



Electromagnetic Radiation in a Uniformly Moving, Homogeneous Medium

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$$\psi_{\omega(\mathbf{k})}^{(+)} = \left[\begin{array}{c} \frac{f^*(\mathbf{k})}{D_+(\omega(\mathbf{k}))} \\ \delta(\mathbf{q} - \mathbf{k}) + \frac{f^*(\mathbf{k})f(\mathbf{q})}{D_+(\omega(\mathbf{k}))[\omega(\mathbf{k}) - \omega(\mathbf{q}) + i\epsilon]} \end{array} \right] \quad (4.2)$$

with

$$D(z) = [z - M] \left(1 + \int \frac{|f(\mathbf{k})|^2}{[M - \omega(\mathbf{k})][z - \omega(\mathbf{k})]} \right), \quad (4.3)$$

where M is the physical mass of the stable V particle. The S matrix determined by taking the inner product of $\psi_{\omega(\mathbf{k})}^{(+)}$ with the corresponding ingoing wave state of energy $\omega(\mathbf{k})$ is

$$\begin{aligned} S(k) &= \langle \text{in}, \mathbf{k} | \text{out}, \mathbf{k} \rangle \\ &= D_-(\omega(\mathbf{k}))/D_+(\omega(\mathbf{k})). \end{aligned} \quad (4.4)$$

Briefly then, we now take nonrelativistic kinematics, $\omega(\mathbf{k}) = \mathbf{k}^2$, and choose a particular $f(\mathbf{k})$:

$$|f(\mathbf{k})|^2 = \frac{2\lambda}{\pi} [M - \omega(\mathbf{k})] \left(\frac{b^2}{1 + k^2 b^2} \right)^2.$$

Thus $D(z)$, $S(k)$, and the continuum components of $\psi_{\omega(\mathbf{k})}^{(+)}$

are essentially the same as the corresponding quantities in first example we considered in the pairwise theory. Thus, the Fourier transform of $\psi_{\omega(\mathbf{k})}^{(+)}$ into coordinate space becomes non- L^2 as $kb \rightarrow i$, in the same way as in the previous example.

5. CONCLUSIONS

The analysis presented in this paper concerning the singularities of the S matrix demonstrates that the non- L^2 character of the poles of $\bar{f}_-(k, 0)$ persists in domains closer to that of an interacting quantum field theory. From the scalar and vector coupling models it is seen that relativization does not destroy the non- L^2 poles of the exponential potential. For the pair theory, classical and quantized, and for the Lee model it is now possible to make the definite statement that poles of the proper Jost function $\bar{f}_-(k, 0)$ correspond to non- L^2 class solutions of the associated dynamical field equation in coordinate space and that zeros of $\bar{f}_+(k, 0)$ to the L^2 class of solutions. A corollary of these conclusions is that non- L^2 states, as "shadow" states, possess inherently a very elegant reason for not appearing in the unitarity relation, and may be called "shadow" fields in complete correspondence with its nonrelativistic counterpart.

¹ C. A. Nelson, A. K. Rajagopal, and C. S. Shastri, *J. Math. Phys.* **12**, 737 (1971). The present paper is its sequel.

² S. T. Ma, *Phys. Rev.* **69**, 668 (1946); **71**, 195 (1947). See Ref. 1, where more recent work on such singularities are cited.

³ R. Jost, *Helv. Phys. Acta* **20**, 256 (1947).

⁴ S. N. Biswas, T. Pradhan, and E. C. G. Sudarshan, University of Texas at Austin preprint (September 1970).

⁵ T. D. Lee and G. C. Wick, *Nucl. Phys.* **B9**, 209 (1969); **B10**, 1 (1969); T. D. Lee, CERN Report No. Th. 914 (unpublished). A. M. Gleeson and E. C. G. Sudarshan, *Phys. Rev. D* **1**, 474 (1970).

⁶ E. C. G. Sudarshan, "Action at a Distance", Report to the Austin Symposium on the Past Decade in Particle Theory, April 1970; *Fields and Quanta*, Vol. II (in press). C. A. Nelson and E. C. G. Sudarshan, University of Texas at Austin, Preprint CPT-94, AEC-44 (March 1970); Austin preprint (April 1970).

⁷ The relativistic wave equations (2.3) and (2.20) have the same differential equation structure as that of the Schrödinger equation and so the approach of Jost may be used. The definitions and notations employed here is that of R. G. Newton, *Scattering Theory of Waves and Particles*, (McGraw-Hill, New York, 1966).

⁸ G. S. Guralnik and C. R. Hagen, *Phys. Rev.* **130**, 1259 (1963).

⁹ See A. Erdélyi (Ed.), *Bateman Manuscript Project: Higher Transcendental Functions* (McGraw Hill, New York, 1953) Vol. I, Chap. VI. Volumes in this series are hereafter referred to as HTFI, II, III.

¹⁰ HTFI-6-13-(6).

