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A SYMMETRIC STATE-SPACE FORMAT FOR MOMENTUM CONSERVING TIME INTEGRATION ALGORITHMS WITH ENERGY DISSIPATION

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ABSTRACT

A symmetric state-space format is presented for momentum conserving time integration algorithms, and two schemes for introduction of algorithmic damping are described. The balanced damping is formulated in terms of the original state-space variables, leading to a linear increase of damping with frequency in the low-frequency region. In the extended state-space algorithm auxiliary variables introduce damping via discrete first order filters. By this method the low-frequency damping is lifted to third order in frequency. Both algorithms have full symmetry in the state-space variables.

1. INTRODUCTION

Many models for dynamics of structures and solid bodies are based on a spatial discretization e.g. in the form of finite elements. These models typically lead to a large number of high-frequency modes, that do not represent the physics of the original problem well. It is therefore desirable to be able to introduce a controlled energy dissipation in the high-frequency regime directly into the time integration algorithm. The precise mechanism for this depends on the type of algorithm. The classic time integration algorithms, exemplified by the Newmark family [1], are based on collocation. When the displacement \mathbf{u}_n , velocity \mathbf{v}_n and acceleration \mathbf{a}_n are available at time t_n three equations for their value at time t_{n+1} are obtained by satisfying the equation of motion at t_{n+1} and introducing two approximate relations, essentially representing truncated forms of a Taylor expansion. Typically these algorithms are analyzed by a spectral analysis [2]. The collocation algorithms are typically of second order accuracy in linear problems, and algorithmic energy dissipation can be introduced by changes in the parameters defining the truncated Taylor series. However, direct introduction of energy dissipation via change of the value of the parameters in the algorithm leads to undesirable damping of the low-frequency components as well. In the case of the Newmark algorithm family the problem with low-frequency damping has been solved by replacing the exact collocation of the equation of motion at time

t_{n+1} with an average, in which inertial and force terms are given different weights, [3–5]. By suitable choice of the parameters the low-frequency algorithmic damping ratio can be changed from the original linear dependence on frequency to cubic dependence. It has recently been demonstrated that the variants of these so-called alpha methods can be obtained by using a first order filter to modify the algorithmic damping in the original Newmark algorithm [6].

Collocation methods are typically derived from asymptotic arguments applied to the low-frequency components. Thus, the response may not obey conservation laws for e.g. momentum and energy, and this may reduce the accuracy for the high-frequency components of the response. In the case of the Newmark algorithm family and its alpha modified forms this shows up in two ways. The introduction of algorithmic energy dissipation leads to a reinterpretation of the energy by the algorithm whereby the elastic part of the energy becomes larger relative to the kinetic energy term, [7, 8]. This may lead to energy oscillations and even to response ‘overshoot’ in high-frequency transients [9, 10]. Secondly, in the case of collocation methods momentum and energy conservation properties associated with central difference methods do not carry over from the linear to the non-linear case. This has prompted an interest in time integration methods based on an integrated form of the equation of motion. The integrated form represents the increment of the momentum of the system, and these methods are associated with momentum, and often also energy, conservation properties. It has recently been shown that for Green strain elasticity the difference between a classic collocation scheme and the momentum and energy conserving formulation shows up as an extra algorithmic damping term formed by the increment of the geometric stiffness [11]. This term typically experiences sign changes during the motion, thus producing intermittent negative as well as positive damping if a collocation algorithm is used. In conservation algorithms there are typically no free parameters that can be adjusted to introduce controlled algorithmic damping. However, monotonic algorithmic damping can be introduced into the conservative scheme by use of a combination of a weighted average for internal forces as well as for the mean velocity in the definition of the displacement increment [12]. The frequency dependence of this damping is identical to that obtained when changing the parameter values in the original Newmark algorithm, but without the undesirable energy oscillations.

The present paper treats the problem of energy dissipation in momentum conserving time integration in two steps. First it is demonstrated, how the state space formulation leads to a simple format in which the balanced algorithmic damping terms are easily introduced and calibrated. It is then shown, how the representation of algorithmic energy dissipation can be improved by introduction of a new set of variables related to the displacement and velocity vectors by a suitable first order filter with scalar coefficients. By this device an algorithmic dissipation can be obtained that is of third order in the low-frequency regime. In both cases the damping properties are identified from an energy balance equation. This indicates the general form of the damping and demonstrates unconditional stability of the algorithms. The detailed balance between the terms and the frequency behavior is characterized via spectral analysis, and the present paper is therefore concentrated on linear systems. An ad hoc extension to non-linear conservative schemes is fairly straightforward, but for non-linear problems the optimal filter formulation requires further consideration, presently in progress. It is demonstrated that the two algorithms presented here do not exhibit energy oscillations and response overshoot for high-frequency transients. It is an important feature of both algorithms that they can be arranged to require in each time step only the solution of a system of equations of the same size of the corresponding quasi-static problem, followed by one or three vector updates with scalar coefficients - the so-called ‘single step – single solve’ property.

