As reference solution, a grid spacing of \( dx = 1 \cdot 10^{-3} \text{ m} \) was used. The CPU-times and error measures are shown in Table 3.4. Again, only the sum of the user and system time is shown, since the system time makes out a tiny fraction of the total time. Each run was processed twice with very small variations in CPU-time. The time series errors are plotted in Figure 3.16. The skewness errors show a

<table>
<thead>
<tr>
<th>( dx \text{ [m]} )</th>
<th>( Err^* )</th>
<th>Mean CPU-time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8\cdot10^{-1}</td>
<td>7.8</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2\cdot10^{-2}</td>
<td>15.2</td>
</tr>
<tr>
<td>0.25</td>
<td>8.1\cdot10^{-4}</td>
<td>29.9</td>
</tr>
<tr>
<td>0.2</td>
<td>3.3\cdot10^{-4}</td>
<td>36.8</td>
</tr>
<tr>
<td>0.1</td>
<td>1.7\cdot10^{-5}</td>
<td>69.3</td>
</tr>
<tr>
<td>0.05</td>
<td>8.7\cdot10^{-7}</td>
<td>145</td>
</tr>
<tr>
<td>0.01</td>
<td>1.5\cdot10^{-9}</td>
<td>736</td>
</tr>
</tbody>
</table>

Table 3.4: CPU-time and accuracy for different step sizes.
similar variation, although with a more spiky behavior, and are not shown here. After the bar, the solution of \( dx = 0.5 \text{ m} \) gives a time series error of around 1\%. The fast convergence for the step size indicates that the wave field is dominantly linear. This is a consequence of the spreading in wave heights for irregular waves. If the highest wave is chosen to reach the breaking limit, a large fraction of the other waves will be of rather small wave height and therefore rather linear in their evolution.

![Figure 3.16: Time series error as function of space for various step lengths.](image)

In Figure 3.17 a high-wave incident of the time series in \( x = 24 \text{ m} \) is shown. We see that the solution has practically converged for \( dx = 0.5 \text{ m} \).

**CPU-time versus accuracy** For the different step sizes, the mean time series error in \( x = [17; 24] \text{ m} \), as defined in (3.43) is plotted against CPU-time in Figure 3.18. The numbers generating the points are listed in Table 3.4. The points form a straight line in the double logarithmic plot, reflecting the fourth-order accuracy of the integration scheme (the slope of the line is \(-4\) and CPU-time and step length are inversely proportional). An error of 1.2\% can be obtained in 15 s of CPU-time with a step length of 0.5 m.

**Raw power spectra at different locations in the channel** In Figure 3.19, energy spectra at different locations in the channel are plotted. The locations are \( x = 0 \text{ m} \), the corners of the bar: \( x = (6,12,14,17) \text{ m} \) and \( x = 24 \text{ m} \). The plots show how higher harmonics are generated through the propagation over the bar and how the spectrum in the down-wave part of the channel possess a large amount of energy around twice the peak frequency of the initial spectrum.
3.4 Irregular waves — a practical test

Figure 3.17: Time series in $x = 24$ m calculated with different step size.

Figure 3.18: CPU-time versus $Err*$ for frequency domain runs.
Figure 3.19: Energy spectrum at various positions in the channel.
3.4.3 Results of the time domain model

To test the efficiency of the time domain model, time series for the spectrum used in the frequency domain simulations were used as the driving boundary condition. The complex values of the spectral amplitudes were the same as for the frequency domain runs. Again, the simulation time was chosen to be 200 peak periods. To warm up the computational domain, the last 40 periods of this time series was added in the beginning of the signal. Due to the periodicity, this extra time series joins the initial time series smoothly. Hence, when analyzing the results, only the last 200 peak periods of the output were considered, but this time series still forms a periodic signal. Hence for all locations of the domain, the time series analyzed is periodic. This would not be the case if the time series was constructed with a total length of 240 peak periods initially, without any repetitions.

For each run, the grid spacing $dx$ and the time step were chosen such that the Courant number $\sqrt{\frac{g}{h}} \frac{dt}{dx}$ in the deep-water part of the domain was 0.40. The five runs made are listed in Table 3.5 along with the time step, the total number of time steps needed for simulating 200 periods, $M_{\text{sim}}$, and the CPU-times. The CPU-times are given as the sum of user time and system time and as an average over two runs. The largest deviation between the two CPU-times of one run was 1.6%. When running the model, output was taken for each meter of the domain. The temporal resolution of the output was 0.08 s, corresponding to 25.25 points within a peak period. Hereby, the highest frequency of the input spectrum is resolved in the output with 6.3125 grid points per wave period.

The output of the time domain model was analyzed by calculating the error measures (3.41)–(3.43). The time series of the output were interpolated to a grid of 8192 points using FFT and IFFT. Doing so, time series with arbitrary ratios between the time steps can be compared. The results of the run of $dx = 5 \cdot 10^{-3}$ m were used as reference solution.

<table>
<thead>
<tr>
<th>$dx$ [m]</th>
<th>$dt$ [s]</th>
<th>$M_{\text{sim}}$</th>
<th>$C_r$</th>
<th>CPU time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.02</td>
<td>20200</td>
<td>0.40</td>
<td>∼ 62</td>
</tr>
<tr>
<td>0.05</td>
<td>0.01</td>
<td>40400</td>
<td>0.40</td>
<td>∼225</td>
</tr>
<tr>
<td>0.02</td>
<td>0.004</td>
<td>101000</td>
<td>0.40</td>
<td>∼1444</td>
</tr>
<tr>
<td>0.01</td>
<td>0.002</td>
<td>202000</td>
<td>0.40</td>
<td>∼5829</td>
</tr>
<tr>
<td>0.005</td>
<td>0.001</td>
<td>404000</td>
<td>0.40</td>
<td>∼23874</td>
</tr>
</tbody>
</table>

Table 3.5: Grid spacing, time step, number of time steps in simulation, Courant number and CPU-time for time domain runs.

The output of the time domain model was analyzed by calculating the error measures (3.41)–(3.43). The time series of the output were interpolated to a grid of 8192 points using FFT and IFFT. Doing so, time series with arbitrary ratios between the time steps can be compared. The results of the run of $dx = 5 \cdot 10^{-3}$ m were used as reference solution.

Time series error In Figure 3.20 the error measures for the time series are plotted against spatial position. The most accurate solution compared to the reference solution starts out in $x = 6$ m with an error of 4.5%. Similar to the results for regular waves, and for irregular waves within the frequency domain model, the time series error increases most rapidly on the
upward slope and on top of the bar. This is due to the high nonlinearity on top of the bar, resulting in inadequate results for the coarse grids. The run of $dx = 1 \cdot 10^{-2} \text{ m}$ shows an error of around 2.5% at the end of the domain, while the solution produced with $dx = 2 \cdot 10^{-2} \text{ m}$ gives an error of around 10% at the end of the domain. The error plot for the skewness

![Error plot for the skewness](image)

Figure 3.20: Time series error as function of space.

shows a more oscillating behavior in comparison with the time series error. However, the errors are of the same magnitude as the time series error, and the plot is therefore not shown here.

In Figure 3.21, a short part of the time series in $x = 24 \text{ m}$ is plotted for the different runs. The differences between the runs of varying grid sizes are clearly seen. The results generated on coarse grids show a phase shift when comparing to the reference solution of $dx = 5 \cdot 10^{-3} \text{ m}$.

**CPU-time versus accuracy** In Figure 3.22, $Err^*$ is plotted against the CPU-time needed for the simulation (including warm up time). In the double-logarithmic plot, the dependence is linear with a slope of about $-1$.

### 3.4.4 Comparison of CPU-times

For irregular waves, the CPU-plots of the evolution equation model and the time domain model are shown in Figures 3.18 and 3.22. We now compare these results by fitting straight lines through the points and calculate the ratio between the CPU-times.
3.4 Irregular waves — a practical test

Figure 3.21: Time series in $x = 24$ m.

Figure 3.22: CPU-time versus accuracy for time domain runs.
Chapter 3. Efficiency of Evolution Equations

The fitted lines are shown in Figure 3.23. The different slope of the lines are due to the accuracy of the numerical schemes used. For the frequency domain model, the integration scheme is fourth-order in the step length while the time domain model is only second-order in space and time. As for regular waves, the ratio between the CPU-times of the models is depicted against $Err^*$ for the interval $Err^* = [0.01; 0.5]$ in Figure 3.24. For a time series error of 10%, the ratio between the CPU-times is 117 or 100 in round numbers. We regard a time series error of 10% to be an acceptable level for the models, reflecting the accuracy demanded for a practical application of the models.

3.5 Summary and conclusions

In this chapter, it has been shown how the calculation of the nonlinear terms within evolution equations can be carried out based on time series, utilizing the FFT-algorithm. This makes it possible to solve a set of evolution equations with a computational work load of $O(N \log N)$, where $N$ is the number of frequencies. Compared to the conventional computational effort of $O(N^2)$, this is a strong improvement of the computational efficiency.

The practical efficiency of evolution equations has been assessed through two practical tests: regular and irregular waves propagating over a submerged bar. CPU-times of the evolution equations were compared to CPU-times of the corresponding time domain model, with respect to deviation from a converged run. The two tests show that the frequency domain model is around 1000 times faster than the time domain model for regular waves, for a time
3.5 Summary and conclusions

Figure 3.24: Ratio of CPU-time between the time domain model and the frequency domain model. Irregular waves.

series error measure of 10%. For irregular waves the frequency domain model is 100 times faster for the same error level. Of course these tests are just two picks in the parameter space for all possible test cases, but nevertheless, the results show that frequency domain models are indeed relevant for engineering applications due to their computational efficiency.

The tests carried out are based on parameters from a laboratory experiment. For a real application, the physical domain may be much larger. For a larger domain, the warm-up time of the time domain model increases and we therefore expect the frequency domain model to perform even better than for the present tests. If the time series modelled is made longer, the $N \log N$ cost of the nonlinear term in the frequency domain model will make the model less superior than for the present test. However, as the log-factor is only slowly increasing with $N$, the superior efficiency of the model is still expected to apply.

For $M$ physical points and $M$ frequencies, the evolution equations can solve a wave problem with a computational effort of $O(MN \log N)$. For a time domain model the work load is $O(MN)$ and it may therefore be a surprise that the evolution equations are so much faster. For regular waves, one obvious reason is the avoidance of the warm-up time of the computational domain. This advantage is not as pronounced for irregular waves, since the warm-up time is only a small fraction of the total simulation time. Another reason for the efficiency is that evolution equations are formulated in the wave amplitudes rather than the free surface elevation. For a linear wave field, the steps of integration can therefore be very large, since only the linear shoaling needs to be resolved. For nonlinear wave fields, the bound wave components must be resolved, but at a scale defined by the difference between the free wave number and the bound wave number, i.e., the phase mismatch. This scale will
always be larger than the scale of the bound wave itself.

In the time domain, the requirement of a limited Courant number dictates a maximum value of the time step, once the spatial discretization is chosen. Usually, this time step corresponds to a Nyquist frequency being much larger than the cut-off frequency of the wave spectrum. When calculating the similar wave field in the frequency domain, these high frequencies can be left out, such that the highest frequency of calculation is simply the highest frequency of the wave field.

These are the main reasons why the evolution equation approach is faster than the time domain approach. Still, however, one must bear in mind that evolution equations generally are less accurate than the corresponding time domain formulations.
Chapter 4

Inclusion of Roller Breaking

Evolution equations are appropriate for describing the changes of a wave field due to nonlinear interactions and shoaling. With the speed-up of the calculation of the nonlinear terms, irregular waves with a large number of frequencies can be treated efficiently and fast in comparison with a similar time domain formulation. The restriction to weak (quadratic) nonlinearity and the limited dispersion for Boussinesq evolution equations, make triad interactions one of the obvious applications of evolution equations. As this process is strongest in shallow water, a natural extension is to include wave breaking in the models.

Obviously, a detailed description of the wave breaking process is beyond the capability of evolution equations. They are derived under the assumption of irrotational motion and no dissipation. However, a simple breaking model, modelling the decay of the wave amplitudes and the change of the profile implied by breaking may be useful for applications.

Such breaking models have been implemented within evolution equations by several authors. We review some of the contributions in Section 4.1. Usually, these formulations introduce a quasi-linear damping term

\[ a_{p,x}^{\text{breaking}} = -d_p a_p, \quad p = 1, \ldots, N, \]  

where \( d_p \) is a real coefficient based on an empirical dissipation formulation. Such formulations do not take the phase information of the frequency components into account, and have therefore a limited chance of modelling the shape of the breaking waves. Also, since \( d_p \) is real, the breaking formulation (4.1) does not allow for any phase shifts of the harmonics due to the breaking process. Further, as the dissipation functions are based on the overall energy density of the total wave field, the effect of wave breaking is distributed uniformly over a time series, rather than locally, affecting just the single wave event.

In time domain Boussinesq formulations, a number of breaking formulations exist. A number of authors (Tao, 1983; Abbot et al., 1983; Zelt, 1991; Sato et al., 1991; Karambas and Koutitas, 1992) have included the effect of wave breaking through an eddy-viscosity concept.
Chapter 4. Inclusion of Roller Breaking

In these works, the eddy viscosity formulation is based on horizontal gradients of the depth-integrated or depth-averaged horizontal velocity. Another approach to include wave breaking is the incorporation of the surface roller concept, leading to an additional pressure term in the momentum equations (Deigaard, 1989; Brocchini et al., 1992) or an additional convective momentum term (Schäffer et al., 1993) in the momentum equations. Recently, Veeramony and Svendsen (2000) have presented a breaking model based on the vorticity generated by the breaking process.

Schäffer et al. (1993) applied their roller breaking model to breaking regular and irregular waves over a submerged bar. Further tests were presented by Madsen et al. (1997a) for uniformly sloping beaches. Since the Boussinesq model is only weakly nonlinear, the full wave height before breaking was not reached for some of the tests. Besides this, however, the model results for wave profiles and setup are very good. Even the delayed onset of the setup just after the initiation of wave breaking is captured by the model. In a companion paper, (Madsen et al., 1997b), results for irregular waves were presented, including modelling of surf beat and run-up in the swash zone. The modelling of these phenomena was facilitated by incorporating a moving shoreline boundary condition at the beach. As for regular waves, a good match for the shape of the wave profiles as well as phase averaged quantities was obtained.

With background in these results, it was decided to incorporate this breaking model into evolution equations. For one dimensional wave propagation, the Boussinesq model used by Madsen et al. (1997a) is identical to the one of Madsen and Sørensen (1992). The corresponding set of evolution equations was presented by Madsen and Sørensen (1993), and we have already used this model in Chapter 3. Consequently, in this chapter, the surface roller breaking model is incorporated into this particular set of evolution equations. As for the efficiency of the evolution equations, the use of FFT is essential for doing this, since a toggling between the frequency domain representation and local time series for the flow variables is needed.

The new breaking formulation is tested on a physical test of spilling breakers of regular waves (Ting and Kirby, 1994). Results of the evolution equations are compared to results of the time domain formulation (3.1) using the time domain version of the roller breaking formulation. It turns out that two problems appear for the results obtained by the new frequency domain formulation: 1) the waves propagate with a too large phase speed and 2) the characteristic saw-tooth shape of the breakers, as these turn into travelling bores in the inner surf zone, is not reproduced. The cause of (1) is the amplitude dispersion within the wave model. An investigation of this is the subject of Chapter 5. The cause of (2) has not yet been found.

The outline of the chapter is as follows. First, in Section 4.1, the conventional breaking formulations of evolution equations are described. Next, in Section 4.2 the principles of the roller breaking model, as it is used in the time domain, are given. Section 4.3 details how the roller formulation is adapted to the frequency domain. An equation for the mean water level is derived in Section 4.4. Results of the new breaking formulation are presented in
4.1 Conventional breaking formulations

Traditionally, breaking in evolution equations is included through a quasi-linear damping term, simply added to the right hand side of the model, that is
\[ a_{p,x} |_{\text{breaking}} = -d_p a_p, \quad p = 1, \ldots, N. \] (4.2)

This has been done by several authors. Below, we describe the principles for determining \( d_p \), referencing the works of Mase and Kirby (1992), Eldeberky and Battjes (1996) and Chen et al. (1997).

4.1.1 Mase and Kirby (1992)

Mase and Kirby (1992) incorporated such a term into a set of evolution equations for the KdV-equation. The magnitude of the damping function was determined using an empirical formulation of the energy dissipation of Thornton and Guza (1983). We detail this below, following considerations similar to those of Mase and Kirby (1992) and Eldeberky and Battjes (1996).

We consider a wave field expanded as \( \eta(x,t) = \sum_{p=-N}^{N} a_p(x) e^{i(\omega_p t - \int k_p dx)} \). To lowest order, the local energy can be written as the sum of the energy at each frequency
\[ E = \sum_{p=1}^{N} \frac{1}{8} \rho g H_p^2 = 2 \rho g \sum_{p=1}^{N} |a_p|^2 \] (4.3)

where \( \rho \) is the density of the fluid. As also noted in Section 3.4.1, the expression \( E = \frac{1}{2} \sum_{p=1}^{N} |A_p|^2 \) is often seen in the literature, and is valid for the alternative expansion \( \eta(x,t) = \sum_{p=-N}^{N} A_p(x) e^{i(\omega_p t - \int k_p dx)} \), except that the factor \( \rho g \) has been left out. Here we use (4.3) throughout, consistent with the expansion convention (1.2) for the definition of \( a_p \). Multiplication of (4.1) with \( 2 \rho g a_p^* \) and subsequent addition of the complex conjugate expression yields
\[ E_{p,x} |_{\text{breaking}} = -2d_p E_p, \] (4.4)

where \( E_p \) is the local energy at frequency \( p \). Assuming slow variation of depth, thus ignoring \( c_{gp,x} \), we can multiply with \( c_{gp} \), the group velocity of frequency \( p \) to obtain
\[ F_{p,x} |_{\text{breaking}} = -2d_p F_p, \] (4.5)
where $F$ is the energy flux given by $F = c_g E$. If an empirical model for the total spatial decrease of energy flux due to breaking is available, stating $F_x = -D$, summation over all frequencies in (4.5) gives

$$F_{\text{tot},x|\text{breaking}} = -2 \sum_{p=1}^{N} d_p F_p = -D. \quad (4.6)$$

This equation must be fulfilled to match the breaking formulation with the empirical formula. The choice $d_p = \frac{1}{2} \sigma_p D / \sum_{n=1}^{N} \sigma_n F_n$ satisfies this requirement, such that the resulting breaking term in the evolution equations can be written as

$$a_{p,x|\text{breaking}} = -\frac{1}{2} \frac{\sigma_p D}{\sum_{n=1}^{N} \sigma_n F_n} a_p, \quad p = 1, \ldots, N. \quad (4.7)$$

This result is valid no matter the definition of $a_p$, since the scaling between the empirical breaking formulation (giving the value of $D$) and the evolution equation model is expressed solely in the energy flux. The coefficient $\sigma_p$ is a weighting function that governs the distribution of the breaking dissipation among the frequencies. Taking $\sigma_p$ to be constant corresponds to dissipating energy flux proportionally to the energy flux at each frequency, as can be seen from (4.4) and (4.5).

To determine the distribution of $\sigma_p$, Mase and Kirby (1992) analyzed laboratory data for a uniformly sloping beach. Energy spectra of the measuring stations were averaged, and compared between consecutive stations. The effects of shoaling and nonlinear interactions were estimated by running the evolution equation model without breaking from one station to the other, such that the spectral change due to breaking could be determined. The analysis showed that $\sigma_p$ is a growing function of frequency, and Mase and Kirby found that an $f_p^2$ dependence was reasonable. They therefore chose the weighting

$$\sigma_p = F + (1 - F)f_p^2 \frac{\sum_{n=1}^{N} |a_n|^2}{\sum_{n=1}^{N} f_p^2 |a_n|^2} \quad (4.8)$$

with $F = 0.5$. The factor $F$, should not be confused with the energy flux, for which the same symbol is used. This breaking model was later used by Kaihatu and Kirby (1995) and Kaihatu and Kirby (1998) to model breaking within a set of fully dispersive evolution equations and a set of evolution equations for Nwogu’s (1993) equations.

### 4.1.2 Eldeberky and Battjes (1996)

A similar breaking formulation was developed by Eldeberky and Battjes (1996). They, however, assumed a uniform distribution of $\sigma_p$, and thus obtained the simple breaking term

$$a_{p,x|\text{breaking}} = -\frac{1}{2} \frac{D}{\sum_{n=1}^{N} F_n} a_p, \quad p = 1, \ldots, N. \quad (4.9)$$
4.1 Conventional breaking formulations

The energy dissipation was determined from the dissipation model of Battjes and Janssen (1978) for irregular waves. This model states that the spatial decay of energy flux due to breaking can be expressed as

\[ D = \rho g \frac{\alpha}{4} f_c Q_b H_{\text{max}}^2 \]  

(4.10)

where \( \alpha \) is a free parameter of order 1, \( f_c \) is a characteristic frequency, and \( H_{\text{max}} \) is the maximum local wave height, estimated as \( H_{\text{max}} = \gamma h \), where \( \gamma \in [0.6; 0.8] \) and \( h \) is the local depth. \( Q_b \) is the so-called fraction of broken waves, which is given by

\[ \frac{1 - Q_b}{\ln Q_b} = - \left( \frac{H_{\text{rms}}}{H_{\text{max}}} \right)^2, \]  

(4.11)

where \( H_{\text{rms}} = \sqrt{2} H_s \). Further details are given in Battjes and Janssen (1978).

The wave breaking formulation was incorporated into the evolution equation model of Madsen and Sørensen (1993). The model was tested on four experiments, two submerged bars, a monotonic beach and a barred beach. Generally, the overall shapes of the power spectra were well predicted by the model, apart from an overestimation of the low-frequency part of the spectra for two of the tests. Also time series, comparing experimental and model results were presented. The match of the model results to the time series was not as good as for the power spectra. Especially for the cases of strong breaking, deviations occurred.

4.1.3 Influence on skewness and asymmetry of breaking

The results of Eldeberky and Battjes (1996) suggest that a uniform distribution of the energy dissipation is sufficient for modelling the power spectra of breaking irregular waves. On the other hand, Mase and Kirby (1992) found that a weighting with \( f^2 \) should be used. This discrepancy was investigated in detail by Chen et al. (1997), who incorporated a breaking model like (4.1) into the evolution equations of Chen and Liu (1995). The empirical breaking formulation was given by the energy balance model of Whitford (1988), which can be seen as a modified version of the model of Thornton and Guza (1983), used by Mase and Kirby (1992). Ten experiments obtained in the laboratory as well as the field were modelled. By minimizing the deviation between the observations and model predictions, the optimal value of the empirical breaking parameters in the model were found. The overall conclusion was that the shape of the power spectra is not sensitive to the spectral distribution of the breaking dissipation. The higher-order measures of skewness and asymmetry, however, are significantly improved by choosing \( F = 0 \) in (4.8). Taking the frequency dependence to be \( f_p^4 \) in (4.8) instead of \( f_p^2 \) did not give significant changes.

It may seem a little surprising that the spectral shape is unaffected by the frequency dependence of the breaking dissipation. As suggested by Chen et al. (1997), a possible explanation is that the nonlinear energy transfer rearranges the spectral energy content during wave breaking. This explanation is supported by the work of Elgar et al. (1997), who analyzed
field measurements of wave breaking. Here, the frequency distribution of the dissipation due to wave breaking was found to be similar to the frequency distribution of the net effect of nonlinear energy exchange. This indicates that the nonlinear interactions restore the spectral energy content in the frequencies where dissipation occur. However, even though the nonlinear interactions may compensate for a wrong distribution of the energy dissipation in a numerical model, the phases are expectedly affected by this. This may explain that the higher-order measures of skewness and asymmetry are not well modelled for a uniform breaking distribution.

4.2 The roller breaking model

In time domain Boussinesq models, the roller breaking model has been found to describe the wave shape and wave height decay due to breaking with good accuracy, see Schäffer et al. (1993) and Madsen et al. (1997a). The roller model was suggested by Svendsen (1984) for a phase averaged model. The basic idea is to divide the breaking wave into a region which is part of the irrotational wave motion and a roller region on top of this, in which a bulk of water is moved passively at the wave front towards the beach. This is sketched in Figure 4.1. The vertical thickness of the roller region is denoted $\delta$.

![Figure 4.1: Definition sketch and velocity profile for the roller model.](image)

The introduction of the roller region leads to a discontinuity in the velocity profile for the horizontal velocity at the lower boundary of the roller region, see Figure 4.1. Schäffer et al. (1993) rederived the nonlinear shallow water equations for this modified velocity profile. The resulting mass equation is identical to (3.1a), while the momentum equation was altered by the addition of a term $\partial R/\partial x$. Further addition of the classical dispersive Boussinesq terms led to a set of equations identical to the classical Boussinesq equations, (Peregrine, 1967), besides the added roller term.

Since the roller term was derived in the absence of the dispersive Boussinesq terms, one may as well use the enhanced dispersive terms of Madsen and Sørensen (1992). This gives the
4.2 The roller breaking model

formulation used by Madsen et al. (1997a), which in one dimension reads

$$
\eta_t + P_x = 0 \quad (4.12a)
$$

$$
P_t + \left( \frac{P^2}{h + \eta} \right)_x + R_x + g(h + \eta)\eta_x - Bh^3\eta_{xxx} - (B + 1/3)h^2P_{xxt}
- h_x \left( 2Bgh^2\eta_{xx} + \frac{1}{3}hP_{xx} \right) = 0, \quad (4.12b)
$$

where $R$ is a function of the roller thickness $\delta$, the depth $h$, the volume flux $P$, the wave celerity $c$ and the free surface elevation $\eta$ and can be written

$$
R = \frac{\delta}{1 - \delta/d} (c - P/d)^2 \quad d = h + \eta. \quad (4.13)
$$

We see that these equations are identical to (3.1a)–(3.1b), except for the addition of the roller term in the momentum equation.

4.2.1 Determination of the roller thickness $\delta$

The roller thickness $\delta$ is determined by a heuristic geometrical approach. A critical slope of the wave front $\tan \phi$ is defined, and the foot of the roller is determined as the first point in which the surface slope exceeds this value. The roller region is then determined as the part of the wave profile lying above a line passing through the roller foot and having the critical slope $\tan \phi$. This is sketched in figure 4.2. To compensate for the lower bound of the roller being a straight line, the roller thickness is multiplied by a shape factor $f_\delta$. Further, the limiting surface slope $\tan \phi$ is given a time dependence starting from $\tan \phi_B$ decaying to $\tan \phi_0$ within a time scale of $t_{1/2}$:

$$
\tan \phi(t) = \tan \phi_0 + (\tan \phi_B - \tan \phi_0) \exp \left[-\ln 2(t - t_0)/t_{1/2}\right] \quad (4.14)
$$

This is based on the observation that regular waves can have rather steep fronts just before breaking, while breaking waves turn into a travelling bore state in the surf zone, with a much smaller limiting angle. For the Boussinesq model (4.12) the parameter values $\phi_B = 20^\circ, \phi_0 = 10^\circ, f_\delta = 1.5$ and $t_{1/2} = T/5$ have been found to be good a good set of standard parameters, see Schäffer et al. (1993) and Madsen et al. (1997a).
The four parameters

The effect of the four parameters is as follows. The initial breaking angle $\phi_B$ governs the onset of breaking in the model, since the initial breaking criterion is

$$\eta > c_{\text{break}} \tan \phi_B = 1.3 \sqrt{gh} \tan \phi_B. \quad (4.15)$$

The lower the values taken by $\phi_B$, the earlier the breaking starts. The parameter $\phi_0$ is the limiting angle of the breaking waves, as they reach the bore state inside the surf zone. Lowering this value increases the breaking dissipation in the inner surf zone. The phase speed of the breaking wave, here taken to be $1.3 \sqrt{gh}$ affects the breaking criterion similarly to the values of $(\phi_B, \phi_0)$, independent of time, however.

The parameter $t_{1/2}$ is the half-life period of the decay of the breaking angle. The default value used is $T_{\text{peak}}/5$, where $T_{\text{peak}}$ is the peak period for irregular waves, and simply the wave period for regular waves. If this value is decreased, the transition from the initial breaking angle of $\phi_B$ to $\phi_0$ happens faster. The parameter can thus be used for calibrating the spatial variation of the wave height within the surf zone.

Furthermore $f_\delta$ is the shape factor multiplied on the geometric determination of the roller. The larger this parameter is, the thicker the roller and the stronger the breaking. However, as the breaking criterion is still governed by the value of $\tan \phi(t)$, a strong breaking due to a large value of $f_\delta$ will usually just result in a smaller wave in the next spatial position, assigned a relatively smaller initial roller thickness $\delta_0$. The sensitivity to a change of this parameter is therefore rather weak. We use $f_\delta = 1.5$ as the default value.

4.3 Adapting the roller model to the frequency domain

When a spatial wave profile is given, the roller thickness is determined as

$$\delta_0(x, t) = \eta(x, t) - \eta_{\text{toe}} - (x_{\text{toe}} - x) \tan \phi(t). \quad (4.16)$$

where $\delta_0$ denotes the roller thickness prior to the multiplication with the shape factor $f_\delta$. When the spatial variation of the wave field is only known for $x < x_0$, as is the case during the integration of a set of evolution equations, the latter term cannot be evaluated. The expression for $\delta$ needs to be expressed only from a time series of $\eta$.

To allow for this, we assume that the roller region of the breaking wave travels with permanent form locally. That is $\delta_0 = \delta_0(x - ct)$ and thereby $\delta_{0,t} = -c \delta_{0,x}$ and similarly for $\eta$. Differentiating (4.16) with respect to $x$, and using the permanent form assumption yields $\delta_{0,t} = \eta_t - c \tan \phi(t)$, which is easily integrated to

$$\delta_0(x_0, t) = \eta - \eta_{\text{toe}} - c \int_{t_{\text{toe}}}^{t} \tan \phi(t) \, dt. \quad (4.17)$$
When \( \delta \) is calculated using (4.17), the breaking criterion is changed to a critical value for \( \eta_t \). Again, assuming that the wave travels with permanent form locally, we use \( \eta_t > c \tan \phi \) as the breaking criterion. For \( c \) we use the estimate \( c = 1.3\sqrt{gh} \) of Stive (1980).

Once \( \delta_0 \) is determined for the breaking wave crests in a time series, \( \delta \) is found as \( \delta = f_\delta \delta_0 \). Further any negative values of \( \delta \) are set to zero. Altogether, this allows for calculating \( R \) defined by (4.13) as a time series.

In the wave equation (3.2), the inclusion of roller breaking results in the addition of the term \( R_{xx} \) at the right hand side. Hence, following the derivation of (3.40), the roller effect can be included in the evolution equations as

\[
a_{p,x} = -i \frac{\beta_0}{\beta_1} a_p - \beta_{s,p} \frac{h_x}{h} a_p + \frac{i}{\beta_1} \left[ \left( \frac{g\eta^2}{2} + \frac{P^2}{h} \right)_{xx} + R[p]_{xx} \right] e^{i \int k_p dx},
\]

(4.18)

where \( R[p]_{xx} \) is defined by

\[
R_{xx}(t) = \sum_{p=-M}^{M} R[p]_{xx} e^{i \omega_t}.
\]

(4.19)

Assuming that \( R \) is moving with constant form with the wave, i.e. \( R = R(x-ct) \), we approximate \( R_{xx} \) by \( R_0/c^2 \). The time differentiation is performed after the Fourier transformation to the frequency domain, such that the breaking term at each frequency enters the model as

\[
a_{p,x} = -i \frac{\beta_0}{\beta_1} a_p - \beta_{s,p} \frac{h_x}{h} a_p + \frac{i}{\beta_1} \left[ \left( \frac{g\eta^2}{2} + \frac{P^2}{h} \right)_{xx} - \frac{\omega^2}{c^2} R[p] \right] e^{i \int k_p dx}.
\]

(4.20)

### 4.3.1 Details on the implementation

The above description shows how the roller effect can be incorporated into evolution equations. We now detail some aspects of the implementation. Especially, the storing of the values of \( \tan \phi \) is given attention.

Imagine that a time series of breaking rollers is passing an observer, who plots the surface elevation as function of time as well as \( \tan \phi \). Just next to him, towards the shore, another observer is plotting the same quantities seen from his position. Now, as it takes a certain time for the waves to propagate between the observers, their records of the surface elevations will be time shifted. The records of \( \tan \phi \), however, will not be time shifted since the roller model assumes that \( \tan \phi \) is constant in space for each roller. Thus, in a frozen instant when a roller is passing the observers, they will see two different surface elevations, but the same breaking angle \( \phi \), and therefore the same value of \( \tan \phi \).

During the integration of the evolution equations, the time series of \( \tan \phi \) can therefore be reused when moving from one position to another. This is done by storing the time series
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Figure 4.3: Calculation of roller thickness, $\delta_0$.

of $\tan \phi$ in a vector. An example of such a time series is shown in the lower panel of Figure 4.3. In the upper panel the free surface elevation is plotted. We now detail, how the roller thickness prior to the multiplication with $f_B$ is found for such a time series.

A cyclic sweep through the time series is started from the first trough. Starting in a trough ensures that no breaking happens in the start nor end of the sweep. In each point, $j$, the breaking criterion $\eta_{t,j} > c \tan \phi_j$ is checked. If no breaking, $\tan \phi_j$ is overwritten with the value $\tan \phi_B$.

If the breaking limit for $\eta_{t,j}$ is exceeded, the exact start time of breaking $t_{toe}$ is found by linear interpolation of the equation $\eta_t = c \tan \phi$. The surface elevation in the break point, $\eta_{toe}$, is found using $\eta$ and $\eta_t$ in points $j$ and $j-1$. Next, the value of $\tan \phi$ midway between $t_{toe}$ and $t_j$ is approximated, such that $\delta_{0,j} = \eta_j - \eta_{toe} + c \int_{t_{toe}}^{t_j} \tan \phi dt$ can be calculated using a midpoint formula.

In the following points, $\tan \phi_j$ is calculated from $\tan \phi_{j-1}$ by the following formula (see (4.14))

$$\tan \phi(t + \Delta t) = (\tan \phi(t) - \tan \phi_0)e^{-\frac{\ln 2}{\Delta t}} + \tan \phi_0, \quad (4.21)$$

where $e^{-\frac{\ln 2}{\Delta t}}$ is constant for each model run. Subsequently $\delta_{0,j}$ is calculated using the trapezoidal rule for the integral of $\tan \phi$:

$$\delta_0(t + \Delta t) = \delta_0(t) - \eta(t) + \eta(t + \Delta t) - c \frac{\Delta t}{2} (\tan \phi(t) + \tan \phi(t + \Delta t)). \quad (4.22)$$
4.4 Inclusion of mean water level variations in the model

The calculation of $\delta_0$ is stopped when a new wave trough is reached.

When the whole sweep has been done, the negative values of $\delta_0$ are replaced by zeros. Next, $\delta = f_0 \delta_0$ and subsequently $R(t)$ are calculated. By carrying on the calculation of $\tan \phi$ all the way to the next trough, it is ensured that $\tan \phi$ has the right values in the next spatial position, where the roller will have moved towards right in Figure 4.3.

4.4 Inclusion of mean water level variations in the model

The set of evolution equations (4.20) does not treat the mean flow variables $\bar{\eta}$ and $\bar{P}$. In all physical situations of one-dimensional wave propagation, $\bar{P}_x = 0$ due to mass flux conservation. Hence, there is no need to establish an equation for $\bar{P}$ as long as it is of second order in the wave steepness, thus not effecting the dispersion relation through a Doppler shift. For waves propagating towards a beach we have $\bar{P} = 0$. The mean water elevation $\bar{\eta}$, however, is a second-order quantity that is non-zero. Outside the surf zone a set-down occurs due to the shoaling of the waves, and inside the surf zone, a set-up is produced by the reduction in radiation stress caused by wave breaking. Below an equation for this variation of the mean water level is derived.

First, since obviously $\omega_0 = k_0 = 0$, we note that the evolution equations (3.9) for $p = 0$ simply reads $a_{0,x} = 0$. It is not a surprise that the evolution equations cannot describe the mean flow, since the mean flow is not harmonic, as is assumed in the derivation of the evolution equations. For this reason, a separate equation for $a_0 = \bar{\eta}$ must be provided.

We can obtain such an equation by time averaging the momentum equation (4.12b) over one period. When the wave field consists of a series of waves of different frequencies, this means the longest period, i.e. $T = 1/\Delta f$. First, however, for completeness, we note that time averaging the mass equation (3.1a) simply yields $\bar{P}_x = 0$ i.e., conservation of mass flux. For the mean water level we assume

$$\bar{\eta} = O(\varepsilon) \quad \bar{\eta} = \bar{\eta}(\varepsilon x),$$

that is, the mean water level is of the same order as the second-order bound waves and is varying slowly in space. The latter assumption allows us to neglect higher derivatives of $\bar{\eta}$ than the first derivative. Further, we use that the time average of a time derivative of a flow variable is zero for a periodic flow. With these considerations, we get from (3.1b)

$$\left\langle \frac{P^2}{h} + \frac{1}{2} g \eta^2 \right\rangle_x + \bar{R}_x + gh \bar{\eta}_x = 0,$$

(4.24)

where the brackets $<>$ and the over-bar $\bar{}$ denote time averaging over one period. A similar equation was derived by Madsen et al. (1997a). As a side remark, we may write this equation
as
\[
\rho g h \bar{\eta}_x = -\frac{\partial}{\partial x} F_{w,\text{Bouss}} - \rho \bar{R}_x \tag{4.25}
\]
\[
F_{w,\text{Bouss}} = \rho \left\langle \frac{P^2}{h} + \frac{1}{2} \eta^2 \right\rangle, \tag{4.26}
\]

following Madsen et al. (1997a). Here \( F_{w,\text{Bouss}} \) is the radiation stress approximation of the Boussinesq model. In shallow water, \( F_{w,\text{Bouss}} = \frac{3}{16} \rho g H^2 \), similar to the fully dispersive result \( F_w = \frac{1}{16} \rho g (1 + 2G) H^2 \), where \( G = 2kh / \sinh 2kh \).

From (4.24) we have
\[
\bar{\eta}_x = -\frac{1}{h} \left\langle \frac{P^2}{gh} + \frac{1}{2} \eta^2 \right\rangle_x - \frac{1}{gh} \bar{R}_x. \tag{4.27}
\]

To express this in the wave amplitudes, we insert the series expansions, (3.3) in the above expression. As an example, we get
\[
\frac{1}{2} \left\langle \eta^2 \right\rangle_x = \langle \eta \bar{\eta}_x \rangle = \left\langle \sum_{p=-N}^{N} a_p e^{i(\omega_p t - \int k_p dx)} \right\rangle \sum_{n=-N}^{N} (-i k_n a_n + a_{n,x}) e^{i(\omega_p t - \int k_p dx)}
\]
\[
= \sum_{p=-N}^{N} i k_p a_p^* a_p^* + a_p a_{p,x} = \sum_{p=1}^{N} a_p a_{p,x}^* + a_{p,x}^* a_p
\]
\[
= \sum_{p=1}^{N} 2 |a_p| |a_p|_x, \tag{4.28}
\]

where the term of type \( a_0 a_{0,x} \) has been neglected consistent with (4.23). With the aid of this calculation we now express (4.27) as
\[
\bar{\eta}_x = -\frac{1}{gh} \bar{R}_x - \frac{1}{h} \sum_{p=1}^{N} \left( \frac{1}{gh} \frac{\omega_p^2}{k_p^2} + 2 \right) |a_p| |a_p|_x, \tag{4.29}
\]

which is an equation for the mean water level. It should be noted that the spatial derivative of the coefficient inside the sum in the above expression has been neglected. To be consistent, this derivative should have been retained.

The equation (4.29) can be integrated numerically along with the integration of the evolution equations. The spatial derivatives of \( |a_p| \) and \( \bar{R} \) are calculated numerically, using a backward finite difference approximation. For \( |a_p|_x \) this is done with fourth-order accuracy, by fitting a polynomial through the actual plus the last four points. For \( \bar{R} \) this approach is not feasible, since \( \bar{R} \) was found to be non-smooth in space. An approximation to \( \bar{R}_x \) was therefore obtained as the mean slope of \( \bar{R} \) through the actual point plus the last four points. A comparison between the two methods for the Ting and Kirby-test, gave graphically identical results, but the latter method could be used with larger spatial integration steps.
4.4 Inclusion of mean water level variations in the model

4.4.1 How the setup affects the wave field

In the above derivation, we have formally assumed that $\bar{\eta} = O(\varepsilon)$. This means that the setup is a second-order quantity, and we can therefore consistently calculate the harmonic wave field without taking the setup into account. However, within the breaking zone, practical observations show that the setup can be of similar magnitude as the harmonic amplitudes, and therefore it may be reasonable to include the effect of the setup into the calculations of the wave field.

The summation ranges of the nonlinear interaction terms in (3.9) allows for inclusion of a setup through the zeroth harmonic. Hence, the model at the current state allows for a constant deviation of the mean water level from the still water level, the effect entering through the nonlinear terms. The interaction coefficient, however, must be modified slightly. We consider the nonlinear term

$$N_p = -\frac{ig}{\beta_{1,p}} \sum_{s=p-N}^{N} (k_s + k_{p-s})^2 \left(\frac{1}{2} + \frac{1}{ghk_s k_{p-s}} a_s a_{p-s}\right) e^{-i\int (k_s + k_{p-s} - k_p) dx},$$

(4.30)

see (3.7). Due to the symmetry, the terms in the sum for $s = 0$ and $p - s = 0$ are identical. For $s = 0$ we get

$$N_p \left|_{\text{term from } s = 0} \right. = -\frac{ig}{\beta_{1,p}} k_{p-s}^2 \left(\frac{1}{2} + 0\right) a_0 a_p = -\frac{ig}{\beta_{1,p}} k_{p-s}^2 \bar{\eta} a_p,$$

(4.31)

where the zero in the interaction coefficient arises because $\bar{P} = 0 \times \bar{\eta} = 0$. With this modification of the coefficient, and a similar modification of the coefficient for $p - s = 0$, the influence of the setup can be treated through the nonlinear terms.

Alternative ways of including the setup  Now, it is interesting to ask how the inclusion of the setup through the nonlinear terms affect the wave field. Does it correspond to simply changing the water depth with the value of the setup?

We investigate this with basis in the wave equation (3.2), which we consider for constant depth. This wave equation is similar to the time domain formulation. In this equation we insert

$$\eta(x, t) = \bar{\eta} + a(x)e^{i\omega t} + a^*(x)e^{-i\omega t}$$

(4.32)

$$P(x, t) = b(x)e^{i\omega t} + b^*(x)e^{-i\omega t}.$$  (4.33)

We hereby assume that the net mass flux is zero. After insertion in the wave equation, we collect terms being coefficients to $e^{i\omega t}$. These terms form an equation for the propagating wave mode. We get

$$\left(1 - (B + 1/3)h^2 \nabla^2 \right) (-\omega^2) a - gh(1 - Bh^2 \nabla^2) \nabla^2 a = g\bar{\eta} \nabla^2 a$$

(4.34)
This equation describes the propagation of a linear wave on a constant setup. With this equation, we can now think of modelling linear wave propagation on a constant setup in two ways. One way is simply to change the depth $h$ to $\hat{h} = h + \bar{\eta}$ throughout and then put $\bar{\eta} = 0$. Alternatively we can leave $h$ unchanged and preserve the term at the right hand side. For the latter case we rewrite the equation to

$$
(1 - (B + 1/3)h^2 \nabla^2) (-\omega^2)a - g(h + \bar{\eta})\nabla^2a - Bh^2\nabla^4a = 0.
$$

(4.35)

We see that this equation differ from (4.34) with $h = \hat{h}$. The difference is that in (4.34) only the non-dispersive term is assigned a new effective depth, whereas all terms of order $\mu^2$ are unchanged. This is, however, not surprising since the effect of the setup enters through a term of order $\varepsilon$. Hence, if the setup should change the linear $O(\mu^2)$-terms, inclusion of $O(\varepsilon \mu^2)$-terms would be necessary. Such terms are not part of the model.