¹¹ R. Jost and A. Pais, *Phys. Rev.* **82**, 840 (1951).

¹² E. C. G. Sudarshan, in *Lectures in Theoretical Physics, Brandeis Summer Institute, 1961* (Benjamin, New York, 1962).

¹³ B. Friedman, *Principles and Techniques of Applied Mathematics*, (Wiley, New York, 1956).

¹⁴ The boundary conditions of $\varphi(k; r)$ and $f_{\pm}(k, r)$ at $r = 0$ and $r \rightarrow \infty$, respectively, are the same as for a local potential.

¹⁵ An alternative procedure is to introduce a proper regular solution $\bar{\varphi}(k, r) = C(k)\varphi(k, r)$ and a proper regular Green's function $\bar{G}^{(+)} = C(k)G^{(+)}$, i.e.,

$$\lim_{r \rightarrow \infty} e^{\mp ikr} \bar{f}_{\pm}(k, r) = C(k),$$

$$\lim_{r \rightarrow 0} \bar{\varphi}(k, r)' = C(k)$$

and to work with these proper solutions as for a local theory, e.g., $\psi^{(+)}(k, r)'|_{r=0} = k\bar{\varphi}(k, r)'|_{r=0}/\bar{f}_+(k, 0)$. It is permissible to modify the boundary conditions for $f_+(k, r)$ and $\varphi(k, r)$ as they are mathematical constructs; however, see the discussion in the text concerning the Wronskian. Note also C. S. Warke and R. K. Bhaduri, *Nucl. Phys.* **A162**, 289 (1971).

¹⁶ T. D. Lee, *Phys. Rev.* **95**, 1329 (1954).

Electromagnetic Radiation in a Uniformly Moving, Homogeneous Medium

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A new method of treating radiation problems in a uniformly moving, homogeneous medium is presented. A certain transformation technique in connection with the four-dimensional Green's function method makes it possible to elaborate the Green's functions of the governing differential equations in the rest system of the medium, whereas the final integrals determining the field may be calculated in the rest system of the source.

I. INTRODUCTION

In recent years several papers have been concerned with electromagnetic radiation in a uniformly moving medium with a more or less complicated structure. The usual approach is to formulate the governing differential equation in the observer system (rest

system of the source) K and then evaluate the solution for this equation.¹⁻¹¹

In two articles^{12,13} the author showed that advantage may be derived from the fact that the governing differential equations are simplest in the rest frame K' of the medium, the medium being simple (i.e.,

homogeneous, isotropic, lossless and non dispersive) and conducting (but otherwise simple), respectively.

It is the purpose of the present paper to show that a certain transformation technique in connection with the four-dimensional Green's function method makes it possible to derive advantage from both the "simplicity" of the governing differential equation in K' and the "simplicity" of the source distribution in K ; the structure of the medium may be rather complicated.

The basic idea of the method can be described in this way: The first step in the Green's function technique is to find a fundamental solution corresponding to a source which is represented by a four-dimensional δ function. Such a space-time impulse is bound to a single space-time point (world point). As far as we are concerned with a single world point, all inertial systems are equally suitable, and therefore, in the first step, the rest system of the medium should be used because the governing differential equations are simplest in this system.

The next step consists of a summation over the world points of the source distribution. This step is in principle independent of the first one, and, therefore, the most convenient inertial system should be used for the second step. Usually, the rest system of the source (if such a system exists, of course) is the most suitable one for the summation procedure.

Mathematically, the formalism provides the definition of a Green's tensor function which can be evaluated in an arbitrary inertial frame. (In K' this function may even be well known from the corresponding radiation problem, where the medium is at rest relatively to the source). This Green's function can be transformed to another arbitrary inertial frame (e.g., the rest frame K of the source), where the final integrals which represent the field may be worked out.

The general formalism is elaborated for a homogeneous medium and then applied to some examples: An isotropic medium and a lossless, electrically, uniaxially, anisotropic medium are considered. The first case has been treated before by Besieris and Compton,⁹ Chen and Yen,¹⁰ and the author,¹³ the second case by S. W. Lee and Lo.⁶ Case one includes a simple medium (Refs. 1, 5, 8, 12). Application of the method will also be made to a dispersive but otherwise simple medium (ionized gas), a case which has been treated before by K. S. H. Lee and Papas.⁴

II. FORMAL SOLUTION IN THE REST SYSTEM OF THE MEDIUM

In this section all quantities refer to the system of inertia K' in which the medium is at rest. The transformation of the results into four-dimensional tensor language will be done in the next section. In preparation for this transformation we will use three-dimensional, Cartesian tensor notation (cf. Ref. 14) in this section. Latin subscripts run from 1 to 4; Greek subscripts run from 1 to 3. The coordinate $x_4 \equiv ict$, where t is the time and c the speed of light in vacuum. Repeated subscripts obey the summation convention, and commas in subscripts denote partial differentiation with respect to coordinates.