2. BASIC EQUATIONS AND THE STATE-SPACE FORMAT

The following analysis is concerned with linear systems described by a discretized model in terms of the nodal displacement vector $\mathbf{u}(t)$. The equation of motion is of the form

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \quad (1)$$

where the system is described by the mass matrix \mathbf{M} , the viscous damping matrix \mathbf{C} and the stiffness matrix \mathbf{K} , while the external load vector is $\mathbf{f}(t)$. Time differentiation is denoted by a dot, e.g. the velocity $\dot{\mathbf{u}} = d\mathbf{u}/dt$.

The energy balance equation corresponding to the equation of motion (1) is obtained by multiplication with $\dot{\mathbf{u}}^T$, followed by integration,

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right) = \dot{\mathbf{u}}^T \mathbf{f} - \dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} \quad (2)$$

The left side is the rate of change of the mechanical energy, consisting of the sum of kinetic and elastic energy, and the right side contains the rate of work of the external load and an energy dissipation term expressed by the viscous damping matrix. In the development of discrete time integration algorithms it is desirable that they satisfy a similar energy balance equation in discretized form, and that the damping term(s) remain representative over the full frequency range in spite of the time discretization. Many models in dynamics include a large number of high-frequency modes, the details of which are not well described by the spatially discretized model. It is therefore of interest to develop algorithms that introduce additional algorithmic dissipation in high-frequency modes.

It is advantageous to recast the second order differential equation of motion into an augmented set of first order differential equations by introduction of a new independent variable representing the velocity. In the present case with constant mass matrix introduction of a new vector $\mathbf{v} = \dot{\mathbf{u}}$ representing the velocity is the most direct choice. When this definition is multiplied by the mass matrix \mathbf{M} the augmented system for the state space variables $[\mathbf{u}, \mathbf{v}]$ takes the symmetric form,

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \quad (3)$$

While traditional collocation based algorithms like the Newmark family with its various modifications introduce an approximate relation between the displacement, velocity and acceleration, the more recent momentum conserving approach to time integration algorithms is based on an integrated form of the state-space equations. For constant system matrices the first term in the state-space equation integrates exactly, while the second term contains integrals of the displacement and velocity over the time interval $[t_n, t_{n+1}] = [t, t + h]$. If the time mean value over the time interval is denoted by an overbar, e.g. $\bar{\mathbf{f}} = h^{-1} \int_h \mathbf{f} d\tau$, the integrated state-space equations take the form

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{v} \end{bmatrix} + h \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} h \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \quad (4)$$

where $\Delta \mathbf{u}$ and $\Delta \mathbf{v}$ are the finite increments over the time interval. The time integrals of the displacement and velocity vectors are generally unknown, and thus $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ must be represented by an approximation. For linear problems second order accurate schemes can be developed

using the arithmetic mean values $\bar{\mathbf{u}} = \frac{1}{2}(\mathbf{u}_{n+1} + \mathbf{u}_n)$ and $\bar{\mathbf{v}} = \frac{1}{2}(\mathbf{v}_{n+1} + \mathbf{v}_n)$. While the load can in principle be introduced by its time integral, in practice it is convenient to use the arithmetic mean also for the load vector. Higher order accuracy schemes can be developed by evaluating the mean value integrals via integration by parts, but this leads to an equation system in the full set of state-space variables [13].

The energy equation for the discretized equations (4) is obtained by pre-multiplication of this equation with $[\Delta \mathbf{u}^T, -\Delta \mathbf{v}^T]$. The mean value terms are rewritten by use of the relation

$$\Delta \mathbf{u}^T \mathbf{K} \bar{\mathbf{u}} = \left[\frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} \quad (5)$$

for the displacement vector and the similar relation for the velocity mean value.

$$\left[\frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} = \Delta \mathbf{u}^T \bar{\mathbf{f}} - \Delta \mathbf{u}^T \mathbf{C} \Delta \mathbf{u} \quad (6)$$

This energy relation has a format that resembles that of its continuous counterpart (2). However, in the discrete form the velocity in the dissipation term is replaced by $\Delta \mathbf{u}$ and this leads to a non-monotonic dependence of energy dissipation on the frequency of the response, see e.g. [2]. This implies that damping represented via a viscous damping matrix \mathbf{C} will not lead to the desired damping of high-frequency modes outside the range represented by the time increment h . In the following section a simple model where two balanced dissipation terms give monotonic frequency dependence is developed. Subsequently this model is further improved by introduction of extended state-space variables.