We therefore conclude that the time domain model does not model a depth change due to a setup similarly as a depth change due to a varying bathymetry. This will also be the case for the evolution equation model (3.9), since it is derived from the time domain model.

In the simulations with the evolution equations, it was chosen to include the effect of the setup through the nonlinear terms, as described by (4.31). The reason for this choice is that this formulation is closer to the time domain formulation than the local change of $h$. Note that the latter approach cannot be used in a time domain formulation, since $\bar{\eta}$ is not known until the end of the simulation.

### 4.5 Results of the new roller breaking model

We here present results for the spilling breaker test of Ting and Kirby (1994). The test was carried out in a wave flume with an initial depth of 0.4 m. The bathymetry is shown in Figure 4.4. In the interval $x = [9.6; 10]$ m, the slope is 1:20, while from $x = 10$ m, the slope is 1:35. A depth of 0 is reached in $x = 23.3$ m. The wave period is 2 s and the initial wave height 0.125 m. In this section we compare the results of the time domain roller formulation
4.5 Results of the new roller breaking model

4.5.1 Results of the time domain formulation

The time domain formulation (4.12) was run on a grid of $dx = 0.02$ m and a time step of $dt = 0.005$ s. The default parameters of the breaking model was used, i.e. $\phi_B = 20^\circ$, $\phi_0 = 10^\circ$, $t_{1/2} = T/5$ and $f_\delta = 1.5$.

Wave envelope In figure 4.5, the elevations of the crest, trough and mean water level are plotted as function of space. A lack of shoaling, leading to an underestimation of the wave height at the breaking point is the most pronounced deviation between the model results and the experimental data. This is caused by the assumption of weak nonlinearity of the model and can be improved by using a fully nonlinear model, thus retaining terms of order $O(\varepsilon^3\mu^2)$, see Sørensen et al. (1998). The breaking point, however, and the decay of the wave crest height is well modelled. For the incident wave, the mean water level is too small, resulting in an overall lowering of the wave profile when comparing to the experiment. In the interval $x = [16; 20]$ m, the mean water level and trough elevation are under-predicted.

Spatial wave profiles Spatial wave profiles for the numerical results are shown in Figure 4.6. A zoom in for the breaking region is shown in Figure 4.7. The waves are seen to lean forward and eventually evolve into saw-tooth shaped bores. The roller regions are shown as well.
Figure 4.6: Spatial profiles of the time domain model.

Figure 4.7: Spatial profiles of the time domain model. Zoom in from Figure 4.6.
Skewness and asymmetry Skewness and asymmetry are higher-order statistical measures of the nonlinearity of a wave field. For a time series record, the skewness is defined as

\[
S = \frac{\langle (\eta - \bar{\eta})^3 \rangle}{\left( \langle (\eta - \bar{\eta})^2 \rangle \right)^{3/2}} \tag{4.36}
\]

and is a measure of the asymmetry around the mean water level for the time series. Thus, a positive value of skewness indicates that the waves generally have a larger crest height than trough depth. The asymmetry is a measure of the departure from symmetry around a vertical plane. Regular waves on a flat bottom have zero asymmetry, while waves on a shoal have a non-zero asymmetry, because they lean forward in space. The asymmetry can be calculated as the skewness of the Hilbert transform of \( \eta \). Another way is to integrate the so-called bispectrum of the wave record. This is detailed in Appendix E.

The skewness and asymmetry of the Ting and Kirby-test are plotted in Figure 4.8. The skewness is positive and increases up till the point of breaking. As for the wave envelope, the experimental values are not reached by the model, but after the onset of breaking, there is a good match. The asymmetry is zero in the flat part of the domain and gets negative, as the wave shoals up the beach. Even though the asymmetry is under-predicted (with respect to absolute value) for the first points in the breaking zone, the general experimental trend is well modelled.

![Figure 4.8: Skewness (upper curve) and asymmetry (lower curve) of the Ting and Kirby-test. Comparison of time domain results and experiment.](image-url)
**Time series** In the measuring stations, time series of the experiment can be compared to time series of the numerical model. This is shown in Figure 4.9. There is a constant phase difference between the experimental time series and the computational results, which is only due to a different zero point of time for the experiment and the model, respectively. Also, the deviation in mean water level in the interval $x = [16; 20]$ m is clearly seen, when comparing the profiles. All in all, however, the shape of the waves is captured well by the model.

![Figure 4.9: Time series of free surface elevation for the Ting and Kirby-test. Comparison of time domain results and experiment.](image_url)

**4.5.2 Results of frequency domain roller model with default parameters**

The comparison of the time domain model to the experiment gives an idea on what to expect from the frequency domain model. Since the frequency domain model is derived from the time domain model, a fair goal is to obtain similar results. In Figure 4.10, the wave envelope
4.5 Results of the new roller breaking model

for a run with default breaking parameters is shown. Seven harmonics were included in the calculation and the roller term was calculated with a resolution of 128 points in the time series. This is the case for all the results of the frequency domain models shown in this chapter. Also the results of the time domain model are plotted in the figure. We see that within the evolution equation results, the wave shoals a little less, resulting in a later initiation of breaking.

![Graph showing envelope for the spilling breaker test of Ting and Kirby (1994). Frequency domain roller model with default parameters and time domain model.](image)

Figure 4.10: Envelope for the spilling breaker test of Ting and Kirby (1994). Frequency domain roller model with default parameters and time domain model.

4.5.3 Results of frequency domain roller model with adjusted parameters

To obtain a better match with the time domain results, the breaking parameters were adjusted to $\phi_B = 16^\circ$, $\phi_0 = 8^\circ$ and $c = 1.1\sqrt{gh}$.

**Wave envelope** The adjustment of the parameters improves considerably on the envelope, which for the new parameters is depicted in Figure 4.11. The slight lack of shoaling before breaking is, of course, not affected by the change of the breaking parameters, but the breaking point and the decay of wave height match the time domain results quite well. The mean water level and the trough level are both increasing inside the breaking zone, with a similar shape as the time domain results, although the growth is 1-2 meters ahead of the time domain model.
Figure 4.11: Envelope for the spilling breaker test of Ting and Kirby (1994). Frequency domain roller model with adjusted parameters and time domain model.

**Spatial wave profiles** The spatial wave profiles are compared to the profiles of the time domain in Figure 4.12. The profiles are aligned such that the wave crests coincide in $x = 17$ m. By comparing the spatial profiles for the five time instants in the plot, we see that the waves of the frequency domain model travel significantly faster than the waves computed by the time domain model. Since the two wave models have identical linear dispersion properties, the mismatch must be caused by a difference in the amplitude dispersion. A zoom in on the breaking wave profiles is shown in Figure 4.13. Until breaking, the profiles of the two models match quite well, but inside the breaking zone, the frequency domain results do not exhibit the same steep wave front as the time domain results. The breaking wave form of the frequency domain model is more symmetric. After $x = 22$ m, the wave seems to have been damped totally in the frequency domain.

**Skewness and asymmetry** The skewness and asymmetry of the two models are compared in Figure 4.14. The skewness of the frequency domain results matches the time domain model result well until the point of breaking. Within the breaking zone, the curves diverge. While the time domain model shows a decay in skewness, the frequency domain model predicts a growing skewness from $x = 19$ m.

In contrast, the asymmetry compares well to the time domain model, also after the initiation of breaking. The good match continues till $x = 21.3$ m, where the asymmetry suddenly increases. However, we note that the time domain results also shows a fast variation here. This variation — although different between the models — may not be physical. Unfortunately,
4.5 Results of the new roller breaking model

Figure 4.12: Profiles in space for frequency domain roller model with adjusted parameters. Comparison with time domain model.

Figure 4.13: Profiles in space for frequency domain roller model with adjusted parameters. Comparison with time domain model. Zoom of Figure 4.12.
Chapter 4. Inclusion of Roller Breaking

Figure 4.14: Skewness and asymmetry for frequency domain roller model with adjusted parameters. Comparison with time domain model.

the experimental results do not reach that far into the breaking zone, so it cannot be judged from the experimental data.

Time series  The skewness and asymmetry are useful measures for irregular waves, since they associate the nonlinearity of the waves with two single numbers. For regular waves, however, it may be easier to look at the time series of the waves directly. In Figure 4.15, time series of six stations in the wave flume are shown for the two models. To ease the comparison of the wave shape, the time series have been shifted along the time axis, such that the wave crest always appears at $t = 1$ s. The conclusion from watching the time series is rather similar to the conclusion obtained from the spatial profiles. Outside the breaking zone, the wave shapes match rather well, but inside the breaking zone, the waves of the frequency domain model tend not to lean as much forward as the waves within the time domain model. Also, we see that while the time domain model maintains a straight ‘back’ to the right of the crest, the frequency domain results have a more humped look. In the last two frames, the time series of the frequency domain results are almost symmetric.
4.5 Results of the new roller breaking model

Figure 4.15: Time series of frequency domain roller model with adjusted parameters and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
Chapter 4. Inclusion of Roller Breaking

4.6 Attempts to improve the model

The above results show that the breaking model in the frequency domain is capable of damping the waves and reproduce the envelope of the waves reasonably well. However, the amplitude dispersion appears to be too strong and also the wave profiles are not as steep and asymmetric as is the case for the time domain model.

A number of different steps have been taken to clarify the reasons for this and to improve on the results. It is now clear that the problem of excessive amplitude dispersion is a part of the evolution equations, independently of the breaking formulation. The investigations leading to this conclusion is the subject of Chapter 5. The lack of asymmetry, however, is not yet clarified.

In the following, examples of some of the attempts to improve the model are given.

4.6.1 Compensating for the second derivative of $a_p$

In the derivation of the evolution equations, the higher-order linear derivatives of $a_p$ where neglected, see (3.7) and (3.9). In the constant depth version of their model, however, Madsen and Sørensen (1993) took the effect of the second derivative into account, using an approximate integration technique of Bryant (1973). This technique was also used in Chapter 2 when deriving the one-equation model for Nwogu’s (1993) equations.

Consider the evolution equations (3.9) for free reference wave numbers and with the second derivative of $a_p$ retained:

$$-\beta_{2,p} a_{p,xx} + i\beta_{1,p} a_{p,x} = -i\beta_{1,p} \beta_{s,p} \frac{h_x}{h} a_p + g \sum_{s=p-N}^{N} \gamma_{s,p-s} a_s a_{p-s} e^{-i \int (k_s + k_{p-s} - k_p) dx}.$$  \hfill (4.37)

The left hand side can be written as $(-\beta_{2,p} \partial_x^2 + i \beta_{1,p}) a_{p,x}$. Assuming that the wave amplitudes are varying slowly in time, we can divide the whole equation by this operator. For each of the terms in the convolution sum, the operator has the scalar value $i\beta_{2,p} (k_s + k_{p-s} - k_p) + i\beta_{1,p}$. For the depth-slope term, there is no fast spatial variation (we assume slowly varying depth), and we may therefore neglect the $\partial_x$ contribution of the operator and simply divide this term by $i\beta_{1,p}$ only. The model can therefore be written

$$a_{p,x} = -\beta_{s,p} \frac{h_x}{h} a_p + g \sum_{s=p-N}^{N} \frac{\gamma_{s,p-s}}{i\beta_{2,p}(k_s + k_{p-s} - k_p) + i\beta_{1,p}} a_s a_{p-s} e^{-i \int (k_s + k_{p-s} - k_p) dx}.$$  \hfill (4.38)

For constant depth, this model reduces to the constant-depth model of Madsen and Sørensen (1993, equation (5.12)).
4.6 Attempts to improve the model

With the above modification, the model was run for the breaking test. The parameters \( \phi_B = 18^\circ, \phi_0 = 8^\circ, f_\delta = 1.5 \) and \( c = 1.2\sqrt{gh} \) were used. The wave envelope is shown in Figure 4.16. The inclusion of the second derivative results in a stronger shoaling of the wave. This is the reason for changing the breaking parameters from the values used in Section 4.5.3. The spatial wave profiles are compared to the time domain model results in Figure 4.16. We see that the inclusion of the second derivative does not improve on the exaggeration of the amplitude dispersion. Similarly, the shape of the waves is not improved by the correction for \( a_{p,xx} \). This can also be seen by inspection of the time series, which are shown in Figure 4.18. At the initiation of breaking, the profile is almost identical to the time domain profile. As the waves move through the breaking zone, however, the profile is seen to be even more symmetric than without the correction. The hump observed in Figure 4.15 is not present, thus resulting in a quite symmetric shape of the breaking wave time series.

4.6.2 Taking out sub-harmonic transfer

When the nonlinear interaction sums of the evolution equations are expressed exclusively in wave amplitudes for positive (and zero) frequencies, they appear as two sums, see e.g., (3.9). In this equation, the first sum represents super-harmonic interaction, where two waves force a wave mode at their sum frequency. The second sum represents sub-harmonic interaction, where two waves force a wave mode at their difference frequency. Generally, the super-harmonic interactions are stronger than sub-harmonic interactions. In Chapter 5, it is demonstrated that the amplitude dispersion in quadratic models is a combined effect of both
Chapter 4. Inclusion of Roller Breaking

Figure 4.17: Profiles in space for frequency domain roller model with correction for $a_{p,xx}$. Comparison with time domain model.

super-harmonic and sub-harmonic interactions. For a single free wave $\hat{\eta}_1$, a bound second harmonic $\hat{\eta}_2$, is generated through super-harmonic interaction. The sub-harmonic feedback of this wave onto $\hat{\eta}_1$ causes a phase modulation of $\hat{\eta}_1$, which is exactly the amplitude dispersion.

Hence, the amplitude dispersion may be eliminated by simply neglecting the sub-harmonic transfer. Since the super-harmonic interactions are the strongest, it may even be anticipated that the other properties of the wave field are not affected much by doing so.

In Figure 4.19, the spatial wave profiles obtained by this alteration are shown. The amplitude dispersion is really eliminated by this truncation of the interactions. The waves are now running slower than the waves of the time domain model — presumably at the linear phase speed. Inside the surf zone, for $x > 21$ m, the waves of the frequency domain model look strange. It seems like the waves are merging together here. This is caused by the truncation of the sub-harmonic interactions. As mentioned when discussing the frequency distribution of the energy dissipation related to wave breaking, the energy is dissipated relatively stronger at higher frequencies. The spectral shape of the waves is then responsible for redistributing the energy among the frequencies. Hence, a blocking of the sub-harmonic energy transfer violates this interplay between the breaking and the nonlinear interactions. Thus, although eliminating the amplitude dispersion efficiently, the blocking of the sub-harmonic transfer cannot be used generally.
4.6 Attempts to improve the model

Figure 4.18: Time series of frequency domain roller model with correction for $a_{p,xx}$. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
Figure 4.19: Profiles in space for frequency domain roller model without sub-harmonic transfer. Comparison with time domain model.

4.6.3 Increasing the number of frequencies

The effect of including more frequencies has also been investigated. A computation with $N = 16$ and 512 points in the time series for calculating the roller term was carried out. Graphically, the results do not differ from the results presented here.
4.7 Comparison with conventional breaking formulations

To compare the results of the new breaking model with conventional breaking formulations, a model using the dissipation formula of Battjes and Janssen (1978) was implemented.

4.7.1 Alterations of the model

The dissipation formula of Battjes and Janssen (1978) was derived for irregular waves. If it is used in its direct formulation for regular waves, it will dissipate energy no matter how small the waves are. Hence, the breaking formulation was first allowed to be active from $x = 16.7$ m for the runs here presented. When calculating the setup, the formula (4.29) was used with $\bar{R}_x = 0$. The reason for this is that the breaking formulation is not a part of the momentum equation, but is introduced in the equations at a later stage. The influence of the breaking term in the momentum equation can probably be traced back, but such a development was not pursued, since only a brief comparison is aimed at.

As we are dealing with regular waves, $H_{\text{rms}}$ is a poor measure of the wave height, since most of the wave energy is contained in the first harmonic. Hence, $H_{\text{rms}}$ was replaced by $H_{\text{regular}} = 4 \sum_{p=1, \text{odd}}^{N} |a_p|$, which is the wave height for a perfectly regular wave.

4.7.2 Results for uniform breaking

We now present results for uniform breaking, i.e., for $F = 1$ in (4.8), corresponding to the model of Eldeberky and Battjes (1996). The parameters $\alpha = 1.2$, $\gamma = 0.8$, see Section 4.1.2, were used for $N = 7$. The spatial profiles within the breaking zone are plotted in Figure 4.20. The wave height decays satisfactorily as the waves travel through the breaking zone. When comparing to Figure 4.13 with results of the frequency domain roller model, the present waves look rather symmetric.

The envelope of the waves is presented in Figure 4.21 along with the envelope obtained with the frequency domain roller model and the time domain model. For the chosen set of parameters, the decay rate of the conventional breaking model is slower than the rate of the roller model. The wave trough is lifted earlier than for the other models. One reason for this may be the omission of the $\bar{R}_x$-term in the calculation of the setup. The isolated effect of this term is a set-down at the initiation of breaking, followed by a smaller setup, see Madsen et al. (1997a). The lack of asymmetry of the wave profiles is clearly seen by comparing time series of the different breaking formulations. Time series of the models are plotted in Figure 4.22. For $x = 16.77$ m, the profiles are very similar, since breaking has just begun. In $x = (17.98, 19.21, 19.82)$ m, the profiles of the conventional breaking formulation are seen
Figure 4.20: Spatial profiles in the breaking zone for conventional breaking with uniformly weighted dissipation ($F = 1$). Comparison with time domain model.

Figure 4.21: Wave envelopes for conventional breaking with uniformly weighted dissipation ($F = 1$), roller breaking and time domain model.
Figure 4.22: Time series for conventional breaking with uniformly weighted dissipation ($F = 1$), roller breaking and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
to be slightly more symmetric than the profiles of the frequency domain roller model. In $x = 22$ m, the waves of both frequency domain models look rather symmetric.

In Figure 4.23, the skewness and asymmetry are plotted. The skewness of the conventional breaking model maintains its growth after the initiation of breaking. The decrease of skewness inside the breaking zone is not captured at all. The asymmetry also deviates somehow from the results of the frequency domain roller breaking. While the asymmetry decreases after breaking for both of the roller models, it increases for the conventional breaking model.

Figure 4.23: Skewness and asymmetry for conventional breaking with uniformly weighted dissipation ($F = 1$), roller breaking and time domain model.

### 4.7.3 Results for $f^2$-weighted breaking

It is well known in the literature that the skewness and asymmetry of breaking irregular waves is improved by weighting the breaking induced energy dissipation according to the squared frequency (Mase and Kirby, 1992; Kaihatu and Kirby, 1995; Chen et al., 1997). To test the effect on regular waves, we let $F = 0$ in (4.8). The other parameters were identical to the one used for uniform breaking. The spatial profiles of the waves are plotted in Figure 4.24. We see that the waves almost travel with the same speed as the waves of the time domain model. This is probably due to the smaller height of the waves, giving less amplitude dispersion. The envelope of the waves is plotted in Figure 4.25. The wave height decays rapidly in the beginning of the breaking zone. This results in a smaller wave height and therefore in a reduction of the amplitude dispersion.
4.7 Comparison with conventional breaking formulations

Figure 4.24: Spatial profiles in the breaking zone for conventional breaking with $f^2$-weighted dissipation ($F = 0$). Comparison with time domain model.

Figure 4.25: Wave envelopes for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking and time domain model.
The time series are plotted in Figure 4.26. For $x = 17.56$ m and $x = 18.42$ m, the smaller wave height is clearly seen. For these smaller waves, the back of the wave is more straight than for the roller model. For the three last frames $x = (19.28, 20.14, 21.0)$ m, the time series are very similar. This includes the hump at the back of the roller in $x = 19.28$ m and $x = 20.14$ m. The skewness and asymmetry are depicted in Figure 4.27. After breaking, the skewness decreases similarly to the time domain results and the results of the frequency domain roller model. However, in $x = 19$ m, the skewness starts to grow. The same behavior is observed for the frequency domain roller model, whereas the time domain skewness continues to decay almost all the way through the surf zone.

All in all, the conventional breaking with $f^2$-weighting gives results of similar quality as the frequency domain roller model. However, we note that this model was given the value of $x_{\text{break}}$ explicitly and that the calculation of $H_{\text{rms}}$ was modified.
4.8 Summary and conclusions

In this chapter, conventional breaking descriptions of evolution equations have been described. The disadvantages of these models is that they do not allow any phase changes induced by the breaking. Further, they work on the whole wave spectrum, thus globally on a time series, and the breaking location for regular waves needs to be given explicitly. To overcome these shortcomings, the roller breaking model known from time domain Boussinesq formulations has been adapted to the frequency domain. Also an equation for the mean water level variation has been derived.

The new breaking scheme is based on a local toggling from the complex amplitude spectrum to a local time series. This is carried out using FFT and allows for calculating the roller thickness and the roller term \( R \) as time series. With an inverse Fourier transform, the contribution to each frequency can be calculated.

The new model has been implemented in the evolution equations of Madsen and Sørensen (1993). Results of the new model for the test of Ting and Kirby (1994) have been presented along with results of the corresponding time domain model. Generally, the waves obtained with the new frequency domain model deviate from the time domain results in two ways: 1) the amplitude dispersion is over-predicted and 2) the waves do not lean as much forward, thus almost failing to give the characteristic saw-tooth shape of the breaking waves in the inner breaking zone.

The amplitude dispersion can be turned off by blocking the sub-harmonic energy transfer.

Figure 4.27: Skewness and asymmetry for conventional breaking with \( f^2 \)-weighted dissipation \((F = 0)\), roller breaking and time domain model.

4.8 Summary and conclusions
This suggests that the over prediction of the amplitude dispersion is a characteristic of the evolution equations themselves, independent of the wave breaking. This turns out to be true and is the subject of Chapter 5. So far, the explanation of the lack of asymmetry of the waves has not been found.

The new breaking formulation has also been compared to results for a conventional breaking formulation based on the dissipation model of Battjes and Janssen (1978). For $F = 1$, i.e., uniform breaking, the lack of asymmetry is more pronounced for this model. For $F = 0$, i.e., $f^2$-weighted breaking, the results match the new breaking model very well. For this test of regular waves, it seems like the $f^2$-distribution of the dissipation captures the characteristics of the breaking waves similarly well as the new breaking model.

However, some modifications of the conventional breaking model was necessary to give a satisfactory breaking description: the location of breaking needed to be pre-specified, and $H_{\text{rms}}$ was replaced by an estimate of $H$ for the regular wave. In contrast, the roller model does not need special adjustments for treating regular waves. Also, when modelling an irregular time series with discrete breaking events, the roller model is expected to improve on the breaking description, because it works locally around the breaking events, rather than globally damping the whole spectrum. A comparison of the roller breaking formulation to the bulk energy dissipation models should therefore be made for irregular waves.
Chapter 5

Amplitude Dispersion in Evolution Equations

The simulations of breaking waves of Chapter 4 shows that the amplitude dispersion of the evolution equations of Madsen and Sørensen (1993) is stronger than for the corresponding time domain formulation. As the deviations in phase speed occur already before the point of breaking, it is reasonable to expect that the deviations are due to properties of the wave models themselves, independent of the breaking formulation. Hence in this chapter, the wave models are analyzed by calculating their nonlinear regular wave solutions on constant depth. This approach reveals significant differences for the amplitude dispersion within the models.

Two methods of analysis are deployed, a weakly nonlinear approach and a fully nonlinear approach. The weakly nonlinear analysis is based on a Stokes type expansion and is shown to match the analysis of Madsen and Sørensen (1993) for their time domain Boussinesq formulation. The purpose of this analysis is two-fold. It establishes an easy way of investigating the third-order properties of a set of evolution equations, and gives insight into the mechanism of amplitude dispersion within evolution equations.

The fully nonlinear analysis is based on calculating finite amplitude solutions of regular waves for the wave models. This is similar to stream function theory for the fully nonlinear and fully dispersive water wave problem. Such an approach has been used by Otta and Schäffer (1999) for three sets of Boussinesq formulations and Kaihatu (2001) for fully dispersive evolution equations. The study of Kaihatu (2001) is described in detail in Chapter 6, Section 6.1.3.

The difference between the amplitude dispersion in evolution equations and the time domain formulation is due to the approximations applied when transforming from the time formulation to the frequency domain. Three approximations are involved, 1) truncation of the linear operator on the left hand side of the equation, 2) neglect of derivatives of the wave amplitudes in the nonlinear terms and 3) approximation of the amplitudes for the flux by linear theory in the convective terms. It is found that the main reason for the increase in
amplitude dispersion is the neglect of derivatives of the wave amplitudes within the nonlinear terms. This conclusion is made with both the weakly nonlinear and fully nonlinear analysis.

Before presenting the two methods of analysis and their results, a section is dedicated to the reference solutions used. For the weakly nonlinear analysis, third-order Stokes wave theory is used, while the results of the fully nonlinear analysis is compared to results of stream function theory. A byproduct of the analysis of amplitude dispersion is results for amplitudes of the second-order and third-order bound harmonics for a monochromatic wave. These results are presented and discussed in Appendix G.

5.1 Reference solutions

5.1.1 The Stokes wave theory of Fenton (1985)

Fenton (1985) presented a new derivation of fifth-order Stokes waves. The expansion parameter chosen is \( \varepsilon = kH/2 \), making the theory appropriate for engineering purposes, where the wave height, rather than the amplitude of the first harmonic, is usually specified. Further, the theory can be applied for an arbitrary value of \( c_E \), the Eulerian mean current velocity below wave trough level, or an arbitrary value of \( c_S \), the net volume flux velocity. Both of these features are new in relation to the classical fifth-order Stokes wave theory of Skjelbreia and Hendrickson (1960). In this theory, the expansion parameter is based on the amplitude of the first harmonic and \( c_E = 0 \) is assumed throughout. The last difference is important for our purpose, since we need a reference solution valid for a zero mass flux, corresponding to one-dimensional wave propagation towards a beach. For \( c_E = 0 \), the theories of Fenton and Skjelbreia and Hendrickson are consistent with each other to fourth order, while at fifth order a sign error in one of the coefficients in Skjelbreia and Hendrickson’s theory gives a deviation, see Fenton (1985) for further details.

In a frame of reference following the wave, Fenton (1985) gives the following expansion for the free surface elevation (in our notation)

\[
k \eta(x, t) = \varepsilon \cos kx + \varepsilon^2 B_{22} \cos 2kx + \varepsilon^3 B_{31} (\cos kx - \cos 3kx) + \cdots
\]  

(5.1)

where \( B_{ij} \) are dimensionless coefficients dependent on \( kh \). The coefficients referred to in this section are listed in the end of this section, page 135. In a stationary frame, the argument of the cos functions changes to \( kx - \omega t \). Further, we note that to third order, \( \varepsilon = kH/2 \) can be replaced by \( \hat{\varepsilon} = \varepsilon + \varepsilon^3 B_{31} \equiv kA_1 \), where \( A_1 \) is the amplitude of the first harmonic. We can therefore write the wave field (5.1) as

\[
\eta(x, t) = A_1 \cos \theta + \tilde{A}_2 \left( \frac{kA_1}{k} \right)^2 \cos 2\theta + \tilde{A}_3 \left( \frac{kA_1}{k} \right)^3 \cos 3\theta , \quad \theta = \omega t - kx
\]  

(5.2)
with
\[
\hat{A}_2 = B_{22} = \coth kh \frac{1 + 2S}{2(1 - S)} \quad \hat{A}_3 = -B_{31} = 3 \frac{1 + 3S + 3S^2 + 2S^3}{8(1 - S)^3}
\] (5.3)

where \( S = \sech^2 kh \).

Fenton gave two versions of the dispersion relation. One is formulated in \( c_E \) and the other in \( c_S \). To third order these read
\[
\sqrt{\frac{k}{g} c_E - \frac{\omega}{\sqrt{gk}}} + C_0 + \varepsilon^2 C_2 = O(\varepsilon^4) \quad (5.4)
\]
\[
\sqrt{\frac{k}{g} c_S - \frac{\omega}{\sqrt{gk}}} + C_0 + \varepsilon^2 \left( C_2 + \frac{D_2}{kh} \right) = O(\varepsilon^4). \quad (5.5)
\]

We consider the two special cases \( c_E = 0 \) and \( c_S = 0 \). Defining \( \omega = \omega_0(1 + \varepsilon^2 \omega_{13}) \) the two dispersion relations give
\[
\omega_{13,c_E} = \frac{C_2}{C_0} \quad \text{for } c_E = 0 \quad (5.6)
\]
\[
\omega_{13,c_S} = \frac{C_2 + D_2/kh}{C_0} \quad \text{for } c_S = 0 \quad (5.7)
\]

These expressions are functions of \( kh \) only. The results (5.3) and (5.6)–(5.7), establishes the reference solution needed for the weakly nonlinear analysis. The coefficients of \( kh \) are defined by
\[
B_{22} = \coth kh \frac{1 + 2S}{2(1 - S)}
\]
\[
B_{31} = -3 \frac{1 + 3S + 3S^2 + 2S^3}{8(1 - S)^3}
\]
\[
C_0 = \sqrt{\tanh kh}
\]
\[
C_2 = \sqrt{\tanh kh} \frac{2 + 7S^2}{4(1 - S)^2}
\]
\[
D_2 = -\frac{1}{2} \sqrt{\coth kh}
\]
with
\[
S = \sech 2kh.
\]

5.1.2 The stream function theory of Fenton (1988)

For the fully nonlinear analysis, the stream function theory of Fenton (1988) is used as reference solution. This formulation allows for the specification of either \( c_S \) or \( c_E \). All the
solutions presented here were calculated with $c_S = 0$, reflecting the physical condition for one-dimensional waves propagating towards a beach.

The stream function program used is that listed in Fenton (1988) and takes the dimensionless wave height and wave period as input parameters. For a given number of modes, the program solves the fully nonlinear water wave problem by discretizing it in a number of points. This results in a set of nonlinear equations, which is solved by the Newton-Raphson method. This requires knowledge of the Jacobian matrix, which is approximated by a finite difference approximation in each iteration. The output of the program includes the amplitudes of the stream function at the free surface, the Fourier coefficients of the free surface elevation and the wave number, all in dimensionless form. When the wave number is known, the phase speed is easily calculated as $c = \omega/k$.

For the calculations presented, 21 modes were used. Changing to 20 modes did not change the results. The stop criterion for the iterations was that the sum of absolute changes of the wave amplitudes should not exceed $10^{-12}$.

### 5.2 Weakly nonlinear analysis of amplitude dispersion

We consider a set of evolution equations for constant depth in the form

$$a_{p,x} = i \sum_{s=p-N}^{N} W_{s,p-s} a_{s} a_{p-s} e^{-i(k_s+k_p-k_s)x}$$  \hspace{1cm} (5.8)

For simplicity of the later analysis, we revert the model to the $\hat{\eta}_p$ amplitudes, see (1.2). This recasts the model to the simple form

$$\left( \frac{\partial}{\partial x} + ik_p \right) \hat{\eta}_p = i \sum_{s=p-N}^{N} W_{s,p-s} \hat{\eta}_s \hat{\eta}_{p-s}. \hspace{1cm} (5.9)$$

We now look for a solution of the form

$$\eta(x,t) = \frac{\varepsilon}{k_1} \cos \theta + \frac{\varepsilon^2}{k_1} \tilde{A}_2 \cos 2 \theta + \frac{\varepsilon^3}{k_1} \tilde{A}_3 \cos 3 \theta \hspace{1cm} (5.10)$$

with

$$\theta = \omega_1 t - \tilde{k}_1 x \hspace{1cm}, \hspace{1cm} \tilde{k}_1 = k_1(1 - \varepsilon^2 k_{13}). \hspace{1cm} (5.11)$$

This is a primary wave with bound harmonics at second and third order, with a modulated wave number. Here $\varepsilon = k_1 A_1$, such that (5.10) also can be written

$$\eta(x,t) = A_1 \cos \theta + \frac{\tilde{A}_2}{k_1} (k_1 A_1)^2 \cos 2 \theta + \frac{\tilde{A}_3}{k_1} (k_1 A_1)^3 \cos 3 \theta. \hspace{1cm} (5.12)$$
The coefficients \((\tilde{A}_1, \tilde{A}_2, k_{13})\) are dimensionless functions of \(kh\). The form of the expansion (5.10) is chosen since it can be used in the dimensional form of the evolution equations, without using artificial small parameters. Further, the results for \((\tilde{A}_1, \tilde{A}_2, k_{13})\) can be compared directly to the Stokes wave solution.

Matching (5.10) with the Fourier expansion (1.2) gives

\[
\begin{align*}
\hat{\eta}_1 &= \frac{\varepsilon}{2k_1} e^{-ik_1 x} \\
\hat{\eta}_2 &= \frac{\varepsilon^2}{2k_1} \tilde{A}_2 e^{-2ik_1 x} \\
\hat{\eta}_3 &= \frac{\varepsilon^3}{2k_1} \tilde{A}_3 e^{-3ik_1 x}.
\end{align*}
\]

We now insert this into (5.9), to find expressions for \((\tilde{A}_1, \tilde{A}_2, k_{13})\). The equations are

\[
\begin{align*}
(\frac{\partial}{\partial x} + ik_1)\hat{\eta}_1 &= 2iW_{2,-1}\hat{\eta}_2 \hat{\eta}_1^* \\
(\frac{\partial}{\partial x} + ik_2)\hat{\eta}_2 &= iW_{1,1}\hat{\eta}_1 \hat{\eta}_1 \\
(\frac{\partial}{\partial x} + ik_3)\hat{\eta}_3 &= 2iW_{2,1}\hat{\eta}_2 \hat{\eta}_1
\end{align*}
\]

which by insertion of (5.13) yield

\[
\begin{align*}
&i(-k_1 + \varepsilon^2 k_1 k_{13} + k_1)\frac{\varepsilon}{2k_1} e^{-ik_1 x} = 2iW_{2,-1} \frac{\varepsilon^2}{2k_1} \tilde{A}_2 \frac{\varepsilon}{2k_1} e^{-ik_1 x} \\
&i(-2k_1 + 2\varepsilon^2 k_1 k_{13} + k_2)\frac{\varepsilon^2}{2k_1} \tilde{A}_2 e^{-2ik_1 x} = iW_{1,1} \frac{\varepsilon^2}{4k_1^2} e^{-2ik_1 x} \\
&i(-3k_1 + 3\varepsilon^2 k_1 k_{13} + k_3)\frac{\varepsilon^3}{2k_1} \tilde{A}_3 e^{-3ik_1 x} = 2iW_{2,1} \frac{\varepsilon^2}{2k_1} \tilde{A}_2 \frac{\varepsilon}{2k_1} e^{-3ik_1 x}.
\end{align*}
\]

We now split this up in orders of \(\varepsilon\). The zeroth-order and first-order problems are identically satisfied. The second-order problem is given by (5.18), which is solved for \(\tilde{A}_2\):

\[
\tilde{A}_2 = \frac{W_{1,1}}{2k_1(k_2 - 2k_1)}. \tag{5.20}
\]

The third-order problem consists of (5.17) and (5.19) which constitute equations for \(k_{13}\) and \(\tilde{A}_3\), respectively. We solve these to obtain

\[
\begin{align*}
k_{13} &= \frac{W_{2,-1} \tilde{A}_2}{2k_1^2(k_2 - 2k_1)} = \frac{W_{2,-1} W_{1,1}}{2k_1^3(k_2 - 2k_1)} \\
\tilde{A}_3 &= \frac{W_{2,1}}{k_1(k_3 - 3k_1)} \tilde{A}_2 = \frac{W_{2,1} W_{1,1}}{2k_1^2(k_2 - 2k_1)(k_3 - 3k_1)}. \tag{5.22}
\end{align*}
\]

For a given set of evolution equations, the above expressions for \((\tilde{A}_2, \tilde{A}_3)\) can be compared directly to the results (5.3). We now derive an expression for \(\omega_{13}\) for evolution equations, based on \(k_{13}\).
5.2.1 The relation between $k_{13}$ and $\omega_{13}$

For Stokes waves, $\omega_{13}$ describes the modulation of the angular frequency of the waves caused by finite amplitude effects. The phase speed of the wave is

$$c_{\text{Stokes}} = \frac{\omega_{\text{tot}}}{k_0} = c_{\text{lin}}(1 + \varepsilon^2 \omega_{13}). \quad (5.23)$$

In evolution equations, $\omega$ is fixed for each frequency, and the amplitude dispersion therefore enters through a modulation of the wave number $k_1$. The phase speed is

$$c_{\text{evo}} = \frac{\omega}{k_1} = \frac{\omega}{k_1(1 - \varepsilon^2 k_{13})} = c_{\text{lin}}(1 + \varepsilon^2 k_{13}). \quad (5.24)$$

Now from these expressions it may be anticipated that $k_{13}$ and $\omega_{13}$ are equal to each other. This, however, is not the case. The reason is that $c_{\text{Stokes}}$ and $c_{\text{evo}}$ are the nonlinear phase speeds of two different waves. The Stokes wave has the linear wave length but a smaller period, while the evolution equation wave has the linear period but a larger length. This is illustrated in Figure 5.1, where the linear dispersion relation and the alterations by the two wave theories are shown.

Figure 5.1: Linear velocity ($c_0$), velocity of a nonlinear wave in evolution equations ($c_1$) and velocity of a third-order Stokes wave ($c_2$).

The point on the dispersion curve is the linear wave, which is the same for the two theories. We see that the correction of $k$ and $\omega$, respectively, produces two nonlinear waves which are not identical. The phase speed of each wave is given by the slope of the secant going through Origo and the point in the dispersion diagram representing the wave. For comparing the two results, for nonlinear waves, we must consider the change of $c_1$, if we changed the wave.
number with the amount $\varepsilon^2 k_0 k_{13}$. This corresponds to moving the point with velocity $c_1$ to the point with velocity $c_2$. We express this movement through a Taylor expansion:

$$c_2 = c_1 + \frac{dc}{dk} \varepsilon^2 k_0 k_{13} = c_1 + \left( \frac{dc}{dk} + \frac{dc}{d\omega} \frac{d\omega}{dk} \right) \varepsilon^2 k_0 k_{13}$$

$$= c_1 + \left( -\frac{\omega_0}{k_0^2} + \frac{1}{k_0} \frac{d\omega}{dk} \right) \varepsilon^2 k_0 k_{13} = c_1 + c_0 \left( \frac{c_0}{c_0} - 1 \right) \varepsilon^2 k_0 k_{13}$$

(5.25)

(5.26)

Insertion of (5.23) and (5.24) for $c_2$ and $c_1$ gives

$$c_0(1 + \varepsilon^2 \omega_{13}) = c_0(1 + \varepsilon^2 k_{13}) + c_0(c_g/c_0 - 1) \varepsilon^2 k_{13}$$

(5.27)

which is readily simplified to

$$\omega_{13} = \frac{c_g}{c_0} k_{13}.$$  

(5.28)

Hence, when the amplitude dispersion of evolution equations is investigated, the values of $k_{13}$ should be converted to $\omega_{13}$ using (5.28), if an analytical comparison with e.g., Stokes waves is made.

### 5.2.2 Results for the evolution equations of Madsen and Sørensen (1993).

With the above analysis, we can now investigate the amplitude dispersion of the evolution equations of Madsen and Sørensen (1993). As we have seen in Chapter 4, the amplitude dispersion of the evolution equations is larger than for the corresponding time domain model. Hence, it is interesting to point out, which approximations in the derivation cause the amplitude dispersion to increase.

For constant depth, the time domain formulation can be recast into the following form, see (3.7)

$$\left[ \beta_{4,p} \partial_x^3 - i \beta_{3,p} \partial_x^2 - \beta_{2,p} \partial_x + i \beta_{1,p} \right] \partial_x a_p$$

$$= -g \sum_{s=p-N}^{N} \left[ \left( \frac{1}{2} a_s a_{p-s} + b_s b_{p-s} \right) e^{-i(k_s + k_{p-s})x} \right] e^{i(k_s x)}$$

(5.29)

where one approximation has been applied: The term $P^2/(h + \eta)$ has been reduced to $P^2/h$, consistent with the quadratic accuracy of the time domain formulation. In the following, we will denote the model with the term $P^2/h$ the ‘simplified’ time domain model, while the model with the term $P^2/(h + \eta)$ is denoted the ‘full’ time domain model. By expanding the term $P^2/(h + \eta) - P^2/h$, the difference between the ‘full’ and ‘simplified’ model is seen to be of third order. The coefficients $b_p$ in (5.29) are the Fourier coefficients of the depth integrated flux $P$, see (3.3). Further, $\partial_x$ is a short hand notation of $\frac{\partial}{\partial x}$. 
We write this as
\[ \eta(x, t) = \sum_{p=-N}^{N} \tilde{a}_p e^{ip(\omega_1 t - \tilde{k}_1 x)} \]  
(5.30)
with \( \tilde{a}_p \in \mathbb{R} \) being constant. Matching this expansion with (3.3) gives
\[ a_p(x) = \tilde{a}_p e^{-i(p\tilde{k}_1 - k_p)x} \]  
(5.31)
and therefore \( \partial_x a_p = -i(p\tilde{k}_1 - k_p)a_p \), or simply \( \partial_x = -i(p\tilde{k}_1 - k_p) \). If (5.31) is inserted into the right hand side of (5.29), one obtains
\[ \left[ \beta_{4,p} \partial_x^3 - i\beta_{3,p} \partial_x^2 - \beta_{2,p} \partial_x + i\beta_{1,p} \right] \partial_x a_p \]
\[ = -g \sum_{s=p-N}^{p} \left[ \left( \frac{1}{2} \tilde{a}_s \tilde{a}_{p-s} + \tilde{b}_s \tilde{b}_{p-s} \right) e^{-ip\tilde{k}_1 x} \right] e^{ik_p x}. \]  
(5.32)
The total exponential variation on the right hand side is \( e^{-i(p\tilde{k}_1 - k_p)x} \), and thus for the right-hand side as a whole, we also have \( \partial_x = -i(p\tilde{k}_1 - k_p) \). Division of the above equation with the operator in the square brackets on the left hand side, gives with \( \partial_x = -i(p\tilde{k}_1 - k_p) \)
\[ a_{p,x} = ig \sum_{s=p-N}^{p} \frac{\left[ \left( \frac{1}{2} \tilde{a}_s \tilde{a}_{p-s} + \tilde{b}_s \tilde{b}_{p-s} \right) e^{-ip\tilde{k}_1 x} \right]_{xx}}{\beta_{4,p}(p\tilde{k}_1 - k_p)^3 + \beta_{3,p}(p\tilde{k}_1 - k_p)^2 + \beta_{2,p}(p\tilde{k}_1 - k_p) + \beta_{1,p}} e^{ik_p x}. \]  
(5.33)
We now carry out three manipulations of this expression: 1) we express the double \( x \)-differentiation on the right hand side by multiplying with \( -(p\tilde{k}_1)^2 \), 2) we express the flux amplitudes as \( \tilde{b}_j = \frac{\omega_j}{jk_1} \tilde{a}_j \) and 3) we back-substitute \( a_j \) for \( \tilde{a}_j \) on the right hand side using (5.31). For a bound wave field of form (5.30), no approximations are involved in these manipulations. The resulting equation is
\[ a_{p,x} = \]
\[ -ig \sum_{s=p-N}^{p} \frac{(p\tilde{k}_1)^2 \left( \frac{1}{2} + \frac{1}{gh} \left( \frac{\omega_1}{k_1} \right)^2 \right)}{\beta_{4,p}(p\tilde{k}_1 - k_p)^3 + \beta_{3,p}(p\tilde{k}_1 - k_p)^2 + \beta_{2,p}(p\tilde{k}_1 - k_p) + \beta_{1,p}} a_{s,p-s} e^{-i(k_s + k_{p-s} - k_p)x}. \]  
(5.34)
This is easily rewritten to the form (5.9), by substituting \( a_p(x) = \tilde{\eta}_p(x) e^{ik_p x} \). The kernel function \( W_{s,p-s} \) is then similar to the above, which we write as
\[ W_{s,p-s} = -g \frac{(k_{s}^{nl} + k_{p-s}^{nl})^2 \left( \frac{1}{2} + \frac{1}{gh} \frac{\omega_s \omega_{p-s}}{k_{s}^{nl} k_{p-s}^{nl}} \right)}{\beta_{4,p}(k_1^s - k_p)^3 + \beta_{3,p}(k_1^s - k_p)^2 + \beta_{2,p}(k_1^s - k_p) + \beta_{1,p}}. \]  
(5.35)
The introduction of the new wave numbers \( k_{s}^{nl}, k_{p-s}^{nl} \) and \( k_1^s \) allows for describing the effects of the approximations involved, when deriving the evolution equations. First, we note that setting \( (k_{s}^{nl}, k_{p-s}^{nl}, k_1^s) = jk_1 \) reproduces the kernel function of (5.34). By varying the choice of wave numbers, the following options can be covered:
5.2 Weakly nonlinear analysis of amplitude dispersion

- \( k_j^l = k_j \): the linear operator is truncated so that only the \( \beta_{1,p} \) term is left
- \( k_j^{nl} = k_j \): the double \( x \)-differentiation in the nonlinear term is approximated by multiplying with \(-(k_s + k_{p-s})^2\), thus neglecting the spatial variation of \( a_s \) and \( a_{p-s} \).
- \( k_j^l = k_j \): the flux amplitudes are calculated by the linear approximation \( b_j = (\omega_j/k_j)a_j \).