The constitutive equations are given by

$$D'_\kappa = \epsilon_{\kappa\lambda} E'_\lambda, \quad (1a)$$

$$B'_\kappa = \mu_{\kappa\lambda} H'_\lambda, \quad (1b)$$

$$J_\kappa^{0'} = \sigma_{\kappa\lambda} E'_\lambda, \quad (1c)$$

where the dyadics $\epsilon_{\kappa\lambda}$, $\mu_{\kappa\lambda}$, and $\sigma_{\kappa\lambda}$ are assumed to be independent of the space-time coordinates x'_ν (i.e., the medium is homogeneous). $J_\kappa^{0'}$ denotes the free current density, whereas the free charge density $\rho^{0'}$ is assumed to be vanishing.

A vector potential A'_κ and a scalar potential $A'_4 \equiv (i/c)\Phi'$ can be introduced in the usual way. The field is given by

$$\mu_{\kappa\lambda} H'_\lambda = \epsilon_{\kappa\lambda\nu} A'_{\nu,\lambda}, \quad (2a)$$

$$E'_\kappa = ic (A'_{4,\kappa} - A'_{\kappa,4}), \quad (2b)$$

where $\epsilon_{\kappa\lambda\nu}$ is the three-dimensional permutation symbol, and the differential equations for the potentials can be written as

$$\epsilon_{\kappa\lambda\nu} \epsilon_{\rho\sigma\tau} \mu_{\nu\tau}^{-1} A_{\sigma,\lambda\rho} + c^2 \epsilon_{\kappa\lambda} (A'_{4,4\lambda} - A'_{\lambda,44}) - ic \sigma_{\kappa\lambda} (A'_{4,\lambda} - A'_{\lambda,4}) = J'_\kappa, \quad (3a)$$

$$c^2 \epsilon_{\lambda\nu} (A'_{\nu,\lambda 4} - A'_{4,\lambda\nu}) = J'_{\lambda 4}. \quad (3b)$$

J'_κ is the current density of the source and $J'_{\lambda 4} \equiv ic\rho'$, where ρ' is the charge density of the source. Finally $\mu_{\nu\tau}^{-1} \mu_{\tau\lambda} = \delta_{\nu\lambda}$ which is the three-dimensional Kronecker symbol. [The matrices representing the constitutive dyadics in (1) are assumed to be regular.]

Introducing the definitions $n^2 \equiv c^2 \mu\epsilon$ and

$$\epsilon_{\kappa\lambda}^* \equiv \epsilon^{-1} \epsilon_{\kappa\lambda} - \delta_{\kappa\lambda}, \quad (4a)$$

$$(\mu_{\kappa\nu}^{-1})^* \equiv \mu \mu_{\kappa\lambda}^{-1} - \delta_{\kappa\lambda}, \quad (4b)$$

$$\sigma_{\kappa\lambda}^* \equiv \sigma^{-1} \sigma_{\kappa\lambda} - \delta_{\kappa\lambda}, \quad (4c)$$

where ϵ, μ, σ are constants (different from zero), we may write the differential equations (3) in the form

$$\begin{aligned} \epsilon_{\kappa\lambda\nu} \epsilon_{\rho\sigma\tau} (\mu_{\nu\tau}^{-1})^* A'_{\sigma,\rho\lambda} + n^2 [\epsilon_{\kappa\lambda}^* (A'_{4,\lambda 4} - A'_{\lambda,44}) \\ - (i/c)(\sigma/\epsilon) \sigma_{\kappa\lambda}^* (A'_{4,\lambda} - A'_{\lambda,4})] - A'_{\kappa,\lambda\lambda} - n^2 \\ \times [A'_{\kappa,44} - (i/c)(\sigma/\epsilon) A'_{\kappa,4}] = \mu J'_\kappa, \end{aligned} \quad (5a)$$

$$\begin{aligned} \epsilon_{\lambda\nu}^* (A'_{\nu,4\lambda} - A'_{4,\nu\lambda}) - A'_{4,\lambda\lambda} - n^2 [A'_{4,44} - (i/c)(\sigma/\epsilon) \\ \times A'_{4,4}] = \mu J'_{\lambda 4}/n^2. \end{aligned} \quad (5b)$$

In (5) we also have applied the Gauge condition

$$A'_{\lambda,\lambda} + n^2 [A'_{4,4} - (i/c)(\sigma/\epsilon) A'_{4,4}] = 0 \quad (6)$$

in the usual way¹⁵ to the isotropic terms of the equations.