3. BALANCED DISSIPATION

It is easily seen that damping can be obtained by introducing terms in the diagonal of the first matrix of the integrated state space equations (4). For a linear system without structural damping introduction of terms of order $O(h)$ leads to the following form,

$$\begin{bmatrix} \frac{1}{2} \alpha h \mathbf{K} & \mathbf{M} \\ \mathbf{M} & -\frac{1}{2} \alpha h \mathbf{M} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} h \bar{\mathbf{u}} \\ h \bar{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} h \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \quad (7)$$

The extra diagonal terms are determined by the fact that only the system matrices \mathbf{K} and \mathbf{M} are available, and the extra terms must be of order $O(h)$. The numerical coefficients α on the two dissipation terms must be identical to produce complex conjugate roots of the characteristic equation in the full frequency interval, [14]. The energy balance equation is obtained by pre-multiplication with $[\Delta \mathbf{u}^T, -\Delta \mathbf{v}^T]$.

$$\left[\frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} = \Delta \mathbf{u}^T \bar{\mathbf{f}} - \alpha \left(\frac{1}{2} \Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{u} - \frac{1}{2} \Delta \mathbf{v}^T \mathbf{M} \Delta \mathbf{v} \right) \quad (8)$$

It is seen that the coefficient $\alpha > 0$ contributes an energy dissipation term implying unconditional stability of the scheme.

3.1 Spectral analysis

The spectral analysis is performed on the free vibrations associated with the discretized state-space equations (7), which for convenience are written in the generic form

$$\mathbf{B} \Delta \mathbf{w} + \mathbf{A} \bar{\mathbf{w}} = \mathbf{0} \quad (9)$$

In this equation the matrices \mathbf{B} and \mathbf{A} represent the block matrices of (7), while the vector \mathbf{w}_n contains all state-space variables at time t_n , $\mathbf{w}_n = [u, hv]^T_n$. The free vibrations satisfy the evolution equation $\mathbf{w}_{n+1} = \lambda \mathbf{w}_n$ in terms of the amplification factor λ . Substitution of the free vibration solution into the generic equation of motion (9) leads to the generalized eigenvalue problem

$$(\mathbf{A} + 2z\mathbf{B}) \mathbf{w}_n = \mathbf{0} \quad (10)$$

where the eigenvalue z is related to the amplification factor λ by the Möbius transformation,

$$z = \frac{\lambda - 1}{\lambda + 1}, \quad \lambda = \frac{1 + z}{1 - z} \quad (11)$$

Stable solutions require the amplification factor to be on or within the unit circle in the complex plane. The Möbius transformation maps the unit circle on the left complex half-plane, and thus in general stability can be analyzed in terms of the variable z by the Routh-Hurwitz criterion.

In the present case the spectral analysis of the algorithm can be carried out explicitly. The time step h and the angular modal frequency ω are combined into the non-dimensional frequency parameter $\Omega = h\omega$, whereby

$$\mathbf{A} = \begin{bmatrix} \Omega^2 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \alpha\Omega^2 & 1 \\ 1 & -\alpha \end{bmatrix} \quad (12)$$

The characteristic equation has the solution

$$\frac{1}{z} = -\alpha \pm \frac{2i}{\Omega} \quad (13)$$

It is seen that $\text{Re}[z] \leq 0$ for all values of Ω , when $\alpha \geq 0$, confirming the unconditional stability of the algorithm. The amplification factor follows from the Möbius transformation (11) as

$$\lambda = \frac{1 \pm \frac{1}{2}(1 - \alpha) i\Omega}{1 \mp \frac{1}{2}(1 + \alpha) i\Omega} \quad (14)$$

The two branches of the locus of the complex amplification factor λ are shown in Fig. 1a for $|\lambda_\infty| = 0.6$. They connect the low-frequency limit $\lambda_0 = 1$ with the high-frequency limit

$$\lambda_\infty = -\frac{1 - \alpha}{1 + \alpha} \quad (15)$$

by semi-circles in the upper and lower complex plane, respectively. For $\alpha = 0$ there is no damping, and the amplification factor λ traces the unit circle.

The influence of the amplification factor is often separated into two parts: the amplification of the response, and the error in the period of free vibration response. These two effects are obtained by representing the amplification factor in factored form as

$$\lambda = |\lambda| e^{i\varphi} = e^{-\zeta\varphi} e^{i\varphi} \quad (16)$$

The maximum absolute value of the amplification factor $|\lambda|$ is the spectral radius, while φ is the phase and ζ the (algorithmic) damping ratio. The spectral radius is shown as function of the non-dimensional frequency $\Omega = \omega h$ in Fig. 1b for $|\lambda_\infty| = 0.6$. Different values of $|\lambda_\infty|$ leads to similar curves. It is characteristic for all the curves that damping sets in well below $\Omega = 1$. The detailed low-frequency behavior follows from the asymptotic analysis below.

