



Hypergeometric Functions with Integral Parameter Differences

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¹ We use the signature $(-, +, +, +)$:

$$\begin{aligned} \Gamma_{\mu\nu}^{\sigma} &\equiv \frac{1}{2}g^{\sigma\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau}), \\ R_{\mu\nu\sigma}{}^{\tau} &\equiv \Gamma_{\nu\sigma,\mu}{}^{\tau} - \Gamma_{\mu\sigma,\nu}{}^{\tau} + \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\rho}{}^{\tau} - \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\rho}{}^{\tau}, \\ R_{\mu\nu} &\equiv R_{\mu\nu}{}^{\tau}{}_{\tau}. \end{aligned}$$

A comma denotes ordinary differentiation; a semicolon denotes covariant differentiation; $16\pi G = 1$, $c = 1$, G is the gravitational constant, and c is the velocity of light. All other notation is defined in the text or is standard.

² For a review of the geometric theory, see *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 9.

³ J. L. Synge, *Relativity: The General Theory* (Interscience, New York, 1960), Chap. VIII; K. S. Thorne, Ph.D. thesis, University Microfilms, Ann Arbor, 1965.

⁴ The general solutions to Cases I and II of the possible field configurations and a particular solution of Case III is given in Ref. 2 and also in a paper by L. Witten, *Colloq. Theor. Relativity*, Centre Belge Rech. Math., Univ. (Louvain, Belgium), p. 59, 1960.

⁵ This differs slightly from the line element given in Ref. 4. $\mu \equiv 0$ requires that we take the limit $l \rightarrow \infty$ in the line elements of Ref. 4. The necessity that l approach ∞ can be argued for on a variety of grounds, which we shall not present here.

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⁷ M. A. Melvin, *Phys. Letters* **8**, 65 (1964).

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⁹ K. S. Thorne, *Phys. Rev.* **138B**, 251 (1965).

¹⁰ K. S. Thorne, *Phys. Rev.* **139B**, 244 (1965).

¹¹ J. L. Safko, *Ann. Phys. (N.Y.)* **58**, 352 (1970).

¹² M. A. Melvin and J. S. Wallingford, *J. Math. Phys.* **7**, 333 (1966).

¹³ J. L. Safko, *Bull. Am. Phys. Soc.* **13**, 1725 (1968).

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Hypergeometric Functions with Integral Parameter Differences

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For a generalized hypergeometric function ${}_pF_q(z)$ with positive integral differences between certain numerator and denominator parameters, a formula expressing the ${}_pF_q(z)$ as a finite sum of lower-order functions is proved. From this formula, Minton's two summation theorems for $p = q + 1$, $z = 1$ are deduced, one of these under less restrictive conditions than assumed by Minton.

This paper deals with generalized hypergeometric functions ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ having the special property that, with suitable enumeration of parameters, $a_i = b_i + m_i$, $i = 1, 2, \dots, n$, where m_1, \dots, m_n are positive integers and $n \leq \min(p, q)$. It is assumed that $p \leq q + 1$ and that no denominator parameter b is a negative integer or zero. A function of this type may be expressed as a finite sum of ${}_{p-n}F_{q-n}$ functions in the following way:

$${}_pF_q \left[\begin{matrix} b_1 + m_1, \dots, b_n + m_n, a_{n+1}, \dots, a_p \\ b_1, \dots, b_n, b_{n+1}, \dots, b_q \end{matrix} ; z \right] = \sum_{j_1=0}^{m_1} \dots \sum_{j_n=0}^{m_n} A(j_1, \dots, j_n) z^{J_n} {}_{p-n}F_{q-n} \left[\begin{matrix} a_{n+1} + J_n, \dots, a_p + J_n \\ b_{n+1} + J_n, \dots, b_q + J_n \end{matrix} ; z \right], \quad (1)$$