Applying fourfold space-time Fourier transforms [cf. (12), Sec. III] to the differential equations (5), we obtain the matrix equation

$$(M'_{rs} + V' \delta_{rs}) a'_s = s'_r, \quad (7)$$

where

$$M'_{\kappa\lambda} \equiv -\epsilon_{\kappa\sigma\nu}\epsilon_{\rho\lambda\tau}(\mu_{\nu\tau}^{-1})^*k'_\rho k'_\sigma + n^2[k'_4{}^2\epsilon_{\kappa\lambda}^* - (i/c)(\sigma/\epsilon)k'_4\sigma_{\kappa\lambda}^*], \quad (8a)$$

$$M'_{\kappa 4} \equiv -n^2[k'_4\epsilon_{\kappa\lambda}^* - (i/c)(\sigma/\epsilon)\sigma_{\kappa\lambda}^*]k'_\lambda, \quad (8b)$$

$$M'_{4\kappa} \equiv -\epsilon_{\lambda\kappa}^*k'_\lambda k'_4, \quad (8c)$$

$$M'_{44} \equiv +\epsilon_{\kappa\lambda}^*k'_\kappa k'_\lambda, \quad (8d)$$

$$V' \equiv k'_\lambda k'_\lambda + n^2[k'_4{}^2 - (1/c)(\sigma/\epsilon)k'_4], \quad (8e)$$

and

$$s'_k \equiv \mu(j'_\kappa, j'_4/n^2). \quad (9)$$

It is noticed that M'_{rs} vanishes when the medium is isotropic.

Defining

$$g'_{rs} \equiv (M'_{rs} + V'\delta_{rs})^{-1}, \quad (10)$$

we finally get

$$a'_r = g'_{rt} s'_t \quad (11)$$

which establishes the formal solution of the problem in the transformed domain.

III. TRANSLATION INTO FOUR-DIMENSIONAL TENSOR FORMALISM

Consider a tensor field $T \dots(x_r)$ of arbitrary order (we use four-dimensional tensors as in Ref. 12) and a fourfold Fourier integral

$$T \dots(x_r) = \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} t \dots(k_r) e^{ik_s x_s} d(k_r) = \int_{M_4} t \dots(k_r) e^{ik_s x_s} d(k_r), \quad (12)$$

where the integration is to be taken over the whole Minkowski 4-space M_4 .

A proper Lorentz transformation can be applied to the integration variables k_r as well as to the space-time variables x_r . The inner tensor product $k_r x_r$ is invariant, and we still have to integrate over the whole space M_4 because a Lorentz transformation is a one-to-one mapping of M_4 on itself. Therefore, the Fourier amplitudes $t \dots$ transform like a tensor if $T \dots$ does, and vice versa (because the transformation matrix does not depend on x_r).

The four-current density J_r transforms like a tensor (cf. Ref. 16) and so does the four-potential A_r (cf. below). Therefore, the Fourier amplitudes j_r and a_r are tensors of order one (4-vectors).

We make also use of the 4-vector

$$S_r \equiv S_{rs} J_s, \quad S_{rs} \equiv \mu(\delta_{rs} + \kappa/n^2 U_s U_r), \quad (13)$$

where $\kappa \equiv (n^2 - 1)/c^2$ and U_r is the 4-velocity, i.e., $U'_r = (0, 0, 0, ic)$ and $S'_r = (J'_\rho, J'_4/n^2)$. It is seen that the Fourier amplitudes of S'_r are given by (9).

Furthermore, we define a second-order tensor g_{rs} by means of the components in K' as given by (10). The

tensor character of Eq. (11) now ensures that

$$a_r = g_{rt} s_t \quad (14)$$

is valid in an arbitrary inertial frame, and, referring to the statements made in the beginning of this section, we are able to give a covariant, formal solution of the problem:

$$A_n(x_r) = \frac{1}{(2\pi)^4} \int_{M_4} G_{ns}(u_r) J_s(z_r) d(z_r), \quad (15)$$

where

$$G_{ns}(u_r) \equiv \int_{M_4} g_{nm}(k_r) S_{ms} e^{ik_t u_t} d(k_r), \quad u_t \equiv x_t - z_t. \quad (16)$$

For a nondispersive medium, n (and therefore also S_{rs}) is independent of the variables k_r . In that case we use a modified definition of the Green's tensor function:

$$A_n(x_r) = \frac{1}{(2\pi)^4} \int_{M_4} G_{ns}(u_r) S_s(z_r) d(z_r), \quad (15')$$

where

$$G_{ns}(u_r) \equiv \int_{M_4} g_{ns}(k_r) e^{ik_t u_t} d(k_r). \quad (16')$$

Since the Green's tensor function (16) is simplest in K' (which is a consequence of the fact that the governing differential equations are simplest in K'), an evaluation of the integral in (16) should be sought (may be well known) in the rest frame of the medium (cf. the following sections).

In Minkowski's theory the field quantities $E_\kappa, B_\kappa, H_\kappa, D_\kappa$ transform in a certain manner from one inertial frame to another, which is called the Lorentz transformation of the field. This can be expressed very briefly by saying that the field quantities constitute the components of two second-order field tensors F_{rs} and H_{rs} (consult Ref. 17 for the definition).