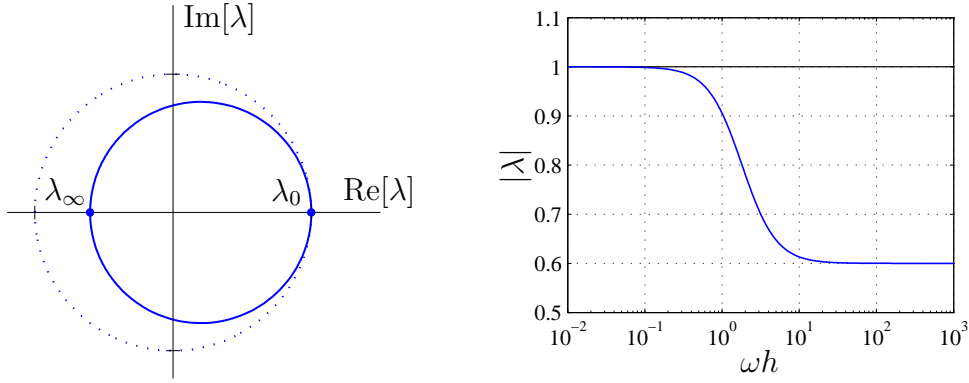


Figure 1: Balanced damping with $\lambda_\infty = 0.6$. a) Amplification factor λ , b) Spectral radius $|\lambda|_{\max}$.

3.2 Low-frequency asymptotics

The damping of the algorithm is characterized by the spectral radius $|\lambda|$. It follows from the solution (14) for the amplification factor that

$$|\lambda|^2 = \frac{1 + \frac{1}{4}(1 - \alpha)^2 \Omega^2}{1 + \frac{1}{4}(1 + \alpha)^2 \Omega^2} \simeq 1 - \alpha \Omega^2 \quad (17)$$

from which the low-frequency algorithmic damping ratio follows as

$$\zeta \simeq \frac{1}{2} \alpha \Omega \quad (18)$$

Thus, the introduction of balanced dissipation into the conservative time integration algorithm introduces a low-frequency damping that increases linearly with the non-dimensional frequency Ω . The behavior of the Newmark algorithm with alpha damping is identical [6], and the reason for introducing various alpha modifications of the Newmark method is precisely to increase the order of the low-frequency damping in order to minimize the effect of damping in the low-frequency regime, [3–6]. A similar reduction of the low-frequency damping of the present algorithm is discussed in Section 4.

The phase angle can be expressed in terms of the variable z by use of the Möbius transformation (11), from which

$$\tan \varphi = \frac{\text{Im}[\lambda]}{\text{Re}[\lambda]} = \frac{\text{Im}[2z]}{1 - |z|^2} = \frac{\Omega}{1 - \frac{1}{4}(1 - \alpha^2)\Omega^2} \quad (19)$$

Substitution of a series representation of the phase angle in terms of powers of Ω into (19) leads to the asymptotic result, [14],

$$\varphi \simeq \Omega - \frac{1}{12}(1 + 3\alpha^2)\Omega^3 \quad (20)$$

From this expression the relative error of the period $T = 2\pi/\omega$ follows as

$$\frac{\Delta T}{T} = \frac{\Omega}{\varphi} - 1 = \frac{1}{12}(1 + 3\alpha^2)\Omega^2 + O(\Omega^4) \quad (21)$$

It is seen that the period error is positive and that algorithmic dissipation increases the period error further. Thus, it is desirable to introduce only modest algorithmic damping.

Table 1. Balanced dissipation algorithm.

1)	System matrices $\mathbf{K}, \mathbf{C}, \mathbf{M}$,
	$\mathbf{K}_* = \kappa \left[\mathbf{K} + \frac{2}{\kappa h} \mathbf{C} + \left(\frac{2}{\kappa h} \right)^2 \mathbf{M} \right]$
2)	Initial conditions: $\mathbf{u}_0, \mathbf{v}_0$.
3)	Increments $n = n + 1$:
	$\Delta \mathbf{u} = \mathbf{K}_*^{-1} \left\{ \mathbf{f}_{n+1} + \mathbf{f}_n - 2\mathbf{K}\mathbf{u}_n + \frac{4}{\kappa h} \mathbf{M}\mathbf{v}_n \right\}$
	$\Delta \mathbf{v} = 2\Delta \mathbf{u} / \kappa h - 2\mathbf{v}_n / \kappa$
4)	Vector updates:
	$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta \mathbf{u}, \quad \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta \mathbf{v}$
5)	Return to 3) for new time step, or stop.

3.3 Balanced dissipation algorithm

The algorithm with consistent energy dissipation can now be expressed for linear systems including structural damping \mathbf{C} . The formulation takes on a particularly compact form when introducing the parameter

$$\kappa = 1 + \alpha \quad (22)$$

The second state-space equation gives the forward velocity \mathbf{v}_{n+1} by a simple vector relation, while the displacement is determined by a system of equations with the ‘effective stiffness matrix’

$$\mathbf{K}_* = \kappa \left[\mathbf{K} + \frac{2}{\kappa h} \mathbf{C} + \frac{4}{(\kappa h)^2} \mathbf{M} \right] \quad (23)$$

It is seen that the terms are scaled as if extending the time increment from h to κh , whereby the dynamic terms get less weight. The algorithm is shown in Table 1.