where

$$J_n = j_1 + \dots + j_n, \quad (2)$$

$$A(j_1, \dots, j_n) = \binom{m_1}{j_1} \dots \binom{m_n}{j_n} \frac{(b_2 + m_2)_{j_1} (b_3 + m_3)_{j_2} \dots (b_n + m_n)_{j_{n-1}} (a_{n+1})_{j_n} \dots (a_p)_{j_n}}{(b_1)_{j_1} (b_2)_{j_2} \dots (b_n)_{j_n} (b_{n+1})_{j_n} \dots (b_q)_{j_n}}, \quad (3)$$

and

$$(c)_r = \Gamma(c + r) / \Gamma(c). \quad (4)$$

By the principle of analytical continuation, Eq. (1) is valid whenever the functions involved are all analytic; restrictions upon the parameters imposed in the proof may thus be removed.

The proof is based upon an Eulerian integral representation given by Erdélyi,¹ viz.,

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \frac{\Gamma(b_1)\Gamma(1 - b_1 + a_1)}{\Gamma(a_1) \exp(i\pi(b_1 - a_1))} \frac{i}{2\pi} \times \int_0^{(1+)} (1 - t)^{b_1 - a_1 - 1} f(t) dt, \quad (5)$$

where

$$f(t) = t^{a_1 - 1} {}_{p-1}F_{q-1}(a_2, \dots, a_p; b_2, \dots, b_q; zt), \quad (6)$$

valid when $\text{Re } a_1 > 0$, b_1 is not a negative integer or zero, and $|\arg(1 - z)| < \pi$ if $p = q + 1$. Now, as $a_1 = b_1 + m_1$, the branch point of the integrand at $t = 1$ disappears, and the integral takes the form $\int_C f(t) dt / (t - 1)^{m_1+1}$, where C is a closed contour encircling the point $t = 1$ counterclockwise and f is analytic within and on C . From Cauchy's integral formula, we then find that Eq. (5) becomes

$${}_pF_q \left[\begin{matrix} b_1 + m_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right] = \frac{D^{m_1} f(1)}{(b_1)_{m_1}}, \quad (7)$$

D denoting differentiation with respect to t . Application of Leibniz's differentiation formula and the well-known expression for the derivative of a ${}_pF_q$ then yields

$$\begin{aligned} &{}_pF_q \left[\begin{matrix} b_1 + m_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right] \\ &= \sum_{j=0}^{m_1} z^j \binom{m_1}{j} \frac{(a_2)_j \dots (a_p)_j}{(b_1)_j \dots (b_q)_j} \\ &\quad \times {}_{p-1}F_{q-1} \left[\begin{matrix} a_2 + j, \dots, a_p + j \\ b_2 + j, \dots, b_q + j \end{matrix} ; z \right]. \quad (8) \end{aligned}$$

This result can itself be applied to each member of its rhs if $a_2 = b_2 + m_2$, etc. It is easily seen that the general result (1) is obtained in this way.

A special case ($p = 3, q = 2$) of Eq. (8) has been derived by Rösler² from the series representation.

From formula (1) we now derive two summation theorems for $p = q + 1, z = 1$. These have been given recently by Minton,³ the first one, however, under more restrictive conditions than those given below.

To deduce the first theorem, we take $q = p - 1 = n + 1, a_{n+1} = b = b_{n+1} - 1, z = 1$, and for brevity $a_{n+2} = a$. The hypergeometric functions in the multiple sum of Eq. (1) then become ${}_2F_1(1)$'s, which all exist provided that

$$\text{Re}(-a) > m_1 + \dots + m_n - 1. \quad (9)$$

By Gauss' summation theorem we then get, after some rearrangements,

$$\begin{aligned} &{}_{n+2}F_{n+1}(b_1 + m_1, \dots, b_n + m_n, b, a; \\ &\quad b_1, \dots, b_n, b + 1; 1) \\ &= \frac{\Gamma(b + 1)\Gamma(1 - a)}{\Gamma(b + 1 - a)} \sum_{j_1=0}^{m_1} \dots \sum_{j_n=0}^{m_n} B_n(j_1, \dots, j_n), \end{aligned}$$

where

$$B_n(j_1, \dots, j_n) = \frac{(b + 1)_{J_n}}{(a)_{J_n}} (-1)^{J_n} A(j_1, \dots, j_n).$$