Define a 4-vector A_r in such a way that the A'_r are identical with the potentials entering Eqs. (2), then the tensor equation

$$F_{rs} = c(A_{s,r} - A_{r,s}) \quad (17)$$

is valid because (17) is equivalent to (2) in the rest system K' , which in turn shows that the primes may be omitted in (2b), whereas the spatial part of (17) is equivalent to the equation $B_\kappa = \epsilon_{\kappa\lambda\nu} A_{\nu,\lambda}$.

In order to express the field quantities H_κ and D_κ by means of the potential, this can only be done via the constitutive relations. Therefore, it might be convenient to have covariant forms of these relations to our disposal.

Following Marx¹⁸ in a slightly deviating way, we define

$$F_r \equiv (1/c) F_{rs} U_s, \quad (18a)$$

$$\tilde{F}_r \equiv (1/ic^2) F^*_{rs} U_s, \quad (18b)$$

$$K_r \equiv (1/c^2) H_{rs} U_s, \quad (18c)$$

$$\tilde{K}_r \equiv (1/ic) H^*_{rs} U_s, \quad (18d)$$

where F^*_{rs}, H^*_{rs} are dual tensors, i.e., $F^*_{rs} \equiv (1/2!) \epsilon_{rsmn} F_{mn}$, say. It is readily seen that $\tilde{F}'_r = (E'_\rho, 0)$, $\tilde{F}'_r = (E'_\rho, 0)$, $K'_r = (D'_\rho, 0)$, and $\tilde{K}'_r = (H'_\rho, 0)$.

By defining tensors e_{rs}, m_{rs} by means of $e'_{\rho\sigma} \equiv \epsilon_{\rho\sigma}$, $m'_{\rho\sigma} \equiv \mu_{\rho\sigma}$, $e'_{4\sigma} \equiv m'_{4\sigma} \equiv 0$ (the remaining components can be chosen arbitrarily), it follows from (1) that

$$K_r = e_{rs} F_s, \tag{19a}$$

$$F_r = m_{rs} \tilde{K}_s. \tag{19b}$$

Using tensor relations inverse to (18) (expressing second-order field tensors in terms of first order field tensors), one can express H_{rs} in terms of F_{rs} , say (cf. Ref. 18). Marx¹⁸ proposed that the components e'_{r4}, m'_{r4} are vanishing. In order to get regular matrices, we choose $e'_{\rho 4} \equiv m'_{\rho 4} \equiv 0$, $\epsilon'_{44} \equiv \epsilon$, and $m'_{44} \equiv \mu$.

If wanted, even a covariant, four-dimensional formulation of the "wave equations" can be obtained from (5) by means of a simple translation technique. (The usual way is to introduce 4-potentials into the 4-tensor formulation of Maxwell-Minkowski equations, as was done by Viglin⁷ and earlier by Jauch and Watson¹⁹ for the isotropic case.) Defining m_{rs}^{-1} such that $m_{rs}^{-1} m_{st} = \delta_{rt}$, we obviously have $(m_{\rho\sigma}^{-1})' = \mu_{\rho\sigma}^{-1}$, $(m_{44}^{-1})' = 1/\mu$. Furthermore, the definitions $e_{rs}^* \equiv (1/\epsilon)e_{rs} - \delta_{rs}$, $(m_{rs}^{-1})^* \equiv \mu(m_{rs}^{-1}) - \delta_{rs}$ imply that the spatial parts of these tensors in K' are given by (4) and that the remaining components in K' are equal to zero.

Consider the first term of Eq. (5a); we get

$$\epsilon_{\kappa\lambda\nu} \epsilon_{\rho\sigma\tau} (\mu_{\nu\tau}^{-1})^* A'_{\sigma,\rho\lambda} = \epsilon_{\kappa\lambda n 4} \epsilon_{rst 4} (m_{nt}^{-1})^* A'_{s,rl} \\ = -(1/c^2) \epsilon_{\kappa\lambda n u} \epsilon_{rstv} (m_{nt}^{-1})^* A'_{s,rl} U'_u U'_v.$$

It is observed that the "fourth component" (κ is replaced by the integer 4 in the final expression) vanishes.

The second term in Eq. (5a) is translated in this way:

$$n^2 \epsilon_{\kappa\lambda}^* (A'_{4,\lambda 4} - A'_{\lambda,44}) = -\frac{n^2}{c^2} e_{\kappa l}^* (A'_{r,ls} - A'_{l,rs}) U'_r U'_s.$$

Again, it is noticed that the fourth component of the final expression vanishes.