4. EXTENDED STATE-SPACE FOR HIGH-FREQUENCY DISSIPATION

In the balanced dissipation algorithm the energy dissipation is introduced via the diagonal terms containing $\Delta \mathbf{u}$ and $\Delta \mathbf{v}$. In the low and mid-frequency regime these terms are roughly equivalent to $h\dot{\mathbf{u}}$ and $h\dot{\mathbf{v}}$. The idea is to replace these terms formulated by use of the state-space variables with similar terms using a new set of variables $\mathbf{s}(t)$ and $\mathbf{t}(t)$ defined via first order filters with scalar coefficients,

$$\alpha h \dot{\mathbf{s}} + \mathbf{s} = h \dot{\mathbf{u}} \quad , \quad \alpha h \dot{\mathbf{t}} + \mathbf{t} = h \dot{\mathbf{v}} \quad (24)$$

The non-dimensional parameter α determines the common time scale of the two filters. At low frequencies the second term in the filter equation will dominate the first, and thus the new variables \mathbf{s} and \mathbf{t} will nearly be out of phase with the corresponding state-space variables \mathbf{s} and \mathbf{t} . Thus, it may be conjectured that representing the dissipation in terms of the increments $\Delta \mathbf{s}$ and $\Delta \mathbf{t}$ instead of the original formulation in terms of $\Delta \mathbf{u}$ and $\Delta \mathbf{v}$ will strongly reduce the damping in the low-frequency regime. In the high-frequency regime discretization errors occur, and intuitive predictions are more uncertain. However, in the high frequency regime the first terms of the filters (24) dominate and thus the new variables \mathbf{s} and \mathbf{t} are expected to follow

the original state-space variables \mathbf{u} and \mathbf{v} closely, and therefore lead to a comparable kind of damping.

A general spectral analysis shows that combination of filters with parameter α and use of the same parameter α for representing the magnitude of the balanced damping algorithm (??) leads to a single quadruple root $-\lambda_\infty$ for the amplification factor at infinite frequency. This corresponds to equal damping of all modes at infinite frequency and is adopted as an optimal property. Introduction of the dissipation terms via the increments $\frac{1}{2}\alpha\Delta\mathbf{s}$ and $\frac{1}{2}\alpha\Delta\mathbf{t}$ leads to the state-space equations

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} + h \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{v}} \end{bmatrix} + \frac{1}{4}\alpha^2 h \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{s} \\ \Delta\mathbf{t} \end{bmatrix} = h \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix} \quad (25)$$

supplemented by the discretized filter equations

$$\alpha \begin{bmatrix} \Delta\mathbf{s} \\ \Delta\mathbf{t} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{s}} \\ \bar{\mathbf{t}} \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{v} \end{bmatrix} \quad (26)$$

The viscous damping matrix \mathbf{C} is indicated in the algorithm to identify its position and weighting but is not included in the energy balance and spectral analysis.

4.1 Energy balance

The energy balance equation of the filter algorithm follows from multiplication of the state-space equations (25) with $[\Delta\mathbf{u}^T, -\Delta\mathbf{v}^T]$,

$$\left[\frac{1}{2}\mathbf{v}^T \mathbf{M} \mathbf{v} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} = \Delta\mathbf{u}^T \bar{\mathbf{f}} - \frac{1}{4}\alpha^2 (\Delta\mathbf{u}^T \mathbf{K} \Delta\mathbf{s} + \Delta\mathbf{v}^T \mathbf{M} \Delta\mathbf{t}) \quad (27)$$

In order to obtain an energy equation in terms of positive definite quadratic terms, the increments $\Delta\mathbf{u}$ and $\Delta\mathbf{v}$ are eliminated in the damping terms by use of the discretized filter equations (26). Hereby the energy balance equation takes the form

$$\begin{aligned} \left[\frac{1}{2}\mathbf{v}^T \mathbf{M} \mathbf{v} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} + \frac{1}{8}\alpha^2 (\mathbf{t}^T \mathbf{M} \mathbf{t} + \mathbf{s}^T \mathbf{K} \mathbf{s}) \right]_n^{n+1} = \\ \Delta\mathbf{u}^T \bar{\mathbf{f}} - \frac{1}{4}\alpha^3 (\Delta\mathbf{s}^T \mathbf{K} \Delta\mathbf{s} + \Delta\mathbf{t}^T \mathbf{M} \Delta\mathbf{t}) \end{aligned} \quad (28)$$

This is the energy balance equation of the total system consisting of the original state-space variables \mathbf{u} and \mathbf{v} together with the corresponding extended state-space variables \mathbf{s} and \mathbf{t} . For $\alpha = 0$ the original energy conservation equation of the classic discretized state-space formulation is recovered. For $\alpha > 0$ the extended energy defined in terms of all variables is positive definite and decreases as determined by a positive definite quadratic form of the increments of the new filtered variables \mathbf{s} and \mathbf{t} . This secures unconditional stability of the extended state-space algorithm given by (25) and (26) for $\alpha \geq 0$.