All three approximations are applied in the derivation of the evolution equations. When the wave field solved for is known (as here where we seek a bound wave solution), all the approximations can be avoided.

With the new kernel, (5.35), \( \omega_{13} \) can be calculated for different combinations of the approximations. We shall look at five combinations of the different wave numbers. These are listed in Table 5.1, along with their labels in the plots. When none of the three approximations are applied, the kernel (5.35) corresponds to the ‘simplified’ time domain formulation. Madsen and Sørensen (1993) made a third order perturbation analysis of this formulation and obtained the result

\[
\omega_{13,MS93} = \frac{9}{16} \frac{1}{\kappa^4} \frac{1 + 2(B + 1/9)\kappa^2 + (B + 1/9)^2\kappa^4}{1 + (2B + 1/3)\kappa^2 + B(B + 1/3)\kappa^4}
\]  

with \( \kappa = kh \). The coefficients of the \( \kappa^4 \) terms were not given in the original paper. \( B \) is the dispersion parameter of the model and should be set equal to \( 1/15 \) for obtaining Padé[2,2] dispersion characteristics, see Section 3.1.

Table 5.1: Different combinations of wave numbers, defining different sets of approximations. The labels cited are corresponding labels in the plots.

<table>
<thead>
<tr>
<th>( k_j^l )</th>
<th>( k_j^{nl} )</th>
<th>( k_j^l )</th>
</tr>
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<tbody>
<tr>
<td>( jk_1 )</td>
<td>( jk_1 )</td>
<td>( jk_1 )</td>
</tr>
<tr>
<td>( k_j )</td>
<td>( k_j )</td>
<td>( k_j )</td>
</tr>
<tr>
<td>( jk_1 )</td>
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<td>( jk_1 )</td>
<td>( jk_1 )</td>
<td>( k_j )</td>
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</tbody>
</table>

In Figure 5.2, the ratio of \( \omega_{13} \) and \( \omega_{13,cE} \) as given by (5.6) is plotted against \( \omega \sqrt{h/g} \) for the time domain formulation and for the evolution equations. The result of Madsen and Sørensen (1993), given by (5.36), is plotted as open circles. There is a perfect match between these results and the results of the present analysis. Further, we clearly see that the amplitude dispersion of the evolution equations is much stronger than for the time domain formulation.
Figure 5.2: $\omega_{13}$ of ‘simplified’ time domain formulation and evolution equations normalized by $\omega_{13}$ for Stokes waves with $c_E = 0$.

For the flat section of the Ting and Kirby (1994) test, we have $\omega \sqrt{h/g} = 0.6$, where the over-prediction is more than 50% relative to the time domain formulation. This explains the observation of Chapter 4, where the waves of the frequency domain model were travelling much faster than the waves calculated by the time domain formulation.

**Results for $c_S = 0$** When waves propagate towards a beach, $\bar{P} = 0$. As the above analysis does not include a zero-frequency component of $P$, the bound wave solution obtained also corresponds to $\bar{P} = 0$. Therefore, the reference solution of Stokes waves should rather be the one for $c_S = 0$ instead of the one for $c_E = 0$, as used above and by Madsen and Sørensen (1993). In Figure 5.3 the results of Figure 5.2 are therefore repeated, but normalized with $\omega_{13,c_S}$. The relative difference between the time domain formulation and the evolution equations is the same as in the other figure, but we now see that the time domain formulation over-predicts the amplitude dispersion in the region $\omega \sqrt{h/g} \leq 0.88$. The largest over-prediction is around 19% and occurs for $\omega \sqrt{h/g} = 0.69$. This is a new finding, since it until now has been believed that the amplitude dispersion was always underestimated for this time domain formulation. Still, however, the amplitude dispersion of the evolution equations is much stronger. Note also that we here analyze the ‘simplified’ model and not the ‘full’ model. The difference between these models is assessed in Section 5.3.

**Effects of the various approximations** For all the variations shown in the Table 5.1, the results for $\omega_{13}$ are shown in Figure 5.4, again normalized with $\omega_{13,c_S}$. We see that
Figure 5.3: $\omega_{13}$ of ‘simplified’ time domain formulation and evolution equations normalized by $\omega_{13}$ for Stokes waves with $c_S = 0$.

Figure 5.4: $\omega_{13}$ for the various combinations of approximations of Table 5.1. The results are normalized by $\omega_{13}$ for Stokes waves with $c_S = 0$. 
the truncation of the linear operator as well as the linear flux approximation decreases the amplitude dispersion when comparing to the ‘simplified’ time domain formulation, while the approximation of the differentiation in the nonlinear terms by multiplying with \(- (k_s + k_{p-s})^2\) increases the amplitude dispersion strongly. We also see that the approximations interact, since the amplitude dispersion of the evolution equations is not just obtained as the sum of the curves of the three approximations. The plot shows that the strong amplitude dispersion of the evolution equations is caused by the neglect of the derivatives of \(a_p(x)\) in the nonlinear terms.

5.3 Fully nonlinear analysis

The weakly nonlinear analysis is valid for small Ursell numbers. Obviously, when the nonlinearity gets too large, the assumption \(\varepsilon \ll 1\) is violated. Further, when the relative water depth decreases and near resonant conditions are approached, the second-order bound wave becomes of similar amplitude as the first harmonic, and the ordering in the \(\varepsilon\)-hierarchy breaks down.

For breaking waves, both of these problems are present. The waves are highly nonlinear and the water depth is small. Hence the above analysis breaks down, and for the same reasons, the Stokes solution is not a good reference solution. As an alternative to the above investigation, a fully nonlinear analysis can be made. The target should then be stream function theory rather than Stokes theory. Such an analysis is presented in the following.

We use the method proposed by Kaihatu (2001), who calculated fully nonlinear solutions to fully dispersive evolution equations expressed in the amplitudes of the still water velocity potential. The starting point is (5.29) in which we insert (5.31). Similarly to the derivation of (5.35), we obtain the equations

\[
\left( \Gamma \beta_{4,p}(p\tilde{k}_1 - k_p) + \Gamma \beta_{3,p}(p\tilde{k}_1 - k_p)^2 + \Gamma \beta_{2,p}(p\tilde{k}_1 - k_p)^3 + \beta_{1,p}(p\tilde{k}_1 - k_p) \right) \tilde{a}_p \\
= -g \sum_{s=p-N}^{p-N} \left( k_n^l + k_{p-s}^l \right)^2 \left( \frac{1}{2} + \frac{1}{gh} k_{p}^f k_{p-s}^f \right) \tilde{a}_s \tilde{a}_{p-s}.
\]

The exponential phase functions vanish identically when \(\tilde{a}\) is inserted. In the above equation, the wave numbers \(k_n^l\) and \(k_f^l\) have the same function as in (5.35). The wave number \(\tilde{k}_1\) is the fully nonlinear wave number of the first harmonic. The parameter \(\Gamma\) is simply a Boolean switch, where \(\Gamma = 1\) corresponds to using the full linear operator, while \(\Gamma = 0\) corresponds to using the truncated linear operator. For \(p = 1, \ldots, N\), the above equation establishes a nonlinear equation system for the unknowns \(\tilde{k}_1, \tilde{a}_1, \ldots, \tilde{a}_N\). An extra equation

\[
H = 4 \sum_{p=1}^{N} \tilde{a}_p
\]

(5.38)
which can be derived from (5.30), relates the wave amplitudes $\tilde{a}_1, \ldots, \tilde{a}_N$ to the wave height $H$. Thereby the equations (5.37) for $p = 1, \ldots, N$ and (5.38) establishes a closed system of nonlinear equations for the unknowns $\tilde{k}_1, \tilde{a}_1, \ldots, \tilde{a}_N$. Solving this system provides a fully nonlinear regular wave solution to the evolution equations for constant depth, with or without approximations. The value of $\tilde{k}_1$ defines the phase speed of the wave through $c_{\text{nonlin}} = \omega_1/\tilde{k}_1$, which differs from the linear phase speed $c_{\text{lin}} = \omega_1/k_1$. The above set of equations were solved using the Newton-Raphson method for nonlinear systems of equations. As convergence criterion it was used that the relative change of each solution variable should not exceed $1 \cdot 10^{-12}$ between two successive iterations.

5.3.1 Maximum wave height

When solving the fully nonlinear set of equations, a finite wave height is needed. To get a measure of the highest wave on a certain relative depth, we use the empirical measure of Battjes and Janssen (1978)

$$H_{\text{max}} = \frac{\gamma_1}{k_1} \tanh \left( \frac{\gamma_2}{\gamma_1} k_1 h \right)$$

(5.39)

with $\gamma_1 = 0.88$ and $\gamma_2 = 0.78$. The shallow water limit of this expression is $H_{\text{max}} = 0.78h$ and the deep water limit $H_{\text{max}} = 0.88/k$, which can be written as $H_{\text{max}}/L = 0.14$. These limits almost agree with the maximum solitary wave height of $0.833h$ and the maximum deep-water steepness of $H_{\text{max}}/L = 0.141$. The expression (5.39) can therefore be seen as a composite expression matching these two limits asymptotically.

A note on the validity and use of (5.39) As a side remark, the validity of the above expression is examined by comparing to the empirical expression of Williams (1981)

$$\frac{H_{\text{max}}}{h} = \frac{0.141063 \frac{L}{h} + 0.0095721 \left( \frac{L}{h} \right)^2 + 0.0077829 \left( \frac{L}{h} \right)^3}{1 + 0.0788340 \frac{L}{h} + 0.0317567 \left( \frac{L}{h} \right)^2 + 0.0093407 \left( \frac{L}{h} \right)^3}.$$  

(5.40)

The two expressions are compared in Figure 5.5 in the interval $kh = [0; 6]$. Apart from the shallow water region $kh < 0.4$, the curves match very well. The largest deviation occurs in the limit $kh \to 0$, where the approximate expression (5.39) under-predicts the maximum wave height by 6%. Changing $\gamma_2$ to 0.833 gives the right wave height in the shallow water limit, but leads to an over-prediction of the maximum wave height outside the shallow water limit. This modification of (5.39) is therefore not recommended.

If the dimensionless frequency $\omega \sqrt{h/g}$ is given instead of the dimensionless wave length $L/h$, the determination of $H_{\text{max}}/h$ becomes an iterative process. For a given guess of $H_{\text{max}}/h$, the fully nonlinear dimensionless wave length $L/h$ must be calculated by stream function theory, which is then inserted into (5.40) or (5.39) to give a new value of $H_{\text{max}}/h$. This process must then be repeated until convergence.
Chapter 5. Amplitude Dispersion in Evolution Equations

In the calculations of this section, a much simpler (but not exact) approach has been taken. For a given dimensionless frequency $\omega \sqrt{h/g}$, $L/h$ was determined using the dispersion relation of linear Stokes theory and $H_{\text{max}}/h$ was then determined by (5.39). The error in doing this is largest for the highest waves, where the linear dispersion relation is inaccurate. However, as the purpose of using (5.39) is just to get a measure of the largest wave at a given frequency, this simple approach is regarded as sufficiently accurate. A small error for the maximum wave height only affects the scaling $H/H_{\text{max}}$.

5.3.2 Results for a large wave

With the fully nonlinear solver, we are now able to calculate solutions to the equations for highly nonlinear waves. First, we consider the behavior of a wave of height $H = 0.85 H_{\text{max}}$, as function of the dimensionless frequency $\omega \sqrt{h/g}$. For each frequency, a fully nonlinear wave solution of the given height was calculated for the evolution equations (i.e., all approximations applied), the ‘simplified’ time domain model and the ‘full’ time domain model. The latter formulation cannot be written in the form (5.37), due to the non-quadratic structure of the nonlinear term. Instead, the nonlinear term was calculated via time series and transformed back to the frequency domain using FFT. The resulting set of equations was solved with the Matlab function \texttt{fsolve} for nonlinear equation systems, which does not require an explicit implementation of the Jacobian of the equation system.

For all models 21 wave modes were used. A calculation with 20 wave modes did not change the results significantly. The convergence criterion used was that the largest change of
the wave amplitudes should not exceed $1 \cdot 10^{-12}$ within an iteration, and further, that the residuals of the equations solved should not exceed $1 \cdot 10^{-12}$.

\[ \omega \sqrt{h/g} = 0.2, \quad \text{nonlinear contribution to the phase speed is 19\%.} \]

It is known that the highest solitary wave has a phase speed of $1.29\sqrt{gh}$, thus with a nonlinear contribution to the phase speed of 29\% of the linear phase speed. This gives an idea of the largest influence of amplitude dispersion. All the wave models studied have phase speeds exceeding the phase speed of linear Stokes waves. For the Boussinesq-based models, a minimum in phase speed is seen around $\omega \sqrt{h/g} = 1.3$. For smaller values of the dimensionless frequency, the larger velocities are due to amplitude dispersion. The phase speeds exceed that of stream function theory for shallow water. For larger values of the dimensionless frequency, the larger velocities are due to the over-prediction of the linear phase speed of the Padé[2,2] dispersion relation. For $\omega \sqrt{h/g} = 1.25$, the phase speed of the Padé[2,2] dispersion relation is 0.4\% larger than for Stokes waves, while at $\omega \sqrt{h/g} = 2.0$, the phase speed is 9\% larger than for Stokes waves. The latter result matches the over-prediction in Figure 5.6 well, thus indicating that the amplitude dispersion for the Boussinesq-based models vanishes for deep and intermediate water.

Figure 5.6: Nonlinear phase speed for waves of height $H = 0.85H_{\text{max}}$ normalized by phase speed of linear Stokes waves.
Chapter 5. Amplitude Dispersion in Evolution Equations

Figure 5.7: $\Delta_a$ defined by (5.41) for time domain Boussinesq formulation and evolution equations. The wave height is $H/H_{\text{max}} = 0.85$.

To examine the amplitude dispersion of the models a little closer, the measure

$$\Delta_a = \frac{(c/c_{\text{lin}})_{\text{model}} - 1}{(c/c_{\text{lin}})_{\text{stream func}} - 1}$$  \hspace{1cm} (5.41)$$

is defined. This compares the nonlinear contribution to the phase speed for a wave model, to the nonlinear contribution to the phase speed for stream function theory. For small wave heights, $\Delta_a$ converges towards the ratio of $\omega_{13,\text{model}}$ to $\omega_{13,\text{Stokes}}$. The formulation of $\Delta_a$ makes it possible to compare the influence of amplitude dispersion without influence of the differences in linear dispersion. Figure 5.7 shows a plot of $\Delta_a$ against dimensionless frequency for the ‘simplified’ and ‘full’ time domain Boussinesq formulations and the evolution equations. The curves confirm the conclusion from the previous figure, that the amplitude dispersion vanishes at deep water. For the ‘full’ time domain formulation, the amplitude dispersion is over-predicted by 14% in $\omega\sqrt{h/g} = 0.2$. For $\Delta_a = 1$, the amplitude dispersion is as strong as that of stream function theory. This value is reached at $\omega\sqrt{h/g} = 0.44$ for the ‘full’ time domain formulation. For $\omega\sqrt{h/g}$ close to 1.45, this model exhibits negative amplitude dispersion. Even though this is a striking behavior, we do not go further into this. The ‘simplified’ time domain formulation has stronger amplitude dispersion than the ‘full’ model. For the most shallow wave condition examined, $\omega\sqrt{h/g} = 0.2$, the amplitude dispersion is 76% stronger than for stream function theory. $\Delta_a = 1$ is reached at $\omega\sqrt{h/g} = 0.83$. The evolution equations show even stronger amplitude dispersion. In $\omega\sqrt{h/g} = 0.2$, the amplitude dispersion is more than three times as large as for stream function theory. $\Delta_a = 1$ is reached at $\omega\sqrt{h/g} = 0.99$. 
5.3 Fully nonlinear analysis

5.3.3 Matching with weakly nonlinear analysis

The fully nonlinear analysis is able to predict the effects of amplitude dispersion for all wave heights. It is therefore interesting to compare the results of this analysis to the results of the weakly nonlinear analysis. This has been done for the frequencies $\omega \sqrt{h/g} = (0.3, 1.0)$ and a perfect match in the limit of small wave height has been obtained. These investigations are reported in Appendix F.

5.3.4 Effects of various approximations

The above mentioned match between the weakly and fully nonlinear analysis constitutes the fully nonlinear analysis method as a reliable tool for analyzing the models. The results confirm that the evolution equations indeed has a stronger amplitude dispersion than the corresponding time domain formulation. We now study the cause of this by analyzing different combinations of the approximations involved in the derivation of the evolution equations. Six formulations are analyzed. These are described below.

The models and approximations analyzed With different choices of the wave numbers and $\Gamma$ in (5.37), different combinations of the approximations can be obtained. The combinations looked at and their labels in the plots are listed in Table 5.2. These models agree with the models analyzed using the weakly nonlinear method, see Figure 5.4, except for two deviations. First, the ‘full’ time domain model was not analyzed by the weakly nonlinear method. The second deviation concerns the ‘xx time’ model, which is used to study the influence of the neglect of derivatives of $a_p(x)$ in the nonlinear terms of the evolution equations. It would have been more natural to use a combination of wave numbers as in the

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$k_0^m$</th>
<th>$k_0^n$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$jk_1$</td>
<td>$jk_1$</td>
<td>‘Time $P^2/h$’. ‘Simplified’ time domain formulation</td>
</tr>
<tr>
<td>1</td>
<td>$\tilde{j}k_1$</td>
<td>$\tilde{j}k_1$</td>
<td>‘Time $P^2/d$. ‘Full’ time domain formulation</td>
</tr>
<tr>
<td>0</td>
<td>$k_j$</td>
<td>$k_j$</td>
<td>‘Evolution, all approx’. Evolution equations</td>
</tr>
<tr>
<td>0</td>
<td>$jk_1$</td>
<td>$jk_1$</td>
<td>‘Truncate linear op’. Effect of truncating linear operator</td>
</tr>
<tr>
<td>1</td>
<td>$j\tilde{k}_1$</td>
<td>$k_j$</td>
<td>‘Linear flux approx’. Effect of linear approximation for flux</td>
</tr>
<tr>
<td>0</td>
<td>$j\tilde{k}_1$</td>
<td>$k_j$</td>
<td>‘xx time’. Evolution equations with true differentiation of nonlinear terms.</td>
</tr>
</tbody>
</table>

Table 5.2: The different combinations of approximations considered for the fully nonlinear analysis.
second last line of Table 5.1 and \( \Gamma = 1 \), i.e., letting the only approximation be the neglect of the derivatives of \( a_p(x) \) in the nonlinear terms. In this way, the effect of all three approximations could be examined by comparing the resulting curves to the curve of the time domain model. This particular combination of approximations however, gave convergence problems for the Newton-Raphson method as well as \texttt{fsolve} of Matlab. Although other methods of solving nonlinear equation systems could have been deployed, the ‘xx time’ combination was used instead. This combination corresponds to the standard evolution equations, but with no neglect of derivatives of \( a_p(x) \) in the nonlinear terms. This model was easily solved by the Newton-Raphson method. The effect of neglecting the derivatives of \( a_p(x) \) in the nonlinear terms can thus be analyzed by comparing the results of the ‘xx-time’ model to the results of the evolution equations.

**Results** In Figure 5.8, \( \Delta_a \), as defined in (5.41), is plotted as function of dimensionless frequency for \( H = 0.45H_{\text{max}} \). Figure 5.9 shows similar curves for \( H/H_{\text{max}} = 0.85 \). The conclusions for the two choices of wave heights are very similar, and we therefore discuss the figures as a whole.

![Figure 5.8: \( \Delta_a \) for \( H/H_{\text{max}} = 0.45 \) for various models and approximations.](image_url)

The deviations between the ‘simplified’ and ‘full’ time domain models and evolution equations have been discussed for \( H/H_{\text{max}} = 0.85 \) in Section 5.3.2. The ‘full’ curves are therefore only included for reference. We focus on the effect of the approximations. We see that truncating the linear operator results in a decrease of the amplitude dispersion, when comparing to the ‘simplified’ time domain model. In comparison with stream function theory, the approximation still produces an over-prediction of the amplitude dispersion in shallow water, while at deeper water the amplitude dispersion decays towards zero.
Figure 5.9: $\Delta_a$ for $H/H_{\text{max}} = 0.85$ for various models and approximations.

Application of the linear approximation for the flux amplitudes results in a decrease of amplitude dispersion as well, when comparing to the ‘simplified’ time domain model. In shallow water the resulting model shows a little over-prediction of the amplitude dispersion. At deeper water an under-prediction of the amplitude dispersion occurs. This combination of approximations gives the best match with stream function theory of the models considered. The transition from over-prediction to under-prediction of the amplitude dispersion occurs at $\omega \sqrt{h/g} \approx 0.6$ for this model as well as for the model involving truncation of the linear operator.

The effect of retaining the derivatives in the nonlinear terms rather than neglecting them can be observed by comparing the curve for evolution equations to the curve ‘xx time’. A large reduction in the amplitude dispersion is obtained. The resulting model consequently under-predicts the amplitude dispersion — also in comparison with stream function theory. Further this curve provides the best match with the ‘full’ time domain model.

From these observations, we deduce that the truncation of the linear operator decreases the amplitude dispersion. The same is the case for the linear flux approximation, while the neglect of the derivatives of $a_p(x)$ in the nonlinear terms results in an over-prediction of the amplitude dispersion. This agrees with the results of the weakly nonlinear analysis, and it is thereby clear, that the large over-prediction of the amplitude dispersion in evolution equations is due to the neglect of derivatives of $a_p(x)$ in the nonlinear terms.
Chapter 5. Amplitude Dispersion in Evolution Equations

5.4 Summary and conclusions

In this chapter the amplitude dispersion within the time domain formulation and evolution equations of Madsen and Sørensen (1993) has been analyzed. Two methods have been undertaken, a weakly nonlinear method based on a Stokes expansion and a fully nonlinear method based on fully nonlinear solutions to the wave equations.

The weakly nonlinear analysis has been shown to match the results of Madsen and Sørensen (1993) for the time domain formulation. However, as this Boussinesq formulation is formulated with the depth integrated volume flux \( P \) as dependent variable, the reference solution for comparison should be third-order Stokes theory with a zero net mass flux and not with a zero Eulerian current velocity below wave trough level as used by Madsen and Sørensen (1993). The comparison to the reference solution of zero net mass flux shows that the ‘simplified’ time domain formulation over-predicts the amplitude dispersion for \( \omega \sqrt{h/g} < 0.88 \), and under-predicts the amplitude dispersion for larger dimensionless frequencies. The main result of the analysis, however, is that amplitude dispersion of evolution equations can be up to twice as large as for Stokes waves, and is thus much stronger than for the time domain formulation. The main cause of the strong amplitude dispersion is the neglect of derivatives of \( a_p(x) \) in the nonlinear terms.

The results of the fully nonlinear analysis confirm the results of the weakly nonlinear analysis. Waves of wave heights up to 0.85\( H_{\text{max}} \) have been analyzed, and the results have been shown to agree with the weakly nonlinear analysis in the limit of small wave height. Within this analysis, the influence of the reduction of the ‘full’ time domain formulation to the ‘simplified’ time domain formulation has been assessed. The influence of this approximation is to increase the amplitude dispersion. Hence, the weakly nonlinear analysis presented shows a larger amplitude dispersion than for the model with \( P^2/(h + \eta) \). Still, however, there is a region in shallow water were the amplitude dispersion is over-predicted.

With respect to evolution equations, the fully nonlinear analysis confirms the results of the weakly nonlinear analysis, of too strong amplitude dispersion. In contrast to the weakly nonlinear analysis, however, the amplitude dispersion is also exaggerated in the shallow water limit — most severely in fact — see Figure 5.6. The reason for this shortcoming of the weakly nonlinear analysis is the invalidity of the Stokes expansion for shallow water. The results are only valid for small wave heights. As soon as the waves obtain a finite amplitude, the second harmonic of the Stokes wave becomes of similar magnitude as the first harmonic, and the analysis breaks down.

With the fully nonlinear analysis it has been found, that the main reason for the general over-prediction of the amplitude dispersion in the evolution equations is the neglect of derivatives of \( a_p(x) \) in the nonlinear terms. This agrees with the findings of the weakly nonlinear analysis.

With basis in this conclusion it is relevant to ask what can be done to improve on the amplitude dispersion problems. The analysis has shown that a set of evolution equations
with the derivatives $a_{p,x}$ and $a_{p,xx}$ retained in the nonlinear terms performs similarly to the ‘full’ time domain formulation. Hence, it is natural to make a model which includes, as a first step, $a_{p,x}$ in the nonlinear terms. This could be evaluated by a backward finite difference approximation, using the values in the previous spatial points. This is the step recommended for improving on the amplitude dispersion within the evolution equations.

However, another track was followed after this conclusion. Instead of trying to improve on the evolution equations of Madsen and Sørensen (1993), fully dispersive evolution equations (Agonon et al., 1993; Kaihatu and Kirby, 1995; Eldeberky and Madsen, 1999) were tried instead. The motivation for this change of wave model was the exact linear dispersion and shoaling characteristics for all depths of these models. Hence, better performance regarding amplitude dispersion and the shape of breaking waves was expectable for these models.

Chapter 6 describes the derivation of fully dispersive evolution equations, the implementation of the speed-up technique of FFT, their second-order properties and third-order properties and the inclusion of roller breaking.
Chapter 6

Fully Dispersive Models, 2nd Order

6.1 Introduction

Evolution equations based on Boussinesq formulations are relatively easy to derive, since the vertical dimension is already eliminated in the equations. Further, as a number of increasingly accurate Boussinesq formulations have been developed during the last 10–15 years, there is a hope of similar accuracy of evolution equations. However, as discussed in Chapter 2, evolution equations are restricted to weak, quadratic nonlinearity. Hence, the full nonlinearity of the new Boussinesq formulations cannot be captured by evolution equations. It is therefore relevant to pursue a formulation of evolution equations being weakly nonlinear but retaining exact linear dispersion and shoaling characteristics.

These features characterize fully dispersive evolution equations. Two derivations are given in the literature, which are briefly reviewed below.

6.1.1 The evolution equations of Agnon et al. (1993)

Agnon et al. (1993) derived a set of weakly nonlinear evolution equations in the complex amplitudes of the still water potential. The equations were valid for one horizontal dimension and initially included a slow spatial as well as temporal evolution of the potential amplitudes. The final, discretized model, however, only retained a spatial evolution, thus resembling the form of evolution equations studied here.

The main steps in the derivation was the following: The free surface boundary conditions were expanded around the still water level and combined into a single equation in $\phi$. A multiple scales expansion of $(x, t)$ was introduced, to separate the fast and slow variation of the wave field. The governing equations were next transformed to Fourier space (with
respect to time) and the Laplace equation depth-integrated. Initially, the vertical structure of bound waves as well as free waves was considered, that is

\[ \phi(x, z, t) = \frac{\cosh k(z + h)}{\cosh kh} \phi(x, z = 0, t) \] (6.1)

where \( k \) can be a free wave number or a bound wave number. The model derived, however, was based on the vertical structure of a free wave. Agnon et al. (1993) defined a detuning parameter

\[ \mu = \frac{(k_s + k_{p-s} - k_p)}{k_p} \] (6.2)

giving a measure of the deviation between bound and free wave numbers. The bound waves within the model are thus described with an error of \( O(\mu) \).

The model was extended to two dimensions in Agnon and Sheremet (1997), following the wide-angle approach as described in Section 1.6.3. They also developed stochastic evolution equations based on the deterministic model.

6.1.2 The evolution equations of Kaihatu and Kirby (1995)

Kaihatu and Kirby (1995) derived a set of evolution equations, essentially being an extension of the model of Agnon et al. (1993) to weakly two-dimensional wave propagation. Their starting point was the Laplace equation, which was depth-integrated assuming a vertical structure of the velocity field corresponding to linear waves. The resulting equation was combined with the free surface conditions, giving a nonlinear mild-slope equation. Utilizing the expansion

\[ \phi(x, y, z = 0, t) = \sum_{p=-N}^{N} -\frac{iq}{\omega_p} a_p(x, y) e^{i(\bar{k}_p dx - \omega_p t)} \] (6.3)

where \( \bar{k}_p \) is a \( y \)-averaged free wave number, the parabolic approximation \( a_p = a_p(\varepsilon^2 x, \varepsilon y) \) was used to eliminate higher-order derivatives of \( a_p \). The resulting model thus only retained the derivatives \( a_{p,x}, a_{p,y} \) and \( a_{p,yy} \), thereby forming a set of evolution equations for \( a_p \). In an appendix, the model was extended to allow for a spatially varying current. For one horizontal dimension and no current, the model is identical to the model of Agnon et al. (1993).

6.1.3 The second-order transformation from \( \phi \) to \( \eta \)

Both of the two above models were compared to experimental results. Agnon et al. (1993) simulated a laboratory test of almost uni-directional irregular waves propagating into a harbor, and also field measurements of shoaling waves in Walker Bay, South Africa. Kaihatu
and Kirby (1995) simulated the test of Whalin (1971) for regular wave propagation over a semicircular shoal and the test of breaking irregular waves on a plane sloping beach of Mase and Kirby (1992). For the latter purpose, the $f^2$-weighted breaking model of Mase and Kirby (1992) was incorporated.

None of the two papers clearly mention how the initial amplitudes of $\phi$ were obtained from the data, or how the computational results in $\phi$ were transformed back to amplitudes of the free surface elevation. Hence, one may assume that a linear transformation was applied, as is also confirmed by Agnon (private communication) and Kaihatu (2001). This inconsistency was pointed out by Eldeberky and Madsen (1999), who derived a second-order transformation between $\eta$ and $\phi$, and transformed the evolution equations of Agnon et al. (1993), as well as Agnon and Sheremet (1997), to a set of evolution equations formulated directly in the complex amplitudes of the free surface elevation $\eta$. The correction of the transformation improves the accuracy of super-harmonic energy transfer significantly. Also stochastic evolution equations in the free surface elevation were derived, correcting the equations of Agnon and Sheremet (1997).

Kaihatu (2001) discussed this correction of the deterministic model as well. The influence of the new second-order terms was examined by deriving fully nonlinear solutions to the equations in $\phi$. The solutions were then transformed to $\eta$, using either the linear transformation or the correct second-order transformation. In shallow water, essentially no difference was found, while at deep water, the effect was more pronounced. Further, from this analysis it was found that the nonlinear phase speed exceeds the phase speed of Stokes third-order waves for $kh < 1.7$, where $k$ is the linear wave number. For larger values of $kh$, the nonlinear phase speed was smaller than for Stokes waves. Comparison with stream function theory confirmed the over-prediction of the phase speed in shallow water, even though the transition between over-prediction of the phase speed (small $kh$) and under-prediction (larger $kh$) occurred at a much smaller wave number, $kh \approx 0.3$, than for the Stokes wave analysis.

### 6.1.4 Objectives of this chapter

In this chapter various aspects of fully dispersive evolution equations are treated. First, in Section 6.2, a new derivation is presented. This derivation does not make use of any assumption for the vertical structure of the velocity field, and therefore produces a model which describes free waves as well as their second-order bound wave components exactly. The second-order bichromatic transfer functions of the new model therefore match Stokes second-order theory exactly. In Section 6.3, the bichromatic transfer functions of the models of Eldeberky and Madsen (1999) as well as Agnon et al. (1993) are investigated. The possibility of speeding up the calculation of the nonlinear terms using FFT is discussed in Section 6.4. While the models of Agnon et al. (1993) and Eldeberky and Madsen (1999) can be speeded up using this technique, it turns out that the new exact model can not. However, it is possible to approximate the nonlinear interaction coefficient of the model to allow for the speed up. Following the approach of Section 5.2, a weakly nonlinear analysis
of the amplitude dispersion as well as the third-order transfer is given in section 6.5. Finally in Section 6.6, incorporation of the roller breaking scheme is presented. Results of this are given in Section 6.7. A summary and conclusions for the chapter are given in Section 6.8.

6.2 Derivation

In the following, a new derivation of fully dispersive evolution equations is presented. The derivation follows the same lines as the derivation of the ‘augmented’ mild-slope equation of Agnon (1999). First, we derive evolution equations for the complex amplitudes of the still water potential $\Phi$. In a second step, we transform the equations to equations in the complex amplitudes of $\eta$. As a starting point, we formulate the governing equations for two horizontal dimensions, while from Section 6.2.4, the derivation is restricted to only one horizontal dimension.

6.2.1 The governing equations

We consider the motion of an inviscid irrotational fluid, as depicted in Figure 6.1. A Cartesian coordinate system $(x, y, z) = (x, z)$ with the z-axis pointing upwards from the still water level is adopted. Here $x = (x, y)$. The surface elevation is denoted by $\eta(x, t)$ and the velocity potential by $\phi(x, z, t)$. The velocity field within the fluid is $(u, v, w)$ and $g$ is the acceleration of gravity. The depth is described by $h(x)$, measuring the distance from the bottom to the still water level.

![Figure 6.1: Definition sketch for derivation of fully dispersive evolution equations.](image-url)
The governing equations can be written

\[
\nabla^2 \phi + \phi_{zz} = 0 \quad -h < z < \eta \\
\phi_z - \nabla \phi \cdot \nabla \eta - \eta_t = 0 \quad z = \eta \\
\phi_t + g \eta + (\nabla \phi)^2 + \phi_z^2)/2 = 0 \quad z = \eta \\
\phi_z + \nabla h \cdot \nabla \phi = 0 \quad z = -h,
\]

stating local continuity (6.4), the kinematic and dynamic free surface conditions (6.5), (6.6), and impermeability of the bottom (6.7). \(\nabla\) is the horizontal gradient operator, i.e., \(\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\). In the remaining part of this section, we derive a single approximate equation in \(\phi(x,0,t)\) from these four equations. This equation provides the basis for the fully dispersive evolution equations.

### 6.2.2 Power series solution to the Laplace equation

The local continuity equation (6.4) is linear, and it is therefore natural to expand the velocity potential as a power series in \(z\):

\[
\phi(x, z, t) = \sum_{n=0}^{\infty} z^n \phi_n(x, t).
\]  

(6.8)

It is easily seen that \(\phi_0 = \phi(x,0,t) \equiv \Phi\) and \(\phi_1 = w(x,0,t) \equiv W\). Further, insertion of (6.8) in the Laplace equation yields the recursion relation \(\phi_{n+2} = -\nabla^2 \phi_n/((n+1)(n+2))\) and thereby the solution

\[
\phi(x, z, t) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}\nabla^{2n}}{(2n)!} \Phi + (-1)^n \frac{z^{2n+1}\nabla^{2n+1}}{(2n+1)!} W.
\]  

(6.9)

We identify the above series in \((z\nabla)\) as the Taylor series of the functions \(\cos(z\nabla)\) and \(\sin(z\nabla)/\nabla\), thus allowing us to write the series as

\[
\phi(x, z, t) = \cos(z\nabla)\Phi + \frac{1}{\nabla}\sin(z\nabla)W;
\]  

(6.10)

where the capitalized form of the trigonometric functions has been used to indicate that they denote operators. The above result is not new. It can be found in Madsen and Schäffer (1998). The compact operator notation is not new either, see e.g. Rayleigh (1876).

### 6.2.3 Expanding the bottom boundary condition around a constant reference depth

We choose to include the effect of a mildly sloping bottom in the model. Consequently, we retain only first-order derivatives of the depth variation. A convenient way of dealing with
Chapter 6. Fully Dispersive Models, 2nd Order

this is to introduce a constant reference depth \( h(x) = h_0 + \delta(x) \) and expand the bottom boundary condition around \( h_0 \). This proves to be advantageous, when combining the sea bed condition with the series solution (6.10) to the Laplace equation.

We Taylor expand the sea bed condition around the constant reference level to obtain

\[
\phi_z - \delta \phi_{zz} + \frac{1}{2} \delta^2 \phi_{zzz} + O(\delta^3) + \nabla \delta \cdot (\nabla \phi - \delta \nabla \phi_z) + O(\delta^2 \nabla \delta) = 0, \quad z = -h_0. 
\]  

(6.11)

Using the Laplace equation, the double \( z \)-differentiations are rewritten to \( \nabla^2 \) differentiations. This gives

\[
\phi_z + \delta \nabla^2 \phi + \frac{1}{2} \delta^2 \nabla^2 \phi_z + \nabla \delta \cdot (\nabla \phi - \delta \nabla \phi_z) = O(\delta^3, \delta^2 \nabla \delta) , \quad z = -h_0. 
\]  

(6.12)

To lowest order, this equation states that \( \phi_z = O(\delta, \nabla \delta) \). Hence the higher-order terms involving \( \phi_z \) can be let out with an error of \( O(\delta^3, \delta^2 \nabla \delta, \delta(\nabla \delta)^2) \). This allows for writing the sea bed condition in the compact form

\[
\phi_z = -\nabla \cdot (\delta \nabla \phi) + O(\delta^3, \delta^2 \nabla \delta, \delta(\nabla \delta)^2) \quad z = -h_0. 
\]  

(6.13)

The above result can also be found in Mei (1985), without the error term. The advantage of this formulation is that it is defined on a constant level \( z = -h_0 \). When the series solution (6.10) is inserted for \( \phi \), the operators will have the argument \( h_0 \nabla \), where \( \nabla \) and \( h_0 \) can be interchanged. This simplifies the derivation considerably. All effects of varying depth are now represented by \( \delta \). The assumption of mildly sloping bottom allows us to neglect all but first-order derivatives of \( \delta \), while as the last step in the derivation, we can replace the reference depth \( h_0 \) with the local depth \( h \), implying \( \delta = 0 \) and \( \nabla \delta = \nabla h \).

Insertion of (6.10) into (6.13) yields

\[
\sin(h_0 \nabla) \cdot \nabla \Phi + \cos(h_0 \nabla)W = -\nabla \cdot \{ \delta (\cos(h_0 \nabla) \nabla \Phi - \sin(h_0 \nabla)W) \}. 
\]  

(6.14)

This equation combines the Laplace equation and the kinematic sea bed condition.

Until now, the derivation has been valid for two horizontal dimensions. In the following, we restrict the derivation to just one horizontal dimension. Hence, in the following \( \nabla \equiv \frac{\partial}{\partial x} \).

6.2.4 The free surface boundary conditions

So far, we have not made use of the free surface boundary conditions (6.5) and (6.6). We do this now, to express \( W = \Phi_z \) by \( \Phi \) and its spatial derivatives. Combined with (6.14) this result can produce a single equation in \( \Phi \) and its \( x \)-derivatives for the wave motion, thus eliminating the vertical coordinate. The steps in the treatment of the surface boundary conditions are the following. 1) Expand the boundary conditions around \( z = 0 \), 2) eliminate \( \phi_z \) and 3) eliminate \( \eta \) and combine into one equation in \( \phi \). We carry out these steps in the following.
Expanding the surface conditions around $z = 0$ We assume that the dominant spatial length scale is given by a reference wave number $k_0$ and the dominant time scale is given by the reference angular frequency $\omega_0^2 = g k_0$. Further, we assume $a_0$ to be a representative wave amplitude. With these assumptions, the following dimensionless variables (denoted by a prime) can be defined

$$x = x'/k_0, \quad z = z'h_0, \quad h = h'h_0 \quad (6.15)$$

$$t = t'/\omega_0, \quad \eta = a_0 \eta', \quad \phi = \frac{g a_0}{\omega_0} \phi'. \quad (6.16)$$

This scaling is the classical scaling of Stokes waves and applies for deep and intermediate water. Insertion into the free surface boundary conditions (6.5) and (6.6) yields

$$\phi'z' - \varepsilon \nabla \phi' \nabla \eta' - \eta' = 0 \quad (6.17)$$

$$\phi' + \eta' + \varepsilon \left( (\nabla' \phi')^2 + \phi_z'^2 \right)/2 = 0, \quad (6.18)$$

where $\varepsilon = a_0 k_0$ is the wave steepness. We will assume $\varepsilon \ll 1$ in the following, allowing a perturbation approach to the equations. However, instead of working with the dimensionless variables, we choose to work with the physical variables directly, retaining the $\varepsilon$-factors on the nonlinear terms. Doing so, $\varepsilon$ is still to be regarded as a small number related to the wave steepness, even though it formally has the value 1.

Expanding (6.5) and (6.6) around $z = 0$ yields

$$\eta_t - \Phi_z + \varepsilon \left( \eta \Phi_{xx} + \eta_x \Phi_z \right) = O(\varepsilon^2) \quad (6.19)$$

$$g \eta + \Phi_t + \varepsilon \left( \Phi_t^2/2 + \Phi_z^2/2 + \eta \Phi_{zt} \right) = O(\varepsilon^2), \quad (6.20)$$

where we have used the Laplace equation to eliminate higher-order derivatives of $\Phi$ with respect to $z$.