In order to translate the third term of (5a), we define tensors s_{rs} and s_{rs}^* by means of $s'_{\rho\sigma} \equiv \sigma_{\rho\sigma}$, $s'_{4\rho} \equiv s'_{\rho 4} \equiv 0$, $s'_{44} \equiv \sigma$, $s_{rs}^* \equiv s_{rs}/\sigma - \delta_{rs}$. Since $s_{\rho\sigma}^* = \sigma_{\kappa\lambda}^*$ [cf. (4)], we conclude that

$$-(i\sigma/c\epsilon) \sigma_{\kappa\lambda}^* (A'_{4,\lambda} - A'_{\lambda,4}) = -(\sigma/c^2 \epsilon) s_{\kappa l}^* (A'_{ul} - A'_{lu}) U'_u$$

and that the fourth component of the final expression is zero.

The first term of (5b) is equal to the fourth component of $-(1/c^2) e_{ln}^* (A'_{n,rl} - A'_{r,nl}) U'_r U'_l$, whereas the first three components are equal to zero.

It is not difficult to translate the remaining, isotropic terms of (5a) and (5b) into the four-dimensional form of the isotropic wave equation.

Omitting primes, we finally get a 4-tensor formulation of Eq.(5):

$$(1/c^2) \{ [\epsilon_{\kappa\lambda n u} \epsilon_{rstv} (m_{nt}^{-1})^* A_{s,rl} + n^2 e_{kl}^* (A_{u,lv} - A_{l,uv})] U_u V_v \\ - (\sigma/\epsilon) s_{kl}^* (A_{u,l} - A_{l,u}) U_u + e_{ln}^* (A_{n,rl} - A_{r,nl}) U_r U_l \\ + A_{k,nn} - [(n^2 - 1)/c^2] A_{k,rs} U_r U_s = -S_k. \tag{20}$$

Finally, we want to make some remarks on dispersive media. In K' the constitutive parameters of a dispersive medium are assumed to be functions of the frequency $\omega' = ck'_4/i$ corresponding to a Fourier component of the field.

Since the formalism provides a decomposition of the field into Fourier components in all inertial frames, we may define a "frequency" referring to an arbitrary inertial system K :

$$\omega \equiv ck_4/i. \tag{21}$$

In order to get covariant forms for the functions which express the dependency of the constitutive parameters on ω' , we observe that

$$\omega' = -k'_r U'_r = -k_r U_r. \tag{22}$$

Clearly, (22) shows that in K (different from K') the constitutive parameters depend not only on k_4 (i.e., ω) but also on the spatial part k_p of the "propagation tensor" k_r .

IV. ISOTROPIC, NONDISPERSIVE MEDIUM

For an isotropic, nondispersive medium it is seen from (4), (8), and (10) that $\epsilon_{\kappa\lambda}^* = 0$, etc., $M'_{rs} = 0$, and $g'_{rs} = (1/V') \delta_{rs}$. Defining a Green's function $G'(u'_i)$ such that $G'_{rs}(u'_i) = G(u'_i) \delta_{rs}$ we find that this function is given by the integral [cf. (16a)]

$$G'(u'_r) = \int_{M_4} \frac{1}{V'(k'_r)} e^{ik'_t u'_t} d(k'_r), \tag{23}$$

where $V'(k'_r)$ is given by (8e).

The substitution $k'_4 + il'_4 \rightarrow k'_4$, $l'_4 \equiv (i/2c)(\sigma/\epsilon)$ leads to

$$G'(u'_r) = e^{l'_4 u'_4} \iiint_{-\infty}^{\infty} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{ik'_t u'_t} \\ \times \frac{e^{ik'_t u'_t}}{k'_\lambda k'_\lambda + n^2(k'_4{}^2 + l'_4{}^2)} d(k'_r), \alpha \equiv -|l'_4|. \tag{24}$$

The integral in (21) is the Fourier integral of the time-dependent Green's function of the Klein-Gordon differential equation. Since G is a tensor of zero order (Lorentz invariant) the Lorentz transformation of this well-known function²⁰ to an arbitrary inertial frame can be achieved by translating the dependency of this function on the variables u_r into covariant tensor language, as was done by the author in Ref. 13. This is easier than evaluating the integrals in the covariant edition of (24) which is given by

$$G(u_r) = e^{l'_t u'_t} \int_{M_4} \frac{e^{ik'_t u'_t}}{k_t k_t - \kappa(k_t U_t)^2 - [(n/c) l_t U_t]^2} d(k_r), \tag{25}$$

where we have defined a 4-vector l_r such that $l'_r = (0, 0, 0, (i/2c)(\sigma/\epsilon))$.

The "difficult" way of evaluating (25) was used by Chen and Yen¹⁰ and, for the lossless case, by Compton.⁵ Earlier K.S.H. Lee and Papas¹ evaluated these integrals for a time-harmonic source in a simple medium. [Harmonic time dependency implies that the k_4 -integration in (25) "disappears". This is readily seen from (15a) in connection with (25):

Assuming that S_s in (15a) is time-harmonic, the z_4 -integration introduces a time-dependent δ function in (24)]

V. IONIZED GAS

Consider a medium which is dispersive but otherwise simple (i.e., isotropic and lossless). Furthermore, assume that $\mu = \mu_0$ and

$$\epsilon(\omega') = \epsilon(-k_r U_r) = \epsilon_0 [1 + (ck_{(p)}/k_r U_r)^2], \quad (26)$$

where μ_0, ϵ_0 are the permeability and the dielectric constant of vacuum, $k_{(p)} \equiv i\omega_{(p)}/c$, and $\omega_{(p)}$ is a real number.