4.2 Spectral analysis

The spectral analysis of the extended state-space algorithm is based on the free vibration response of a single mode of the equations (25) and (26). In the non-dimensional generic format

(9) this corresponds to the extended modal amplitude vector $\mathbf{w} = [u, hv, s, ht]^T$ and the matrices

$$\mathbf{A} = \begin{bmatrix} \Omega^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & \frac{1}{4}\alpha\Omega^2 & 0 \\ 1 & 0 & 0 & -\frac{1}{4}\alpha\Omega \\ -1 & 0 & \alpha & 0 \\ 0 & -1 & 0 & \alpha \end{bmatrix} \quad (29)$$

As shown in [14] the solution to the quartic characteristic equation is

$$\frac{1}{z} = -\left[\alpha \mp \frac{1}{i\Omega}\right](\pm)\sqrt{\left[\alpha \pm \frac{1}{i\Omega}\right]^2 - \alpha^2} \quad (30)$$

where the notation (\pm) is introduced to denote a sign combination that is independent of the previously introduced \pm . The following expression is found after replacing the sign combination $(\pm)\pm$ with (\pm) ,

$$\lambda = \frac{1 \pm i(1 - \alpha)\Omega (\pm) \sqrt{1 \pm 2i\alpha\Omega}}{1 \mp i(1 + \alpha)\Omega (\pm) \sqrt{1 \pm 2i\alpha\Omega}} \quad (31)$$

The sign $(+)$ corresponds to the two branches with origin in $\lambda_0 = 1$, whereas the sign $(-)$ corresponds to the two additional branches with origin at λ_f introduced by the filters.

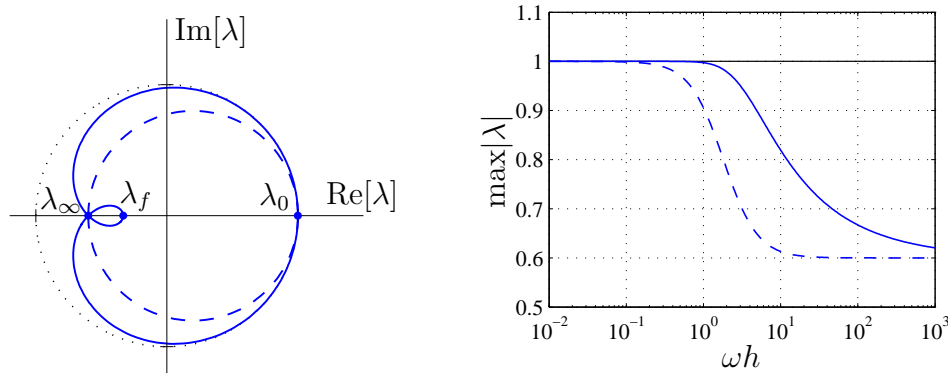


Figure 2. Filter damping with $|\lambda_\infty| = 0.6$. a) Amplification factor λ , b) Spectral radius $|\lambda|_{\max}$.

The trace of the locus of the amplification factor in the complex plane is illustrated in Fig. 2 a for $|\lambda_\infty| = 0.6$. In the low-frequency limit $\Omega = 0$ there are two double roots corresponding to the amplification factors

$$\lambda_0 = 1, \quad \lambda_f = -\frac{1 - 2\alpha}{1 + 2\alpha} \quad (32)$$

where λ_f is the low-frequency origin of the branches introduced by the filters. The corresponding result for the balanced alpha-damping is shown as dashed curves. It is seen that the filter damping stays much closer to the unit circle to the left of the imaginary axis. This is in close correspondence with the effect of ‘generalized alpha damping’ for the Newmark algorithm, illustrated in [6]. The corresponding spectral radius $\max|\lambda|$ is shown in Fig. 2b. It is seen that the spectral radius remains close to unity right up to $\Omega \simeq 1$ before decreasing towards its value $|\lambda_\infty|$. The branches forming the small loop connecting λ_f and λ_∞ are generated by the extended state-space variables. These branches are equivalent to the extra real-valued root appearing in the generalized alpha method [5, 6]. Within the extended state-space formulation they represent the passive response of the extended state-space variables.

4.3 Asymptotic properties

The low-frequency behavior of the algorithm is obtained by an asymptotic analysis, [14]. The spectral radius follows from the Möbius transformation (11b) as

$$|\lambda|^2 = \frac{1+z}{1-z} \frac{1+\bar{z}}{1-\bar{z}} = \frac{1+|z|^2 + \text{Re}[2z]}{1+|z|^2 - \text{Re}[2z]} \simeq 1 - \frac{1}{2}\alpha^3 \Omega^4 \quad (33)$$

By the definition of the damping ratio (16) this corresponds to the algorithmic damping ratio

$$\zeta \simeq \frac{1}{4}\alpha^3 \Omega^3 \quad (34)$$

It is seen that the introduction of the extended state-space variables has changed the leading term of the low-frequency damping from $\frac{1}{2}(\alpha\Omega)$ to $\frac{1}{4}(\alpha\Omega)^3$, i.e. from a linear to a cubic term in $\alpha\Omega$. The damping ratio of the extended state-space algorithm is illustrated by the fully drawn curve in Fig. 3a for $|\lambda_\infty| = 0.8$, while the similar result for the balanced dissipation algorithm is shown by the dashed curve. It is seen that while the damping ratio of the extended state-space algorithm remains below 0.5 pct. for all values of Ω below the Nyquist frequency, the damping ratio of the balanced dissipation algorithm initially grows linearly with Ω before levelling off at a value of 8 pct. at the Nyquist frequency.