Next, the definition (3) is applied, the binomial coefficients being written in the form $(-1)^j (-m)_j / j!$;

this leads to

$$\begin{aligned} B_n(j_1, \dots, j_n) &= B_{n-1}(j_1, \dots, j_{n-1}) \\ &\quad \times \frac{(b_n + m_n)_{J_{n-1}} (-m_n)_{j_n} (b + J_{n-1})_{j_n}}{(b_n)_{J_{n-1}} j_n! (b_n + J_{n-1})_{j_n}}. \end{aligned}$$

The terms containing j_n obviously constitute a terminating ${}_2F_1(1)$, which is summed by Gauss' theorem. After some rearrangements we obtain (summation limits understood)

$$\begin{aligned} &\sum_{j_1, \dots, j_n} B_n(j_1, \dots, j_n) \\ &= \frac{(b_n + m_n)_{-b}}{(b_n)_{-b}} \sum_{j_1, \dots, j_{n-1}} B_{n-1}(j_1, \dots, j_{n-1}). \end{aligned}$$

Repeating this procedure, we finally arrive at Minton's first theorem,

$$\begin{aligned} &{}_{n+2}F_{n+1}(b_1 + m_1, \dots, b_n + m_n, b, a; \\ &\quad b_1, \dots, b_n, b + 1; 1) \\ &= \frac{\Gamma(b + 1)\Gamma(1 - a)}{\Gamma(b + 1 - a)} \prod_{k=1}^n \frac{(b_k + m_k)_{-b}}{(b_k)_{-b}}, \quad (10) \end{aligned}$$

valid under the condition (9), i.e., if the lhs of (10) exists at all. In Minton's proof,³ a was required to be negative integral.

The particular case $n = 1$ of Eq. (10) was obtained by Mitra⁴ by series manipulations.

The second summation theorem may be deduced from the first³ by letting $b \rightarrow \infty$. It may, however, also be deduced directly from Eq. (1) by taking $q = p - 1 = n, a_{n+1} = -(m_1 + \dots + m_n)$, and $z \rightarrow 1$. The hypergeometric functions in the multiple sum of Eq. (1) then reduce to power functions $(1 - z)^h$, where $h = -a_{n+1} - J_n$ and $h \geq 0$ for all terms. When $z \rightarrow 1$, all terms of the multiple sum will thus tend to zero, except the one for which $-a_{n+1} = J_n$, i.e., $j_i = m_i, i = 1, 2, \dots, n$; the limit of this term is $A(m_1, \dots, m_n)$. After some reductions we find the summation formula

$$\begin{aligned} &{}_{n+1}F_n(b_1 + m_1, \dots, b_n + m_n, -(m_1 + \dots + m_n); \\ &\quad b_1, \dots, b_n; 1) \\ &= \frac{(-1)^{m_1 + \dots + m_n} (m_1 + \dots + m_n)!}{(b_1)_{m_1} \dots (b_n)_{m_n}}, \quad (11) \end{aligned}$$

which is easily transformed to the form given by Minton.³

It may be of interest to compare Eq. (11) with the special case of (10) obtained by taking $b_n = b, m_n = 1$ and then replacing $n - 1$ by n , viz.,

$$\begin{aligned} &{}_{n+1}F_n(b_1 + m_1, \dots, b_n + m_n, a; b_1, \dots, b_n; 1) = 0, \\ &\quad \text{Re}(-a) > m_1 + \dots + m_n. \quad (12) \end{aligned}$$

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