Eliminating $\Phi_z$ and $\eta$ The kinematic boundary condition (6.19) yields to lowest order $\Phi_z = \eta_t + O(\varepsilon)$. Substitution of this into the quadratic terms of (6.20) gives

$$g \eta + \Phi_t + \varepsilon \left( \Phi_t^2/2 + \eta_t^2/2 + \eta_{tt} \right) = O(\varepsilon^2) \quad (6.21)$$

To lowest order, this equation states $\eta = -\Phi_t/g + O(\varepsilon)$. We use this to rewrite the quadratic terms. Further, we manipulate the quadratic terms of (6.19) similarly. We hereby obtain

$$- \Phi_z - \Phi_{tt}/g - \varepsilon \left[ \frac{2 \Phi_t \Phi_{tt}}{g^3} + \frac{\Phi_t \Phi_{ttt}}{g^3} + 2 \frac{\Phi_z \Phi_{zt}}{g} + \frac{\Phi_t \Phi_{xx}}{g} \right] = O(\varepsilon^2) \quad (6.22)$$

$$g \eta + \Phi_t + \varepsilon \left[ \frac{\Phi_t^2}{2g^2} + \frac{\Phi_t \Phi_{tt}}{g^2} + \frac{\Phi_z^2}{2} \right] = O(\varepsilon^2). \quad (6.23)$$

These equations express $W = \Phi_z$ and $\eta$ in terms of $\Phi$ and its spatial and temporal derivatives.
Rewriting to symmetric expressions The above results for \( \Phi_z \) and \( \eta \) leaves some freedom in writing the nonlinear terms. In the frequency domain, the quadratic terms take the form of convolution sums. Here, symmetry among the two factors in each nonlinear term is a computational advantage, since the range of summation can then be halved. Further, if some of the terms can be written as a time derivative, the averaged equations take a particularly simple form. This is an advantage when deriving equations for the mean flow. Thus in (6.22) we use the identity \( \frac{1}{2} \Phi^2_x + \Phi_x \Phi_{txt} = \frac{1}{2} (\Phi_t^2)_{tt} - \frac{1}{2} \Phi^2_t \) and in (6.23) the identity \( 2 \Phi_t \Phi_{txt} + \Phi_x \Phi_{ttt} = \frac{1}{2} (\Phi_t^2)_{tt} - \frac{1}{2} (\Phi^2_t)_{tt} \). This yields the expressions

\[
\begin{align*}
g \eta + \Phi_t &+ \varepsilon \left[ \frac{1}{2} \Phi^2_x + \frac{1}{2g^2} (\Phi^2_t)_{tt} - \frac{1}{2g^2} \Phi^2_t \right] = O(\varepsilon^2) \quad \text{(6.24)} \\
- \Phi_z - \frac{1}{g} \Phi_t &+ \varepsilon \left[ -\frac{1}{2g^3} (\Phi^2_t)_{tt} + \frac{1}{2g^3} (\Phi^2_t)_{tt} - \frac{1}{g} (\Phi^2_x)_{t} - \frac{1}{g} \Phi_{xx} \Phi_t \right] = O(\varepsilon^2). \quad \text{(6.25)}
\end{align*}
\]

6.2.5 Transforming to the frequency domain

Until now, the equations have been formulated in the time domain. We now transform them to the frequency domain, utilizing the expansions

\[
\begin{align*}
\eta(x,t) &= \sum_{p=-N}^{N} \tilde{\eta}_p(x) e^{i \omega_p t} = \sum_{p=-N}^{N} a_p(x) e^{i (\omega_p t - \int k_p dx)} \quad \text{(6.26)} \\
\Phi(x,t) &= \sum_{p=-N}^{N} \tilde{\phi}_p(x) e^{i \omega_p t} = \sum_{p=-N}^{N} b_p(x) e^{i (\omega_p t - \int k_p dx)}. \quad \text{(6.27)}
\end{align*}
\]

The expansion for \( \eta \) is identical to the expansion (3.28), and the expansion for \( \Phi \) follows the same idea. With these expansions, the expressions (6.25), (6.24) for \( \Phi_z \) and \( \eta \) can be written

\[
\begin{align*}
\tilde{\eta}_p &= -\frac{i \omega_p}{g} \phi_p + \varepsilon \sum_{s=p-N}^{N} \left( \frac{1}{2g^3} k_s k_{p-s} - \frac{1}{2g^3} \omega_s \omega_{p-s} \omega_p^2 + \frac{1}{2g^3} \omega_s^2 \omega_{p-s} \right) \tilde{\phi}_s \tilde{\phi}_{p-s} \quad \text{(6.28)} \\
\tilde{\phi}_{p,z} &= \frac{\omega_p^2}{g} \phi_p + \varepsilon \sum_{s=p-N}^{N} f^{(2)}_{s, p-s} \tilde{\phi}_s \tilde{\phi}_{p-s} \quad \text{(6.29)}
\end{align*}
\]

with

\[
f^{(2)}_{s, p-s} = \frac{i}{2g^3} \omega_s^2 \omega_{p-s} \omega_p - \frac{i}{2g^3} \omega_s \omega_{p-s} \omega_p^3 + \frac{i}{g} k_s k_{p-s} \omega_p + \frac{i}{2g} k_s \omega_{p-s} + \frac{i}{2g} k_{p-s} \omega_s. \quad \text{(6.30)}
\]

Here we have used that in the nonlinear terms, \( \frac{\partial}{\partial x} = -ik_p + O(\varepsilon) \), following the assumption of weak nonlinearity and mildly sloping bottom.

We will make use of the expression for \( \tilde{\eta}_p \) when transforming the evolution equations in \( \hat{\phi} \) to evolution equations in \( \hat{\eta} \) at a later stage. First, we transform the combined bottom condition
(6.14) to the frequency domain and insert (6.29) to obtain
\[
\sin(h_0 \nabla) \nabla \hat{\phi}_p + \cos(h_0 \nabla) \left[ \frac{\omega_p^2}{g} \hat{\phi}_p + \varepsilon \sum_{s=p-N}^{N} f^{(2)} \hat{\phi}_s \hat{\phi}_{p-s} \right] \\
= -\nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \hat{\phi}_p - \sin(h_0 \nabla) \frac{\omega_p^2}{g} \hat{\phi}_p \right) \right\}. \tag{6.31}
\]
Here we have only retained the linear part of $W$ at the right hand side, since we are only interested in linear bottom slope terms. We operate on both sides with the operator $\sec(h_0 \nabla)$ to obtain
\[
\left( \nabla \tan(h_0 \nabla) + \frac{\omega_p^2}{g} \right) \hat{\phi}_p \\
= -\sec(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \hat{\phi}_p - \sin(h_0 \nabla) \frac{\omega_p^2}{g} \hat{\phi}_p \right) \right\} - \varepsilon \sum_{s=p-N}^{N} f^{(2)} \hat{\phi}_s \hat{\phi}_{p-s}. \tag{6.32}
\]

**Splitting the dispersion operator** We refer to the operator on the left hand side of (6.32) as ‘the dispersion operator’. In the following we split this operator, to obtain a left hand side appropriate for evolution equations. We follow the lines of Agnon (1999), who splitted the dispersion operator to obtain a mild-slope equation.

The $\tan$-operator is to be interpreted through its infinite Taylor series, and the operator as a whole can therefore be considered as a polynomial in $\nabla$ of infinite order. Two of the roots are the progressive linear wave numbers $\nabla = \pm ik_p$. This can be seen simply by insertion, using the rules for complex arguments of trigonometric functions, see Appendix H. Doing so, the linear dispersion relation $\omega_p^2 = gk_p \tanh k_p h_0$ is recovered. Besides the progressive wave number solutions, there is an infinite set of evanescent wave modes, represented by the roots $\nabla = (\pm k_1^e, \pm k_2^e, \ldots)$. The polynomial can be factorized, with each of the factors being $\left( \nabla - \nabla_{\text{root}} \right)$. One of these factors is $(\nabla + ik_p)$, and we define the remaining factor by
\[
\left( \nabla \tan(h_0 \nabla) + \frac{\omega_p^2}{g} \right) \equiv \frac{\nabla + ik_p}{H(h_0 \nabla, k_p h_0)} \tag{6.33}
\]

which implies
\[
H(h_0 \nabla, k_p h_0) = \frac{h_0 \nabla + ik_p h_0}{h_0 \nabla \tan(h_0 \nabla) + k_p h_0 \tanh k_p h_0}. \tag{6.34}
\]

We now apply this result to (6.32) to obtain
\[
\left( \nabla + ik_p \right) \hat{\phi}_p = -H(h_0 \nabla, k_p h_0) \sec(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \hat{\phi}_p - \sin(h_0 \nabla) \frac{\omega_p^2}{g} \hat{\phi}_p \right) \right\} \\
- \varepsilon \sum_{s=p-N}^{N} H(h_0 \nabla, k_p h_0) f^{(2)} \hat{\phi}_s \hat{\phi}_{p-s}. \tag{6.35}
\]
Chapter 6. Fully Dispersive Models, 2nd Order

This is a set of evolution equations for the potential amplitudes $\hat{\phi}_p$.

It should be noted that the operator $H(h_0 \nabla, k_p h_0)$ is non-singular for $\nabla = -ik_p$. Insertion of $\nabla = -i\hat{k}_p$ gives

$$H(-i\hat{k}_p h_0, k_p h_0) = \frac{-i\hat{k}_p + ik_p}{-\hat{k}_p \tanh k_p h_0 + k_p \tanh k_p h_0} = ig \frac{\Delta k_p}{2 \Delta \omega_p} \Delta \phi_p$$

$$\rightarrow ig \frac{\partial k_p}{2 \omega_p \partial \omega_p} = \frac{ig}{2 \omega_p c_{gp}} \quad \text{for } \hat{k}_p \rightarrow k_p. \quad (6.36)$$

We use this result later in the analysis.

**Simplifying the bed-slope term**  We have assumed that the bottom is mildly sloping. This means that we retain only first-order derivatives of $\delta$. Further, we only consider the interaction of linear waves with the bottom. Hence, in the bottom slope term, we can assume that $\phi_p, x = -i k_p \frac{\phi_p}{h_0}$. We write the depth slope term as

$$T_{\text{bed}} = -H(h_0 \nabla, k_p h_0) \text{Sec}(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \frac{\omega^2_p}{g} \hat{\phi}_p \right) - \sin(h_0 \nabla) \frac{\omega^2_p}{g} \hat{\phi}_p \right\}$$

$$\equiv -\text{OP}_1(h_0 \nabla) \delta \text{OP}_2(h_0 \nabla) \hat{\phi}_p. \quad (6.37)$$

where

$$\text{OP}_1(h_0 \nabla) = H(h_0 \nabla, k_p h_0) \text{Sec}(h_0 \nabla) \nabla, \quad \text{OP}_2(h_0 \nabla) = \cos(h_0 \nabla) \nabla - \sin(h_0 \nabla) \frac{\omega^2_p}{g}. \quad (6.38)$$

Assuming that the bed slope is $O(\varepsilon)$, we express $\nabla$ by

$$\nabla = -ik_p + \varepsilon \partial_x^{(b)} \quad (6.39)$$

where it is understood that $\partial_x^{(b)}$ operates only on $\delta$. We insert this into (6.37) and Taylor expand the operators around $\nabla = -ik_p$. Retaining only the first order of $\varepsilon$ gives

$$T_{\text{bed}} = -\text{OP}_1(-i k_p h_0) \delta \text{OP}_2(-i k_p h_0) \hat{\phi}_p - \varepsilon \frac{\partial \text{OP}_1}{\partial \nabla}(-i k_p h_0) \delta_x \text{OP}_2(-i k_p h_0) \hat{\phi}_p + O(\varepsilon^2). \quad (6.40)$$

Next, we set the reference level $h_0$ equal to the local depth $h$. This implies setting $h_0 = h(x)$ and $\delta = 0$. In this process the first term in the above result vanishes, and we are left with the term

$$T_{\text{bed}} = -\varepsilon \text{OP}_1(-i k_p h) \delta_x \text{OP}_2(-i k_p h) \hat{\phi}_p$$

$$= -\frac{4 \kappa \sinh \kappa (\kappa \sinh \kappa - \cosh \kappa) h_x}{(2 \kappa + \sinh 2 \kappa)^2} \frac{h}{h} \hat{\phi}_p = -\frac{c_{gp,x}}{2 c_{gp}} \hat{\phi}_p \quad (6.41)$$
where $\kappa = k_p h$ and $c_{gp} = \frac{\partial \omega_p}{\partial k_p}$ is the group velocity at frequency $p$. The last identity was shown using Mathematica.

With these manipulations we have the model

$$(\nabla + ik_p) \hat{\phi}_p = -\varepsilon c_{gp,x} \frac{c_{gp}}{2} \hat{\phi}_p - \varepsilon \sum_{s=p-N}^{N} H(h \nabla, k_p h) f^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s}. \quad (6.42)$$

### 6.2.6 The resonant and exact model

In the above equation, we have not yet inserted an expression for $\nabla$ in the H-operator within the nonlinear terms. Two choices are natural.

**The exact model** We may approximate the derivative of each of the participating wave amplitudes by the values associated by their linear wave numbers. This corresponds to setting $\nabla = -i(k_s + k_{p-s})$ in each term in the sum. This results in a model which has exact second-order transfer functions. Hence, to second order, the model describes Stokes waves correctly.

**The resonant model** Another approach is to assume resonance of the forcing nonlinear terms with the free wave mode at the receiving frequency. This amounts to assuming $\nabla = -ik_p$. Doing so, the models of Agnon et al. (1993) and Kaihatu and Kirby (1995) are recovered. The resonance assumption is only true in shallow water, where the dispersion is vanishing. As mentioned in the introduction, Agnon et al. (1993) defined the detuning parameter

$$\mu = (k_s + k_{p-s} - k_p)/k_p, \quad (6.43)$$

and stated that their model describes the second-order bound waves with an error of $O(\mu)$. The detuning parameter $\mu$ is zero in the shallow water limit and increases with relative depth.

We look closer at the two different interaction kernels (exact, resonant) in Section 6.3. Note that in our derivation, no assumption on the vertical structure of the velocity field has been made, except the assumption of weak nonlinearity, affecting $\Phi_z$ through the kinematic free surface condition. No further assumptions are needed, since we do not depth integrate the Laplace equation.

The model (6.42) defines the fully dispersive evolution equations in the amplitudes of the velocity potential at the still water level.
6.2.7 Evolution equations in $\eta$

The evolution equations (6.42) can be transformed into evolution equations in the complex amplitudes of the free surface elevation, $\eta$. Such a transformation was presented by Eldeberky and Madsen (1999) for the models of Agnon et al. (1993) and Kaihatu and Kirby (1995). We follow the same route here.

To lowest order, (6.28) reads $\hat{\phi}_p = \left(\frac{ig}{\omega_p}\right)\hat{\eta}_p + O(\varepsilon)$. This relation can be used in the quadratic terms of the equation, thus producing the result

$$\hat{\phi}_p = \left(\frac{ig}{\omega_p}\right)\hat{\eta}_p + i \sum_{s=p-N}^{N} T_{s,p-s}^{(1)} \hat{\eta}_s \hat{\eta}_{p-s} + O(\varepsilon^2), \quad T_{s,p-s}^{(1)} = \frac{g^2 k_s k_{p-s}}{2\omega_p \omega_s \omega_{p-s}} - \frac{1}{2}\frac{\omega_p}{\omega_s} + \frac{\omega_p \omega_{p-s}}{2\omega_p}.$$  \hspace{1cm} (6.44)

This result is also found in Eldeberky and Madsen (1999, equation (3.4)–(3.6)). We can easily rewrite the quadratic terms of (6.42) to be formulated in $\hat{\eta}$, using just the linear part of this relation. For transforming the left hand side, we need the linear as well as nonlinear terms. For each of the nonlinear terms in the summation of (6.44), we use the linear approximation $\nabla + ik_p = (-i(k_s + k_{p-s}) + ik_p)$. We hereby get

$$(\nabla + ik_p)\hat{\phi}_p = \left(\frac{ig}{\omega_p}\right)\nabla \hat{\eta}_p + \varepsilon \sum_{s=p-N}^{N} \left( k_s + k_{p-s} - k_p \right) T_{s,p-s}^{(1)} \hat{\eta}_s \hat{\eta}_{p-s}. \quad (6.45)$$

We now combine this result with (6.42) to obtain

$$(\nabla + ik_p)\hat{\eta}_p = -\varepsilon \frac{c_{gp,x}}{2c_{gp}} \hat{\eta}_p + \varepsilon \sum_{s=p-N}^{N} W_{s,p-s} \hat{\eta}_s \hat{\eta}_{p-s} \quad (6.46)$$

or in more compact form

$$(\nabla + ik_p)\hat{\eta}_p = -\varepsilon \frac{c_{gp,x}}{2c_{gp}} \hat{\eta}_p + \varepsilon \sum_{s=p-N}^{N} W_{s,p-s} \hat{\eta}_s \hat{\eta}_{p-s} \quad (6.47)$$

with

$$W_{s,p-s} = \frac{\omega_p}{g} \left( k_s + k_{p-s} - k_p \right) T_{s,p-s}^{(1)} - \frac{\omega_p}{\omega_s \omega_{p-s}} \left( H(h \nabla, k_p h) \right) T_{s,p-s}^{(2)}, \quad (6.48)$$

where $T_{s,p-s}^{(1)}$, $T_{s,p-s}^{(2)}$ and $H(h \nabla, k_p h)$ are defined in (6.44), (6.30) and (6.34), respectively.

Using the resonance assumption $\nabla = -ik_p$ in the H-term, this model is identical to the model of Eldeberky and Madsen (1999). Using the linear approximation, $\nabla = -i(k_s + k_{p-s})$, the ‘exact’ model is obtained, producing correct second-order Stokes waves. We analyze this in the next section.
6.3 Second-order bichromatic transfer functions

The model (6.47) is derived under the assumptions of weak nonlinearity, slowly varying bathymetry and only forward propagating waves. Hence, on constant depth, it should match Stokes second-order theory exactly. In this section we compare the evolution equations just derived to Stokes wave theory. We derive the second-order transfer functions for bichromatic waves of the evolution equations and compare them to the results of Stokes wave theory.

Next, we examine the influence of neglecting the second-order terms in the transformation from $\hat{\phi}$ to $\hat{\eta}$.

6.3.1 Bichromatic transfer functions for Stokes waves

The exact second-order bichromatic transfer functions are defined below. The present form is taken from Madsen and Schäffer (1999) who cited it from Schäffer (1996). We consider a first-order wave field of the form

$$\eta^{(1)}(x, t) = A_n \cos \theta_n + B_n \sin \theta_n + A_m \cos \theta_m + B_m \sin \theta_m$$

where $\theta_i = \omega_i t - k_i x$. The bound second-order wave field can be written as

$$\eta^{(2)} = \eta_{nm}^- + \eta_{nm}^+ + \eta_{nm}^{++} + \eta_{nm}^{++}$$

with

$$\eta_{nm}^\pm = \varepsilon \delta_{nm} G_{nm}^\pm (A_p \cos \theta_p + B_p \sin \theta_p)$$

where

$$A_p = \frac{1}{h} (A_n A_m \mp B_n B_m)$$

$$B_p = \frac{1}{h} (A_m B_n \pm A_n B_m)$$

$$\omega_p = \omega_n \pm \omega_m$$

$$k_p = k_n \pm k_m$$

$$\delta = \begin{cases} \frac{1}{2} & \text{for } n = m \\ 1 & \text{for } n \neq m \end{cases}$$

$$G_{nm}^\pm = \frac{h}{g} \left( (\omega_n \pm \omega_m) \left( \frac{H_{nm}^\pm}{D_{nm}^\pm} - \frac{L_{nm}^\pm}{2} \right) \right)$$

$$H_{nm}^\pm = (\omega_n \pm \omega_m) \left( \pm \omega_n \omega_m - \frac{g^2 k_n k_m}{\omega_n \omega_m} \right)$$

$$D_{nm}^\pm = g(k_n \pm k_m) \tanh(k_n \pm k_m) h - (\omega_n \pm \omega_m)^2$$

$$L_{nm}^\pm = \frac{1}{2} \left( \mp \omega_n \omega_m + \frac{g^2 k_n k_m}{\omega_n \omega_m} - (\omega_n^2 + \omega_m^2) \right)$$
Equation (6.50) shows that the quadratic interaction of two waves generates one sub-harmonic wave at the difference frequency and three super-harmonic waves at the sum frequency and the double frequencies of the generating waves.

If the frequencies are considered as both positive and negative, all the four waves produced can be viewed as super-harmonics. The sub-harmonic wave is then a super-harmonic wave produced by the waves \( n \) and \(-m\). One then has to use \( \omega_m = -\omega_m \) and \( k_m = -k_m \). In this way, the ±-signs in the above expressions can be avoided, and one can use the upper part of these signs only. In Figure 6.2, \( G_{nm} \) is plotted as function of the dimensionless frequency

\[
\Omega_n = \frac{\omega_n}{2\pi} \sqrt{h/g}.
\]  

(6.59)

Figure 6.2: Second-order transfer coefficient \( G_{nm} \) as function of dimensionless frequency.

Reformulation to complex wave amplitudes We now reformulate the above results to a complex Fourier representation of the wave field. The purpose is to ease the comparison with transfer functions of evolution equations. We describe the linear wave field as

\[
\eta(x, t) = a_n e^{i\theta_n} + a_m e^{i\theta_m} + c.c.
\]  

(6.60)

such that the relations

\[
A_j = 2\text{Re}\{a_j\}, \quad B_j = -2\text{Im}\{a_j\}, \quad a_j = \frac{1}{2}(A_j - iB_j)
\]  

(6.61)
6.3 Second-order bichromatic transfer functions

hold. The indices $m$ and $n$ are allowed to be negative. The second-order generated wave at frequency $\omega_p = \omega_n + \omega_m$ can be written as $a_p e^{i(\theta_n + \theta_m)} + c.c.$ Insertion into (6.51)–(6.58) and using (6.61) leads after some manipulations to

$$a_p = 2\varepsilon \delta_{nm} \frac{1}{\hbar} G^+ a_n a_m.$$  \hspace{1cm} (6.62)

Here $G^+$ is given by the upper-sign version of (6.55). For $m$ negative, the above expression yields the same result as if the ‘lower’ part of (6.51)–(6.58) is used. The above expression is useful for comparison with transfer functions of evolution equations.

6.3.2 Bichromatic transfer functions for evolution equations

For two forcing waves $\hat{\eta}_n e^{i\omega_n t} + c.c.$ and $\hat{\eta}_m e^{i\omega_m t} + c.c.$, the evolution equations give at the receiving frequency $\omega_p = \omega_n + \omega_m$

$$(\nabla + i k_p) \hat{\eta}_p = i \varepsilon \sum_{s=p-N}^N W_{s,p-s} \hat{\eta}_s \hat{\eta}_{p-s} = 2i \varepsilon \delta_{nm} W_{n,m} \hat{\eta}_n \hat{\eta}_m.$$  \hspace{1cm} (6.63)

For the linear forcing waves we have $\nabla_n = -ik_n$ and $\nabla_m = -ik_m$. We can therefore rewrite the left hand side to $-i(k_n + k_m - k_p) \hat{\eta}_p$ and thereby obtain

$$\hat{\eta}_p = -2\varepsilon \delta_{nm} \frac{W_{n,m}}{k_n + k_m - k_p} \hat{\eta}_n \hat{\eta}_m.$$  \hspace{1cm} (6.64)

In this result we can substitute

$$\hat{\eta}_n = a_n e^{-ik_n x} \quad \hat{\eta}_m = a_m e^{-ik_m x} \quad \hat{\eta}_p = a_p e^{-i(k_n + k_m)x}$$  \hspace{1cm} (6.65)

to obtain an expression similar to (6.62), with $a_n, a_m, a_p$ being constant complex numbers.

$$a_p = -2\varepsilon \delta_{nm} \frac{W_{n,m}}{k_n + k_m - k_p} a_n a_m.$$  \hspace{1cm} (6.66)

By comparing (6.62) and (6.66), we can now define a $G$-function for the evolution equations:

$$G_{evo} = -\frac{\hbar}{k_n + k_m - k_p} W_{n,m}.$$  \hspace{1cm} (6.67)

Hence, given a kernel $W$ of a set of evolution equations, the above formula expresses the bichromatic transfer functions.
6.3.3 Comparison of transfer functions

As a first check, the transfer functions of the ‘exact’ evolution equations were compared to the transfer functions of Stokes waves. This was done using a Mathematica sheet, and it was found that the transfer functions are identical. Hence, for the ‘exact’ evolution equations, the second-order bound waves are identical to second-order Stokes waves for small wave heights.

Next, we compare the transfer functions of the ‘resonant’ evolution equations to Stokes wave theory. As another check, the kernel of the resonant evolution equations was checked against the kernel of Eldeberky and Madsen (1999). Also here the expressions were found to be identical.

In Figure 6.3 $G_{11 \text{ Evo}}$, the transfer function for self-self interaction, is plotted against dimensionless frequency for the resonant model. The result is normalized with the transfer function of Stokes waves.

![Figure 6.3: Second-order self-self interaction transfer function for the quadratic model with the resonance assumption. The values are normalized with the exact transfer function.](image)

We see that the self-self interaction transfer function is 3.5% too large as the largest deviation. All in all the resonance assumption is seen to be rather good.

In Figure 6.4, the ratio between $G_{nm \text{ Evo}}$ and $G_{nm \text{ Stokes}}$ are plotted in the parameter space $(\Omega_n, \Omega_m) \in [0; 1] \times [0; 1]$. The super harmonic transfer of the evolution equations is very close...
to the target of Stokes theory. The small variation observed for the self-self interaction is seen as two small curves close to the diagonal. The sub-harmonic transfer is under-predicted in a region parallel to the diagonal and over-predicted along the diagonal. Lines being parallel to the diagonal represent constant receiving frequencies. Parallel lines close to and below the diagonal represent long waves forced sub-harmonically by waves having close frequencies.

![Second-order bichromatic transfer functions](image)

Figure 6.4: Second-order bichromatic transfer functions for the quadratic model with the resonance assumption. The values are normalized with the exact transfer function.

**Transfer functions when applying linear transformation between $\phi$ and $\eta$** The evolution equation models of Agnon et al. (1993) and Kaihatu and Kirby (1995) were formulated in the amplitudes of the velocity potential. In both papers, results were presented in terms of the free surface elevation, and the transformation applied between $\phi$ and $\eta$ was linear. This corresponds to setting $T_{s,p-s}^{(1)} = 0$ in the interaction kernel (6.48), as well as using the resonance assumption $\nabla = -ik_p$ in the H-operator. In Figure 6.5, the corresponding transfer coefficient for self-self interaction is plotted. We see that the transfer decays fast towards zero away from shallow water. In $\Omega \approx 0.18$, the transfer is half the value of the Stokes wave result. A similar curve is presented in Eldeberky and Madsen (1999), the independent parameter being $kh$ rather than $\Omega$.

The bichromatic transfer function is depicted in Figure 6.6. The decay towards zero applies for a large area in the super-harmonic region. For the sub-harmonic transfer, the transfer is over-predicted in a large part of the region. Eldeberky and Madsen (1999) present a
Figure 6.5: Second-order self-self interaction transfer function for the quadratic model with the resonance assumption and linear transformation from $\phi$ to $\eta$. The values are normalized with the exact transfer function.

A similar figure, comparing the interaction kernel of the resonant model with and without the $T_{s,p-s}^{(1)}$-term. As the resonant kernel including this term is very close to Stokes theory in the super-harmonic region, the plot is similar to the plot in Figure 6.6 for this region. For the sub-harmonic region, some deviations occur, because the reference solutions are not identical. The reference solution of Eldeberky and Madsen (1999) is the resonant kernel, while the reference solution in Figure 6.6 corresponds to the exact kernel. Thus for $\Omega_m < 0.2$, the plot of Eldeberky and Madsen shows an underestimation of the transfer, while the plot in our Figure 6.6 shows a constant over-prediction of the transfer for the sub-harmonic range.
Figure 6.6: Second-order bichromatic transfer functions for the quadratic model with the resonance assumption and linear transformation from $\phi$ to $\eta$. The values are normalized with the exact transfer function.
6.4 Speeding up the calculations using FFT

As for the Boussinesq based evolution equations treated in Chapter 3, the fully dispersive evolution equations can be speeded up computationally using FFT. This, however, only applies for the resonant model. We go a bit deeper into this in this section.

For the Boussinesq based evolution equations, the nonlinear terms arise in the time domain as a well-defined physical time series. Hence, obviously, this time series can be calculated for each spatial step during the integration of the evolution equations. For the fully dispersive evolution equations, the nonlinear terms arise from the convolutions involved in (6.24) and (6.25), but are manipulated further in the derivation. However, no matter how a nonlinear term arises, it can be treated using FFT’s if it can be expressed as a product of quantities belonging to the two forcing frequencies and the receiving frequency. As an example, consider the very first nonlinear term in (6.47)

\[
S_1 = i \frac{\omega_p}{g} \sum_{s=p-N}^{N} k_s g^2 \frac{k_s k_{p-s}}{2 \omega_s^2 \omega_{p-s}} \tilde{\eta}_s \tilde{\eta}_{p-s} = i \frac{g}{2} \sum_{s=p-N}^{N} \left\{ \frac{k_s^2}{\omega_s^2} \tilde{\eta}_s \right\} \left\{ \frac{k_{p-s}}{\omega_{p-s}} \tilde{\eta}_{p-s} \right\}
\]

(6.68)

The last sum can be seen as being the p'th Fourier component of the product of two time series \( \tilde{\eta}_1 \) and \( \tilde{\eta}_2 \) defined by the Fourier series

\[
\tilde{\eta}_1 = \sum_{p=-N}^{N} k_p^2 \hat{\eta}_p e^{i \omega_p t}, \quad \tilde{\eta}_2 = \sum_{p=-N}^{N} k_p \hat{\eta}_p e^{i \omega_p t}.
\]

(6.69)

Hence writing their product as

\[
\tilde{\eta}_1 \tilde{\eta}_2 = \sum_{p=-2N}^{2N} \left[ \tilde{\eta}_1 \tilde{\eta}_2 \right]_p e^{i \omega_p t}
\]

(6.70)

allows us to express \( S_1 \) simply as

\[
S_1 = i \frac{g}{2} \left[ \tilde{\eta}_1 \tilde{\eta}_2 \right]_p.
\]

(6.71)

This allows for a speed up of the computation of \( S_1 \), since the time series \( \tilde{\eta}_1, \tilde{\eta}_2 \) can be calculated from the wave amplitudes \( \hat{\eta} \) using inverse FFT. The product \( \tilde{\eta}_1 \tilde{\eta}_2 \) can then be calculated as a product of two time series, and the Fourier components of this product, \( \left[ \tilde{\eta}_1 \tilde{\eta}_2 \right]_{1,...,N} \), be extracted by a forward FFT. As mentioned in Chapter 3, more than \( 3N \) points must be used in the time series to avoid aliasing within the \( N \) positive frequencies modelled. Still, however, the sums can be obtained at a computational effort of \( O(N \log N) \) due to the efficiency of the FFT algorithm.

The above procedure requires that each term in the summations can be written as products of quantities of index \( s, p-s \) and \( p \), respectively. This applies for all terms within the resonant model, and the speed up is therefore easily implemented. For the exact model, however, the function \( H(\nabla h, k_p h) \), with \( \nabla = -i(k_s + k_{p-s}) \), does not allow for such a factorization, and the trick therefore cannot be used for this model.
6.4 Speeding up the calculations using FFT

6.4.1 Approximations to the exact kernel

Although the exact kernel cannot be factorized, it may be possible to approximate it in a form, allowing for factorization. We here formulate this problem mathematically and show two attempts of such approximations. As already mentioned, the reason for the failure of factorization is the H-operator with \( \nabla = -i(k_s + k_{p-s}) \), since all other parts of the kernel are easily factorized. We can therefore concentrate on the H-operator.

From (6.34) we have

\[
H(-i(k_s + k_{p-s})h, k_p h) = \frac{-i(k_s + k_{p-s}) + ik_p}{-(k_s + k_{p-s}) \tanh(h(k_s + k_{p-s})) + k_p \tanh k_p h}.
\]  

(6.72)

Defining \( \kappa = k_p h \) and \( R = (k_s + k_{p-s})h \) we write this as

\[
H(-i(k_s + k_{p-s})h, k_p h) = i \frac{R - \kappa}{R \tanh R - \kappa \tanh \kappa}.
\]  

(6.73)

Hence, the problem of factorizing H can be stated as finding \( f_{1j} \) and \( f_{2j} \) such that

\[
F(R, \kappa) = \frac{R - \kappa}{R \tanh R - \kappa \tanh \kappa} \approx \sum_{j=1}^{M} f_{1j}(R) f_{2j}(\kappa).
\]  

(6.74)

The functions \( f_{2j}(\kappa) \) can be of any form. For \( R = r_1 + r_2 \), the functions \( f_{1j}(R) \) must consist of terms of the form \( g_1(r_1)g_2(r_2) \). Good choices for \( f_{1j} \) are therefore polynomials in \( R \) or polynomials in \( e^R \). The number of terms, \( M \), in the approximation of H should not be too large, since the model then gets too complicated compared to the approximations involved (no reflections, weak nonlinearity).

Regarding the approximation of \( F(R, \kappa) \), we note that in the super-harmonic region, the contribution to the transfer functions from the Hf\(^2\)-term in the kernel vanishes away from the smallest forcing frequencies. This can be deduced from Figure 6.6, where the transfer function without the contribution from the transformation to \( \eta \) is compared to the exact transfer function of Stokes theory. Consequently a good approximation of \( F(R, \kappa) \) does not need to be accurate in the super-harmonic region away from Origo.

A first idea of approximation A first idea of approximating \( F(R, \kappa) \) is to Taylor expand it around \( R = \kappa \). The expansion is

\[
F(R, \kappa) = F(\kappa, \kappa) + \frac{\partial F}{\partial R}(\kappa, \kappa)(R - \kappa) + O((R - \kappa)^2)
\]  

(6.75)

where the first term corresponds to the resonance approximation. Further, the approximation is a polynomial in \( R \) and is therefore easily factorized.
In Figure 6.7 the ratio of $G_{11}$ for the approximation (6.75) and Stokes theory is plotted. The extra term in the expansion compared to the pure resonance assumption improves the results slightly, see Figure 6.3. However, inspection of the ratio of the transfer functions for the two-dimensional frequency range gives poorer results than for the pure resonance assumption. The ratio of $G_{nm}$ to the transfer functions of Stokes theory are shown in Figure 6.8. The super-harmonic range is almost exact, and better than in Figure 6.4, but the sub-harmonic range exhibits larger errors than in Figure 6.4.

**A second idea** Instead of using the resonance assumption as starting point for an approximation, we may take a closer look at the function $F(R, \kappa)$, to see if this gives some ideas. This function is plotted in Figure 6.9. We see that $F(R, \kappa)$ equals unity for a very large part of the region plotted. Only for the very long waves (close to Origo in the super-harmonic region and close to the diagonal in the sub-harmonic region), there is a deviation from unity. Hence, $F = 1$ can be used in a large part of the plot as a very good approximation. The corresponding transfer function for self-self interaction is plotted in Figure 6.10. Close to $\Omega = 0$ there is a large under-prediction and for $\Omega = 0.1$ there is an over-prediction of around 20%. The transfer function as a whole is depicted in Figure 6.11. The quality of the transfer function is similar to the deviation of $F(R, \kappa)$ from unity, see Figure 6.9. The super-harmonic transfer is incorrect in a region close to Origo, and the sub-harmonic transfer is under-predicted close to the diagonal. A large region of the sub-harmonic area, however, is determined with great accuracy. Comparing to the transfer functions of the
6.4 Speeding up the calculations using FFT

Figure 6.8: $G_{nm}$ for the approximation (6.75). The result is normalized with the exact transfer function.

Figure 6.9: $F(R, \kappa)$ as function of $(\Omega_m, \Omega_n)$. 
Figure 6.10: $G_{11}$, i.e., transfer function for self-self interaction for the approximation $F(R, \kappa) = 1$. The result is normalized with the exact transfer function.

Figure 6.11: $G_{nm}$ for the approximation $F(R, \kappa) = 1$. The result is normalized with the exact transfer function.
resonant kernel, Figure 6.4, we see that the region of under-predicted sub-harmonic transfer below the diagonal is not present. Hence a combination of these two approximations may be provide a good fit to $F(R, \kappa)$.

A simple choice of combination is the composite expression

$$F(R, \kappa) = e^{-\nu R} F(\kappa, \kappa) + (1 - e^{-\nu R}) \times 1.$$  \hfill (6.76)

For small $R$, thus for the region close to Origo in the super-harmonic region and close to the diagonal in the sub-harmonic region, this expression resembles the resonance approximation, while for larger values of $R$, the unity approximation is approached. For $\nu = 0.2$, the transfer function for self-self interaction is depicted in Figure 6.12. There is an under-prediction of 3% close to the shallow water limit, followed by a 6% over-prediction. This is not as good as for the resonance approximation alone.

![Graph](image)

Figure 6.12: $G_{11}$, i.e., transfer function for self-self interaction for the composite approximation (6.76). The result is normalized with the exact transfer function.

The bichromatic transfer functions are depicted in Figure 6.13. In the region $\Omega_m, \Omega_n \in [0.2] \times [0.2]$, the super-harmonic errors are of same magnitude or smaller than the error observed for the self-self interaction. Comparing to the transfer functions for the model with the resonance assumption, Figure 6.4, the new transfer function does not have the region of under-predicted sub-harmonic transfer parallel to the diagonal. This is an improvement. Close to the diagonal, where the composite expansion resembles the resonant kernel, the sub-harmonic interaction is still over-predicted.
Figure 6.13: $G_{nm}$ for the composite approximation (6.76). The result is normalized with the exact transfer function.

**Summing up** The above approximation and its results are encouraging. The approximation $F(R, \kappa) = 1$ is indeed simple, and a successful refinement is therefore likely to exist. From the plot of $F(R, \kappa)$, Figure 6.9, we see that a good approximation should resemble unity in a large region and only differ close to the diagonal in the sub-harmonic region. The slow variation of $F(R, \kappa)$ also explains the success of the resonance assumption. When $F(R, \kappa)$ is not much sensitive to its arguments, a large range of approximations will do.

Although the second-order transfer functions of the combined approximation (6.76) is an improvement in comparison with the results of the resonant kernel, the remaining parts of this chapter are based on the resonant kernel and the exact kernel. We note that it is possible to solve the model with the exact kernel. The interest in approximating the kernel is therefore related to the speed-up of the model rather than to the physics of the equations.

### 6.5 Amplitude dispersion

Having analyzed the second-order properties of the fully dispersive evolution equations, it is natural to extend the analysis to third order. This enlights the characteristics of amplitude dispersion as well as the third-order transfer. The analysis presented follows the weakly nonlinear analysis, developed in Chapter 5.
6.5 Amplitude dispersion

6.5.1 Weakly nonlinear analysis

The weakly nonlinear analysis is based on the method given in Chapter 5. Thus given the kernel (6.48), the frequency dependence of $\omega_{13}$ is readily found using (5.21) and (5.28). We analyze the equations formulated in $\eta$ as well as the equations formulated in $\phi$, (6.42). The latter model results are obtained simply by setting $T_{s,p-s}^{(1)} = 0$ in (6.48), that is, neglecting the second-order contribution from the transformation between $\eta$ and $\phi$.

Results for $\omega_{13}$, exact models In Figure 6.14, the results for $\omega_{13}$ for the exact models are shown. The results are normalized with $\omega_{13,C_s}$ of Stokes waves. Thus the comparison covers the situation with a zero net mass flux, like waves propagating towards a beach. For easy comparison with the results of Chapter 5, the results of the ‘simplified’ time domain model and the evolution equations of Madsen and Sørensen (1993) are plotted as well. It is noted that the time domain model was already derived in Madsen et al. (1991), while the third-order analysis was given in Madsen and Sørensen (1993). This is the reason for the labeling ‘MS93’ in the plots.

![Figure 6.14: $\omega_{13}$ for fully dispersive evolution equations, normalized by $\omega_{13}$ for Stokes waves with $c_S = 0$. The curves of the ‘simplified’ time domain model and the evolution equations of Madsen and Sørensen (1993) are repeated from Chapter 5.](image)

The plot shows that the fully dispersive evolution equations over-predict the amplitude dispersion severely. The models formulated in $\phi$ and $\eta$ are only equivalent to second order, and the amplitude dispersion, being a third-order property, therefore deviates between the models. The equations formulated in $\eta$ over-predict the amplitude dispersion by a factor
of 5 in $\omega \sqrt{h/g} = 1.03$. Towards deep water, the amplitude dispersion decays, but still in $\omega \sqrt{h/g} = 2\pi$, corresponding to $kh = 39.5$, $\omega_{13}$ is more than 1.5 times as large as for Stokes waves. The results for the evolution equations in $\phi$ show a similar over-prediction in shallow and intermediate water, while for deeper water the amplitude dispersion decays towards zero. The largest over-prediction occurs for $\omega \sqrt{h/g} = 0.95$, where $\omega_{13}$ is nearly 4 times as large as for Stokes waves. The transition between over-prediction and under-prediction occurs for $\omega \sqrt{h/g} = 1.36$. We also see that the over-prediction of $\omega_{13}$ is much larger for the fully dispersive models than for both of the Boussinesq based models.

Given that the second-order transfer functions of the fully dispersive models are exact when comparing to Stokes waves, it may seem surprising that the amplitude dispersion is deviating strongly. However, as the amplitude dispersion is a third-order characteristic, there is no guarantee that a model with correct second-order characteristics should provide the correct amplitude dispersion. First, some third-order terms have been let out from the governing equations, when expanding the free surface conditions around the still water level, and second, the manipulations have only been carried out consistently to second order, thus with the possibility of modifying the third-order properties. Therefore, a good reproduction of the amplitude dispersion, or any other third-order property, cannot be expected. This also explains the difference in amplitude dispersion between the model in $\phi$ and the model in $\eta$. The transformation between the two models is not consistent to third order, and therefore changes the third-order properties. The reason that $\omega_{13}$ vanishes in deep water for the model formulated in $\phi$, is the decay of the super-harmonic transfer function for self-self interaction, see Figure 6.5. Thus, in deep water, there is no second-order bound harmonic to interact sub-harmonically with the primary wave. This sub-harmonic interaction produces the modulation of the linear wave number of the primary wave, and thereby the amplitude dispersion. In the equations this corresponds to setting $W_{1,1} = 0$ in the expression for $k_{13}$, see (5.21).

It should be noted that comparison with $\omega_{13,cS}$ gives a larger over-prediction of the amplitude dispersion than if $\omega_{13,cE}$ was used as reference solution. The effect can be deduced by comparing Figures 6.14 and 6.15, where in the latter figure results normalized with $\omega_{13,cE}$ are shown. The difference between the two reference solutions can be up to a factor of 2.5. Still however, the model in $\eta$ over-predicts the amplitude dispersion, while the model for $\phi$ over-predicts the amplitude dispersion for shallow and intermediate water and under-predicts the amplitude dispersion in deep water.

**Effect of resonance approximation** In Figure 6.16, the influence of the resonance assumption is examined. The curves for the exact kernel are repeated, and the similar results obtained under the resonance assumption are shown as dotted curves. The effect is most pronounced for the model in $\eta$. For both models, the amplitude dispersion is reduced, but with a relatively small amount that does not change the overall model behavior.
6.5 Amplitude dispersion

Figure 6.15: \( \omega_{13} \) for fully dispersive evolution equations, normalized by \( \omega_{13} \) for Stokes waves with \( c_E = 0 \). The curves of the ‘simplified’ time domain model and the evolution equations of Madsen and Sørensen (1993) are repeated from Chapter 5.

Figure 6.16: \( \omega_{13} \) for fully dispersive evolution equations, normalized by \( \omega_{13} \) for Stokes waves with \( c_S = 0 \). Results for the resonance assumption are shown as the dotted curves.
Results for $\tilde{A}_3$  Having carried out the third-order analysis it is natural to present the results for $\tilde{A}_3$, i.e., the transfer function to the third harmonic as well. The expression for $\tilde{A}_3$ is given in (5.22). In Figure 6.17, the results of the exact models in $\eta$ and $\phi$ are compared to the results of the ‘simplified’ time domain model and the evolution equations of Madsen and Sørensen (1993). All curves are normalized with the result of Stokes wave theory. We see that the transfer to the third harmonic is over-predicted for the model in $\eta$. In shallow water, the model agrees with Stokes wave theory, but in deep water, the transfer function converges towards a value of around 2.2 times that of Stokes waves. For the model in $\phi$, the transfer is first over-predicted and then under-predicted, decaying to zero in deep water. The largest over-prediction is of 11% and occurs for $\omega\sqrt{h/g} = 0.41$, while the transition to the region of under-prediction occurs in $\omega\sqrt{h/g} = 0.61$. For both of the fully dispersive models, the transfer to the third harmonic is larger than for the Boussinesq based models.