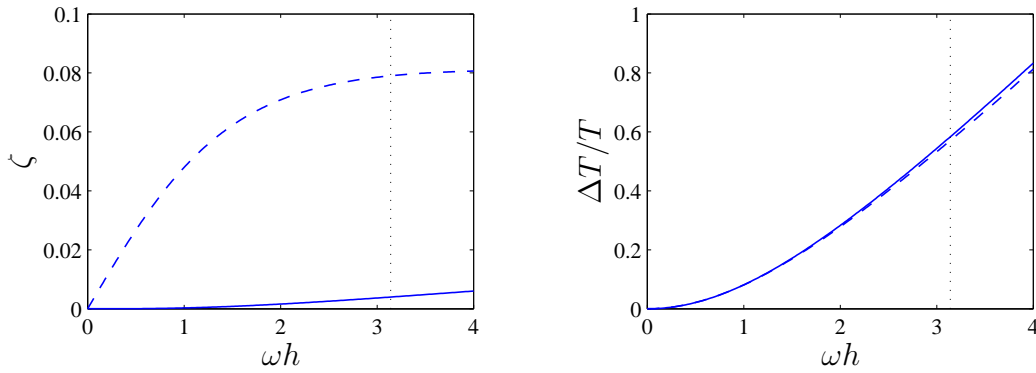


Figure 3: Response parameters for $|\lambda_\infty| = 0.8$. Balanced dissipation (---), Extended state-space (—); a) Damping ratio, b) Period error

The low-frequency expansion of the phase angle for the extended state-space algorithm is identical to (20) derived for the balanced dissipation algorithm. Thus, the asymptotic period error of the extended state-space algorithm is also given by (21). This result is identical to that of the alpha damped Newmark method, while the generalized alpha modification of the Newmark method leads to an increased effect of damping of 50 pct. [6]. The relative period error for the extended state-space algorithm based on the exact amplification factor (31) is shown in Fig. 3b by the fully drawn curve for $|\lambda_\infty| = 0.8$. The similar result for the balanced dissipation algorithm is shown by the dashed curve. It is seen that the difference is negligible.

4.4 Extended state-space algorithm

The extended state-space algorithm is given by the system equations (25) and the filter equations (26). It is an important feature of this set of equations that it can be advanced one time step by solution of a single set of equations of the size of the original system matrices \mathbf{K} and \mathbf{M} , followed by three simple vector updates without matrix operations. The equation system is

Table 2. Extended state-space algorithm.

1)	System matrices $\mathbf{K}, \mathbf{C}, \mathbf{M}$,	
		$\mathbf{K}_* = \kappa \left[\mathbf{K} + \frac{2}{\kappa h} \mathbf{C} + \left(\frac{2}{\kappa h} \right)^2 \mathbf{M} \right]$
2)	Initial conditions: $\mathbf{u}_0, \mathbf{v}_0$,	
		$\mathbf{s}_0 = \mathbf{0}, \quad \mathbf{t}_0 = \mathbf{0}$
3)	Increments $n = n + 1$:	
		$\Delta \mathbf{u} = \mathbf{K}_*^{-1} \left\{ \mathbf{f}_{n+1} + \mathbf{f}_n - \mathbf{K} [2\mathbf{u}_n - (\kappa - 1) \mathbf{s}_n] + \frac{2}{\kappa h} \mathbf{M} [2\mathbf{v}_n - (\kappa - 1) \mathbf{t}_n] \right\}$
		$\Delta \mathbf{v} = 2\Delta \mathbf{u} / \kappa h - [2\mathbf{v}_n - (\kappa - 1) \mathbf{t}_n] / \kappa$
		$\Delta \mathbf{s} = (\Delta \mathbf{u} - \mathbf{s}_n) / (\frac{1}{2} + \alpha), \quad \Delta \mathbf{t} = (\Delta \mathbf{v} - \mathbf{t}_n) / (\frac{1}{2} + \alpha)$
4)	Vector updates:	
		$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta \mathbf{u}, \quad \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta \mathbf{v}$
		$\mathbf{s}_{n+1} = \mathbf{s}_n + \Delta \mathbf{s}, \quad \mathbf{t}_{n+1} = \mathbf{t}_n + \Delta \mathbf{t}$
5)	Return to 3) for new time step, or stop.	

obtained by first eliminating the forward values of the filter variables \mathbf{s}_{n+1} and \mathbf{t}_{n+1} from the system equations, and then eliminate the forward value of the velocity \mathbf{v}_{n+1} . This results in an equation system for the displacement vector increment $\Delta \mathbf{u}$, and the eliminated variables are subsequently recovered by simple vector relations.