Following Ref. 4, we wish to evaluate the Green's function for the unbounded space in the form of a one-dimensional integral, but our approach is different from that in Ref. 4: A well-known result referring to the rest frame K' of the medium is transformed to K . In the case which is under consideration Eq. (16) reduces to

$$G'_{ns}(u'_i) = \mu_0 \int_{M_4} \frac{\delta_{ns} - \kappa/n^2 U'_n U'_s}{k'_\lambda k'_\lambda + n^2 k_4'^2} e^{ik'_i u'_i} d(k'_r), \quad (27)$$

where we have made use of (13).

It is well known from the corresponding radiation problem in a nonmoving and nondispersive medium that the threefold space integral in (27) can be reduced to a single integral (of course, this result can be obtained by exactly the same steps which are used in Ref. 4 but with the simplification that v is zero).

We have

$$G'_{ns}(u'_i) = i\pi^2 \mu_0 \int_{-i\infty}^{i\infty} \int_{-\infty}^{\infty} + [\delta_{ns} + (\kappa/n^2) U'_n U'_s] H_0^1[\rho' \sqrt{-(k_3'^2 + n^2 k_4'^2)}] \times e^{i(k_3' u_3' + k_4' u_4')} dk_3' dk_4', \quad (28)$$

where $\rho' \equiv (u_1'^2 + u_2'^2)$ and H_0^1 is the Hankel function of zero order and first kind.

By means of (26), the argument of the Hankel function can be written as $\rho' [k_{(p)}^2 - (k_3'^2 + k_4'^2)]^{1/2}$.

As in Ref. 4 we choose (without loss of generality) the velocity of K relative to K' to be directed along the $x_3(x'_3)$ axis. This implies that the first and the second component of a 4-vector are unaffected by the Lorentz transformation so that $\rho' = \rho$. Furthermore, $k_3'^2 + k_4'^2 = k_3^2 + k_4^2$ and $k_3' u_3' + k_4' u_4' = k_3 u_3 + k_4 u_4$ (because these expressions are tensors of zero order, i.e., invariants).

The transformation of (27) to an arbitrary inertial frame K is now almost trivial. Eliminating κ/n^2 by means of (26), we obtain

$$G_{ns}(u_i) = i\pi^2 \mu_0 \int_{-i\infty}^{i\infty} \int_{-\infty}^{\infty} \left(\delta_{ns} + \frac{k_{(p)}^2 U_n U_s}{c^2 k_{(p)}^2 + (k_r U_r)^2} \right) \times H_0^1[\rho \sqrt{k_{(p)}^2 - (k_3^2 + k_4^2)}] e^{i(k_3 u_3 + k_4 u_4)} dk_3 dk_4. \quad (29)$$

The first term of this expression (containing δ_{ns}) may be integrated with respect to k_3 , leading to the result of Ref. 4.

On the other hand, it can be done in this way: Since $k'_\lambda k'_\lambda + n^2 k_4'^2 = k'_r k'_r - k_{(p)}^2 = k_r k_r - k_{(p)}^2$, the first term of (27) can be written as

$$\mu_0 \delta_{ns} \int_{-i\infty}^{i\infty} e^{ik_4 u_4} \iiint_{-\infty}^{\infty} \frac{e^{ik_r u_r}}{k_r k_r - (k_{(p)}^2 - k_4^2)} d(k_p) dk_4.$$

This expression is covariant (!), and the spatial integral of this term is well known (from the simple wave equation) and is equal to $\exp[i\tau(k_{(p)}^2 - k_4^2)^{1/2}]/4\pi r$ in agreement with Ref. 4.²¹

VI. LOSSLESS, ELECTRICALLY, UNIAXIALLY ANISOTROPIC MEDIUM

Consider a medium for which $\sigma, \sigma_{\kappa\lambda}^*$, and $(\mu_{\kappa\lambda}^{-1})$ are vanishing and

$$\epsilon_{\kappa\lambda}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau - 1 \end{pmatrix}, \quad (30)$$

where τ is a given positive constant [cf. (4)].