The influence of the resonance approximation can be examined in Figure 6.18, where the results obtained using the resonance assumption are shown as dotted curves. The influence is very small. This is to be expected, since the transfer to the third harmonic is a combined result of self-self interaction and interaction between the first and the second harmonic. Both of these are super-harmonic interactions, and we have seen that the resonance assumption does not imply significant changes to these transfer functions, see Figure 6.4.
6.6 Roller breaking in fully dispersive models

We now incorporate roller breaking into the model. The method for calculating the roller term $R_{xx}$ was developed in Chapter 4, and the main challenge is therefore to clarify how the roller term should be included into the fully dispersive model. This is treated in the following. After this, results for the Ting and Kirby (1994) test are presented and compared to uniform as well as $f^2$-weighted breaking.

6.6.1 Inclusion of the roller term

The evolution equations of Madsen and Sørensen (1993) are based on the wave equation (3.2). The effect of roller breaking is incorporated by adding the term $R_{xx}$ to the right hand side. We write the equation as

$$ \eta_t - gh \eta_{xx} + D = S + N + R_{xx} $$

(6.77)

where ($D, S, N$) symbolize dispersive terms, bottom slope terms and nonlinear terms, respectively. We now derive a similar expression for fully dispersive theory, thus establishing an expression where the roller term can be included.

Figure 6.18: $\tilde{A}_3$ for fully dispersive evolution equations, normalized by $\tilde{A}_3$ for Stokes waves. Results for the resonance assumption are shown as the dotted curves.
Chapter 6. Fully Dispersive Models, 2nd Order

We differentiate (6.14) with respect to time and operate with \( \text{Sec}(h_0 \nabla) \) on both sides to obtain

\[
\nabla \text{Tan}(h_0 \nabla) \Phi_t + W_t = -\text{Sec}(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \Phi_t - \sin(h_0 \nabla) W_t \right) \right\}.
\]

(6.78)

The free surface conditions (6.19) and (6.20) can be written

\[
\Phi_t = -g \eta - \varepsilon N_1
\]

(6.79)

\[
W_t = \Phi_{zt} = \eta_{tt} + \varepsilon N_2
\]

(6.80)

where \( N_1 \) and \( N_2 \) are quadratic nonlinear terms defined through (6.19) and (6.20). Insertion into (6.78) gives

\[
\nabla \text{Tan}(h_0 \nabla) \left( -g \eta - \varepsilon N_1 \right) + \eta_{tt} + \varepsilon N_2
= \text{Sec}(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) g \nabla \eta + \sin(h_0 \nabla) (\eta_{tt} + \varepsilon N_2) \right) \right\}.
\]

(6.81)

The first term of the Tan-operator is \( -gh_0 \nabla^2 \eta \), the next terms being higher-order dispersive terms. This term is also found in the wave equation (6.77). Also the \( \eta_{tt} \) term appears in the above equation as well as in (6.77). The above wave equation is therefore the wave equation of fully dispersive theory, corresponding to the wave equation (6.77) of Boussinesq theory. In shallow water, the Boussinesq based wave equation will approach the equation of fully dispersive theory. Hence, in the fully dispersive model, the effect of wave breaking may be added, simply by adding \( R_{xx} \) to the right hand side of (6.81). The wave breaking formulation is thereby only valid in shallow water, where the original Boussinesq formulation and the breaking model are valid. The breaking scheme is therefore appropriate for breaking in the surf zone and expectedly not appropriate for deep water breaking.

We carry out the following five steps to turn (6.81) into a set of evolution equations: 1) add \( R_{xx} \) to the right hand side, 2) neglect the nonlinear terms on the right hand side (quadratic bed slope terms), 3) multiply the equation with \( -1/g \), 4) transform to the frequency domain and 5) operate with the H-operator as defined in (6.34). The resulting equation is

\[
(\nabla + ik_p) \hat{\eta}_p = -H(h_0 \nabla, k_p h_0) \text{Sec}(h_0 \nabla) \nabla \left\{ \delta \left( \cos(h_0 \nabla) g \nabla \eta + \sin(h_0 \nabla) \eta_{tt} \right) \right\}
+ \frac{\varepsilon}{g} H(h_0 \nabla, k_p h_0) N_{2,p} - \frac{\varepsilon}{g} (\nabla + ik_p) N_{1,p} - \frac{1}{g} H(h_0 \nabla, k_p h_0) R^{(p)}_{xx},
\]

(6.82)

where \( R^{(p)}_{xx} \) is the \( p \)'th Fourier component of \( R_{xx} \), as defined in e.g., (6.71). For \( h_0 \to h \), i.e., \( \delta \to 0 \), the above model can be shown to be identical to (6.47), except for the incorporation of the roller term. Hence, the roller term can be included in the evolution equations for \( \eta \) by adding the term \( -\frac{1}{g} H(h_0 \nabla, k_p h_0) R^{(p)}_{xx} \) to the right hand side.

Details on the implementation  The implementation of the roller breaking scheme is identical to the one described in Chapter 4, and is therefore not detailed here. However, a note on the choice of \( \nabla \) in the H-operator multiplying on the roller term should be given. Two choices are obvious
• $\nabla = -ik_p$, i.e. the free wave number of each harmonic. This choice is in line with the resonance assumption.

• $\nabla = -i\omega_p/c_{\text{break}}$. Here $c_{\text{break}}$ is the phase speed of the breaking wave, used to determine the breaking criterion and in the calculation of $R$. $c_{\text{break}}$ is independent of the frequency and the associated choice of $\nabla$ therefore corresponds to assuming that the roller is a sum of coupled bound waves, moving with the same phase speed.

In the calculations presented, the latter of these possibilities was used. We have already seen that $H$ is a very monotonous function, and the influence of the above choice is therefore expected to be rather small.

### 6.6.2 Inclusion of mean water variation

Inside the surf zone, the setup can be rather large compared to the wave amplitudes themselves, see e.g. Figure 4.5. Therefore it is chosen to incorporate mean water variations into the model.

Utilizing that $\bar{W}_t = 0$, the time averaged version of (6.81) can be written

$$\nabla \text{Tan}(h_0 \nabla)(-g\bar{\eta} - \varepsilon \bar{N}_1) = \text{Sec}(h_0 \nabla)\{ \delta \left( \text{Cos}(h_0 \nabla)(g \nabla \bar{\eta} + \varepsilon \nabla \bar{N}_1) \right) \}.$$  \hspace{1cm} (6.83)

We assume that $\bar{\eta}$ is slowly varying in space, thus $\nabla \bar{\eta} = O(\varepsilon \bar{\eta})$. This allows us to expand the infinite series operators and retain only the first term in them. We hereby get

$$-gh_0 \nabla^2 \bar{\eta} - h_0 \nabla^2 \bar{N}_1 = \nabla \{ \delta (g \nabla \bar{\eta} + \nabla \bar{N}_1) \}.$$  \hspace{1cm} (6.84)

Next we set $h_0 = h(x)$, implying $\nabla \delta = \nabla h$ and $\delta = 0$. This corresponds to setting the reference depth equal to the water depth, locally, as described on page 160. From the right hand side we thereby get two terms having $\nabla h$ as a factor, and we can combine them with the terms at the left hand side as

$$-g (h\bar{\eta}_x)_x - (h \bar{N}_{1x})_x = 0$$  \hspace{1cm} (6.85)

which can be integrated to the results

$$\bar{\eta}_x = -\frac{1}{g} \bar{N}_{1x} + c_1$$  \hspace{1cm} (6.86)

$$\bar{\eta} = -\frac{1}{g} \bar{N}_1 + c_1 x + c_2$$  \hspace{1cm} (6.87)

where $(c_1, c_2) \in \mathbb{R}$. The latter of these equations can also be derived directly from the dynamic free surface condition. Doing so, one obtains $c_1 = 0$. This derivation is given in
Chapter 7. We now calculate $\bar{N}_1$. In (6.79) we insert the expansion (6.27), and retain only the lowest-order terms, thus expressing $\hat{\varphi}_{p,x}$ in terms of the free wave number $k_p$. We obtain

$$\bar{N}_1 = \left\langle \frac{1}{2} \Phi^2_x - \frac{1}{2g^2} \Phi^2_{tt} \right\rangle$$

$$= \frac{1}{2} \sum_{p=-\infty}^{\infty} \left( k_p^2 - \frac{\omega_p^2}{g^2} \right) \hat{\varphi}_p \hat{\varphi}_p^* = \sum_{p=1}^{N} \left( k_p^2 - \frac{\omega_p^4}{g^2} \right) |\hat{\varphi}_p|^2 = \sum_{p=1}^{N} \frac{k_p^2}{\cosh^2 k_p h} |\hat{\varphi}_p|^2$$

where the last result was obtained using the linear dispersion relation $\omega_p^2 = gk_p \tanh k_p h$.

We now utilize the linear part of (6.44) to rewrite the above result to the amplitudes of $\eta$.

Utilizing the dispersion relation once again, we obtain

$$\bar{N}_1 = \sum_{p=1}^{N} \frac{k_p}{\sinh k_p h \cosh k_p h} |\hat{\eta}_p|^2 = \sum_{p=1}^{N} \frac{g}{h} G_p |\hat{\eta}_p|^2$$

with

$$G_p = \frac{2k_p h}{\sinh 2k_p h}.$$  

Using (6.87), we hereby get

$$\bar{\eta} = -\frac{1}{g} \bar{N}_1 = -\sum_{p=1}^{N} \frac{G_p}{h} |\hat{\eta}_p|^2$$

which for a single wave reduces to the classical result

$$\bar{\eta} = -\frac{1}{16} G \frac{H^2}{h}$$

see e.g. Mei (1983) Section 10.5.

With the knowledge $c_1 = 0$, (6.86) provides a differential equation for the mean water level variation due to pure wave motion. From the Boussinesq model, we know that the inclusion of the roller breaking model results in an extra term $\bar{R}_x/gh$, added to the right hand side, see (4.25). Utilizing (6.89) we therefore have

$$\bar{\eta}_x = -\left[ \sum_{p=1}^{N} \frac{1}{h} G_p |\hat{\eta}_p|^2 \right]_x - \frac{1}{gh} \bar{R}_x,$$

which can be integrated along with the wave amplitudes, to obtain the mean water level variation.
Feedback of the setup to the wave field  In the absence of breaking, the deviation of the mean water level from $z = 0$ is of second order. This can be seen from (6.91). Hence within the second-order accuracy of the model, the feedback of the set-down on the wave field can be consistently neglected. However, as the setup induced by breaking can be of the same order of magnitude as the wave amplitudes themselves, it is reasonable to include the influence of the wave field explicitly in the model. As discussed in Section 4.4, the setup can be included in the model in two ways. One way is to include $\hat{\eta}_0 = \bar{\eta}$ in the summations in the nonlinear terms. For the Boussinesq evolution equations of Chapter 4 this required a modification of the interaction coefficient, since the rewriting $\hat{P}_p = (\omega_p/k_p)\hat{\eta}_p$ is not valid for $p = 0$. Similarly, care must be taken in the fully dispersive model of this section. The value of $\hat{\phi}_0$ cannot be replaced by $\hat{\eta}_0$ using the transformation (6.44). Also note that $\hat{\phi}_{0,t} \neq 0$, but is given by the dynamic free surface condition: $\hat{\phi}_{0,t} = -g\bar{\eta} + O(\varepsilon)$. To derive the interaction coefficient describing the interaction of the harmonic amplitudes with the harmonic wave components, one must thus go through the derivation of the model, taking care of these specialties.

An alternative way is simply to change the local depth in each spatial position, i.e. setting $h := h_{\text{still water}} + \bar{\eta}$. In this way the change of depth due to the set-down or set-up is taken into account explicitly, affecting the wave numbers, and all depth-dependent coefficients of the model.

In Chapter 4, the first of these approaches was taken since it is the closest approach to the time domain formulation. In this case, for the fully dispersive wave model, we do not aim at matching a time domain model. We therefore choose the latter approach. This further has the advantage that the waves then can be followed further up the beach than just to the still water intersection point.

In contrast to the Boussinesq evolution equations of Chapter 4, we note that the fully dispersive evolution equations do not automatically have a zero net mass flux. Outside the breaking zone, the feedback of the net mass flux on the wave field is beyond the quadratic accuracy of the model, but inside the surf zone the presence of the surface rollers will generate a return current. An additional equation for the net mass flux taking the presence of surface rollers into account is therefore a recommended extension of the model. In the present study, however, such an extension has not been pursued.

6.7 Results of the breaking model

In this section results of the roller breaking model within fully dispersive evolution equations are presented. The wave model used is the model formulated in $\eta$, (6.47), together with (6.93) for the mean water level variation. Like in Chapter 4, we use the results of the time domain Boussinesq model as reference.
6.7.1 Results for default parameters

Within the time domain Boussinesq model, the parameters $\phi_B = 20^\circ$, $\phi_0 = 10^\circ$, $t_{1/2} = T/5$ and $f_\delta = 1.5$ were used. These are the recommended values for this model. In Figure 6.19, the wave envelope of the fully dispersive model using these parameters are shown. Further, $c_{\text{break}} = 1.3\sqrt{gh}$ was used. The equations were solved for 7 harmonics using the exact kernel. The equations were integrated using the integration scheme of Scraton (1964), described in Appendix C. The grid spacing was $dx = 0.02$ m. The incoming wave height was set to $H = 0.123$ m, and the boundary condition was determined by solving a system of fully nonlinear equations, similar to the approach of Section 5.3. The figure shows that the wave of the fully dispersive model is more skew than that of the time domain Boussinesq model. At the flat section, the mean water level is chosen to be identical for the two models, while the crest and trough level of the fully dispersive model are lifted in comparison with the time domain model. The decay of the crest elevation is seen to be too slow, when comparing to the Boussinesq model.

Figure 6.19: Envelope for the spilling breaker test of Ting and Kirby (1994). Roller model with default parameters and time domain model.

6.7.2 Results for adjusted parameters

To obtain a better match with the wave envelope of the time domain model, the parameters $\phi_B = 19^\circ$, $\phi_0 = 6^\circ$ were chosen. All other parameters were chosen identically to the above test. The wave envelope of the new run is depicted in Figure 6.20. The decay of the crest
height is still slower than the decay of the time domain model, but matches the overall variation better. The mean water level variation is very close to the results of the time domain model, while the trough envelope is lifted a little. As the initial crest elevation is larger than for the time domain model, and the wave height at the break point is slightly smaller, we see that the shoaling is less strong than for the time domain model.

The spatial wave profiles are depicted in Figure 6.21. First we note that the amplitude dispersion is over-predicted in the model, agreeing with the expectation of the weakly nonlinear analysis. The waves have broader crests than the time domain model. As wave breaking initiates, the crest point seems to be moving backward within the profile, thus producing a crest being asymmetric in the opposite direction (leaning away from the beach) than expected. Although the crest is not resembling the triangular shape of the time domain model, the backs of the breakers do have an upward slope, resembling some of the overall shape of the time domain results. A zoom in of Figure 6.21 is provided in Figure 6.22. Here the local ‘opposite asymmetry’ of the wave crest is clearly seen. This shape of the crest explains the need for adjusting $\phi_0$ to $6^\circ$ rather than using the value of $10^\circ$ of the time domain model.

The skewness and asymmetry are plotted in Figure 6.23. As anticipated by observing the wave envelopes, the skewness of the incident wave is larger than for the time domain model. The overall variation of skewness and asymmetry follows the time domain results until $x=17$, which is shortly after the initiation of wave breaking. Further towards the beach, the skewness and asymmetry are not well modelled.

Time series of the fully dispersive model are shown in Figure 6.24 for six stations evenly spaced in the interval $x = 16.7; 21$ m. The plots confirm the above conclusions, that the
Figure 6.21: Profiles in space for roller model with adjusted parameters. Comparison with time domain model.

Figure 6.22: Profiles in space for roller model with adjusted parameters. Comparison with time domain model. Zoom of Figure 6.21.
wave of the fully dispersive model have a broader crest, with the highest point positioned towards the back of the profile. The back of the wave does show some asymmetry, even though not resembling the almost straight, down-sloping back of the time domain waves.

### 6.7.3 Influence of resonance assumption

The influence of the resonance assumption was examined by carrying out a simulation identical to the above, but using the resonant kernel. Time series of this run, as well as the above run are shown in Figure 6.25. There is almost no difference between the two sets of results. For the flat section of the bathymetry, at the front of the sloping beach, we have $\Omega = \omega \sqrt{h/g}/(2\pi) = 0.1$. For this dimensionless frequency, the difference in transfer functions of the two kernels is not large, see Figure 6.4. Hence, the small difference between the results of the two kernels agrees with the expectation.

### 6.7.4 Comparison to Boussinesq evolution equations

Having compared the results of the fully dispersive model to the time domain Boussinesq formulation, it is interesting to compare it to the Boussinesq evolution equations, as used in Chapter 4. We use the simulation of Section 4.5.3 and compare it to the above simulation with the exact kernel.
Figure 6.24: Time series of roller model with adjusted parameters and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
Figure 6.25: Time series of roller model with adjusted parameters, roller model with resonant kernel and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
The wave envelopes are shown in Figure 6.26. The decay rate of the fully dispersive model is larger for the fully dispersive model short after breaking, and smaller for $x \geq 19$ m, compared to the Boussinesq evolution equations.

Figure 6.26: Envelope for the spilling breaker test of Ting and Kirby (1994). Roller model with adjusted parameters. Comparison to Boussinesq roller model (frequency domain) and time domain model.

Wave profiles of the models inside the surf zone are shown in Figure 6.27. The figure shows that the amplitude dispersion is strongest for the Boussinesq frequency domain model. The weakly nonlinear analysis shows that for $\omega \sqrt{h/g} \lesssim 0.5$ the amplitude dispersion is stronger for the Boussinesq evolution equations. Since for the deepest part of the domain $\omega \sqrt{h/g} = 0.6$, the results thus agree with the expectation from the analysis. The special shape of the crest for the fully dispersive model, the crest point leaning backwards in the profile, is not found for the Boussinesq frequency domain model. For $x \geq 21$ m, the crest of the fully dispersive wave model is almost double peaked. This is not the case for the Boussinesq waves generated in the frequency domain. The above findings are confirmed by comparing plots of the time series. These plots are shown in Appendix I, Figure (I.1).

### 6.7.5 Comparison to conventional breaking formulations

We now compare the results of the roller model in fully dispersive evolution equations to results of a conventional bulk dissipation model. The dissipation model used, is the same as described in Section 4.1, i.e., the dissipation model of Battjes and Janssen (1978) with uniform or $f^2$-weighted energy dissipation. We first present results for uniform weighting.
6.7 Results of the breaking model

Figure 6.27: Profiles in space for roller model with adjusted parameters. Comparison with Boussinesq roller model (frequency domain) and time domain model.

Results for uniformly weighted energy dissipation  In Figure 6.28, wave envelopes for the fully dispersive model with the conventional breaking formulation, uniformly weighted, are compared to the results of the roller model for fully dispersive evolution equations and the time domain Boussinesq model. The parameters $\alpha = 1.0, \gamma = 0.8$ and $F = 1$ were used. The decay of the crest elevation is seen to be a little weaker for the conventional breaking formulation compared to the roller model, and the setup is also over-predicted. The latter deviation is expected to be due to the absence of the $R_x$ term in (6.93), when using the conventional dissipation formulation. The spatial wave profiles are shown in Figure 6.29. The wave shape obtained by the conventional breaking formulation almost produces a double peaked crest. This lack of asymmetry is confirmed by the time series, which are shown in Appendix I, Figure I.2. In this appendix, the skewness and asymmetry are shown as well, Figure I.3. After initiation of breaking, the asymmetry is completely off, showing even positive values.

Returning to Figure 6.29, the amplitude dispersion is seen to be weaker for the conventional breaking model. This is expected to be due to the smaller wave height through the surf zone (due to the higher trough elevation).

Results for $f^2$-weighted breaking Next, we present results for $f^2$-weighted dissipation. The simulation here shown was carried out with the parameters $\alpha = 0.7, \gamma = 0.7$ and $F = 0$, thus weighting all of the energy dissipation according to the squared frequency. The wave envelopes are shown in Figure 6.30. The wave height obtained is smaller than the one for the roller model. This particular run was chosen, since results for weaker dissipation were...
Figure 6.28: Wave envelopes for conventional breaking with uniform dissipation \((F = 1)\), roller breaking and time domain model.

Figure 6.29: Spatial profiles in the breaking zone for conventional breaking with uniform dissipation \((F = 1)\). Comparison with frequency domain roller breaking and time domain model.
found to overestimate the wave height for $x \geq 21$. The trough elevation and the mean water level variation match the results of the roller model better than for uniform breaking, see Figure 6.28.

![Wave envelopes for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking and time domain model.](image)

Figure 6.30: Wave envelopes for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking and time domain model.

The spatial wave profiles of the models are compared in Figure 6.31. The results for $f^2$-weighted energy dissipation still produce a broad crest, having two distinct maxima in the wave profile. The amplitude dispersion is weaker, again expectedly due to the smaller wave height, when comparing to the results of the roller model.

Time series of the models are compared in Figure 6.32. For $x \geq 17.56$ m, the broad shape of the crest is clearly seen. However, the back of the profile follows the straight shape of the time domain model rather well. All in all, the time series of the $f^2$-weighted bulk dissipation model is judged to match the time domain Boussinesq results better than the roller model in fully dispersive evolution equations.

This conclusion also holds for the skewness and asymmetry, depicted in Figure 6.33. These measures are reproduced qualitatively well until $x \approx 19$ m — about 2 meters longer than for the roller model.
Figure 6.31: Spatial profiles in the breaking zone for conventional breaking with $f^2$-weighted dissipation ($F = 0$). Comparison with time domain model.
6.7 Results of the breaking model

![Graphs showing time series for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking, and time domain model. The time series are aligned horizontally, so the crest is always in $t = 1$ s.]

Figure 6.32: Time series for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking, and time domain model. The time series are aligned horizontally, so the crest is always in $t = 1$ s.
Figure 6.33: Skewness and asymmetry for conventional breaking with $f^2$-weighted dissipation ($F = 0$), roller breaking and time domain model.
6.8 Summary and conclusions

In this chapter a new derivation of fully dispersive quadratic evolution equations has been presented. This derivation does not require an assumption for the vertical structure of the flow. The equations derived possess exact second-order bichromatic transfer functions, when comparing to Stokes wave theory. Models formulated in the Fourier amplitudes of the still water potential as well as the free surface elevation have been derived. In the nonlinear terms, an operator $H(h\nabla, kp_h)$ appears. In the exact models, $\nabla$ is chosen as $\nabla = -i(k_s + kp)$, corresponding to a linear approximation for each pair of interacting waves. The choice $\nabla = -ik_p$ corresponds to assuming that the forcing of each pair of interacting waves is resonant with the receiving wave mode. Utilizing this assumption, the models of Agnon et al. (1993), Kaihatu and Kirby (1995) and Eldeberky and Madsen (1999) are recovered.

The influence of the resonance assumption has been examined in terms of the second-order transfer functions. The resonant model has almost perfect super-harmonic transfer, while the sub-harmonic transfer functions are over-predicted for the longest waves, and under-predicted in a band around $\omega \sqrt{h/g}/2\pi = 0.18$ for the generated wave.

The use of FFT for speeding up the calculation of the nonlinear terms can be applied to the resonant models. For the exact models, the trick cannot be applied directly, due to the form of the interaction coefficient. However, as the resonant kernel produces acceptable second-order transfer, an approximation to the exact kernel can be made, based on the resonant kernel. A first idea of such an approximation has been presented, being closer to the exact kernel than the resonant kernel.

Amplitude dispersion within fully dispersive models has been investigated using a weakly nonlinear perturbation approach. It is found that fully dispersive models in $\eta$ over-predict the amplitude dispersion with as much as a factor of 5, when Stokes waves with a zero mass flux are used as reference solution. For a zero Eulerian velocity below wave trough level, the overestimation is a factor 2 at maximum. The evolution equations in $\phi$ over-predict the amplitude dispersion for shallow and intermediate water depth, while the amplitude dispersion decay towards zero in deep water.

The roller breaking scheme has been incorporated into the fully dispersive evolution equations in $\eta$, and an equation for the variation of the mean water level has been derived. The generation of a return current due to the surface rollers should in principle be included in the model as well, but such an extension has not been made. The model is able to describe the wave height decay and setup satisfactorily. However, the shape of the breaking waves is not well modelled. The highest point of the crest is positioned towards the back of the profile, almost producing a double-peaked crest. This enforces the need for choosing $\phi_0$ as low as 6°, due to the poor steepness of the wave front. A comparison with a conventional bulk energy dissipation model has been made. When the dissipation is uniformly distributed among the frequencies, the shape of the breaking waves is almost symmetric, thus giving poorer results than for the roller breaking scheme. For an $f^2$-weighting of the dissipation,
the shape of the breaking waves improves and becomes better than for the roller breaking model.

The results of the roller model in fully dispersive evolution equations have also been compared to the results of the evolution equations of Madsen and Sørensen (1993), as described in Chapter 4. The results of the Boussinesq based model lacks asymmetry as well, but not to the same extent as for the fully dispersive model. One reason for this may be the difference in the second-order and third-order transfer functions of the two models. While the second-order and third-order transfer of the Boussinesq based evolution equations are smaller than for Stokes waves, the second-order transfer for the fully dispersive equations matches Stokes theory, and the third-order transfer is over-predicted. This leads to differences in the profile shape.

Fully dispersive wave models should therefore not be used due to a good prediction of the profile shape in the surf zone. The motivation for using them should rather be their accurate linear and second-order characteristics spanning from deep to shallow water. Further, at least for the wave parameters of the Ting and Kirby (1994) test, the roller breaking scheme does not lead to better results than a conventional bulk energy dissipation model with $f^2$-weighted dissipation among the frequencies.
Chapter 7

Fully Dispersive Models, 3rd Order

The scope the three preceding chapters has been the description of breaking waves in the surf zone. The fully dispersive models of Chapter 6, have exact first-order and second-order characteristics for all water depths, and can therefore model wave propagation on deep water as well. On deep water, however, triad interactions do not occur due to the frequency dispersion. The quadratic nonlinearity produces a bound second-order wave field, but there is no resonant interaction between the first-order and second-order wave fields. At the next order however, the cubic interaction may become resonant. In one horizontal dimension an example of such a resonance is the Benjamin-Feir instability, where small amplitudes at the side-bands of a primary wave are amplified. Another familiar example is the amplitude dispersion of a Stokes wave, where the cubic interaction leads to a modulation of the angular frequency.

In this chapter the fully dispersive evolution equations of Chapter 6 are extended to third order. We only treat the equations formulated in $\phi$, since these contain fewer nonlinear terms than the corresponding equations in $\eta$. Similar to the derivation of the quadratic evolution equations, we allow for a mildly sloping sea bed.

The model derived is able to describe the shoaling of a wave spectrum including cubic effects. The model is deterministic and no assumptions on the spectral width is made. The model can therefore be seen as a spatial version of the Zakharov equation, valid for a mildly sloping bottom. Recently, Shemer et al. (2001) presented a spatial version of the Zakharov model for constant depth. This model was derived by transforming the temporal Zakharov model.

The model of this chapter is derived directly from the governing hydrodynamic equations.

The chapter is divided as follows. First, in Section 7.1, the governing equations are presented and the quadratic model of Chapter 6 extended to cubic order. As for the quadratic model, the new model incorporates infinite series operators, operating on products of $\phi$. To be able to express the arguments of the operators by free wave numbers, it is necessary to express all terms in the model by the linear wave amplitudes. This is the objective of Section 7.2. In Section 7.3, two versions of the resulting model are presented, a so-called reduced model
and an unreduced model. To cubic order, the feedback of the mean flow to the wave field must be taken into account. This is described in Section 7.4. In Section 7.5, a comparison of the reduced model to the Nonlinear Schrödinger Equation for constant depth is given. For a spectrum consisting of just a single wave, the reduced model is identical to the Nonlinear Schrödinger Equation. In Section 7.6, the reduced model is used for simulating the spatial evolution of wave group propagation on constant depth, following Shemer et al. (2001). Unfortunately, the results do not match the experimental results. The reason is believed to be an error in the numerical implementation. The findings of the chapter are summarized in Section 7.7.

7.1 The governing equations

The derivation of the cubic model follows the same principles as the derivation of the quadratic model of Section 6.2. A sketch of the physical system is given in Figure 7.1, repeated from Figure 6.1, but for just one horizontal dimension. The only difference to the derivation of Chapter 6 is that we now go to third order in the nonlinear terms, and thus retain all cubic terms in the model.

We restrict the derivation to one horizontal dimension. With the convention $\nabla \equiv \frac{\partial}{\partial x}$, the governing equations are therefore given by (6.4)–(6.7). Again we define the nonlinearity parameter $\varepsilon = k_0 a_0$, where $(k_0, a_0)$ are a reference wave number and amplitude, respectively. As for the derivation in Section 6.2, we work in the dimensional equations, with $\varepsilon$ artificially multiplied on the nonlinear terms. Hence, in the equations, $\varepsilon$ should be regarded as a small number, although its formal value is unity. For the bottom slope, we assume that $h = h(\varepsilon x)$, equivalent to a mild-slope assumption. This implies that the first-order bottom-slope terms are formally at the same order as the quadratic nonlinear terms. The cubic terms are of magnitude $O(\varepsilon^2)$, and it would therefore be consistent to retain terms of type $h_{xx}, h_x^2$ having.

Figure 7.1: Definition sketch for model derivation.
7.1 The governing equations

a magnitude of \( O(\varepsilon^2) \). However, in this derivation, these terms are left out for the sake of simplicity. We note that since these terms appear at the highest order, they can be added to the resulting equations of the present derivation, without modifying the terms derived here.

Following the derivation of Section 6.2, the basis of the derivation of the cubic model is the infinite series solution of the Laplace equation, combined with the kinematic sea bed condition, here repeated from (6.14):

\[
\sin(h_0 \nabla) \nabla \phi + \cos(h_0 \nabla) W = -\nabla \{ \delta(\cos(h_0 \nabla) \nabla \phi - \sin(h_0 \nabla) W) \}. \tag{7.1}
\]

The kinematic and dynamic free surface conditions are Taylor expanded around \( z = 0 \). By retaining cubic terms we obtain

\[
\begin{align*}
\eta_t - \Phi_z + \varepsilon(\eta \Phi_{xx} + \eta_x \Phi_z) + \varepsilon^2(\eta^2 \Phi_{xxx}/2 + \eta \eta_x \Phi_{xz}) &= O(\varepsilon^3) \tag{7.2} \\
g\eta + \phi_t + \varepsilon(\Phi_z^2/2 + \Phi_z^2/2 + \eta \Phi_{zt}) + \varepsilon^3(\eta \Phi_z \Phi_{xx} - \eta \Phi_z \Phi_z - \eta^2 \Phi_{xxt}/2) &= O(\varepsilon^3) \tag{7.3}
\end{align*}
\]

Here we have used the Laplace equation to eliminate higher-order derivatives of \( \Phi \) with respect to \( z \). The next steps towards a closed expression for \( W = \Phi_z \) are to eliminate \( \Phi_z \) from the nonlinear terms and \( \eta \) from both of the equations. These manipulations were carried out in a Mathematica sheet, and the results are given in the following.

**Eliminating** \( \Phi_z \) The kinematic boundary condition (7.2) yields to lowest order \( \Phi_z = \eta_t + O(\varepsilon) \). Substitution of this into the cubic terms gives

\[
\begin{align*}
\eta_t - \Phi_z + \varepsilon(\eta \Phi_{xx} + \eta_x \Phi_z) + \varepsilon^2(\eta^2 \eta_{xxt}/2 + \eta \eta_x \eta_{xzt}) &= O(\varepsilon^3). \tag{7.4}
\end{align*}
\]

This cubic approximation of \( \Phi_z \) is substituted into the dynamic free surface condition giving

\[
\begin{align*}
g\eta + \phi_t + \varepsilon(\Phi_z^2/2 + \eta_t^2/2 + \eta \eta_t) + \varepsilon^2(\eta \eta_x \Phi_z + 2 \eta \eta_x \Phi_x + \eta \eta_x \Phi_{xt} + \eta \eta_x \Phi_{xx} + \eta^2 \Phi_{xxt}/2) &= O(\varepsilon^3). \tag{7.5}
\end{align*}
\]

**Eliminating** \( \eta \) We now eliminate \( \eta \) from the surface boundary conditions. First we establish an explicit expression for \( \eta \) from the dynamic boundary condition (7.5). To lowest order, we have \( g\eta + \Phi_t = O(\varepsilon) \) which is substituted into the quadratic terms of (7.5). The resulting equation relates \( \eta \) to \( \Phi \) with an error of \( O(\varepsilon^2) \), and can therefore be substituted into the nonlinear terms of (7.5) with cubic accuracy. The resulting equation is

\[
\begin{align*}
g\eta + \phi_t + \varepsilon \left( \frac{\Phi_{tt}^2}{2g^2} + \frac{\Phi_{tt} \Phi_{ttt}}{g^2} + \frac{\Phi_{x}^2}{2} \right) + \varepsilon^2 \left( \frac{5\Phi_{tt} \Phi_{ttt}}{2g^4} + \frac{3\Phi_{tt} \Phi_{ttt}^2}{g^4} + \frac{4\Phi_{tt} \Phi_{ttt} \Phi_{tttt}}{g^4} \right) + \frac{3\Phi_{x} \Phi_{xtt}}{g^2} + \frac{3\Phi_{x} \Phi_{xxt}}{g^2} + \frac{\Phi_{t} \Phi_{x} \Phi_{xx}}{g^2} + \frac{\Phi_{t} \Phi_{x} \Phi_{xxt}}{2g^2} \right) &= O(\varepsilon^3). \tag{7.6}
\end{align*}
\]
With this result we can eliminate \( \eta \) from the kinematic boundary condition (7.2). The resulting equation is

\[
- \Phi_z - \Phi_t / g + \epsilon \left[ -\frac{2 \Phi_t \Phi_{tt}}{g^3} - \frac{\Phi_t \Phi_{uu}}{g^3} - \frac{2 \Phi_x \Phi_{xt}}{g} - \frac{\Phi_t \Phi_x}{g} \right] \\
+ \epsilon^2 \left[ -\frac{8 \Phi_t \Phi_{tt}^2}{g^5} - \frac{13 \Phi_t \Phi_{tt} \Phi_{uu}}{g^5} - \frac{10 \Phi_t \Phi_{tt} \Phi_{ttt}}{g^5} - \frac{6 \Phi_t \Phi_{tt} \Phi_{uuu}}{g^5} - \frac{2 \Phi_t \Phi_{uuu}}{g^5} \\
- \frac{\Phi_{uuu} \Phi_x^2}{2 g^3} - \frac{4 \Phi_{tt \Phi_x} \Phi_{xt}}{g^3} - \frac{4 \Phi_{tt} \Phi_{xt}^2}{g^3} - \frac{4 \Phi_{tt} \Phi_x \Phi_{xtt}}{g^3} - \frac{\Phi_t \Phi_{tt} \Phi_{xxx}}{2 g^3} - \frac{\Phi_t \Phi_{tt} \Phi_{xxt}}{2 g^3} - \frac{\Phi_t \Phi_{tt} \Phi_{xxx}}{2 g^3} \\
- \frac{3 \Phi_x^2 \Phi_{xx}}{2 g} - \frac{2 \Phi_t \Phi_t \Phi_{xx}}{g^3} - \frac{\Phi_t \Phi_{tt} \Phi_{xxx}}{g^3} \right] = O(\epsilon^3), \quad (7.7)
\]

expressing \( \Phi_z \) from \( x \) and \( t \) derivatives of \( \Phi \) with cubic accuracy. Note, that no assumptions about the vertical variation of the velocity field have been made to derive this result, except for the approximations involved with the perturbation approach.

**Rewriting to symmetric expressions**  As in Section 6.2, we rewrite the above expressions to symmetric forms, such that in the quadratic products, the interacting frequency components appear symmetrically, while in the cubic products two of the three interacting frequency components appear in symmetric form. Further, products involving time derivatives are rewritten in a large extent to achieve terms that as a whole are time derivatives. The symmetry is an advantage in the numerical implementation of the equations, since the summation ranges of the convolution sums can then be reduced. Terms being time derivatives are appropriate for treating the mean flow, since they cancel out when averaging the equations over one wave period. The rewritten forms of the above equations are

\[
g\eta + \Phi_t + \epsilon \left[ \frac{1}{2} \Phi_x^2 + \frac{1}{2 g^2} (\Phi_t^2)_{tt} - \frac{1}{2 g^2} \Phi_t^2 \right] + \epsilon^2 \left[ \frac{1}{2 g^4} (\Phi_t^2 \Phi_{tt})_t - \frac{5}{6 g^4} (\Phi_t^3)_t \right] \\
+ \frac{1}{g^4} (\Phi_t \Phi_t^3)_{tt} + \frac{1}{g^4} (\Phi_t^2 \Phi_{ttt})_t - \frac{2}{g^2} (\Phi_x^2 \Phi_x^2)_t + \frac{3}{2 g^2} (\Phi_x^2 \Phi_t)_t + \frac{1}{g^2} \Phi_x^2 \Phi_t + \frac{1}{g^2} \Phi_x^2 \Phi_{tt} + \frac{1}{g^2} \Phi_x^2 \Phi_t \right] = O(\epsilon^3) \quad (7.8)
\]
The governing equations

7.1 The governing equations

7.1.1 Transforming to the frequency domain

We now transform the two above expressions to the frequency domain. We insert the expansions (6.26) and (6.27) to obtain the third-order result

\[
\hat{\phi}_{p,z} = \frac{\omega^2}{g} + \varepsilon \sum_{s=-N}^{N} \tilde{\tau}_{s,p-s}^{(2)} \hat{\phi}_s \hat{\phi}_{p-s} + \varepsilon^2 \sum_{s=-N}^{N} \sum_{t_{1}}^{t_{0}} \tilde{\tau}_{s,t,p-s-t}^{(3)} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t}
\]

(7.10)

with

\[
\tilde{\tau}_{s,p-s}^{(2)} = -\frac{1}{2g^3} \partial_{s,t} \partial_{p-s,t} \omega^2 + \frac{1}{2g^3} \partial_{s,t} \partial_{p-s,t} \partial_t - \frac{1}{g} \nabla_s \nabla_{p-s} \partial_t - \frac{1}{2g^2} \nabla_{p-s} \partial_t - \frac{1}{2g^2} \nabla_{p-s} \partial_s
\]

(7.11)

and

\[
\tilde{\tau}_{s,t,p-s-t}^{(3)} = \frac{1}{2g^3} \omega_s \omega_l \omega_p \omega^3_{s-t} \omega_p - \frac{5}{2g^5} \omega_s^2 \omega_l^2 \omega^3_{s-t} \omega_p - \frac{3}{g^3} \omega_s^2 \omega_l^3 \omega_{s-t} \omega_p
\]

\[
- \frac{2}{g^5} \omega_s^3 \omega_l \omega_p^4 \omega^2_{s-t} \omega_p - \frac{2}{g^5} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p - \frac{1}{g^4} \omega_s \omega_l^2 \partial_{s-t} \omega_p - \frac{3}{g^3} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p
\]

\[
+ \frac{2}{g^3} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p + \frac{1}{g^3} \partial_{s,t} \partial_{s-t} \omega_p - \frac{3}{g^3} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p
\]

\[
+ \frac{2}{g^3} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p + \frac{1}{g^3} \partial_{s,t} \partial_{s-t} \omega_p - \frac{3}{g^3} \partial_{s,t} \omega_s^3 \omega_l \omega_p^2 \omega^2_{s-t} \omega_p
\]

(7.12)

If one inserts \( \partial_{j,t} = i \omega_j \) and \( \nabla_j = -ik_j \) into \( \tilde{\tau}_{s,p-s}^{(2)} \), it reduces to \( \tilde{\tau}_{s,p-s}^{(2)} \) as given in (6.30). The reason that the above form is used, is that \( \nabla \hat{\phi}_p = -ik_p \hat{\phi}_p + O(\varepsilon) \). When retaining
cubic accuracy, the error term in this approximation cannot be neglected. In (7.10), the summation ranges for \( t \) in the cubic sum are not written explicitly. The limits are functions of \((p, s, N)\) and are given in Appendix J. For simplicity, no summation ranges in the sums are given in the following. For the quadratic sums, the ranges are always \((p - N, \ldots, N)\) as described in Appendix A, while for the cubic sums the reader is referred to Appendix J.

Insertion of (7.10) into the Fourier transformed sea bed condition, (7.1), gives

\[
\sin(h_0 \nabla) \nabla \hat{\phi}_p + \cos(h_0 \nabla) \left[ \frac{\omega_p^2}{g} \hat{\phi}_p + \varepsilon \sum_s \hat{\tilde{f}}^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s} + \varepsilon^2 \sum_s \sum_t \hat{\tilde{f}}^{(3)}_{s,t,p-s-t} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t} \right]
= -\nabla \left\{ \delta \left( \cos(h_0 \nabla) \nabla \hat{\phi}_p - \sin(h_0 \nabla) \left[ \frac{\omega_p^2}{g} \hat{\phi}_p + \varepsilon \sum_{s=p-N}^N \hat{\tilde{f}}^{(2)} s \hat{\phi}_p \right] \right) \right\}. \tag{7.13}
\]

At the right-hand side, only the nonlinear terms related to \( \tilde{f}^2 \) have been retained, since the terms involving \( \delta \) become bottom-slope terms of magnitude \( O(\varepsilon) \) at the end of the derivation. Operation on the above equation with \( \sec(h_0 \nabla) \) yields

\[
\left( \nabla \tan(h_0 \nabla) + \frac{\omega_p^2}{g} \right) \hat{\phi}_p = -\sec(h_0 \nabla) \nabla \delta \left\{ \left( \cos(h_0 \nabla) \nabla - \frac{\omega_p^2}{g} \sin(h_0 \nabla) \right) \hat{\phi}_p \right\} + \sec(h_0 \nabla) \nabla \delta \sin(h_0 \nabla) \sum_s \hat{\tilde{f}}^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s}
- \varepsilon \sum_s \hat{\tilde{f}}^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s} - \varepsilon^2 \sum_s \sum_t \hat{\tilde{f}}^{(3)}_{s,t,p-s-t} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t}. \tag{7.14}
\]

The operator on the left-hand side is the dispersion operator. We split this operator in the same way as in Section 6.2.5. Thus, operation on both sides of the above equation with \( H(h_0 \nabla, k_p h_0) \) yields

\[
\left( \frac{\partial}{\partial x} + ik_p \right) \hat{\phi}_p = -H(h_0 \nabla, k_p h_0) \sec(h_0 \nabla) \nabla \delta \left\{ \left( \cos(h_0 \nabla) \nabla - \frac{\omega_p^2}{g} \sin(h_0 \nabla) \right) \hat{\phi}_p \right\}
+ \varepsilon \sum_s H(h_0 \nabla, k_p h_0) \sec(h_0 \nabla) \nabla \delta \sin(h_0 \nabla) \hat{\tilde{f}}^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s}
- \varepsilon \sum_s H(h_0 \nabla, k_p h_0) \hat{\tilde{f}}^{(2)}_{s,p-s} \hat{\phi}_s \hat{\phi}_{p-s}
- \varepsilon^2 \sum_s \sum_t H(h_0 \nabla, k_p h_0) \hat{\tilde{f}}^{(3)}_{s,t,p-s-t} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t}. \tag{7.15}
\]

For each harmonic amplitude \( \hat{\phi}_p \), this is a first-order ODE. The term in the first line of the right-hand side represents linear shoaling, while the term in the second line represents the interaction of the bound quadratic waves with the sloping sea bed. The third and fourth line represent quadratic and cubic interactions. The first-order form of the operator on the left-hand side allows for solving for the wave amplitudes by integrating in the \( x \)-direction.
7.2 Expanding $\hat{\phi}_p$

However, since the differential operators are infinite, the equation cannot be applied directly. In the following sections, we shall replace the infinite operators with local approximations, thus expressing all the $\nabla$’s by local free wave numbers. This step results in a model suited for practical application.