The resulting algorithm is shown in Table 2 with the parameter κ defined by

$$\kappa = \frac{(1 + \alpha)^2}{1 + 2\alpha} = 1 + \frac{\alpha^2}{1 + 2\alpha} \quad (35)$$

in the present case. The algorithmic damping effect is obtained by sequential updating of the extended state-space variables \mathbf{s} and \mathbf{t} . The filter has deliberately been formulated as a direct vector relation without matrix operations, and thus the introduction of these variables is essentially free of computational overhead.

5. TRANSIENT HIGH-FREQUENCY RESPONSE

One of the problems associated with introducing algorithmic energy dissipation into the Newmark family of algorithms is, that the known techniques lead to a reinterpretation of the energy as registered by the algorithm, [7, 8]. In alpha damping of the original Newmark algorithm an extra term appears in the energy, equivalent to an increase in the stiffness. This implies that the usual circular constant energy contours in the normalized two-dimensional $(u, v/\omega)$ phase plane become elliptical. As a consequence, free response started with zero velocity will have an equivalent energy larger than the mechanical energy of the system. At times where the kinetic energy dominates this corresponds to a too large velocity. For severely undersampled modes, i.e. modes with $\Omega \gg \frac{1}{2}\pi$, this may lead to the so-called ‘overshoot effect’, illustrated e.g. in [9, 10]. In the generalized alpha method the stiffness modification of the energy in the algorithm remains, and a new term representing a filtered velocity is added. This term does not cure the ‘overshoot’ problem, but may enhance it further [8].

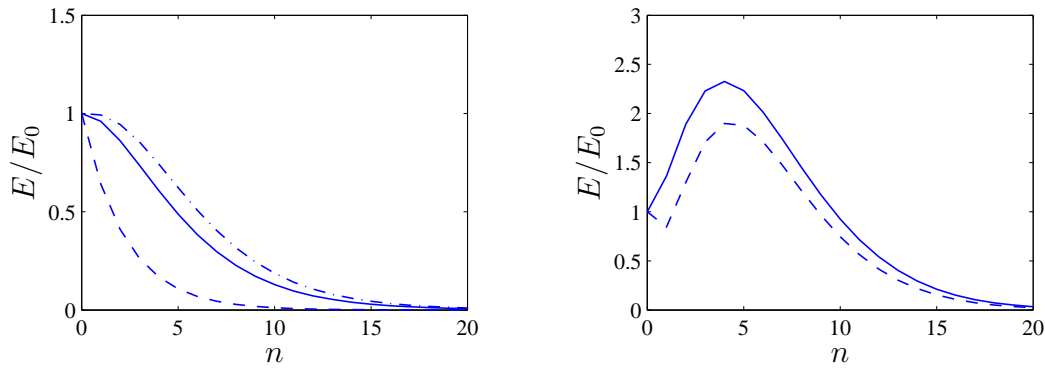


Figure 4: Response for $(u_0, v_0) = (1, 0)$ and $h = 10T$. a) Conservative (present): balanced dissipation (---), extended state-space (—), extended system energy (- · -). b) Newmark based: alpha damping (---), generalized alpha (—).

Neither of the two algorithms developed in this paper exhibit energy oscillations and ‘overshoot’. This is illustrated in Fig. 4, showing the free response to unit initial displacement of a very high-frequency vibration mode with period $T = h/10$ and $|\lambda_\infty| = 0.8$. Figure 4a shows the energy of the response calculated by the balanced dissipation algorithm in dashed line. It is seen that the energy displays a regular exponential decay. The similar result from using the alpha damped form of the Newmark algorithm is shown in dashed line in Fig. 4b. In this case the energy increases to double its initial magnitude before starting to decrease, exhibiting the so-called ‘overshoot’ phenomenon. The regular free vibration response of the present algorithms is due to a complete symmetry in the modal variables u and v/ω – and the auxiliary variables – for any single mode, see [14].

6. DISCUSSION

Two procedures have been developed for incorporating algorithmic energy dissipation into conservative time integration algorithms for linear systems. In the balanced dissipation algorithm the dissipation is introduced explicitly in the form of two damping terms. A spectral analysis reveals the correct balance between the terms, and describes the frequency dependence of the dissipation. The extended state-space algorithm is then developed by replacing the damping terms in the first algorithm with a representation in terms of auxiliary filtered variables. This leads to an extended state-space, and an energy relation for the full system in the extended state-space. Both algorithms are given a compact implementation in ‘single step – single solve’ format. The principles can be extended to non-linear problems, but the process involves proper choice of the matrices by which the dissipation effect is introduced into the system.

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