M_{kl} as given by (8) reduces considerably, and g'_{rs} is readily found from (10). We get

$$g'_{rs}(k'_i) = \begin{pmatrix} 1/V' & 0 & 0 \\ 0 & 1/V' & 0 \\ 0 & 0 & \frac{X'}{V'Z'} \frac{(\tau-1)n^2 k_3' k_4'}{V'Z'} \\ 0 & 0 & \frac{(\tau-1)k_3' k_4'}{V'Z'} \frac{Y'}{V'Z'} \end{pmatrix}, \quad (31)$$

where V' is given by (8e) and

$$X' \equiv k_1'^2 + k_2'^2 + \tau k_3'^2 + (nk_4')^2, \quad (32a)$$

$$Y' \equiv k_1'^2 + k_2'^2 + k_3'^2 + \tau(nk_4')^2, \quad (32b)$$

$$Z' \equiv k_1'^2 + k_2'^2 + \tau k_3'^2 + \tau(nk_4')^2. \quad (32c)$$

If we restrict the discussion to inertial frames moving on the axis of symmetry (i.e., the x_3 direction) it is consistent (i.e., independent of the inertial frame) to define two 4-vectors s_r^o and s_r^e in such a way that $s_r = s_r^o + s_r^e$ and $s_3^o = s_4^o = s_1^e = s_2^e = 0$.

Case I: Consider a source for which $s_r^e = 0$, e.g., a longitudinally orientated current loop (magnetic dipole). Obviously, $s_1 = s_1'$, $s_2 = s_2'$, and it is immediately seen from (31) that the only mode which is excited is identical with the isotropic case. This mode is called the ordinary or the first mode (cf. Ref. 11).

Case II: Next, consider a source for which $s_r^o = 0$, e.g., a longitudinally orientated, electrical dipole. Using the continuity equation $s_3^o k_3^o + s_4^o k_4^o = 0$, we derive, by means of (14) and (31), the equation

$$g_{rt}(k_n) s_t = g^e(k_n) \delta_{rt} s_t, \quad g^e(k_n) \equiv 1/Z(k_n). \quad (33)$$

Obviously, only one mode is excited. The mode is of a

different form than in Case I and is called the extraordinary or second mode (cf. Ref. 11).

It is not difficult to find the corresponding extraordinary Green's function G^e from the ordinary Green's function $G^o \equiv G$ as given by (23), (24) (with $\sigma = 0$).

The connection between the Green's functions is given by

$$G^e(u_r) = \tau^{-1} G^o(u_1, u_2, u_3/\sqrt{\tau}, u_4/\sqrt{\tau}). \quad (34)$$

In K' , (34) is verified just by inspection of the corresponding fourfold Fourier integrals, and it is easily seen that the relation is covariant under those special Lorentz transformations which we are dealing with in this section.

$G(u_r)$ may be found in Ref. 13. Applying the extraordinary Green's function $G^e(u_r)$ to the case of a time-harmonic source, we readily derive the extraordinary Green's function as given by Lee and Lo.⁶

Case III: Finally, consider an arbitrary source distribution. From the continuity equation $s'_\rho k'_\rho + n^2 s'_4 k'_4 = 0$ in connection with (14) and (31), we get

$$a'_r = \tilde{g}'_{rt} s'_t, \quad (35)$$

where we have defined a tensor \tilde{g}'_{rt} by means of the components in K' :

$$\tilde{g}'_{rt} \equiv \begin{pmatrix} 1/V' & 0 & 0 & 0 \\ 0 & 1/V' & 0 & 0 \\ 0 & 0 & 1/Z' & 0 \\ 0 & 0 & 0 & 1/Z' \end{pmatrix}$$

$$+ \frac{1-\tau}{V'Z'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k'_1 k'_3 & k'_2 k'_3 & 0 & 0 \\ k'_1 k'_4 & k'_2 k'_4 & 0 & 0 \end{pmatrix} \quad (36)$$

The contribution to the Green's function \tilde{G}'_{rt} corresponding to the first matrix term in (36) can immediately be obtained from Cases I and II. As to the second matrix term, the evaluation of the Fourier integral is more complicated. But, if \tilde{G}'_{rt} can be found in K' , the transformation to another inertial system K is simple. In fact, the second matrix in (27) is not affected at all by Lorentz transformations between inertial frames moving on the axis of symmetry. (Of course, the way in which the matrix depends on k_r is affected.)

Though there is no need for working in K at all (as was done in Ref. 6), it is easily verified (in K') that

$$V = k_r k_r + (1 - n^2)(k_r U_r/c)^2, \quad (37a)$$

$$Z = k_r k_r + (\tau - 1)(k_r T_r)^2 + (1 - \tau n^2)(k_r U_r/c)^2, \quad (37b)$$

where we have defined a 4-vector T_r such that $T'_r = (0, 0, 1, 0)$.

If the frequency ω is introduced by means of (21), the equations $V = 0$ and $Z = 0$ represent surfaces called the ordinary and extraordinary dispersion surface, respectively. These surfaces are of fundamental significance when asymptotic expressions of the Fourier integrals are to be evaluated (as was done in Ref. 6), but this can more easily be done in K' (cf. Refs. 22, 23), whereupon the result can be translated to K as outlined here.

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