7.2 Expanding $\hat{\phi}_p$

The equation just derived involves infinite operators in $\nabla$. We now expand $\phi$ in order to express the right hand side of the equation by simple free wave modes for which $\nabla$ can be expressed by local wave numbers. We furthermore invoke the mild-slope approximation, thus only retaining first-order derivatives of $\delta$.

We write (7.15) as

$$
\left(\frac{\partial}{\partial x} + ik_p\right)\hat{\phi}_p = - OP_1 \delta OP_2 \hat{\phi}_p
+ \varepsilon \sum_s OP_3 \delta OP_{3,s,p-s} \hat{\phi}_s \hat{\phi}_{p-s}
+ \varepsilon \sum_s OP_4 \delta \hat{\phi}_s \hat{\phi}_{p-s}
- \varepsilon^2 \sum_s \sum_t OP_5 \delta \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t}
$$

(7.16)

with

$$
OP_1(h_0 \nabla) = H(h_0 \nabla, k_p h_0) \sec(h_0 \nabla) \nabla
$$

(7.17)

$$
OP_2(h_0 \nabla) = \cos(h_0 \nabla) \nabla - \frac{\omega^2}{g} \sin(h_0 \nabla)
$$

(7.18)

$$
OP_3(h_0 \nabla_s, h_0 \nabla_{p-s}) = \sin(h_0(\nabla_s + \nabla_{p-s})) \tilde{f}^{(2)}_{s,p-s}
$$

(7.19)

$$
OP_4(h_0 \nabla_s, h_0 \nabla_{p-s}) = H(h_0(\nabla_s + \nabla_{p-s}), k_p h_0) \tilde{f}_{s,p-s}
$$

(7.20)

$$
OP_5(h_0 \nabla_s, h_0 \nabla_t, h_0 \nabla_{p-s-t}) = H(h_0(\nabla_s + \nabla_t + \nabla_{p-s-t}), k_p h_0) f^{(3)}_{s,t,p-s-t}
$$

(7.21)

and expand $\hat{\phi}_p(x)$ by the regular expansion

$$
\hat{\phi}_p(x) = \left(a_p^{(0)}(x) + \varepsilon a_p^{(1)}(x) + \varepsilon^2 a_p^{(2)}(x)\right) \xi_p(\varepsilon x) e^{-\int k_p(\varepsilon x) dx}.
$$

(7.22)

The exponential term factors out the fast variation of the wave field whereas the factor $\xi(\varepsilon x) = c^{-1/2}$ accounts for the lowest-order varying depth effects, i.e., linear shoaling caused by the first derivative of $h$ ($\delta$). The slow spatial variation of $\xi_p$ and $k_p$ is justified by the mild-slope assumption.
7.2.1 Expansion of operators

Before substituting the expansion (7.22) into (7.16) we establish a few results with respect to expanding the infinite operators around \( \varepsilon = 0 \). First, we examine the expansion of a term of the following type

\[
T = \text{OP}(\nabla) a(\varepsilon x) e^{-i \int k(\varepsilon x) dx} = \text{OP}(\nabla)f
\]

(7.23)

with

\[
f = a(\varepsilon x) e^{-i \int k(\varepsilon x) dx}.
\]

(7.24)

Here \( a(\varepsilon x) \) represents any product of slowly varying functions appearing in front of the fast varying exponential function, e.g. \( \delta(\varepsilon x) a_p^{(0)}(\varepsilon x) \xi_p(\varepsilon x) \). A Taylor expansion of the operator yields

\[
T = \left[ \sum_{p=0}^{\infty} \frac{1}{p!} \text{OP}^{(p)}_0 \nabla^p \right] f,
\]

(7.25)

where \( \text{OP}^{(p)}_0 \) is the \( p \)'th derivative of the operator evaluated at \( \nabla = 0 \). We calculate the first derivatives of \( f \) by hand to

\[
\nabla^1 f = ( (-i k) a + \varepsilon a_x ) e^{-i \int k(\varepsilon x) dx}
\]

(7.26)

\[
\nabla^2 f = ( (-i k)^2 a + 2\varepsilon (-i k) a_x + \varepsilon (-i k_x) a + \varepsilon^2 a_{xx} ) e^{-i \int k(\varepsilon x) dx}
\]

(7.27)

\[
\nabla^3 f = ( (-i k)^3 a + 3\varepsilon (-i k)^2 a_x + 3\varepsilon (-i k) a_x (-i k_x) a + \varepsilon^2 (-i k)^3 a_x + 3\varepsilon^2 (-i k) a_x (-i k_x) a + 3\varepsilon^2 (-i k_x) a_x + 3\varepsilon^2 (-i k_x) a_x + 3\varepsilon^2 (-i k_x) a_x + 3\varepsilon^2 (-i k_x) a_x + O(\varepsilon^3)
\]

(7.28)

\[
\nabla^4 f = ( (-i k)^4 + 4\varepsilon (-i k)^3 a_x + 6\varepsilon (-i k)^2 (-i k_x) a_x + 12\varepsilon^2 (-i k)^3 a_x + 6\varepsilon^2 (-i k)^2 (-i k_x) a_x + 6\varepsilon^2 (-i k_x) a_x + 6\varepsilon^2 (-i k_x) a_x + 6\varepsilon^2 (-i k_x) a_x + O(\varepsilon^3)
\]

(7.29)

where only first-order derivatives of \( k \) have been retained. This sequence inspires the following explicit expression for \( \nabla^p f \) which can be proved by induction:

\[
\nabla^p f = ( (-i k)^p + \varepsilon p (-i k)^{p-1} a_x + \frac{1}{2} \varepsilon p (p-1) (-i k)^{p-2} (-i k_x) a_x + \frac{1}{2} \varepsilon^2 p (p-1) (p-2) (-i k_x) a_x + \frac{1}{2} \varepsilon^2 p (p-1) (p-2) (-i k_x) a_x + O(\varepsilon^3) + O(\varepsilon^3)
\]

(7.30)

Insertion into (7.25) now yields

\[
\text{OP}(\nabla) f = \left[ \text{OP}(-i k) a + \varepsilon \text{OP}'(-i k) a_x - \frac{1}{2} \varepsilon i \text{OP}''(-i k) k_x a_x - \frac{1}{2} \varepsilon^2 i \text{OP}'''(-i k) k_x a_x + \frac{1}{2} \varepsilon^2 \text{OP}''(-i k) a_{xx} \right] e^{-i \int k(\varepsilon x) dx} + O(\varepsilon^3)
\]

(7.31)

The terms of order \( O(\varepsilon) \) can be found in Agnon et al. (1999) as well. This result shows how an infinite operator in \( \nabla \) operating on \( f \) can be expressed by scalar functions of the local
wave number $k$ and derivatives of $a$ and $k$. Again we emphasize that $k_{xx}$-terms have been neglected due to the mild-slope assumption.

Some of the operators in (7.16) take two different gradients as arguments. For this case we now establish a result similar to (7.31). We want to expand a term of type

$$T_2 = \text{OP}(\nabla_1, \nabla_2)a_1(\varepsilon x)a_2(\varepsilon x)e^{-i\int (k_1(\varepsilon x)+k_2(\varepsilon x))dx} = \text{OP}(\nabla_1, \nabla_2)f_2$$

(7.32)

with

$$f_2 = a_1(\varepsilon x)a_2(\varepsilon x)e^{-i\int (k_1(\varepsilon x)+k_2(\varepsilon x))dx}$$

(7.33)

around the wave numbers $k_1$ and $k_2$. Here it is understood that $\nabla_1$ operates on $a_1(\varepsilon x)e^{-i\int k_1(\varepsilon x)dx}$ only, and likewise for $\nabla_2$. Similarly to the previous calculation, we write the operator as a Taylor series — now with two independent variables. This yields

$$T_2 = \left[\sum_{p=0}^{\infty} \frac{1}{p!}(\nabla_1 \frac{\partial}{\partial \nabla_1} + \nabla_2 \frac{\partial}{\partial \nabla_2})^p \text{OP}_0\right] f_2$$

(7.34)

where $\text{OP}_0$ is a short hand notation for $\text{OP}(0,0)$. Calculating $(\nabla_1 \frac{\partial}{\partial \nabla_1} + \nabla_2 \frac{\partial}{\partial \nabla_2})^p \text{OP}_0 f$ for a few $p$’s leads to the formula

$$(\nabla_1 \frac{\partial}{\partial \nabla_1} + \nabla_2 \frac{\partial}{\partial \nabla_2})^p \text{OP}_0 f_2 = \left[\sigma^p \text{OP}_0 a_1 a_2 + \frac{1}{2} \varepsilon p(p-1) \sigma^{p-2} \frac{\partial^2}{\partial \nabla_1^2} \text{OP}_0 (-ik_{1,x}) a_1 a_2 + \varepsilon p \sigma^{p-1} \frac{\partial}{\partial \nabla_1} \text{OP}_0 a_{1,x} a_2 + \varepsilon \sigma^{p-1} \frac{\partial}{\partial \nabla_2} \text{OP}_0 a_{1} a_{2,x}\right] e^{-i\int (k_1(\varepsilon x)+k_2(\varepsilon x))dx} + O(\varepsilon^2)$$

(7.35)

with $\sigma = \left(-ik_1 \frac{\partial}{\partial \nabla_1} - ik_2 \frac{\partial}{\partial \nabla_2}\right)$. This result was proven by induction. Insertion in (7.34) now yields

$$\text{OP}(\nabla_1, \nabla_2)f_2 = \left[\text{OP}_0 a_1 a_2 + \frac{1}{2} \varepsilon \frac{\partial^2 \text{OP}}{\partial \nabla_1^2} (-ik_{1,x}) a_1 a_2 + \frac{1}{2} \varepsilon \frac{\partial^2 \text{OP}}{\partial \nabla_2^2} (-ik_{2,x}) a_1 a_2 + \frac{\partial \text{OP}}{\partial \nabla_1} a_{1,x} a_2 + \frac{\partial \text{OP}}{\partial \nabla_2} a_{1} a_{2,x}\right] e^{-i\int (k_1(\varepsilon x)+k_2(\varepsilon x))dx} + O(\varepsilon^2),$$

(7.36)

where all operators are evaluated at $(-ik_1, -ik_2)$. We see that the $O(\varepsilon)$-terms of (7.31) is a special case of the above result. We only need the $O(\varepsilon)$ terms of the above result, and the $O(\varepsilon^2)$ terms are therefore not derived.

### 7.2.2 The spatial variation of $a^{(0)}$

With the above results, we are nearly ready to insert the expansion (7.22) into (7.16). Before doing this, however, we shall make a short remark on the spatial variation of $a^{(0)}$, from the
expansion (7.22). Substitution of the expansion into the equation yields at order $O(1)$

$$a_{p,x}^{(0)}\xi_{p,x}e^{-i\int k_{p}dx} = 0$$

i.e.

$$a_{x}^{(0)} = 0 \tag{7.37}$$

For linear waves this is true, since we have accounted for the oscillatory spatial variation by the exponential function in the expansion. However, we are still interested in modelling a slow modulation of the linear wave, caused by the cubic interactions. Hence, the above expansion can seem insufficient, since it already at order $O(1)$ yields $a_{x}^{(0)} = 0$ for the linear waves. However, since we are dealing with a perturbation hierarchy, $a_{x}^{(0)} = 0$ at $O(1)$ simply means $a_{x}^{(0)} = O(\varepsilon^p)$ with $p > 0$.

To investigate this further, a multiple scale expansion was applied. The expansion is

$$\phi_{p}(x) = (a_{0}^{(0)})_{p}(x,x_{1},x_{2}) + \varepsilon a_{1}^{(1)}(x,x_{1},x_{2}) + \varepsilon^2 a_{2}^{(2)}(x,x_{1},x_{2})\xi_{p}(x_{1},x_{2})e^{-i\int k_{p}(x_{1},x_{2})dx} \tag{7.38}$$

with $x_{1} = \varepsilon x$ and $x_{2} = \varepsilon^2 x$. With this expansion, the same results as for the regular expansion were obtained, furthermore showing that $a_{0}^{(0)} = a_{0}^{(0)}(\varepsilon^2 x)$. Hence, a spatial modulation of the linear waves is still within reach of the model. Since the two expansions give the same results for the model derived, we use the simpler regular expansion (7.22) in the following.

### 7.2.3 Insertion of expansion

We insert the expansion (7.22) into (7.16). For the bottom-slope terms, we retain only $\delta_{x}$-terms, while putting $\delta = 0$ after applying the expanded operators. Setting $\delta = 0$ corresponds to using $h_{0} = h(x)$ as the locally constant reference depth. We also neglect all terms proportional to $h_{x}^2$, e.g., terms like $\delta_{x}\xi_{x}$. Moreover, we make use of our new knowledge $a_{x}^{(0)} = O(\varepsilon^2)$, gained from the above argument.

The $O(1)$-equation and its solution is given by (7.37). At $O(\varepsilon)$ we get

$$e^{-i\int k_{p}dx}(\xi_{p,x}a_{p}^{(0)} + \xi_{p}a_{p,x}^{(1)}) = -\delta_{x} OP_{1}^{'OP_{2}} a_{p}^{(0)}\xi_{p} e^{-i\int k_{p}dx} \tag{7.39} - \sum_{s} OP_{4,s,p-s} a_{s}^{(0)}a_{p-s}^{(0)}\xi_{p-s} e^{-i(\int (k_{s} + k_{p-s})dx)}$$

where the operators are evaluated with the local wave numbers as arguments. With a bit of algebra one can show that $\delta_{x} OP_{1}^{'OP_{2}}|_{\nabla = -ik_{p}} = c_{gp,x}/(2c_{gp})$ where $c_{gp}$ is the group velocity of the $p^{th}$ frequency. Thus with the choice $\xi_{p} = c_{gp}^{-1/2}$ the depth-slope term on the right hand side cancels with the term $\xi_{p,x}a_{p}^{(0)}$ at the left hand side. The transformation associated with this choice of $\xi_{p}$ is the so-called Liouville transformation, representing the lowest-order effect of shoaling.
7.3 Two practical models

Having made use of the Liouville transform, we calculate an explicit solution for \( a^{(1)} \). We integrate (7.39), ignoring the slow spatial variation of \((a^{(0)}, k, \xi)\) to obtain

\[
a^{(1)}_p = \sum_s \frac{i \mathsf{OP}_{4,s,p-s} \{k_s, k_{p-s}\}}{k_p - k_s - k_{p-s}} \frac{\xi_s \xi_{p-s}}{\xi_p} e^{-i (k_s + k_{p-s} - k_p)dx}. \tag{7.40}
\]

Here, we have introduced the convention \( \mathsf{OP}\{k_j\} = \mathsf{OP}|_{\nabla = -ik_j} \), to ease the reading of the result. We will make use of this convention in the following.

At \( O(\varepsilon^2) \) (7.16) gives

\[
e^{-i \int k_p dx} (\xi_{p,x} a^{(1)}_p + \xi_p a^{(2)}_p) = -\delta_x \mathsf{OP}'_1 \mathsf{OP}'_2 a^{(1)}_p \xi_p e^{-i \int k_p dx}
\]

\[
+ \delta_x \sum_s \mathsf{OP}'_1 \mathsf{OP}_3 \frac{a^{(0)}_p}{a^{(0)}_s} \xi_s \xi_{p-s} e^{-i \int (k_s + k_{p-s}) dx}
\]

\[
- \delta_x \sum_s \left( \frac{\partial \mathsf{OP}_4}{\partial \nabla_s} \alpha_{p-s} + \frac{\partial \mathsf{OP}_4}{\partial \nabla_{p-s}} \alpha_{p-s}
\right)
\]

\[
- \frac{i}{2} \frac{\partial^2 \mathsf{OP}_4}{\partial \nabla^2_s} \beta_s - \frac{i}{2} \frac{\partial^2 \mathsf{OP}_4}{\partial \nabla^2_{p-s}} \beta_{p-s}
\]

\[
a^{(0)}_p \xi_s \xi_{p-s} e^{-i \int (k_s + k_{p-s}) dx}
\]

\[
- \sum_s \mathsf{OP}_4 \frac{a^{(0)}_s}{a^{(0)}_{p-s}} \frac{a^{(1)}_p}{a^{(1)}_{p-s}} \xi_s \xi_{p-s} \xi_{p-s} e^{-i \int (k_s + k_{p-s}) dx}
\]

\[
- \sum \sum_t \mathsf{OP}_5 \frac{a^{(0)}_t}{a^{(0)}_{s-t}} \frac{a^{(0)}_s}{a^{(0)}_{p-s-t}} \xi_s \xi_{p-s-t} \xi_{p-s-t} e^{-i \int (k_s + k_{p-s-t}) dx}
\]

where we have defined

\[
\alpha_j = \frac{\xi_{j,x}}{(\delta_x \xi_j)} \quad \text{and} \quad \beta_j = \frac{k_{j,x}}{\delta_x}. \tag{7.42}
\]

The above results make it possible to express all terms in (7.15) by products of linear wave components. For a product of linear wave components, the fast spatial variation is known through the free wave numbers, and the arguments to the infinite series operators can therefore be expressed in terms of free wave numbers, turning the operators into scalar functions. In the next section, we derive a reduced and an unreduced model from the above results, both suited for practical application.

7.3 Two practical models

We now finish the derivation of the model, following two slightly different tracks. First, we derive a reduced model in the new variable \( \tilde{a}^{(0)} = a^{(0)} + \varepsilon^2 a^{(2)} \). This equation only contains terms of order \( O(\varepsilon^2) \) at the right hand side. Second, we derive an unreduced equation in \( \varphi \), having terms of first and second order at its right hand side. Before doing
the actual derivations, it is natural to comment on the difference between the reduced and the unreduced model.

The unreduced model is formulated simply in \( \hat{\phi}_p \), and is therefore the generalization of the quadratic model in \( \phi \) of Chapter 6. Solving this model implies that the bound wave field must be resolved and handled as part of the discretized spectrum.

The reduced model is formulated in a \( \tilde{a}^{(0)} = a^{(0)} + \varepsilon^2 a^{(2)} \), which apart from the \( \varepsilon^2 \)-term and the Liouville transform is the free wave amplitude. In the reduced model, the second-order bound wave field does not appear directly and is not resolved in the spectrum. Given the spectrum of \( \tilde{a}^{(0)} \), the second-order bound wave field can be calculated explicitly through (7.40). The interaction of a bound second-order wave with a free wave can therefore be expressed as a product of three free wave amplitudes, where the interaction coefficient contains the transfer function for the bound wave. This makes it possible to use a smaller bandwidth for the frequency spectrum, thereby saving computational time. Thus by resolving just the free waves in the spectrum for \( \tilde{a}^{(0)} \), the third-order effects on the free waves can be calculated by the model. The bound second-order wave field can then be added as postprocessing. In this approach, the bound superharmonic third-order harmonics, belonging to frequencies beyond the maximum frequency of the free spectrum, are discarded. Practically, however, these harmonics are less important than the third-order modulation of the free waves. The methodology of the reduced model is similar to the formulation of the Zakharov equation, besides the bottom-slope terms, which are not part of the Zakharov equation.

### 7.3.1 A reduced model

To derive the reduced model, we add the \( O(1) \) and \( O(\varepsilon^2) \) equations, i.e., (7.37) and (7.41). The left hand side of the new equation is

\[
\text{LHS} = e^{-i \int k_p dx} \left( \xi_p a^{(0)}_{p,x} + \varepsilon^2 \xi_{p,x} a^{(1)}_{p,x} + \varepsilon^2 a^{(2)}_{p,x} \right) \\
= e^{-i \int k_p dx} \xi_p \left[ a^{(0)}_{p,x} + \varepsilon^2 a^{(2)}_{p,x} \right] + \varepsilon^2 \xi_{p,x} a^{(1)}_{p,x} e^{-i \int k_p dx}.
\]

(7.43)

We now define the new variable

\[
\tilde{a}^{(0)} = a^{(0)} + \varepsilon^2 a^{(2)}.
\]

(7.44)

As we see from the above equation, the left hand side of the equation can be expressed by this new variable at the cost of adding a term of type \( \varepsilon^2 \delta a^{(1)} \) to the right hand side. Because the right hand side of the \( O(1) \)-equation (7.37) is zero, all terms at the right hand side of the new equation is of order \( O(\varepsilon^2) \). These terms can be expressed from \( a^{(0)} \) (through the explicit solution for \( a^{(1)} \)), thus making the local wave numbers of all terms known. Hence, all the operators can be expressed as scalar functions of free wave numbers. Furthermore, we are free to do the substitution \( a^{(0)} \rightarrow \tilde{a}^{(0)} \) in all terms at the right hand side, since the terms hereby neglected are of order \( O(\varepsilon^4) \).
Thus by adding the right hand sides of (7.37) and (7.41) and the extra term from the left hand side, inserting the solution for \( a^{(1)} \) and transforming to \( a^{(0)} \) we get the equation

\[
\frac{\partial}{\partial x} \tilde{a}_p^{(0)} = \delta_x \sum_s \left[ \frac{-i \Omega_{4,s,p-s} \{ k_s, k_{p-s} \} \alpha_p}{k_p - k_s - k_{p-s}} \right] \tag{7.45}
\]

\[
- \frac{i \Omega_p \Omega_{4,s,p-s} \{ k_s, k_{p-s} \} \Omega_{4,s,p-s} \{ k_s, k_{p-s} \}}{k_p - k_s - k_{p-s}}
\]

\[
+ \Omega_p \{ k_s, k_{p-s} \} \Omega_{3,s,p-s} \{ k_s, k_{p-s} \}
\]

\[
- \left( \frac{\partial \Omega_4}{\partial \xi_s} \alpha_s + \frac{\partial \Omega_4}{\partial \xi_{p-s}} \alpha_{p-s} \right) + \frac{i}{2} \left( \frac{\partial^2 \Omega_4}{\partial \xi_s^2} \beta_s + \frac{\partial^2 \Omega_4}{\partial \xi_{p-s}^2} \beta_{p-s} \right)
\]

\[
\tilde{a}_s^{(0)} \tilde{a}_{p-s}^{(0)} e^{-i f (k_s + k_{p-s} - k_p)} dx
\]

\[
- \sum_s \sum_t \Omega_{4,s,p-s} \{ k_t + k_{p-s-t} \} \Omega_{4,s,p-s-t} \{ k_t, k_{p-s-t} \}
\]

\[
\tilde{a}_s^{(0)} \tilde{a}_{p-s-t}^{(0)} e^{-i f (k_s + k_{p-s-t} - k_p)} dx
\]

\[
\tilde{a}_{s}^{(0)} \tilde{a}_{p-s-t}^{(0)} e^{-i f (k_s + k_{p} - k_p)} dx
\]

\[
- \sum_t \sum \Omega_{5,s,t,p-s-t} \tilde{a}_s^{(0)} \tilde{a}_{p-s-t}^{(0)} e^{-i f (k_s + k_{p-s-t} - k_p)} dx.
\]

This equation describes the slow spatial evolution of the wave spectrum, caused by processes of order \( O(\varepsilon^2) \). It is suited for direct numerical integration, since the right hand side is formulated explicitly in \( \tilde{a}^{(0)} \) and the operators are represented by scalar functions of free wave numbers. The first sum on the right hand side represents the interaction of second-order bound waves with the sloping sea bed. The first double sum represents the interaction of second-order bound waves with the free wave field, while the last double sum represents the cubic interaction among free waves.

The solution variable \( \tilde{a}^{(0)} \) can be transformed to \( \phi \) using the explicit solution for \( a^{(1)} \), in which the error of using \( \tilde{a}^{(0)} \) rather than \( a^{(0)} \) is \( O(\varepsilon^3) \) and therefore can be neglected.

### 7.3.2 An unreduced model

It is also possible to derive a model formulated directly in the Fourier amplitudes of the potential, \( \phi_p \). We do this now. Addition of the \( O(1) \), \( O(\varepsilon) \) and \( O(\varepsilon^2) \) equations yields a cubic shoaling model in \( a^{(0)}, a^{(1)} \) and \( a^{(2)} \), and we can easily collapse the left hand side to the original appearance \( (\frac{\partial}{\partial x} + ik_p) \phi_p \). At the right hand side, we use

\[
\varepsilon^2 \tilde{a}_p^{(0)} \Omega_\xi_p e^{-i f k_p} dx = \varepsilon^2 \tilde{\phi}_p + O(\varepsilon^3)
\]

\[
(7.46)
\]
in the highest order terms, whereas the terms of order \(O(\varepsilon)\) are expressed in terms of \(\phi\) using
\[
a_p^{(0)} \xi_p e^{-i k_p x} = (a_p^{(0)} + \varepsilon a_p^{(1)}) \xi_p e^{-i k_p x} - \varepsilon a_p^{(1)} \xi_p e^{-i k_p x}
\]
\[
= \hat{\phi}_p - \varepsilon \sum_s \frac{i \text{OP}_{4,s,p-s} \hat{\phi}_s}{k_p - k_s - k_{p-s}} \hat{\phi}_{p-s} + O(\varepsilon^2). \tag{7.47}
\]

That is, the rewriting of the \(O(\varepsilon)\)-terms from \(a_p^{(0)}\) to \(\hat{\phi}_p\) gives rise to correction terms, effectively of order \(O(\varepsilon^2)\).

Thus, addition of (7.37), (7.39) and (7.41), substitution of the solution for \(a^{(1)}\), (7.40) and application of the above rewritings gives the model

\[
\frac{\partial}{\partial x} + ik_p \hat{\phi}_p =
\]

(1) \[-\varepsilon \delta_x \text{OP}_1^2 \text{OP}_2 \{k_p\} \hat{\phi}_p\]

(2) \[\varepsilon \sum_s \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \hat{\phi}_s \hat{\phi}_{p-s}\]

(3) \[\varepsilon^2 \delta_x \sum_s \left[ \frac{i \text{OP}_1^2 \text{OP}_2 \{k_p\} \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\}}{k_p - k_s - k_{p-s}} \right.\]

(4) \[\left. \text{OP}_1^2 \{k_s + k_{p-s}\} \text{OP}_{3,s,p-s} \{k_s, k_{p-s}\} \right.\]

(5) \[\left. \begin{array}{c}
\left( \frac{\partial \text{OP}_4}{\partial \phi} \right)_s + \left( \frac{\partial \text{OP}_4}{\partial \phi} \right)_{p-s} \end{array} \right) + \frac{i}{2} \left( \frac{\partial^2 \text{OP}_4}{\partial \phi^2} \right)_{p-s} \hat{\phi}_s \hat{\phi}_{p-s} \]

(7.48) \[\varepsilon^2 \sum_s \sum_t \frac{i \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \text{OP}_{4,t,p-s-t} \{k_t, k_{p-s-t}\} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t}}{k_p - k_t - k_{p-s-t}} \]

(9) \[-i \text{OP}_{4,s,p-s} \{k_t, k_{p-s-t}\} \text{OP}_{4,t,p-s-t} \{k_t, k_{p-s-t}\} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t} \]

(10) \[-i \text{OP}_{4,s,p-s} \{k_t, k_{p-s-t}\} \text{OP}_{4,t,s,p-t} \{k_t, k_{s-t}\} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t} \]

(11) \[-\varepsilon^2 \sum_s \sum_t \text{OP}_{5,s,t,p-s-t} \hat{\phi}_s \hat{\phi}_t \hat{\phi}_{p-s-t} \]

Here the role of each term are as follows: Term (1)–(2) can be shown to be identical to the right hand side of the quadratic model (6.42). Term (4)–(6) represent the interaction of the second-order bound wave field with the sloping sea bed. Term (9)–(10) represent the interaction of second-order bound waves with free waves. Term (11) represents the cubic interaction of free waves. Term (3) is a correction term, compensating for the bound wave part of \(\hat{\phi}_p\) in term (1). Similarly, term (7)–(8) are correction terms compensating for the bound wave part of \(\hat{\phi}_s\) and \(\hat{\phi}_{p-s}\) in term (2).
7.3 Two practical models

As an alternative, we can formulate a model in the variable $\psi$ defined as

$$\psi_p = a_p^{(0)} + \varepsilon a_p^{(1)} + \varepsilon^2 a_p^{(2)} \quad \text{i.e.,} \quad \hat{\phi}_p = \psi_p \xi_p e^{-i f k_p dx}. \quad (7.49)$$

This transformation takes out the fast spatial variation of the wave field as well as the lowest order effects of a sloping bottom. In terms of $\psi_p$, the model is

$$\frac{\partial}{\partial x} \psi_p =$$

$$- \varepsilon \sum_s \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \psi_s \psi_p e^{-i f (k_s + k_{p-s} - k_p) dx}$$

$$+ \varepsilon^2 \delta_x \sum_s \left[ - \frac{i \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \alpha_p}{k_p - k_s - k_{p-s}} \right]$$

$$+ \frac{i \text{OP}_1 \text{OP}_2 \{k_s + k_{p-s}\} \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\}}{k_p - k_s - k_{p-s}}$$

$$+ \text{OP}_1 \text{OP}_3 \{k_s, k_{p-s}\} - \left( \frac{\partial \text{OP}_4}{\partial s} \alpha_s + \frac{\partial \text{OP}_4}{\partial \nabla_{p-s}} \alpha_{p-s} \right)$$

$$+ \left( \frac{\partial^2 \text{OP}_4}{\partial s^2} \beta_s + \frac{\partial^2 \text{OP}_4}{\partial \nabla_{p-s}^2} \beta_{p-s} \right) \psi_s \psi_p e^{-i f (k_s + k_{p-s} - k_p) dx}$$

$$+ \varepsilon^2 \sum_s \sum_t \frac{i \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \text{OP}_{4,t,p-s-t} \{k_t, k_{p-s-t}\}}{k_p - k_t - k_{p-s-t}}$$

$$\psi_t \psi_p \psi_{p-s-t} \xi_s \xi_{p-s-t} e^{-i f (k_s + k_{p-s-t} - k_p) dx} \quad (7.50)$$

$$+ \frac{i \text{OP}_{4,s,p-s} \{k_s, k_{p-s}\} \text{OP}_{4,t,s-t} \{k_t, k_{s-t}\}}{k_s - k_t - k_{s-t}}$$

$$\psi_s \psi_t \psi_p e^{-i f (k_{s-t} + k_t + k_{p-s-t} - k_p) dx}$$

$$- \frac{i \text{OP}_{4,s,p-s} \{k_s, k_{p-s-t}\} \text{OP}_{4,t,p-s-t} \{k_t, k_{p-s-t}\}}{k_p - k_t - k_{p-s-t}}$$

$$\psi_s \psi_t \psi_{p-s-t} \xi_s \xi_{p-s-t} e^{-i f (k_s + k_{p-s-t} - k_p) dx}$$

$$- \frac{i \text{OP}_{4,s,p-s} \{k_t + k_{s-t}, k_{p-s-t}\} \text{OP}_{4,t,s-t} \{k_t, k_{s-t}\}}{k_s - k_t - k_{s-t}}$$

$$\psi_s \psi_t \psi_p e^{-i f (k_{s-t} + k_t + k_{p-s-t} - k_p) dx}$$

$$- \frac{i \text{OP}_{4,s,p-s} \{k_t + k_{s-t}, k_{p-s-t}\} \text{OP}_{4,t,s-t} \{k_t, k_{s-t}\}}{k_s - k_t - k_{s-t}}$$

$$\psi_s \psi_t \psi_p e^{-i f (k_{s-t} + k_t + k_{p-s-t} - k_p) dx}$$

$$- \varepsilon^2 \sum_s \sum_t \text{OP}_{5,s,t,p-s-t} \psi_s \psi_t \psi_{p-s-t} \xi_s \xi_{p-s-t} e^{-i f (k_s + k_{p-s-t} - k_p) dx}.$$
7.4 Including wave induced mean flow in the model

When waves propagate over varying depth, the shoaling leads to a wave induced set-down of the mean water level. Also, the mass transport involved with the wave motion above the wave trough level results in a nonzero mean velocity below the wave trough level. As the wave field is transformed, this mass flux changes, and the mean mass flux below wave trough level must adjust accordingly to ensure a constant net mass flux.

In this section we derive explicit expressions for the wave induced variation of the mean water level \( \bar{\eta} \) and the mean flux below wave trough level. The mean flow variables enter the evolution equations for the spectral amplitudes through the quadratic interaction terms.

7.4.1 The set-down

For the wave induced set-down \( \bar{\eta} \), we anticipate that \( \bar{\eta} = O(\varepsilon) \) and \( \bar{\eta} = \bar{\eta}(\varepsilon x) \). As a consequence \( \bar{\eta}_x \) must be \( O(\varepsilon^2) \). This turns out to be true as can be seen from the derivation below, and it is emphasized that the above considerations are not used in the derivation.

We derive an expression for \( \bar{\eta} \) by time averaging the dynamic free surface condition (7.8). First, however, we state a few considerations about time averaging.

A few remarks on time averaging  The dynamic free surface condition in the form (7.8) is suited for time averaging, since a large number of the terms are time derivatives, and are thus likely to cancel out in the averaging. If \( F \) is a product of flow variables in a periodic flow, we have \( \langle F \rangle \approx 0 \). However, as \( \Phi \) itself is not a physical flow variable, a little care needs to be taken when averaging products involving \( \Phi \) and \( \Phi_t \). The equations we average in this section (7.8) and (7.9) do not contain products involving \( \Phi \) itself. Hence, we can concentrate on pure \( t \)-derivatives of \( \Phi \), since spatial derivatives of \( \Phi \) are physical flow variables.

Inspection of the dynamic free surface condition in the form (7.2), shows that \( \Phi_t \) can be expressed by physical flow variables to any order desired. Hence averaging any time derivative of a product involving \( \Phi_t \) and other flow variables gives zero to any order. This also holds for time derivatives of products involving \( \Phi_t^2 \) and \( \Phi_u \). Further, since \( \Phi_u \) is \( \frac{\partial}{\partial t} \Phi_t \), its time average is zero to any order. The same argument holds for higher temporal derivatives of \( \Phi \).

For these reasons, all the nonlinear terms within (7.8) and (7.9) being time derivatives average to zero.
An equation for the set-down We can now establish an expression for the set-down by time averaging the dynamic free surface condition (7.8). With the above remarks we get

\[ g\bar{\eta} = -\langle \Phi_t \rangle - \varepsilon \left( \frac{\langle \Phi_x^2 \rangle}{2} - \langle \Phi_u^2 \rangle \right) \]

\[ - \varepsilon^2 \left( \frac{\langle \Phi_{ut}^2 \rangle}{g^4} + \frac{\langle \Phi_x^2 \Phi_{ttt} \rangle}{g^2} - \frac{\langle \Phi_{xt}^2 \Phi_t \rangle}{g^2} \right). \tag{7.51} \]

The first term \( \langle \Phi_t \rangle \) does not depend on \( x \) for a periodic flow. This is easily seen by calculating its gradient:

\[ \langle \Phi_t \rangle = \langle \Phi_{xt} \rangle = \langle U_t \rangle = 0 \]

with \( U_x \) being the horizontal velocity at the still water level. \( U \) is a periodic flow variable and its time derivative therefore averages to zero. This finding allows us to write (7.51) as

\[ \bar{\eta} = \bar{\eta}(x_0) - \varepsilon \left[ \frac{\langle \Phi_x^2 \rangle}{2g} - \langle \Phi_u^2 \rangle \right] \]

\[ - \varepsilon^2 \left( \frac{\langle \Phi_{ut}^2 \rangle}{g^4} + \frac{\langle \Phi_x^2 \Phi_{ttt} \rangle}{g^2} - \frac{\langle \Phi_{xt}^2 \Phi_t \rangle}{g^2} \right) \]

\[ x \]

which is an explicit expression for the set-down based on its initial value and the wave conditions in \( x \) and \( x_0 \). The products involved in this expression are easily calculated from the time series of \( \Phi \) and its derivatives. The toggling from the Fourier amplitudes to the time domain can be done using Fast Fourier Transform, thereby saving some computational time compared to evaluating the cubic terms as double convolutions in the frequency domain. Note, however, that in the quadratic term, the approximation \( \hat{\phi}_{px} = -ikp\hat{\phi}_p + O(\varepsilon) \) would violate the \( O(\varepsilon^2) \)-terms. Hence, information of the bound wave field should be taken into account also.

To second order, the above result can be written

\[ \bar{\eta} = \bar{\eta}(x_0) - \varepsilon \left[ \sum_{p=1}^{N} \left( \frac{k_p^2}{g} - \frac{\omega_p^2}{g^2} \right) |\hat{\phi}_p|^2 \right] \]

\[ x \]

\[ x_0 \]

agreeing with (6.88) combined with (6.91).

7.4.2 Mean drift below wave trough level

We choose to consider a mean drift below wave trough level of the same order of magnitude as the wave generated return current for a total net mass flux of zero. We therefore assume \( \nabla\Phi_0 = O(\varepsilon) \) and \( \nabla\Phi_0 = \nabla\Phi_0(\varepsilon x) \). The time average of the bottom boundary condition, (7.1) is

\[ \text{Sin}(h_0 \nabla)\nabla \Phi + \text{Cos}(h_0 \nabla)\dot{W} = -\nabla \{ \delta(\text{Cos}(h_0 \nabla)\nabla \Phi - \text{Sin}(h_0 \nabla)\dot{W}) \} \tag{7.54} \]
which we expand in $\varepsilon$. Further, we let $h_0 = h$ and thus $\delta = 0$, retaining $\nabla\delta = \nabla h$. The lowest-order terms are

$$h\nabla^2 \Phi + \tilde{W} = -\nabla h (\nabla \Phi - h \nabla \tilde{W}).$$

(7.55)

$\tilde{W}$ can be found from (7.9). $\langle \Phi_{tt} \rangle$ is zero for the reasons argued in the paragraph on averaging. We therefore have

$$\tilde{W} = \Phi_z = -\frac{\varepsilon}{g} \langle \Phi_{xx} \Phi_t \rangle + O(\varepsilon^2).$$

(7.56)

In the quadratic term, the fast variation of the wave field does not contribute to the time averaged quantity. Thus $\tilde{W} = O(\varepsilon^2)$ and the last term in (7.55) is therefore of higher order than the other terms in the equation. We therefore neglect this term and proceed by defining the utility variable $Q = h \nabla \Phi$. We expand this as

$$Q = \varepsilon q_1 + \varepsilon^2 q_2 + \varepsilon^3 q_3$$

and see that (7.55) can be written

$$\nabla q_1 = -\tilde{W} + O(\varepsilon^3)$$

(7.57)

**An expression for $\tilde{W}$** We now express $\tilde{W}$ as a gradient of wave field quantities. We use (7.4) as our starting point, which we write as

$$W = \Phi_z = \eta_t + \frac{\partial}{\partial x} \left\{ \varepsilon \eta \Phi_x + \frac{1}{2} \varepsilon^2 \eta^2 \eta_{xt} \right\} + O(\varepsilon^3).$$

(7.58)

To eliminate $\eta$, we substitute (7.6) leading to the result

$$W = \Phi_z = \eta_t - \frac{\partial}{\partial x} \left\{ \varepsilon \Phi_x \Phi_t / g + \varepsilon^2 \left( \Phi_{tt} \Phi_x / 2g^3 + \Phi_t \Phi_{xtt} / 2g^3 + \Phi_t^2 / 2g \right) \right\} + O(\varepsilon^3).$$

(7.59)

To obtain a simpler result for the time average of this equation, we rewrite the cubic products to a form involving a large number of terms being time derivatives. The rewritten expression reads

$$W = \Phi_z = \eta_t - \frac{\partial}{\partial x} \left\{ \varepsilon \Phi_x \Phi_t / g + \varepsilon^2 \left( -\frac{1}{g^3} (\Phi_t^2 \Phi_{xt})_t 
+ \frac{1}{2g^3} (\Phi_t^2 \Phi_x)_{tt} - \frac{1}{2g^3} (\Phi_{tt}^2 \Phi_x)
+ \frac{1}{g^3} (\Phi_t^2 \Phi_{xt})_t 
+ \frac{1}{2g} (\Phi_{ttt}^2 \Phi_x) \right) \right\} + O(\varepsilon^3).$$

(7.60)

We now substitute the time average of this result into (7.57) and integrate once in space. We hereby obtain

$$q_1(x) = q_1(x_0) + \left[ \varepsilon \frac{1}{g} \langle \Phi_x \Phi_t \rangle + \varepsilon^2 \left( -\frac{1}{2g^3} \Phi_{ttt}^2 \Phi_x + \frac{1}{g^3} \Phi_t^2 \Phi_{xtt} + \frac{1}{2g} \Phi_{ttt} \right) \right]_{x_0}^x,$$

(7.61)
which expresses $q_1$ in terms of its initial value and the wave conditions in $x$ and $x_0$. The products of the expression can be evaluated in the time domain in the same manner as for the set-down.

To second order, we can express $q_1$ as

$$q_1(x) = q_1(x_0) - \varepsilon \left[ \frac{2}{g} \sum_{p=1}^{N} \omega_p k_p |\hat{\phi}_p|^2 \right]_{x_0}^x + O(\varepsilon^2).$$  \hspace{1cm} (7.62)

For a single wave of height $H$, we may insert $|\hat{\phi}_p| = Hg/(4\omega_p)$ to obtain

$$q_1(x) = q_1(x_0) - \left[ \frac{1}{8} \frac{g}{c} \frac{H^2}{h} \right]_{x_0}^x. \hspace{1cm} (7.63)$$

Division with $h$ yields

$$U(x) = U(x_0) - \left[ \frac{1}{8} \frac{g}{c} \frac{H^2}{h} \right]_{x_0}^x \hspace{1cm} (7.64)$$

which is the classical formula for the return current of Stokes second-order waves.

### 7.4.3 How the mean flow quantities enter the model

Having derived expressions for the mean flow, we now consider its feedback on the wave field. As we have seen above, the mean flux below wave trough level $q_1$ and the set-down $\bar{\eta}$ are one order of magnitude smaller than the oscillating wave field. Hence, these quantities only enter the terms of magnitude $O(\varepsilon)$ at the right hand side of the models (i.e., (7.45),(7.48) and (7.50)). Further, since the depth-slope term of magnitude $O(\varepsilon)$ is a linear term and we already have established the equations for the mean flow variables, we only need to consider the quadratic term of magnitude $O(\varepsilon^2)$. The contribution of the mean flow in this term gives rise to terms of magnitude $O(\varepsilon^2)$.

For the unreduced model in $\hat{\phi}_p$ the quadratic term has the form

$$\text{term}_2 = -\varepsilon \sum_s \text{OP}_{4,s,p-s} \{ k_s, k_{p-s} \} \hat{\phi}_s \hat{\phi}_{p-s}. \hspace{1cm} (7.65)$$

We now consider the term having $s = 0$ and denote this as

$$\text{term}_{2,0} = -\varepsilon \text{OP}_{4,s,p-s} \{ k_s, k_{p-s} \} \hat{\phi}_s \hat{\phi}_{p-s} |_{s=0}. \hspace{1cm} (7.66)$$

For $s = 0$ we do not have $\partial_t \phi_0 = \omega_0 \phi_0 = 0$, see e.g. (7.3). We therefore retain the operator-based form of $\hat{\gamma}^{(2)}_{s,p-s}$ as given in (7.11), rather than rewriting derivatives to factors of $(k, \omega)$. To lowest order, $\phi_0$ does not contribute to (7.66), since it has no fast spatial or temporal
variation. The contribution is therefore found by Taylor expanding the operator around 
\(\nabla_s = 0\) and \(\partial_{s,t} = 0\)

\[
\text{term}_{2,0} = -\varepsilon \left( \left. \frac{\partial \text{OP}_4 \text{s.p.s}}{\partial \nabla_s} \right|_{s=0} \nabla \phi_0 \hat{\phi} + \left. \frac{\partial \text{OP}_4 \text{s.p.s}}{\partial (\partial_{s,t})} \right|_{s=0} \phi_0 \epsilon \hat{\phi}_p \right). \tag{7.67}
\]

We only retain the first term in the series, since the higher-order terms do not contribute within the models accuracy. We now calculate each term in detail. For \(\left. \frac{\partial \text{OP}_4 \text{s.p.s}}{\partial \nabla_s} \right|_{s=0}\) we get

\[
\left. \frac{\partial \text{OP}_4 \text{s.p.s}}{\partial \nabla_s} \right|_{s=0} = H(-i\hbar \omega_k) \left. f^{(2)} \right|_{s=0} + H(-i\hbar \omega_k) \left. \frac{\partial f^{(2)}}{\partial \nabla_s} \right|_{s=0} = H(-i\hbar \omega_k) \left( - \frac{1}{2g^3} \partial_{p,t}^4 - \frac{1}{2g} \nabla_p^2 \right) + \text{h.o.t.} \tag{7.68}
\]

For the other term we have

\[
\left. \frac{\partial \text{OP}_4 \text{s.p.s}}{\partial (\partial_{s,t})} \right|_{s=0} = H(-i\hbar \omega_k) \left. \frac{\partial f^{(2)}}{\partial (\partial_{s,t})} \right|_{s=0} = H(-i\hbar \omega_k) \left( - \omega_k^4 \frac{1}{2g^3} + \frac{k_p^2}{2g} \right) + \text{h.o.t.} = -\frac{i}{4\omega_p c_{gp} \cosh^2 k_p h} + \text{h.o.t.} \tag{7.69}
\]

The time derivative of \(\phi_0\) can be expressed from (7.3). To lowest order this gives

\[
\phi_{0,t} = -g \hat{\eta} - \frac{1}{2} \varepsilon \left< \Phi_x^2 \right> + \varepsilon \frac{1}{2g^2} < \Phi_{tt} > = -g \hat{\eta} - \varepsilon \sum_{s=1}^{N} (k_s^2 - \frac{\omega_s^2}{g^2}) |\phi_s|^2 \tag{7.70}
\]

Further, as argued on page 221, \(\phi_{0,t}\) is constant in space. Hence the above expression only needs to be calculated initially, when solving the model equations.

As the last step we express \(\nabla \phi_0\) in terms of \(q_1\) and rewrite \(H(-i\hbar \omega_k)\) using (6.36). Putting it all together we obtain

\[
\text{term}_{2,0} = \varepsilon^2 \frac{ik_p}{2c_{gp}^2 \hbar} q_1 \phi_p - i\varepsilon^2 \frac{k_p^2}{4\omega_p c_{gp}^2 \cosh^2 k_p h} \phi_{0,t} \phi_p \tag{7.71}
\]

For \(p - s = 0\) we get exactly the same result, since \(\text{OP}_4\) is symmetric in \(s, p - s\). Hereby, the contributions from the mean flow to the spatial evolution of the spectral amplitudes of the potential can be incorporated into the model. The above result is easily transferred to the reduced model (7.45) and the model formulated in \(\psi\), (7.50), since it is of magnitude \(O(\varepsilon^2)\).

### 7.5 Comparison with the NLS-equation

As an analytical check of the model derived, we compare it to the nonlinear Schrödinger equation. We use the formulation of Mei (1983), section 12.2 for the comparison.
7.5 Comparison with the NLS-equation

7.5.1 The NLS-equation as written in Mei (1983)

We first outline the formulation of the NLS-equation in Mei (1983). The starting point for deriving the NLS-equation is the boundary value problem for the Laplace equation in two horizontal dimensions with a flat sea bed and the kinematic and dynamic free surface boundary conditions. The free surface conditions are expanded around the still water level, \(z = 0\), in \(\varepsilon = ka\) to cubic order. Here \(a\) is a measure of the free surface elevation. To allow for a slow modulation of the wave field, the independent variables are expanded by a multiple scales approach

\[
\begin{align*}
x & \to x + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \quad (7.71) \\
y & \to \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \quad (7.72) \\
t & \to t + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots \quad (7.73)
\end{align*}
\]

Here, \(y\) is the horizontal direction perpendicular to the \(x\)-direction. The reason for not having a zeroth-order dependence in the \(y\)-direction is that the dominant direction of wave propagation is taken to be the \(x\)-direction. In the comparison with our model, we leave out all terms connected to the \(y\)-dependence of the wave field.

Also the dependent variables are scaled. Due to the nonlinearity of the problem, the primary harmonic wave train will generate higher harmonics in the higher orders of the solution. Therefore, a double expansion of the velocity potential (and other variables) is assumed

\[
\Phi(x, z, t) = \sum_{n=1}^{\infty} \varepsilon^n \phi_n = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \varepsilon^n e^{im\psi} \phi_{nm},
\]  

(7.74)

where \(\psi = kx - \omega t\) is the phase, \((\omega, k)\) satisfying the linear dispersion relation. The free surface elevation \(\eta\) has a similar expansion as \(\Phi\), even though the different harmonics are not separated explicitly:

\[
\eta(x, t) = \sum_{n=1}^{\infty} \varepsilon^n \eta_n
\]  

(7.75)

The first-order solution is

\[
\begin{align*}
\eta_1 & = \frac{1}{2}(Ae^{i\psi} + \text{c.c.}) \quad (7.76) \\
\phi_1 & = \phi_{10} - g \frac{\cosh k(z + h)}{2\omega \cosh kh} (iAe^{i\psi} + \text{c.c.}), \quad (7.77)
\end{align*}
\]

where \(A\) and \(\phi_{10}\) to this order are arbitrary functions of the slow variables. The second-order
solution is given by

\[ \phi_2 = \phi_{20} - \frac{\omega k(z + h) \sinh k(z + h)}{2k^2 \sinh kh} (A_{x_1} e^{i\psi} + c.c.) \]
\[ - \frac{3}{16} \frac{\omega \cosh 2k(z + h)}{\sinh^4 kh} (iA^2 e^{2i\psi} + c.c.) \]

(7.78)

\[ \eta_2 = - \frac{1}{g} \phi_{10_1} - \frac{k}{2 \sinh 2kh} |A|^2 \]
\[ + \frac{1}{2\omega} (iA_{t_1} e^{i\psi} + c.c.) - \frac{kh \sinh kh}{2k \cosh kh} (iA_{x_1} e^{i\psi} + c.c.) \]
\[ + \frac{k(2 \cosh^2 kh + 1) \cosh kh}{8 \sinh^3 kh} (A^2 e^{2i\psi} + c.c.) \]

(7.79)

and the condition

\[ A_{t_1} + c_g A_{x_1} = 0 \]

(7.80)

which must be fulfilled in order to avoid secular terms. At third order, a similar requirement of vanishing secular terms gives the two equations

\[ \left( \frac{\partial}{\partial t_1} + c_g \frac{\partial}{\partial x_1} \right) A + i\varepsilon \left\{ - \frac{1}{2k^2} \frac{\partial^2 \omega}{\partial x_1^2} A + \frac{\omega k^2 (\cosh 4kh + 8 - 2 \tanh^2 kh)}{16 \sinh^4 kh} |A|^2 A \right. \]
\[ \left. - \left( \frac{k^2}{2\omega \cosh^2 kh} \frac{\partial \phi_{10}}{\partial t_1} - k \frac{\partial \phi_{10}}{\partial x_1} \right) A \right\} = 0 \]

(7.81)

\[ \frac{\partial^2 \phi_{10}}{\partial t_1^2} - g \frac{\partial^2 \phi_{10}}{\partial x_1^2} = \frac{\omega^3 \cosh^3 kh}{2k \sinh^2 kh} (AA^*)_{x_1} - \frac{\omega^3}{4 \sinh^2 kh} (AA^*)_{t_1}. \]

(7.82)

The first of these results, (7.81), is the nonlinear Schrödinger equation for water wave propagation, describing the modulation of a wave train by cubic interactions and interaction with the mean flow. The second equations describes the evolution of the mean flow potential forced by the wave train.

### 7.5.2 Comparison with the equation for \( A \)

We now compare the equation for \( A \), (7.81), to the reduced model, (7.45). First, we rewrite (7.81) to the case of periodic flow and substitute the amplitude of the potential for the amplitude of the surface elevation.

**Rewriting to periodic flow** For a periodic flow, the physical variables have no dependence of the slow time variable \( t_1 \). The first term in (7.81) is therefore zero and multiplication
with the operator \( L = 1 + i \varepsilon \frac{1}{2} \frac{\partial^2 \omega}{\partial x^2} \frac{\partial}{\partial x_1} \) makes the \( A_{x_1} \)-term disappear as a higher-order term. The equation can therefore be written

\[
c_g \frac{\partial}{\partial x_1} A + i \varepsilon \left( \frac{\omega k^2 (\cosh 4kh + 8 - 2 \tanh^2 kh)}{16 \sinh^4 kh} |A|^2 A \right.
- \left. \left( \frac{k^2}{2\omega \cosh^2 kh} \frac{\partial \phi_{10}}{\partial t_1} - k \frac{\partial \phi_{10}}{\partial x_1} \right) A \right) = 0. \tag{7.83}
\]

Next, we match the first-order solution of the potential with the potential within our model by combining (7.22) and (7.77), further utilizing that the mean flow potential \( \phi_{10} \) is not a part of the first-order solution within our model. We get

\[
- \frac{g}{2\omega} \left( i A e^{i(kx-\omega t)} + \text{c.c.} \right) = \xi_1 a_1^{(0)} e^{i(\omega t-kx)} + \text{c.c.} \tag{7.84}
\]

We now introduce

\[
b_1 = \xi_1 a_1^{(0)} \tag{7.85}
\]

in agreement with the expansion (6.27). We then get

\[
A^* = -2i \frac{\omega}{g} \xi_1 a_1^{(0)} = -i \frac{2\omega}{g} b_1. \tag{7.86}
\]

As a last intermediate rewriting, we express \( \phi_{10, x_1} \), the current velocity associated with the mean flow, as \( q_1/h \).

Insertion of the above results into the complex conjugate of (7.83) gives

\[
\xi_1 b_{1,x_1} = \frac{4i \varepsilon \omega^2 \omega k^2 (\cosh 4kh + 8 - 2 \tanh^2 kh)}{c_g g^2 16 \sinh^4 kh} |b_1|^2 b_1
- i \varepsilon \left( \frac{k^2}{2\omega \cosh^2 kh} \phi_{10,t_1} - k \frac{q_1}{h} \right) b_1. \tag{7.87}
\]

This equation can now be compared to the reduced model of Section 7.3.1.

**Comparison with the reduced model**  The nonlinear Schrödinger equation describes the slow modulation of a uniform wave train due to interactions with the second-order bound wave components and cubic self-interaction. To model that within the reduced model, we simply insert a wave field with just one frequency:

\[
\phi = (a^{(0)} + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)}) \xi_1 e^{i(\omega t-k_1x)} + \text{c.c.} \tag{7.88}
\]
Writing \(a^{(0)} \xi_1 = b_1\) the reduced model (7.45) now gives

\[
b_{1,x} = 2 \left( \frac{i k_1}{2 c_{g,1} h} q_1 - \frac{i k_1^2}{4 \omega_1 c_{g,1} \cosh^2 k_1 h} \phi_{0,t} \right) \hat{\phi}_1 \]

\[
- 2i \text{OP}_{4,2,-1} \{-2k_1, k_1\} \frac{\text{OP}_{4,1,1} \{-k_1, -k_1\}}{k_2 - 2k_1} |b_1|^2 b_1 \]

\[
- \left( \text{OP}_{5,1,1,-1} \{-k_1, -k_1, k_1\} + \text{OP}_{5,1,-1,1} \{-k_1, k_1, -k_1\} \right) |b_1|^2 b_1 \quad (7.89)
\]

The first line describes the interaction with the mean flow, the second the subharmonic interaction between the second-order bound harmonic and the primary wave train and the third and fourth line the cubic self-interaction of the wave train with itself. The first line matches the mean flow terms in (7.87) exactly. For the cubic terms, the operators are evaluated to the total term

\[
b_{1,x \text{ cubic}} = \frac{i \kappa^4 (20 + 13 \cosh 2 \kappa + 2 \cosh 4 \kappa + \cosh 6 \kappa)}{4gh^4(2\kappa + \sinh 2\kappa) \sinh^2 \kappa \cosh^2 \kappa} |b_1|^2 b_1 \quad (7.90)
\]

with \(\kappa \equiv k_1 h\). This expression can be shown to be identical to the term in (7.87). The evaluation of the operators and the comparison of the two cubic terms were carried out in Mathematica sheet.

### 7.5.3 Comparison with the mean flow equation

Having compared the reduced model successfully to the governing equation for the wave amplitude \(A\), we now compare the expression for \(q_1\), (7.61), to the equation for the mean flow (7.82).

**Rewriting to periodic flow** For a periodic flow, both the first and the last terms in (7.82) are zero. For the first term this is easily seen by taking the \(t_1\)-derivative of (7.79). All other terms than \(\phi_{10,t_1 t_1}\) in the resulting equation are \(t_1\)-derivatives of physical variables and therefore vanish.

We now integrate (7.82) with respect to \(x_1\). Substituting \(h\phi_{10,x_1} = q_1\) we obtain

\[
[q_1]^{x_1}_{x_0} = - \left[ \frac{\omega^3 \cosh^3 k h |A|^2}{2gk \sinh^2 k h} \right]^{x_1}_{x_0} \quad (7.91)
\]

Further substitution of (7.86) leads to

\[
q_1(x) = q_1(x_0) - \left[ \frac{2\omega k}{g} |b_1|^2 \right]^{x_1}_{x_0} \quad (7.92)
\]

This matches (7.62) of the present model, since \(|b_1| = |\hat{\phi}_1|\).
7.5.4 Comparison of set-down expressions

Within the nonlinear Schrödinger equation, the set-down is given as part of the second-order solution for the free surface elevation (7.79). Time averaging this result yields

\[ \bar{\eta} = \langle \bar{\eta} \rangle = -\frac{1}{g} \langle \phi_{10,t_1} \rangle - \frac{k}{2\sinh 2kh}|A|^2. \] (7.93)

As we have argued, \( \phi_{10,t_1} \) is constant in space. Using (7.86) we can therefore rewrite (7.93) to

\[ \bar{\eta}(x) = \bar{\eta}(x_0) - \left[ \frac{2\omega^2 k}{g \sinh 2kh} |\hat{\phi}_1|^2 \right]_{x_0} x = \bar{\eta}(x_0) - \left[ \frac{k^2}{g \cosh^2 k h} |\hat{\phi}_1|^2 \right]_{x_0}, \] (7.94)

where we have used \( \omega^2 = g k \tanh kh \) and identities for the hyperbolic functions.

Within our model, the set-down is given by (7.52). Again, the cubic terms do not contribute to the mean flow when only one frequency is present, and we can therefore use (7.53). We get

\[ \bar{\eta} = \bar{\eta}(x_0) + \varepsilon \left[ \left( -\frac{k^2}{g} + \frac{\omega^4}{g^3} \right) |\hat{\phi}_1|^2 \right]_{x_0} = \bar{\eta}(x_0) - \varepsilon \left[ \frac{k^2}{g \cosh^2 k_1 h} |\hat{\phi}_1|^2 \right]_{x_0}, \] (7.95)

which is identical to the above result for the NLS-equation.

With the above comparison we have checked the cubic terms for the wave field and the quadratic terms within the expressions for the mean flow variables against the NLS-equation. For all tests identical results have been found. The quadratic bottom-slope terms are not part of this test and have therefore not been checked by the above calculations.

7.6 Wave group evolution on constant depth

Here a first attempt of repeating test 1 of Shemer et al. (2001) is presented. First, the test and its results are outlined, citing Shemer et al. Next, the simulation with the reduced model and the results of this are presented.

7.6.1 Test 1 of Shemer et al. (2001)

Shemer et al. (2001) studied the evolution of wave groups in a wave flume of constant depth, \( h = 0.6 \) m. The waves were generated by a paddle wave generator, hinged near the bottom. For test 1, the signal driving this generator was

\[ s(t) = s_0 \cos(\Omega t) \cos(\omega_0 t), \quad \Omega = \omega_0/20. \] (7.96)
The carrier frequency was $\omega_0 = 2\pi/T_0$ with $T_0 = 0.9$ s. The forcing signal (7.96) generates a wave signal with a bimodal spectrum. The spectral peaks have the frequencies $\omega = \omega_0 \pm \omega_0/20$.

The measured spectrum of the free surface elevation is depicted in Figure 7.2 for $x = 0.24$ m (upper left frame). The two peaks do not have the same amplitude, probably because the transformation from the paddle signal to the free surface elevation is frequency dependent. Besides the dominant linear peaks, super-harmonic wave components can be seen at the double frequencies and at the sum frequency of the two linear peaks. As the wave groups travel down the wave flume, the frequencies close to the initial peaks gain in energy and the spectrum becomes more wide. This can be seen in the upper right frame of Figure 7.2 for $x = 9.47$ m.

Figure 7.2: Figure 5 of Shemer et al. (2001). Left: $x = 0.24$ m, right $x = 9.47$ m. Upper: measured, lower: simulated.

The corresponding time series are depicted in Figure 7.3. For $x = 0.24$ m, the time series measured is shown in the upper left frame. The spatial evolution of the time series can
be followed frame by frame in the left column. The bottom frame is for $x = 9.47$ m, and the wave groups are seen to have developed a highly asymmetric envelope. The maximum surface elevation for this asymmetric group is around 6 cm, thus being significantly larger than for the initial signal with a maximum elevation of around 4 cm.

Figure 7.3: Figure 7 of Shemer et al. (2001). Left: measured, right: simulated. Upper: $x = 0.24$ m, lower: $x = 9.47$ m.

Simulations of Shemer et al. (2001) The experiments were reproduced numerically, using a spatial version of the Zakharov equation. The transformation of the Zakharov equation from temporal to spatial evolution is described in the same paper.

The incoming wave was described by

$$\eta(x, t) = a \cos(2\pi(19\Delta f)t) + a \cos(2\pi(21\Delta f)t)$$

(7.97)
with \( a = 0.022 \text{ m} \) (measured from Figure 5 of the paper) and \( \Delta f = f_1 = \frac{1}{20}2\pi/T_0 \) with \( T_0 = 0.9 \text{ s} \). The linear spectrum was resolved with the 12 frequencies
\[
f = (13, 15, \ldots, 35)\Delta f.
\]
The convergence of the results as function of the number of frequencies was investigated in the paper, and it was found that the above 12 frequencies were sufficient for the present test.

Shemer et al. considered the influence of modelling the incoming wave spectrum with different amplitudes of the initial peaks, corresponding to the initial spectrum measured, see Figure 7.2. The spectra of such a simulation are shown in the middle panels of Figure 7.2. They noted that the asymmetric form of the initial spectrum did not give much different results than if a spectrum with equal peak amplitudes was used. Using equal initial amplitudes correspond to (7.97), and the resulting spectra are shown in the bottom panels of Figure 7.2. Qualitatively, the simulated spectra agree well with the experimental results.

Time series of the simulated wave groups are shown as the right column of Figure 7.3. The simulations reproduce the evolution of the envelope quite well, even though the forms are not identical in \( x = 9.47 \text{ m} \). For this location, however, the maximum value of the free surface elevation is around 6 cm, agreeing with the experimental value.

### 7.6.2 Simulations with the reduced model

We now describe a simulation similar to the above, carried out with the reduced model. Following Shemer et al. (2001), the range of frequencies in the resolved spectrum is taken to be
\[
f_{\text{free}} = (13, 15, \ldots, 35)\Delta f
\]
that is, 12 free wave modes. These wave modes generate bound waves within the frequency range
\[
f_{\text{bound}} = (2, 4, \ldots, 70)\Delta f,
\]
which are taken into account in the cubic summations.

To get an initial condition for the amplitude spectrum of the potential, we transform (7.97) linearly. The linearized free surface boundary condition is \( g\eta = -\phi_t \), which in Fourier space reads
\[
g\hat{\eta} = g\frac{a}{2} = -i\omega\hat{\phi}.
\]
From this we get
\[
\hat{\phi}_{19} = \frac{ig}{2\omega_{19}}a = 1.627 \cdot 10^{-2}\text{m}^2/\text{s}
\]
\[
\hat{\phi}_{21} = \frac{ig}{2\omega_{21}}a = 1.472 \cdot 10^{-2}\text{m}^2/\text{s}
\]
The set-down was taken to be zero, initially, while the mean flux below wave trough level, \( q_1 \), was set to
\[
q_1(x) = -\frac{2}{g} \sum_{p=\{19,21\}} \omega_p k_p |\hat{\phi}_p|^2
\]
according to (7.62). This corresponds to a zero net mass flux in the simulations. All other free wave modes are initiated with zero amplitude.

**Calculating \( \eta \)** The solution variable of the reduced model is \( \tilde{a}(0) = a(0) + \varepsilon^2 a^{(2)} \). With an error of magnitude \( O(\varepsilon^3) \), the linear amplitudes of \( \phi \) can be found directly from this by multiplying with \( \xi_p e^{-i\psi_p} \). Given the amplitude spectrum of the linear modes of \( \phi \) in a location, the second-order bound components can be calculated using (7.40). Next, the amplitude spectrum of the surface elevation \( \eta \) can be calculated from (7.8). This transformation can be carried out in the time domain, by multiplying time series of the various temporal and spatial derivatives of \( \phi \). In all terms one can use
\[
\hat{\phi}_{p,t} = i\omega_p \hat{\phi}_p
\]
since this is an exact relation. For the \( x \)-derivatives we have on constant depth
\[
\hat{\phi}_{p,x} = \left( \hat{\phi}_{p}^{(0)} + \varepsilon \hat{\phi}_{p}^{(1)} + \varepsilon^2 \hat{\phi}_{p}^{(2)} \right)_x = -ik_p \hat{\phi}_{p}^{(0)} + \varepsilon \hat{\phi}_{p,x}^{(1)} + O(\varepsilon^2),
\]
where the superscripts \((0), (1) \) and \((2) \) denote the linear, second-order and third-order parts of \( \phi \), respectively. In the cubic terms of (7.8) the first term of the above equation suffices, but in the quadratic term of (7.8), the second term in the above result must be taken into account. The quadratic term of (7.8) is thus expressed as
\[
\frac{1}{2} \varepsilon \phi_x^2 = \frac{1}{2} \varepsilon \left( \phi_x^{(0)} \right)^2 + \varepsilon^2 \phi_x^{(0)} \phi_x^{(1)} + O(\varepsilon^3),
\]
where \( \phi_x^{(1)} \) is expressed through (7.40), multiplied by \(-i(k_s + k_{p-s}) \) in each term.

**Results** In Figure 7.4 the amplitude spectra of \( \phi \) and \( \eta \) are shown for \( x = 0.24 \) m and \( x = 9.47 \) m. For the spectrum in \( \phi \) (upper frames) a distinction between the bound and free wave field can be made, since the model returns the free wave amplitudes (plus the cubic contribution at the frequencies resolved), and the bound second-order wave field must be calculated separately. For the results in \( \eta \), however, no distinction is possible. We see that in the transformation from \( \phi \) to \( \eta \), the superharmonics become larger, while the subharmonics become smaller. The reason for this is the linear part of the transformation (7.8), which involves multiplication with \( \omega \).

Comparing the spectra between the two stations, we see that energy is transferred to frequencies higher than the those of the initial peaks. The spectra become wider. Qualitatively,
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Figure 7.4: Spectra of $\phi$ (upper frames) and $\eta$ (lower frames) for $x = 0.24$ m and $x = 9.47$ m.

the spectrum for $\eta$ in $x = 9.47$ m matches the spectrum of Shemer et al. (2001) (Figure 7.2, lower right frame) quite well. Around $f = 2$ Hz, the amplitudes of the reduced model are slightly larger than those of Shemer et al.

Time series for $x = (0.24, 3.69, 6.58, 9.47)$ m are shown in Figure 7.5. We see that the envelope of the wave groups is becoming increasingly asymmetric down the channel. This agrees qualitatively with the experimental and numerical results of Shemer et al. The steepening of the envelope, however, is not as strong as in the results of Shemer et al. The maximum amplitude in $x = 9.47$ m, predicted by the reduced model, is about 4.4 cm, which is much smaller than the 6 cm of the experimental and numerical result of Shemer et al.

It therefore seems that the numerical model is not calculating the evolution of the wave field correctly. The error may exist in the implementation of the evolution equations as well as in the transformation from $\phi$ to $\eta$. There may, of course, also be an error in the derivation of the reduced model (7.45). For a single wave, this model has been checked analytically against the Nonlinear Schrödinger Equation, and a perfect match has been found. Further, it has been checked that the numerical implementation of the right hand side of the reduced model (7.45) agrees with the right hand side of the Nonlinear Schrödinger Equation in the form (7.87). Also here a perfect match was found. It therefore seems that the error is related to the interaction of waves at different frequencies. Given that the simulated spectra match the results of Shemer et al. reasonably well, it seems like the error is related to the phases of the wave field.

Unfortunately, this error has not been found within the time limits of this Ph.D. study. More work should be put into finding the reason for this mistake.
Figure 7.5: Time series for $\eta$ obtained with the reduced model.
7.7 Summary and conclusions

In this chapter, a cubic model for shoaling of waves has been derived. The model is valid for a mildly sloping bottom and no assumptions on the width of the spectrum are made. The model can therefore be seen as a spatial version of the Zakharov model, valid for a mildly sloping bottom. To the authors' knowledge, such a model has not earlier been presented in the literature.

The model is given in two different versions. The so-called reduced model is formulated in the free wave amplitudes plus the cubic bound wave field. This corresponds to the Zakharov model's formulation in the free wave amplitudes. The resulting model only contains terms of magnitude $O(\varepsilon^2)$, and is computationally advantageous, since only the free wave field's frequencies need to be treated. Once the model is solved, the second-order bound wave field can be calculated and added in the locations desired. Alternatively, an unreduced model, formulated in the total wave field at each frequency has been presented. This model includes terms of magnitude $O(\varepsilon)$ as well as $O(\varepsilon^2)$.

The mean flow has been described through expressions for the set-down and the mean flux below wave trough level. To second order, the expressions match Stokes second-order theory. The feedback of the mean flow on the wave field has been treated as well and matches results for Stokes theory to second order. For a single wave, the reduced model has been compared to the Nonlinear Schrödinger Equation, including the mean flow results. A perfect agreement was found.

A first attempt of simulating wave group evolution on constant depth has been presented, following Shemer et al. (2001). The reduced model was used for this, and the evolution of the spectra matches the experimental and numerical results of Shemer et al. qualitatively well. However, it appears that there is an error in the phases of the wave field, since the time series in the down-wave part of the wave flume only match the results of Shemer et al. in a qualitative sense. For future work it is therefore recommended to trace back this discrepancy.
Chapter 8

Conclusions

In this study, various aspects of deterministic evolution equations have been treated. The findings of the various chapters are outlined below. After this, appropriate further developments are discussed and suggestions for future work are given.

**Boussinesq evolution equations** The initial motivation of the study was the possibility of incorporating the increasing accuracy of time domain Boussinesq formulations into corresponding evolution equations. Thus in Chapter 2, a set of evolution equations for Nwogu’s (1993) equations has been derived, and the possibility of avoiding the various approximations in the transformation from the time domain formulation to the frequency domain investigated. It has been found that it is not possible to retain the higher-order linear derivatives of the formulation, if the parabolic solution method is to be used. It is possible to formulate a two-equation model, using the complex amplitudes of the free surface elevation and the complex wave amplitudes of the velocity variable as dependent variable. This formulation corresponds to retaining the second derivative in a one-equation formulation. However, as the two-equation model suffers from spurious reflections on varying depth, such a model is not attractive. Further, as the evolution equations incorporate only quadratic nonlinearity, the overall conclusion of Chapter 2 is that evolution equations should be used due to their simplicity and fast numerical treatment, rather than their accuracy.

**Efficiency of evolution equations** With basis in this conclusion, attention has been paid to the numerical efficiency of evolution equations. Thus in Chapter 3 it has been demonstrated how the nonlinear terms of the Boussinesq evolution equations of Madsen and Sørensen (1993) can be calculated by the aid of FFT. If $N$ frequencies are modelled, this technique reduces the computational effort from $O(N^2)$ to $O(N \log N)$. The practical performance of the evolution equations with this speed-up and the corresponding time domain model has been investigated for regular and irregular waves passing a submerged bar. For each model, a curve relating the accuracy to the CPU-time has been made. By comparing
these curves, it has been found that evolution equations are around 1000 times faster than the time domain model for regular waves and around 100 times faster for irregular waves, for the test cases investigated. The main reason for this difference is the prescription of the linear phase variation within the evolution equations, making a larger grid spacing possible. Further, the Courant number criterion within the time domain model enforces a much smaller time step than the time step corresponding to describing the highest frequency of interest at the Nyquist frequency. Moreover, for regular waves, a considerable warm-up time of the computational domain is needed for the time domain model.

**Incorporation of surface roller breaking**  Quadratic evolution equations are appropriate for modelling triad interactions among water waves in shallow water. A natural extension is therefore to incorporate a model for wave breaking. The use of FFT to calculate time series of the wave field locally, makes it possible to incorporate breaking formulations from time domain Boussinesq models. In Chapter 4 the incorporation of surface roller breaking into the Boussinesq evolution equations of Madsen and Sørensen (1993) has therefore been pursued. The resulting breaking model has been validated against experimental data of regular spilling breaking waves, and results of the time domain Boussinesq model of Madsen et al. (1997a) with surface roller breaking. The new model is able to describe the wave height decay and variation of the mean water level satisfactorily, but two problems appear in the results: 1) the phase speed of the waves is too large and 2) the shape of the breaking waves does not resemble the saw-tooth shape of the experimental results and the results of the time domain model. The surface roller breaking model for evolution equations has been compared to a conventional bulk dissipation formulation as well. When the breaking dissipation was uniformly distributed over the frequencies, even poorer results for the profile shape were obtained, while a weighting of the dissipation according to the squared frequency, gave profiles of similar quality as for the roller formulation in the frequency domain.

**Amplitude dispersion in evolution equations**  The over-prediction of the phase speed in evolution equations, observed for the breaking wave tests, motivates an investigation of amplitude dispersion within evolution equations. A third-order weakly nonlinear analysis as well as a fully nonlinear analysis have thus been presented in Chapter 5. The weakly nonlinear analysis shows that the evolution equations of Madsen and Sørensen (1993) over-predict the amplitude dispersion in the shallow water zone with up to a factor of 2, when comparing to third-order Stokes waves with a zero net mass flux. At deeper water, the amplitude dispersion decays towards zero. Compared to the same reference solution, also the corresponding time domain model, in it’s ‘simplified’ form (i.e., strictly quadratic nonlinear terms), over-predicts the amplitude dispersion in the shallow water zone. This overprediction, however, is much smaller than for the evolution equations. The deviation in amplitude dispersion between the models has been found to be caused by the neglect of the derivatives of the wave amplitudes in the nonlinear terms of the evolution equations. This finding was supported by the results of the fully nonlinear analysis. The fully nonlinear analysis also shows that the over-prediction of the amplitude dispersion becomes extra strong in the shallow water limit.
Fully dispersive evolution equations, 2nd order  Even though the problems of amplitude dispersion may be eliminated by taking the derivatives of the wave amplitudes in the nonlinear terms into account, another track was followed instead. As an alternative to the Boussinesq evolution equations of Madsen and Sørensen (1993), fully dispersive evolution equations have thus been considered. In Chapter 6, a new derivation of these models has been given, making no assumptions on the vertical structure of the velocity field. The resulting formulation has exact second-order bichromatic transfer functions, when comparing to Stokes-type wave theory. This is a new development in the field of quadratic evolution equations. Unfortunately it turns out that the interaction coefficient of this model does not allow for the computational speed-up of using FFT to calculate the nonlinear terms. However, it is possible to develop an approximation to the exact interaction coefficient that allows for the speed-up. A suggestion for such an approximation has been given. Also, for the so-called resonant models (Agnon et al., 1993; Kaihatu and Kirby, 1995; Eldeberky and Madsen, 1999), the FFT-speed up can be used. The amplitude dispersion of fully dispersive evolution equations has been analyzed using the weakly nonlinear approach of Chapter 5. For the models formulated in $\eta$, the amplitude dispersion can be over-predicted with as much as a factor of 5, when comparing to third-order Stokes waves with a zero net mass flux. While the models in $\eta$ over-predict the amplitude dispersion for all frequencies, the models formulated in $\phi$ over-predict the amplitude dispersion in shallow and intermediate water, while the amplitude dispersion decays towards zero in deep water. The transfer to the third harmonic is found to be over-predicted for the models in $\eta$ for all depths, while the models in $\phi$ show an over-prediction in shallow water and an under-prediction in intermediate and deep water.

Although fully dispersive models in $\eta$ do not have better amplitude dispersion properties than the Boussinesq evolution equations of Madsen and Sørensen (1993), roller breaking has been incorporated into these models. A differential equation for the mean water level has been derived as well, while an extension to ensure the conservation of the net mass flux in the surf zone has not been pursued. The breaking model is able to describe the wave height decay and setup of the wave field, but the model fails to describe the saw-tooth shape of breaking waves. The point of maximum elevation in the crest appears to be moved backwards within the wave profile, thus producing a local tilting away from the beach. This behavior is not observed for the Boussinesq evolution equations with roller breaking, and it is suggested that the deviation is due to differences in the nonlinear transfer functions of the models. Comparisons to a conventional bulk dissipation model have been made as well. While a uniform distribution of the dissipation results in highly symmetric waves, a weighting of the dissipation according to the squared frequency produces results of slightly better quality than those obtained with the roller model. Fully dispersive evolution equations in $\eta$ should therefore be used due to their accurate first-order and second-order characteristics rather than due to an accurate description of the wave profiles for high waves. For wave breaking, a bulk dissipation model with $f^2$-weighted breaking can be used.

Fully dispersive evolution equations, 3rd order  As an extension to the fully dispersive quadratic evolution equations, a cubic formulation in $\phi$ has been derived. The model is valid
on a mildly sloping sea bed and describes the deterministic evolution of a discrete spectrum of any width. The model can therefore be seen as a spatial Zakharov-type model, being valid on a sloping sea bed. Such a model has not been derived earlier in the literature. Two versions of the model have been presented: an unreduced model, resolving the spectrum of the free waves as well as the bound waves and a reduced model, describing the evolution of the free wave amplitudes plus the cubic contribution at each frequency. This model can be solved with a spectrum of smaller bandwidth than the unreduced model, since the second-order bound wave field is accounted for implicitly. Equations for the wave-induced variations of the mean water level and the mean flux below wave trough level have been derived, and the feedback of these quantities to the wave spectrum has been accounted for. As an analytical check, the reduced model has been compared to the Nonlinear Schrödinger Equation for constant depth and time periodic wave motion. A perfect agreement has been found.

A numerical simulation of wave group evolution on constant depth, following Shemer et al. (2001) has been presented for the reduced model. Although the model results for the spectral amplitudes resemble the results of Shemer et al., there appears to be an error in the description of the phases. The time series of the free surface elevation calculated with the reduced model do not show the same degree of asymmetry in the wave group envelopes as the experimental and numerical results of Shemer et al. The value of the maximum surface elevation for the modulated wave groups is underestimated as well. These discrepancies are believed to be due to an error in the numerical implementation of the model.

8.1 Suggestions for future work

The results of this thesis show that the asset of evolution equations is their computational efficiency rather than their accuracy. The use of FFT for calculating the nonlinear terms makes simulation of irregular seas with a large number of frequencies possible. Fully dispersive evolution equations have exact linear characteristics for all depths, and although the models with exact second-order characteristics cannot be speeded up using FFT, an approximate version of the models can be used, having almost exact second-order characteristics.

To make the models attractive for engineering use, a two-dimensional version of the models would be desirable. The model of Kaihatu and Kirby (1995) incorporates the effect of weakly two-dimensional wave propagation. The computational effort of solving this model is proportional to the number of points in the long-shore direction. The model of Eldeberky and Madsen (1999) is valid for two-dimensional wave propagation described through a directional spectrum. The depth is assumed to be uniform in the $y$-direction. Given $N$ frequencies and $M$ directional wave modes, the computational effort of solving this model is $O(M^2N \log N)$, utilizing FFT for the frequency convolution. A similar treatment of the directional modes should be straightforward, thus resulting in a directional model, requiring a computational effort of $O(M \log MN \log N)$. Further, a weak $y$-dependence of the depth has been treated.
8.1 Suggestions for future work

in wide-angle models (Dalrymple et al., 1989; Chen and Liu, 1995) and incorporation of a weakly \( y \)-dependent depth is therefore possible as well. A development of such a wide-angle model would be of great practical interest due to its computational efficiency for calculating the evolution of a directional spectrum.

The problems of amplitude dispersion may be solved within the Boussinesq evolution equations by retaining derivatives of the wave amplitudes in the nonlinear terms. For fully dispersive evolution equations, an improved description of the amplitude dispersion would be attractive, since this would lead to a better description of the high waves.

For the spilling breaker test of Ting and Kirby (1994) for regular waves, the surface roller formulation of wave breaking does not lead to better results than a conventional bulk dissipation model with \( f^2 \)-weighting. For an irregular sea, the roller breaking formulation may perform better than a conventional breaking model, since it works directly on the actual breaking events in a time series rather than on the global spectrum. A test of irregular waves, comparing the performance of the roller model and a conventional breaking model is therefore recommended. A breaking test on a submerged bar would be in favour of the roller breaking model, if not all waves in a time series break as they pass the bar.

Within the fully dispersive equations, an equation describing the mean volume flux below wave trough level in the presence of surface rollers is a recommended extension as well. The purpose of such an extension is to ensure that the net mass flux is preserved through the surf zone. For the Boussinesq evolution equations of Madsen and Sørensen (1993), the conservation of the net mass flux is automatically satisfied, and therefore such an extension is not needed for this model.

The cubic model derived is the first model in the literature to describe the spatial evolution of a wide spectrum to cubic order on a sloping sea bed. The failure of the present implementation calls for a further investigation. Also numerical tests on a sloping bottom are recommended, to investigate the influence of the new bottom-slope terms.

Summarizing the above discussion, the following points are thus suggested for future work:

- Develop a 2-dimensional fully dispersive model for directional wave spectra, utilizing FFT for both the frequency spectrum and the directional spectrum.
- Carry out a comparison of the surface roller breaking model and a conventional breaking model in evolution equations for irregular waves passing a submerged bar.
- Develop an enhancement of the fully dispersive equations that improves the description of amplitude dispersion.
- Incorporate an equation for the return current in the fully dispersive model which is valid during wave breaking.
- Trace back the error in the cubic model and carry out calculations on sloping sea beds.
Chapter 8. Conclusions
References


References


Appendix A

The Product of Two Fourier Series

In this appendix we will calculate the product of the two Fourier series

\[ S_1 = \sum_{n=-N}^{N} A_n e^{in\omega t}, \quad A_{-n} = A_n^* \]  
(A.1)

\[ S_2 = \sum_{n=-N}^{N} B_n e^{in\omega t}, \quad B_{-n} = B_n^* \]  
(A.2)

The product is, by simple multiplication

\[ S_1 S_2 = \sum_{n=-N}^{N} \sum_{m=-N}^{N} A_n B_m e^{i(n+m)\omega t} \]  
(A.3)

which can be rewritten by inspecting Figure A.1. The figure shows the \((n, m)\) space, and a line satisfying the relation \(n + m = p\) is drawn. We here only consider positive (or zero) values of \(p\). All pairs of \((A_n, B_m)\) along such a line give rise to terms having the argument \(ip\omega t\) in the exponential function in (A.3). We can therefore write the product (A.3) as

\[ S_1 S_2 = \sum_{p=-2N}^{2N} e^{ip\omega t} \sum_{s=p-N}^{N} A_s B_{p-s}. \]  
(A.4)

**Rewriting the inner sum** The inner sum in (A.4) contains both positive and negative indices in the subscripts of \(A\) and \(B\). This short form is useful when deriving the equations, but in practical applications, the negative valued coefficients will not be available. Their
Appendix A. The Product of Two Fourier Series

Figure A.1: Left: Space of \( m \) and \( n \) for calculating the product of two Fourier series. Right: Splitting of inner sum.

values can be expressed by the symmetry relations in (A.1)–(A.2). We now express the product by positive valued coefficients only. First we split up the summation:

\[
\sum_{s=p-N}^{N} A_s B_{p-s} = \sum_{s=p-N}^{-1} A_s B_{p-s} + \sum_{s=0}^{p} A_s B_{p-s} + \sum_{s=p+1}^{N} A_s B_{p-s} \tag{A.5}
\]

This is illustrated in Figure A.1, where the upper arrow is equivalent to the first sum, the middle arrow equivalent to the second sum, and the lower arrow to the last sum. Note that the first and third sum vanishes when \( p = N \).

We now manipulate the first sum by substituting \( s = -s \) and the last sum by shifting \( s \) with the amount \( -p \). This yields

\[
\sum_{s=p-N}^{N} A_s B_{p-s} = \sum_{s=1}^{N-p} A_{-s} B_{p+s} + \sum_{s=0}^{p} A_s B_{p-s} + \sum_{s=1}^{N-p} A_{p+s} B_{-s} \tag{A.6}
\]

which we rewrite using \( A_{-p} = A_{p}^*, B_{-p} = B_{p}^* \). The result is

\[
\sum_{s=p-N}^{N} A_s B_{p-s} = \sum_{s=0}^{p} A_s B_{p-s} + \sum_{s=1}^{N-p} [A_s^* B_{p+s} + A_{p+s} B_s^*] \tag{A.7}
\]

Again it is emphasized that the last sum should not be calculated when \( p = N \).
Appendix B

More Results of the Boundary Value Problem Formulation

We here bring results for harmonic generation for all three test cases investigated. The wave parameters of each test is given in Table 2.1, which is repeated here for convenience.

<table>
<thead>
<tr>
<th>Case</th>
<th>$h$</th>
<th>$T$</th>
<th>$a_0$</th>
<th>$l_0$</th>
<th>$\varepsilon$</th>
<th>$\mu$</th>
<th>$U$</th>
<th>$L_{\text{beat, est}}$</th>
<th>$L_{\text{beat, obs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test A</td>
<td>0.4</td>
<td>2.5</td>
<td>0.042</td>
<td>4.74</td>
<td>0.11</td>
<td>0.084</td>
<td>14.8</td>
<td>15.4</td>
<td>14</td>
</tr>
<tr>
<td>Shallow</td>
<td>0.4</td>
<td>4.0</td>
<td>0.021</td>
<td>7.79</td>
<td>0.053</td>
<td>0.051</td>
<td>19.9</td>
<td>72.0</td>
<td>60</td>
</tr>
<tr>
<td>Nonlinear</td>
<td>0.4</td>
<td>2.5</td>
<td>0.084</td>
<td>4.74</td>
<td>0.21</td>
<td>0.084</td>
<td>29.5</td>
<td>15.4</td>
<td>12</td>
</tr>
</tbody>
</table>

Table B.1: Parameters for test on harmonic generation. Repeated from Table 2.1.

The approximations investigated and the model comparisons used for this are outlined in Table B.2. The descriptions of the results are given after the figures.
Table B.2: Approximations, models used for investigating them and figure numbers for the three tests. ‘u’ and ‘l’ indicate upper and lower panel of figures.
Appendix B. More Results of the Boundary Value Problem Formulation

Figure B.5: Combined influence of approximation 1, 2 and 3: neglect of nonlinear derivatives and higher-order linear derivatives and elimination of $a_p$. Boundary value problem formulation and one-equation model. Upper: test A. Lower: shallow test.
Figure B.6: Combined influence of approximation 1, 2 and 3 and further neglect of $b_{p,xx}$. Boundary value problem formulation and crude one-equation model. Upper: test A. Lower: shallow test.
Figure B.7: Influence of approximation 1: neglect of derivatives in nonlinear terms. Boundary value problem formulation with and without nonlinear derivatives. Nonlinear test.
Figure B.8: Influence of approximation 2: neglect of higher-order linear derivatives. Boundary value problem formulation without nonlinear derivatives and two-equation model. Non-linear test.
Figure B.9: Combined influence of approximation 1 and 2: neglect of nonlinear derivatives and higher-order linear derivatives. Boundary value problem formulation and two-equation model. Nonlinear test.
Figure B.11: Combined influence of approximation 1, 2 and 3: neglect of nonlinear derivatives and higher-order linear derivatives and elimination of $a_p$. Boundary value problem formulation and one-equation model. Nonlinear test.
Figure B.12: Combined influence of approximation 1, 2 and 3 and further neglect of $b_{p,xx}$. Boundary value problem formulation and crude one-equation model. Nonlinear test.
B.0.1 Results for ‘test A’

The results of test A are described in Section 2.5.

B.0.2 Results for the shallow water test

The results of the boundary value problem model was obtained with 5 harmonics and a spatial step length of 0.4 m. The length of the domain was 80 m, with the sponge layer starting in $x = 60$ m. We see that approximation 1 increases the energy exchange between the harmonics and the beat length. Approximation 2 has almost no influence on the results for this case. The combined effect of approximation 1 and 2 is therefore practically identical to the isolated influence of approximation 1. The influence of eliminating $a_p$, can be studied in Figure B.5l and is not significant at all, when comparing to the results of the two-equation model, Figure B.4. Finally, neglecting $b_{p,xx}$ is seen to decrease the energy exchange and the beat length. The overall comparison of the crude one-equation model to the boundary value problem formulation is surprisingly good.

B.0.3 Results for the nonlinear test

The results of the boundary value problem model was obtained with 4 harmonics and a spatial step length of 0.2 m. The length of the domain was 45 m, with the sponge layer starting in $x = 30$ m. The number of harmonics could not be increased due to convergence problems. Contrary to the other two tests, bound wave numbers were used for the run of the full model (including nonlinear derivatives). For free wave numbers, no convergence was obtained.

The effect of approximation 1 is to increase the energy exchange and beat length. Approximation 2 has almost no influence on the energy exchange, while the beat length is reduced with a similar amount as the increase implied by approximation 1. The combined influence of approximation 1 and 2 is to increase the energy exchange for the first, second and third harmonics (approximation 1 dominates), while the fourth harmonics amplitude and the beat length are reduced (approximation 2 dominates). The consequence of introducing approximation 3 is a reduction in the energy exchange and an increase in beat length. We further see that the one-equation model has some phase problems. Further neglect of $b_{p,xx}$ results in a reduction in the energy exchange and a reduction in the beat length. For the crude one-equation model, the phases of the harmonics are very close to the phases of the boundary value problem formulation, while the amplitudes of the harmonics are under-predicted.

The influence of the approximations are summarized in Section 2.5.6.
Appendix C

The Fourth-Order Integration Scheme of Scraton (1964)

In Chapter 3, the fourth-order integration scheme of Scraton (1964) is used to integrate the evolution equation of Madsen and Sørensen (1993). The advantage of the scheme is that it does not evaluate the right-hand side of the ODE-system at the end point of the local integration interval. Hence, if a discontinuity in the right hand side appears in a point, only the solution after the point will be affected, thus preserving the fourth-order accuracy of the scheme. We here show the definition of the scheme.

The scheme is mentioned in Lambert (1973), page 132. We consider a system of ordinary differential equations defined by

\[ y' = f(x, y) \quad (C.1) \]

The solution is discretized on the grid \( x = (j - 1)h \), such that \( (x_j, y_j) \) defines points of the numerical solution. The integration scheme of Scraton (1964) is then

\[
y_{n+1} = y_n + h\left[ \frac{17}{162} k_1 + \frac{81}{170} k_3 + \frac{32}{135} k_4 + \frac{250}{1377} k_5 \right]
\]

with

\[
k_1 = f(x_n, y_n) \\
k_2 = f(x_n + \frac{3}{8}h, y_n + \frac{3}{8}hk_1) \\
k_3 = f(x_n + \frac{\frac{3}{8}h}{2}, y_n + \frac{1}{12}hk_1 + \frac{1}{4}hk_2) \\
k_4 = f(x_n + \frac{3}{8}h, y_n + \frac{3}{28}h(23k_1 - 81k_2 + 90k_3)) \\
k_5 = f(x_n + \frac{9}{40}h, y_n + \frac{9}{10000}h(-345k_1 + 2025k_2 - 1224k_3 + 544k_4)).
\]

The method requires five evaluations of the right hand side for taking a step. This is one more than the standard fourth-order Runge-Kutta method. However, for this method, an
error estimate can be expressed without extra evaluations of the right hand side. The local truncation error of the method can be estimated as

\[ T_{n+1} = hqr/s \]

with

\[ q = -\frac{1}{18}k_1 + \frac{27}{170}k_3 - \frac{4}{15}k_4 + \frac{25}{153}k_5 \]
\[ r = \frac{19}{24}k_1 - \frac{27}{8}k_2 + \frac{57}{20}k_3 - \frac{4}{15}k_4 \]
\[ s = k_4 - k_1 \]

The error estimator has not been used in the implementation of the integration scheme, since it was implemented with a fixed step length.
Appendix D

Beat Lengths of the Bar Test

The results in Figure 3.6 connects a given accuracy to a given CPU-time. For a given number of harmonics, we have seen that the wave field is fully resolved for a certain step size. This step size is rather easily related to the beat lengths of the harmonics.

A free wave is characterized by a constant wave amplitude. The amplitude of a bound wave, forced by a single pair of free waves, forms a circular path in the complex plane. This can be seen from the solution (2.20) of the crude one-equation model. The spatial length of one circulation along the circular path is given by the beat length, see (2.21). As an example, the wave amplitude of the fifth harmonic on top of the bar is plotted in Figure D.1. The left panel shows the real part of $a_5$, while the right panel shows the path in the complex plane of $a_5$. The fifth harmonic is not a pure bound wave, since the radius of the circle is not constant. This agrees with the amplitude plot in Figure 3.3, which shows that the harmonic amplitudes form a recurrence pattern. However, a large part of the amplitude variation is bound and has a well defined beat length.

The beat length can be estimated as the spatial difference between two successive maxima in the left panel of Figure D.1. We estimate this as $L_{\text{beat, obs}} = 0.7$ m. Since the wave field is dominated by the first harmonic on the bar top, it is reasonable to assume that the fifth harmonic is a bound harmonic of the first harmonic. A linear estimate of the beat length is therefore $L_{\text{beat, est}} = 2\pi / (k_5 - 5k_1) = 0.74$ m, which matches the observed beat length well. In Figure 3.6, we see that for five harmonics, the wave field is properly resolved by a step length of $dx = 0.25$ m.

In Table D.1, similar numbers for the other harmonics are given. From the table we see that 1) the observed beat length of a harmonic is always larger than the free wave length, 2) the linear estimate of the beat length, assuming that the harmonic is a bound wave of the primary wave, matches the observed beat length well, 3) the beat length decreases with the number of harmonics, and 4) the step size needed to resolve the wave field is around one half or one third of the beat length.
Figure D.1: Real and complex variation of $a_5$ on top of bar.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$k_p$ [m$^{-1}$]</th>
<th>$L_{\text{free}}$ [m]</th>
<th>$L_{\text{beat,obs}}$ [m]</th>
<th>$\frac{2\pi}{k_p - pk_{1}}$ [m]</th>
<th>$dx_{\text{max}}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.19</td>
<td>1.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6.72</td>
<td>0.93</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>11.05</td>
<td>0.57</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16.82</td>
<td>0.37</td>
<td>1.4</td>
<td>1.55</td>
<td>0.5</td>
</tr>
<tr>
<td>5</td>
<td>24.45</td>
<td>0.26</td>
<td>0.7</td>
<td>0.74</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>33.32</td>
<td>0.19</td>
<td>0.4</td>
<td>0.44</td>
<td>0.2 / 0.1</td>
</tr>
<tr>
<td>7</td>
<td>42.45</td>
<td>0.15</td>
<td>0.3</td>
<td>0.31</td>
<td>0.1</td>
</tr>
<tr>
<td>8</td>
<td>51.43</td>
<td>0.12</td>
<td>0.2</td>
<td>0.24</td>
<td>0.1 / 0.01</td>
</tr>
</tbody>
</table>

Table D.1: Observed and estimated beat lengths on top of the bar and largest resolving step size.
Hence we conclude that the step size required to resolve the wave field is dictated by the
beat length of the highest harmonic retained in the calculations. Two or three spatial steps per beat length is sufficient.

In this connection it should be mentioned that the fourth-order integration scheme used, calculates the right-hand-side of the ODE-system five times at each step. Hence, covering an oscillation with e.g., two spatial steps actually corresponds to evaluating the right-hand-side 10 times per oscillation.

Still, however, the relation between beat length and sufficient step size is clear.
Appendix D. Beat Lengths of the Bar Test
Appendix E

Calculation of Skewness and Asymmetry

The skewness and asymmetry of a wave record are higher-order statistical measures of non-linearity. The skewness measures the departure from symmetry around the mean water level (is simply the third moment around the mean water level), while the asymmetry is a measure of the asymmetry around a vertical line.

Skewness and asymmetry can be calculated by integrating the so-called bispectrum of the wave field. In this appendix, the bispectrum is defined, and it is shown how this integration can be reduced from a sum of $3N^2$ terms to a sum of $N^2/4$ terms using symmetry properties of the Fourier coefficients. Skewness can also be calculated directly from a time series. We derive a similar expression for the asymmetry.

E.1 Introduction

The bispectrum is a useful property of an irregular wave field. It is sometimes used as dependent variable in stochastic evolution equations. Defining the Fourier amplitudes of a wave field by

$$\eta(x, t) = \sum_{p=-\infty}^{\infty} a_p e^{ipw t},$$

(E.1)

the bispectrum in each spatial point is defined as

$$B(m, p) = a_m a_p a^*_m + p.$$  

(E.2)
The skewness of the wave field can be calculated in the time domain and from the Fourier coefficients by

\[ S = \frac{\langle \eta^3 \rangle}{(\langle \eta^2 \rangle)^{3/2}} = \frac{\sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} B(m, p)}{\left(\sum_{p=-\infty}^{\infty} a_p a_p^*\right)^{3/2}}. \]  

(E.3)

The equality sign in this equation is obtained by inserting (E.1) directly into the time domain expression and carrying out the averaging procedure.

The bispectrum has some symmetry properties e.g., \( B(p, m) = B(m, p) \), \( B(-m, -p) = B(m, p)^* \), \( B(p, -m - p) = B(m, p) \). This makes it possible to reduce the calculation of (E.3) considerably. Further, since \( \eta \in \mathbb{R} \) we have \( a_{-p} = a_p^* \). For a finite number of frequencies, \( N \), it can thus be shown that the skewness can be calculated as

\[ S = 3\sqrt{2} \frac{\text{Re} \left\{ \sum_{m=1}^{\text{floor}(N/2)} \sum_{p=m}^{N-m} \alpha(m, p) B(m, p) \right\}}{\left(\sum_{p=1}^{N} a_p a_p^*\right)^{3/2}}, \]  

(E.4)

where ‘\( \text{floor}(N/2) \)’ means the truncated integer value of \( N/2 \) and \( \alpha = 1/2 \) for \( m = p \) and \( \alpha = 1 \) otherwise. Similarly to the skewness, another property, the asymmetry, is defined as

\[ A = 3\sqrt{2} \frac{\text{Im} \left\{ \sum_{m=1}^{\text{floor}(N/2)} \sum_{p=m}^{N-m} \alpha(m, p) B(m, p) \right\}}{\left(\sum_{p=1}^{N} a_p a_p^*\right)^{3/2}}. \]  

(E.5)

The asymmetry is related to the Hilbert transform of \( \eta(t) \), as stated in e.g. Elgar and Guza (1985). However, as this statement is not very clear, we find the exact relation in this appendix. First, however, we prove the two above expressions for skewness and asymmetry.

E.2 Integrating the bispectrum

We now show how the summation intervals can be reduced considerably by utilizing the symmetry properties of the Fourier coefficients and the bispectrum. We assume \( a_0 = 0 \) throughout. As for the case of an infinite number of frequencies, we define the integrated bispectrum by

\[ \Gamma = \sum_{m=-N}^{N} \sum_{p=-N}^{N} B(m, p) = \sum_{m=-N}^{N} \sum_{p=-N}^{N} a_m a_p a_p^* \]  

(E.6)

where \( N \) is the number of positive frequencies. In the above equation, we have not yet taken into account that \(|m + p|\) exceeds \( N \) for some of the terms. We do this later on. First, we change the summation intervals to positive values of \( m \) and \( p \). We get

\[ \Gamma = \sum_{m=1}^{N} \sum_{p=1}^{N} a_p a_m a_{m+p}^* + a_p^* a_m^* a_{p+m} + a_p^* a_m a_{p-m} + a_p a_m^* a_{p-m}^*. \]  

(E.7)
For the first two terms, the summations are restricted by \( p + m \leq N \). This gives \( p \leq N - m \) as the upper limit of the inner summation. In the two last terms, \( p - m \) can take both positive and negative values. We split up the summation accordingly. When \( p - m \) is negative we require

\[
-N \leq p - m \leq -1
\]

and thus \( m - N \leq p \leq m - 1 \), where the lower limit is automatically satisfied. Similarly \( p - m \) positive imposes

\[
1 \leq p - m \leq N
\]

and thus \( m + 1 \leq p \leq N + m \). In this case, the upper limit is satisfied automatically. With these observations, we split (E.7) up in three sums:

\[
\Gamma = \sum_{m=1}^{N} \sum_{p=1}^{N-m} a_p a_m a_{p+m}^* + a_p^* a_m^* a_{p+m} + \sum_{m=1}^{N} \sum_{p=1}^{m-1} a_p a_m a_{m-p}^* + a_p^* a_m a_{m-p}^* + \sum_{m=1}^{N} \sum_{p=m+1}^{N} a_p a_{m-p}^* + a_p^* a_m a_{p-m}^*
\]

\[
= \Gamma_1 + \Gamma_2 + \Gamma_3.
\]

The number of terms in the above summation is \( 3N^2 \). In the last sum, we add \( m \) to \( p \) in all terms, and account for that in the summation limits. This gives

\[
\Gamma_3 = \sum_{m=1}^{N} \sum_{p=1}^{N-m} a_{p+m} a_m a_p + a_{p+m}^* a_m a_p = \Gamma_1.
\]

In the second sum, we interchange the order of summation in \( m \) and \( p \). Originally, the terms where calculated in the region \( 1 \leq (m, p) \leq N \) under the restriction (E.8). Solving (E.8) for \( m \) gives \( p + 1 \leq m \leq N + p \), where the upper limit is automatically satisfied. We can therefore write the second sum as

\[
\Gamma_2 = \sum_{p=1}^{N} \sum_{m=p+1}^{N} a_p a_{m-p} a_{m-p} + a_p a_{m-p} a_{m-p} = \Gamma_3.
\]

The equality with \( \Gamma_3 \) can be seen by interchanging \( m \) and \( p \) throughout the above expression.

By these manipulations we therefore have \( \Gamma_1 = \Gamma_2 = \Gamma_3 \) and we can write

\[
\Gamma = 3\Gamma_1 = 3 \sum_{m=1}^{N} \sum_{p=1}^{N-m} a_p a_m a_{p+m}^* + a_p^* a_m a_{p+m} = 6\text{Re} \left\{ \sum_{m=1}^{N} \sum_{p=1}^{N-m} B(m, p) \right\},
\]

and utilizing that \( a_0 = 0 \) we arrive at

\[
\Gamma = 6\text{Re} \left\{ \sum_{m=1}^{N-1} \sum_{p=1}^{N-m} B(m, p) \right\}.
\]
E.2.1 Using the symmetry of the bispectrum

Until now we have only used the symmetry $a_{-p} = a_p^*$ to map the integration of the bispectrum into the first quadrant in the $(m, p)$-plane. From the definition of the bispectrum, we recall that $B(m, p) = B(p, m)$. Hence there is symmetry around the line $m = p$ in the $(m, p)$-plane. This can be used to reduce the terms in the calculation of $\Gamma$ by another factor of 2.

In Figure E.1 the summation range of (E.15) is showed in the $(m, p)$ plane for $N$ odd. The symmetry-line $m = p$ is shown and we see that it is sufficient to carry out the summation in, say, the region above the line and multiply by two. However, a little care must be taken for the terms obeying $m = p$. These should only be taken into account once. For $N$ odd, we therefore have

$$
\Gamma = 12 \text{Re} \left\{ \sum_{m=1}^{(N-1)/2} \sum_{p=m}^{N-m} \alpha(m, p) B(m, p) \right\},
\tag{E.16}
$$

with $\alpha$ defined by

$$
\alpha = \begin{cases} 
1/2 & m = p \\
1 & m \neq p.
\end{cases}
\tag{E.17}
$$

For $N$ even, the situation is sketched in Figure E.2. Here the upper limit in the $p$-summation should be $N/2$. We can use floor$(N/2)$ as the general upper limit for both even and odd values of $N$, where floor means the nearest lower integer value. We therefore have

$$
\Gamma = 12 \text{Re} \left\{ \sum_{m=1}^{\text{floor}(N/2)} \sum_{p=m}^{N-m} \alpha(m, p) B(m, p) \right\}.
\tag{E.18}
$$
The above result agrees with the result stated in Elgar and Guza (1985), except that in their result the region below rather than above the symmetry line is used. This does not change the value of $\Gamma$ at all. We see that the number of terms in the sum involved is $N^2/4$. The work of calculation is hereby reduced by a factor of 12.

### E.3 Establishing the results for skewness and asymmetry

We now establish expressions for skewness and asymmetry, using the above results. This is easily done, since we now have a reduced expression for calculating the integrated bispectrum. We can therefore concentrate on the denominator of (E.3). Again, using that $a_0 = 0$ the summation range can be changed to only positive frequencies. Moving the resulting factor of 2 outside the sum gives the results (E.4) and (E.5).

### E.4 Calculating asymmetry from a pseudo time series

As we have seen, the skewness of a wave field can be calculated both from the integrated bispectrum using (E.4) and directly from the time series using (E.3). Using the bispectrum requires a computational effort of $O(N^2)$, while using the time series requires a work amount of $O(N)$. Hence it is interesting to pursue, if a similar expression can be found for the asymmetry.

From the calculations made above, we can express the skewness as

$$S = \frac{\langle \eta^3 \rangle}{(\langle \eta^2 \rangle)^{3/2}} = \frac{3\sum_{m=1}^{N} \sum_{p=1}^{N-m} a_p a_m a_{p+m}^* + a_p^* a_m^* a_{p+m}}{(\langle \eta^2 \rangle)^{3/2}}$$

$$= \frac{6 \text{Re} \left\{ \sum_{m=1}^{N} \sum_{p=1}^{N-m} B(m,p) \right\}}{(\langle \eta^2 \rangle)^{3/2}}.$$  \hspace{1cm} (E.19)

The asymmetry is obtained by taking the imaginary part rather than the real part in the last expression. We now consider the intermediate result

$$\langle \eta^3 \rangle = 3 \sum_{m=1}^{N} \sum_{p=1}^{N-m} a_p a_m a_{p+m}^* + a_p^* a_m^* a_{p+m}$$  \hspace{1cm} (E.20)

and the time series $\tilde{\eta} = \sum_{p=-N}^{N} b_p e^{ip\omega t}$ with $b_p$ satisfying

$$b_p = -ia_p \quad \text{for } p = 1, \ldots, N$$

$$b_{-p} = b_p^* \quad \text{for } p = 1, \ldots, N$$  \hspace{1cm} (E.21)
Appendix E. Calculation of Skewness and Asymmetry

From (E.21) we get
\[ < \tilde{\eta}^3 > = 3 \sum_{m=1}^{N} \sum_{p=1}^{N-m} (-i) a_p a_m a_{p+m}^* + i a_p^* a_m^* a_{p+m} = 6 \text{Im} \left\{ \sum_{m=1}^{N} \sum_{p=1}^{N-m} B(m, p) \right\}. \] (E.24)

Hence, the asymmetry can be calculated as
\[ A = \frac{< \tilde{\eta}^3 >}{(< \tilde{\eta}^2 >)^{3/2}}, \] (E.25)
i.e., as the skewness of \( \tilde{\eta} \). The time series \( \tilde{\eta} \) is obtained by multiplying the positive Fourier coefficients of \( \eta \) by \( -i \) and transforming from Fourier space to the time domain. Starting from the physical time series \( \eta \), the asymmetry can be obtained in this way with a computational effort of \( O(N \log N) \) using FFT. The operation (E.22) is exactly the Hilbert transform of a signal. Hence the remark in Elgar and Guza (1985).
Appendix F

Amplitude Dispersion: Match of Fully and Weakly Nonlinear Analysis

The fully nonlinear analysis of Section 5.3 is able to predict the effects of amplitude dispersion for all wave heights. It is therefore interesting to compare the results of this analysis to the results of the weakly nonlinear analysis of Section 5.2. This has been done for the frequencies $\omega \sqrt{h/g} = (0.3, 1.0)$. We first discuss the results for $\omega \sqrt{h/g} = 0.3$. The phase speeds of stream function theory, the ‘simplified’ time domain model and the evolution equations are plotted as function of $H/H_{\text{max}}$ in Figure F.1, normalized by the linear velocity of a Stokes wave. For all three models the phase speed increases with wave height and approaches the phase speed of a Stokes wave in the limit $H \to 0$. As expected, the time domain formulation and the evolution equations over-predict the phase speed of stream function theory.

To check the agreement with the weakly nonlinear analysis, the quantity

$$\Omega_{13} = \left( \frac{c_g}{c_{\text{lin}}^{\text{model}}} \right) \left( \frac{(c/c_{\text{lin}})^{\text{model}} - 1}{(k_{\text{lin}} H)^2} \right)$$

was calculated as function of wave height. Here, $k_{\text{lin}}$ is the linear wave number of the wave model considered, and the denominator is therefore equal to $\varepsilon = kA_1$ for small wave height. Therefore $\Omega_{13}$ approaches $\omega_{13}$ in the limit $H \to 0$. In Figure F.2, $\Omega_{13}$ is plotted as function of wave height, normalized by $\omega_{13, \text{St}}$ of Stokes waves. In $H/H_{\text{max}} = 0$, the value of $\omega_{13}/\omega_{13, \text{St}}$ of the weakly nonlinear analysis is marked with a circle. We see that the results of the fully nonlinear analysis matches the results of the weakly nonlinear analysis for $H \to 0$. It should be noted that $\Omega_{13}$, (F.1), could as well have been defined with e.g., the nonlinear wave number in the denominator rather than the linear wave number and with the amplitude of the first harmonic in the denominator rather than $H/2$. Any expression that converges towards $\omega_{13}$ for small wave heights can be used. In shallow water, where the amplitude dispersion is strongest, these choices affects the shape of the curves strongly. The main reason is that the Fourier series needed to represent an almost solitary wave converges slowly. The amplitudes for e.g., the first and third harmonic are therefore of the same magnitude, thus resulting in
Appendix F. Amplitude Dispersion: Match of Fully and Weakly Nonlinear Analysis

Figure F.1: Nonlinear phase speed for $\omega \sqrt{h/g} = 0.3$, normalized by phase speed of linear Stokes waves.

Figure F.2: $\Omega_{13}/\omega_{13_{\text{cg}}}$ for $\omega \sqrt{h/g} = 0.3$. The circles in $H = 0$ mark the results of weakly nonlinear analysis, see Figure 5.3.
a large deviation between $H/2$ and $a_1$. Hence, even though $\Omega_{13}$ shows a decreasing behavior for larger wave heights in Figure F.2, the phase speed is still increasing with wave height, as can be seen in Figure F.1.

In Figure F.3 the phase speeds and $\Omega_{13}$ are depicted for $\omega \sqrt{h/g} = 1.0$. At this larger frequency, the ‘simplified’ time domain model has a smaller phase speed than stream function theory, while the evolution equations have a phase speed being a little larger than that of stream function theory, except for the highest waves, where the phase speed is a little smaller. For $H \rightarrow 0$, there is a small difference in phase speed between the Boussinesq models and stream function theory, due to the difference in linear dispersion.

![Figure F.3: Nonlinear phase speed for $\omega \sqrt{h/g} = 1.0$, normalized by phase speed of linear Stokes waves.](image)

The values for $\Omega_{13}$ converges towards the results of the weakly nonlinear analysis for $H \rightarrow 0$, as can be seen in Figure F.4. For $H/H_{\text{max}} = 0.77$, $\Omega_{13}$ for the evolution equations becomes smaller than for stream function theory. This agrees with the results in Figure F.3, where the phase speed of the evolution equations becomes smaller than for stream function theory. Hence for this frequency, the very high waves behaves differently than the weakly nonlinear analysis predicts.
Figure F.4: $\Omega_{13}/\omega_{13\text{cs}}$ for $\omega\sqrt{h/g} = 1.0$. The circles in $H = 0$ mark the results of weakly nonlinear analysis, see Figure 5.3.
Appendix G

Transfer Functions for Higher Harmonics for Evolution Equations

In Chapter 5, the phase speed of the evolution equations and time domain formulation of Madsen and Sørensen (1993) was investigated. When carrying out the analysis, results for the amplitudes of the second and third bound harmonics are obtained as well, and it is therefore natural to present the results for these quantities also. This is the objective of this appendix. First, the results of the weakly nonlinear analysis are presented and next, the results of the fully nonlinear analysis. The matching of the two analyses for small wave heights is demonstrated.

G.1 Weakly nonlinear analysis

The weakly nonlinear analysis is presented in Section 5.2. For the bound wave solution

\[ \eta(x, t) = A_1 \cos \theta + \frac{\tilde{A}_2}{k_1}(k_1 A_1)^2 \cos 2\theta + \frac{\tilde{A}_3}{k_1}(k_1 A_1)^3 \cos 3\theta \]  

(G.1)

with

\[ \theta = \omega_1 t - \tilde{k}_1 x \quad \quad \tilde{k}_1 = k_1(1 - (k_1 A_1)^2 k_{13}), \]  

(G.2)

the expressions for \( \tilde{A}_2 \) and \( \tilde{A}_3 \) are given in (5.20) and (5.22). Madsen and Sørensen (1993) analyzed the ‘simplified’ time domain formulation, that is, the time domain formulation with the term \( P^2/(h + \eta) \) approximated by \( P^2/h \). We denote the model with the original term \( P^2/(h + \eta) \) the ‘full’ time domain formulation. For the simplified time domain formulation Madsen and Sørensen obtained

\[ \tilde{A}_{2, MS93} = \frac{3}{4k^3} \left( 1 + \frac{B + 1/9}{\kappa^2} \right) \]  

(G.3)

\[ \tilde{A}_{3, MS93} = \frac{27}{64k^6} \left( 1 + 2(B + 1/9)\kappa^2 + (B + 1/9)^2 \kappa^4 \right). \]  

(G.4)
Appendix G. Transfer Functions for Higher Harmonics for Evolution Equations

The result for $\tilde{A}_2$ is identical for the full and simplified time domain formulations.

In Figure G.1, $\tilde{A}_2$ for the time domain formulation and the evolution equations are plotted as functions of dimensionless frequency. The results are normalized by $\tilde{A}_2$ of Stokes theory, see Section 5.1. The result of Madsen and Sørensen is plotted as open circles. There is perfect agreement between the these results and the present analysis of the time domain formulation. We see that the second harmonic transfer of the evolution equations is smaller than for the time domain formulation. For both models the transfer is under-predicted and decays towards zero at deep water. The transfer of the evolution equations is half as large as for the time domain model around $\omega \sqrt{h/g} = 1$.

In Figure G.2, the influence of the various approximations are shown. The only approximation which influences the second-order transfer in the weakly nonlinear analysis is the truncation of the linear operator. This is easily explained, since $\tilde{A}_2$ is determined from the kernel component

$$W_{1,1} = -g \frac{(2k_1^{nl})^2}{\beta_{4,2}(2k_1^{lin} - k_2)^3 + \beta_{3,2}(2k_1^{lin} - k_2)^2 + \beta_{2,2}(2k_1^{lin} - k_2) + \beta_{1,2}}. \quad \text{(G.5)}$$

For the primary wave, $k_1^{nl} = k_1^{fl} = k_1$, and there is therefore no effect of replacing $k_1^{nl}$ and $k_1^{fl}$ with their linear approximations.

The results for $\tilde{A}_3$ are shown in Figure G.3. The results of Madsen and Sorensen (1993) are shown as well and match the present results perfectly. The evolution equations show a
smaller transfer to the third harmonic than the time domain formulation. Both models have a smaller transfer than Stokes waves. The transfer of the evolution equations is half as large as for the time domain model at around $\omega \sqrt{h/g} = 0.6$.

The influence of the various approximations is shown in Figure G.2. We see that the truncation of the linear operator and the linear flux approximation reduce the third-harmonic transfer, while the neglect of derivatives of $a_p(x)$ in the nonlinear terms increases the transfer. All the results, however, still show a smaller transfer than that of Stokes waves.

**G.2 Fully nonlinear analysis**

Results for the second and third harmonics of the fully nonlinear analysis is given in this section. Before analyzing the effect of each approximation, we demonstrate that the fully nonlinear analysis converges to the results of the weakly nonlinear analysis in the limit of small wave height.
Figure G.3: $\tilde{A}_3$ for the time domain formulation and the evolution equations.

Figure G.4: $\tilde{A}_3$ for the time domain formulation, evolution equations and the three approximations.
G.2.1 Matching with weakly nonlinear analysis

To test if the fully nonlinear analysis matches the weakly nonlinear analysis, the quantity

$$\Lambda_2 = \frac{2\tilde{a}_2}{k \left( \frac{H}{2} \right)^2} \quad (G.6)$$

was calculated for varying wave heights at $\omega \sqrt{h/g} = (0.3, 1.0)$. Here $k$ is the linear wave number of the wave model investigated. This measure converges towards $\tilde{A}_2$ in the limit of small wave height. In Figure G.5, $\Lambda_2$ is plotted towards $H/H_{\text{max}}$ for stream function theory, the simplified time domain formulation and the evolution equations. The dimensionless frequency is $\omega \sqrt{h/g} = 0.3$. The results are normalized with $\tilde{A}_2$ of Stokes waves, and the results of the weakly nonlinear analysis is marked as circles in $H/H_{\text{max}} = 0$. There is a fine convergence towards the results of the weakly nonlinear analysis for $H \to 0$.

For the third harmonic, the measure

$$\Lambda_3 = \frac{2\tilde{a}_3}{k^2 \left( \frac{H}{2} \right)^3} \quad (G.7)$$

was defined. Again, $k$ is the linear wave number of the model considered. $\Lambda_3$ converges towards $A_3$ for small wave heights. The variation of $\Lambda_3$ with wave height is depicted in Figure G.5: $\Lambda_2$ for $\omega \sqrt{h/g} = 0.3$ for stream function theory, the simplified time domain model and the evolution equations. The results of weakly nonlinear analysis are marked with circles.
Figure G.6: $\Lambda_3$ for $\omega\sqrt{h/g} = 0.3$ for stream function theory, the simplified time domain model and the evolution equations. The results of weakly nonlinear analysis are marked with circles.

Figure G.6. Again, there is a fine convergence towards the results of the weakly nonlinear analysis marked as open circles.

For $\omega\sqrt{h/g} = 1.0$, similar results for $\Lambda_2$ and $\Lambda_3$ are shown in Figures G.7 and G.8. The convergence towards the results of the weakly nonlinear analysis is clearly seen.
Figure G.7: $\Lambda_2$ for $\omega \sqrt{h/g} = 1.0$ for stream function theory, the simplified time domain model and the evolution equations. The results of weakly nonlinear analysis are marked with circles.

Figure G.8: $\Lambda_3$ for $\omega \sqrt{h/g} = 1.0$ for stream function theory, the simplified time domain model and the evolution equations. The results of weakly nonlinear analysis are marked with circles.
G.2.2 Effects of approximations

Having demonstrated that the fully nonlinear analysis agree with the weakly nonlinear analysis for small wave heights, we turn the focus towards the effects of the approximations involved for deriving the evolution equations.

For $H/H_{\text{max}} = 0.85$, the values of $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are plotted against dimensionless frequency in Figures G.9–G.11. The values of $\tilde{a}_1$ are not identical, because at each frequency for all wave models, the wave height is fixed, not $\tilde{a}_1$. Hence, if the models distribute the energy between the higher harmonics differently, as is the case when the transfer functions differ, the fixed wave height leads to different values of $\tilde{a}_1$. One may of course choose $\tilde{a}_1$ to be of the same value for all the models, but then as a consequence, the wave height is not preserved between the models. In this study fixed wave heights are chosen, because this reflects the use of the models from an engineering point of view.

The varying values of $\tilde{a}_1$ makes it difficult to interpret the amplitudes of the higher harmonics as measures of the transfer of energy from the first harmonic. Still, however, the spectral shape i.e., the distribution of the wave height over the wave amplitudes, for each wave model, can be discussed.

For all models and for all frequencies, the amplitude of the first harmonic is larger than that of stream function theory. $H/H_{\text{max}} = 0.85$.

The varying values of $\tilde{a}_1$ makes it difficult to interpret the amplitudes of the higher harmonics as measures of the transfer of energy from the first harmonic. Still, however, the spectral shape i.e., the distribution of the wave height over the wave amplitudes, for each wave model, can be discussed.

For all models and for all frequencies, the amplitude of the first harmonic is larger than that of stream function theory. Further, the amplitude of the second and third harmonics are smaller than for stream function theory. This shows that the decay in magnitude of the harmonic amplitudes is stronger for all the wave models than for stream function theory.
Figure G.10: Ratio of amplitude of second harmonic $\tilde{a}_2$ for various models, to amplitude of second harmonic for stream function theory. $H/H_{\text{max}} = 0.85$.

Figure G.11: Ratio of amplitude of third harmonic $\tilde{a}_3$ for various models, to amplitude of third harmonic for stream function theory. $H/H_{\text{max}} = 0.85$. 
Appendix G. Transfer Functions for Higher Harmonics for Evolution Equations

The energy transfer to higher harmonics is therefore smaller than that of stream function theory for all the wave models considered.

We now discuss the difference in energy transfer between the wave models. As the curves for \( \hat{a}_2 \) and \( \hat{a}_3 \) crosses in \( \omega \sqrt{h/g} \approx 0.5 \), we separate the discussion for deep/intermediate water and shallow water.

**Deep and intermediate water** In Table G.1, the wave models are listed after largest amplitude for \( \omega \sqrt{h/g} > 0.6 \). Thus in the column denoted \( \hat{a}_1 \), the model with the largest value of \( \hat{a}_1 \) is listed first and the model with the smallest value of \( \hat{a}_1 \) is listed last. Similar sortings apply for the columns \( \hat{a}_2 \) and \( \hat{a}_3 \).

<table>
<thead>
<tr>
<th>( \hat{a}_1 )</th>
<th>( \hat{a}_2 )</th>
<th>( \hat{a}_3 )</th>
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<tbody>
<tr>
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<td>time domain ( P^2/d )</td>
<td>time domain ( P^2/h )</td>
</tr>
<tr>
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<td>truncate linear op</td>
<td>time domain ( P^2/d )</td>
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<td>time domain ( P^2/h )</td>
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Table G.1: Magnitude of harmonic amplitudes for the different wave models. \( \omega \sqrt{h/g} > 0.6 \). The models are listed after their magnitude of each harmonic.

We see that the sequence of largest amplitudes is reversed when going from the first to the second harmonic. The ordering of amplitudes for the second and third harmonic is identical. This swapping of the sequence indicates that small amplitudes of the higher harmonics is compensated by a strong amplitude of the first harmonic. This is due to the requirement of identical wave height between the models. The first wave model of the first column can therefore be deduced to have the smallest energy transfer to higher harmonics, while the last model in the first column has the strongest energy transfer. Thus the strongest transfer occurs for the simplified time domain formulation, followed by the full time domain formulation, the model affected by the linear flux approximation, the evolution equations, the model affected by the truncation of the linear operator, and the ‘xx time’ model.

The effect of truncating the linear operator as well as approximating the flux amplitudes linearly, is deduced by comparing to the simplified time domain formulation. Both of the approximations result in a reduction of the energy transfer. The influence of the neglect of derivatives of \( a_p(x) \) in the nonlinear terms is found by comparing the ‘xx-time’ model to the evolution equations. The ‘xx time’ model has the smallest energy transfer. Hence the effect of neglecting the derivatives of \( a_p(x) \) in the nonlinear terms is to increase the energy transfer. These conclusions agree with the conclusions of the weakly nonlinear analysis. Further, the reduction from the full time domain formulation to the ‘simplified’ time domain formulation is seen to increase the energy transfer.
Shallow water For $\sqrt{\omega/h/g} < 0.4$, the ordering of the amplitudes between the models is different from the sequence just listed. The shallow water ordering is listed in Table G.2. The sequence of decaying magnitude is almost identical for all three harmonics. Now, as the

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<td>time domain $P^2/d$</td>
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Table G.2: Magnitude of harmonic amplitudes for the different wave models. Shallow water. The models are listed after their magnitude of each harmonic.

wave height is the same for all the wave models, the differences in harmonic magnitudes must be compensated at harmonics being higher than the third harmonic. With this interpretation, the first column of the table can be interpreted similarly as for the intermediate/deep water case: The first model listed has the smallest energy transfer and vice versa. Hence, the largest energy transfer occurs in the ‘simplified’ time domain model, followed by the ‘full’ time domain model, the evolution equations, the model influenced by the linear flux approximation, the model influenced by the truncation of the linear operator and the ‘xx time’ model. This ordering leads to identical conclusions as for the intermediate/deep water case above.

Results for $H/H_{max} = 0.45$ For the smaller wave height of $H/H_{max} = 0.45$, an analysis similar to the above gives the same conclusions for the effects of the various approximations. Plots for $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are shown in Figures G.12–G.14.

G.3 Conclusions

The results of the weakly nonlinear and fully nonlinear analyses show that the evolution equations have smaller energy transfer to the second and third harmonic than the time domain formulation. For the time domain formulation, the effect of reducing the nonlinear term $P^2/(h + \eta)$ to $P^2/h$ is to increase the energy transfer to higher harmonics. The truncation of the linear operator and the linear flux approximation decreases the energy transfer, while the neglect of linear derivatives of $a_p(x)$ in the nonlinear terms increases the energy transfer. The findings that the truncation of the linear operator decreases the energy transfer and that the neglect of derivatives of $a_p(x)$ in the nonlinear terms increases the energy transfer agree with the findings of Chapter 2 for Nwogu’s (1993) equations.
Appendix G. Transfer Functions for Higher Harmonics for Evolution Equations

Figure G.12: Ratio of amplitude of first harmonic $\tilde{a}_1$ for various models, to amplitude of first harmonic for stream function theory. $H/H_{\text{max}} = 0.45$.

Figure G.13: Ratio of amplitude of second harmonic $\tilde{a}_2$ for various models, to amplitude of second harmonic for stream function theory. $H/H_{\text{max}} = 0.45$. 
Figure G.14: Ratio of amplitude of third harmonic $\tilde{a}_3$ for various models, to amplitude of third harmonic for stream function theory. $H/H_{\text{max}} = 0.45$. 
Appendix H

Relation Between Hyperbolic and Trigonometric Functions

The following rules apply for complex arguments in trigonometric and hyperbolic functions.

\[
\begin{align*}
\sin(ix) &= i \sinh x \\
\csc(ix) &= -i \csch x \\
\sinh(ix) &= i \sin x \\
\csch(ix) &= -i \csc x
\end{align*}
\]

\[
\begin{align*}
\cos(ix) &= \cosh x \\
\sec(ix) &= \sech x \\
\cosh(ix) &= \cos x \\
\sech(ix) &= \sec x
\end{align*}
\]

\[
\begin{align*}
\tan(ix) &= i \tanh x \\
\cot(ix) &= -i \coth x \\
\tanh(ix) &= i \tan x \\
\coth(ix) &= -i \cot x
\end{align*}
\]

The table originates from Spiegel (1968).
Appendix H. Relation Between Hyperbolic and Trigonometric Functions
Appendix I

Extra Plots for Section 6.7

Figure I.1: Time series of roller model with adjusted parameters, Boussinesq roller model (frequency domain) and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
Figure I.2: Time series for conventional breaking with uniform dissipation ($F = 1$), roller breaking and time domain model. The time series are aligned horizontally, so the crest is always at $t = 1$ s.
Figure I.3: Skewness and asymmetry for conventional breaking with uniform dissipation ($F = 1$), roller breaking and time domain model.
Appendix J

Calculation of a Cubic Product in Fourier Space

We consider a complex Fourier series

\[ v = \sum_{p=-N}^{N} a_p e^{i\omega_pt} \]  

where \( N \) is the number of positive frequencies. A cubic product of three Fourier series of the above type can be written

\[ v_1v_2v_3 = \sum_{p=-3N}^{3N} \left\{ \sum_s \sum_t \Gamma_{s,t,p-s-t} a_s a_t a_{p-s-t} \right\} e^{i\omega_pt} \equiv \sum_{p=-3N}^{3N} S_p e^{i\omega_pt} \]  

where \( \Gamma_{s,t,p-s-t} \) depends on \((s, t, p - s - t)\). If the cubic product is part of a transformation, \( \Gamma_{s,t,p-s-t} \) is often denoted ‘the interaction coefficient’ or ‘the kernel’. We now express the limits of the summations for \( S_p \).

All the indices on the \( a \)’s must be within the range \(-N, \ldots, N\). This gives the three requirements

\[ -N \leq s \leq N \]  

\[ -N \leq t \leq N \]  

\[ -N \leq p - s - t \leq N. \]  

The latter relation is rewritten to

\[ -N + p - s \leq t \leq N + p - s, \]  

and thus (J.4) and (J.5) can be combined to the requirement

\[ \max\{-N, -N + p - s\} \leq t \leq \min\{N, N + p - s\}. \]
Appendix J. Calculation of a Cubic Product in Fourier Space

We therefore have

\[ S_p = \sum_{s=-N}^{N} \min\{N,N+p-s\} \sum_{t=max\{-N,-N+p-s\}} \Gamma_{s,t,p-s-t}a_s a_t a_{p-s-t}. \]  

\[ (J.8) \]

**Utilizing the symmetry among \( s \) and \( t \)** The above result can be simplified if there is symmetry among \( s \) and \( t \). This implies

\[ \Gamma_{s,t,p-s-t} = \Gamma_{t,s,p-s-t}. \]  

\[ (J.9) \]

In Figure J.1 an example of the extent of the summation range for \( s \) and \( t \) in (J.8) is shown as the shaded area in the \( (s,t) \) plane. The symmetry line \( s = t \) divides the shaded area into two halfs. Due to the symmetry in \( (s,t) \) we may carry out the summation in just one of the half planes and multiply the result by two. We can restrict the summation to the upper plane by imposing the additional requirement \( t \geq s \). In this way we can calculate \( S_p \) as

\[ S_p = 2 \sum_{s=-N}^{N} \min\{N,N+p-s\} \sum_{t=max\{-N,-N+p-s,s\}} \delta_{s,t} \Gamma_{s,t,p-s-t}a_s a_t a_{p-s-t}. \]  

\[ (J.10) \]

with

\[ \delta_{s,t} = \begin{cases} 1 & \text{for } s \neq t \\ \frac{1}{2} & \text{for } s = t \end{cases}. \]  

\[ (J.11) \]

Figure J.1: The \( (s,t) \) plane of the summation in (J.8).
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