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Published in:
Physical Review E

Link to article, DOI:
10.1103/PhysRevE.100.023104

Publication date:
2019

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Bulk-driven acoustic streaming at resonance in closed microcavities

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(Received 22 May 2019; published 7 August 2019)

Bulk-driven acoustic (Eckart) streaming is the steady flow resulting from the time-averaged acoustic energy flux density in the bulk of a viscous fluid. In simple cases, like the one-dimensional single standing-wave resonance, this energy flux is negligible, and therefore the bulk-driven streaming is often ignored relative to the boundary-driven (Rayleigh) streaming in the analysis of resonating acoustofluidic devices with length scales comparable to the acoustic wavelength. However, in closed acoustic microcavities with viscous dissipation, two overlapping resonances may be excited at the same frequency as a double mode. In contrast to single modes, the double modes can support a steady rotating acoustic energy flux density and thus a corresponding rotating bulk-driven acoustic streaming. Remarkably, the rotating bulk-driven streaming may be excited by a nonrotating actuation, and we determine the optimal geometry that maximizes this excitation. In the optimal geometry, we finally simulate a horizontal 2 streaming pattern, which is dominated by the boundary-driven streaming.

I. INTRODUCTION

Acoustophoresis, particle migration by ultrasound in a microfluidic setting, is an increasingly popular method for non-contact and label-free handling of microparticles suspended in a fluid. Examples include acoustic separation [1–3], trapping [4,5], and tweezing [6–8], as well as enrichment of cancer cells [9,10] and bacteria [11,12], and size-independent sorting of cells [13].

The acoustic field induces the acoustic radiation force that acts on suspended particles and scales with the particle volume [14–18], as well as the acoustic streaming in the fluid [19–22] that gives rise to a viscous Stokes drag force on the particles which scales with the linear size of the particles [23]. Consequently, the streaming-induced drag force is the dominating acoustic force for particles smaller than a critical size. For example, for an aqueous suspension of spherical polystyrene particles in a 1-MHz ultrasound field, the critical radius has been determined to be around 2 μm [24,25]. To extend the application of acoustophoresis into the regime of submicrometer particles, such as bacteria, viruses, and exosomes, a deep understanding of acoustic streaming is therefore important.

The first theoretical analysis of acoustic streaming dates back to Lord Rayleigh [19], who investigated the streaming generated by the viscous boundary layers close to the walls in planar and cylindrical systems. This analysis was later generalized by Nyborg [22], who formulated a slip velocity condition for the acoustic streaming outside the viscous boundary layers at weakly curved boundaries oscillating in the normal direction. Further extensions to this slip-velocity theory involve curvilinear corrections for curved boundary [26], general motion of a flat boundary [27], and general motion of a curved boundary [28]. This kind of streaming is caused by the velocity mismatch between the boundary and the acoustic wave in the bulk, and we refer to it as boundary-driven acoustic streaming.

Acoustic streaming can also be generated by attenuation of the acoustic wave in the bulk as shown by Eckart [29], who established the basic theory and investigated the steady flow caused by a sound beam. This kind of streaming is generated by a force, which is proportional to the acoustic energy flux density (or intensity) such that, say, a sound beam generates acoustic streaming in its direction of propagation. We refer to this kind of streaming as bulk-driven acoustic streaming.

While bulk-driven streaming is known to have a clear effect in systems much larger than the acoustic wavelength [30], it is often neglected in studies of resonating devices with length scales comparable with the acoustic wavelength. However, the bulk-driven streaming was taken into account implicitly by Antfolk et al. in their study of a long straight microchannel with a nearly square cross section [11], as they made a direct numerical simulation of the acoustofluidic properties of the system. They showed numerically that the experimentally observed single-vortex streaming pattern, which remarkably appeared in their microchannel instead of the usual quadrupolar Rayleigh-vortex pattern, could be explained if two orthogonal acoustic resonances with nearly identical resonance frequencies were excited with a quarter-period phase difference at the same frequency.

Such orthogonal phase-shifted acoustic waves have been used in other experiments to generate an acoustic radiation
torque acting on suspended particles [26,31–35]. Also traveling Bessel beams [36,37] as well as arrays of holographic acoustic elements [38] have been used for this purpose, but not for generating specific streaming patterns.

Inspired by the work of Antfolk et al. [11], we study in this paper bulk-driven acoustic streaming at resonance in rectangular-box-shaped cavities with acoustically hard walls and side lengths comparable to the acoustic wavelength. The paper is organized as follows: In Sec. II we present the governing equations for the acoustic pressure and the acoustic streaming at resonance, where the viscous boundary layers are taken into account using our recent boundary-layer analysis [28]. We apply these equations in Sec. III to derive analytical solutions for the acoustic single-mode pressure resonances in the rectangular cavity. We also introduce the so-called double-mode resonances, where two overlapping single modes are excited simultaneously at the same frequency. In Sec. IV we establish the so-called overlap and phase conditions for double-mode resonances that lead to a strong acoustic body force, which causes the bulk-driven streaming. A main result is presented in Sec. V, where we show how a weak symmetry breaking of a perfect square cavity can lead to a rotating body force even for a nonrotating actuation. We validate our theory in Sec. VI by direct numerical simulation in the case of a shallow nearly square cavity similar to the geometry studied experimentally by Hagsäter et al. [39]. We predict that the bulk-driven streaming in closed microcavities is enhanced by increasing the frequency and the bulk viscosity, just as is the case for Eckart streaming in open systems [29]. We propose to test this prediction by replacing water with pyridine in the case for Eckart streaming in open systems [29], which takes the viscous boundary layer into account at the domain boundary ∂Ω. We introduce the inward normal derivative ∂n = −n · n, n being the outward-pointing surface normal vector, and the effective inward normal displacement U11(r) of the wall in terms of the actual displacement u1(r):

\[
\nabla^2 p_1 + k_c^2 p_1 = 0, \quad k_c = \left(1 + \frac{i}{2}\right) k_0, \quad r \in \Omega, \tag{3a}
\]

\[
\partial_n p_1 + \frac{i}{k_s} (k_0 \rho_1 + \partial_\perp^2 p_1) = \frac{k_s^2 U_{11}(r)}{k_0}, \quad r \in \partial \Omega, \tag{3b}
\]

\[
U_{11}(r) = \frac{1}{1 - i \Gamma_\perp} \left(-\nabla - \frac{i}{k_s} \nabla\right) \cdot u_1, \quad r \in \partial \Omega. \tag{3c}
\]

Here, \(i = \sqrt{-1}, \Gamma_\perp\) is the weak bulk damping coefficient, and \(k_s\) is the viscous boundary layer wave number, given in terms of the viscous boundary-layer width \(\delta\), the dynamic viscosity \(\eta_\perp\), the bulk viscosity \(\eta_\parallel\), and the isentropic compressibility \(\kappa_\parallel = \frac{1}{\rho_\parallel} \left(\frac{\partial p}{\partial \rho}\right)_T = \frac{1}{\rho_\parallel^{\gamma-1}}\) of the fluid,

\[
\Gamma_\perp = \left(\frac{4}{3} + \frac{\eta_\parallel}{\eta_\perp}\right) \eta_\parallel k_0 \delta \ll 1, \quad k_s = \frac{1 + i}{\delta}, \quad \delta = \sqrt{\frac{2\eta_\parallel}{\rho_\parallel \omega^2}}. \tag{4}
\]

Equation (3) for the acoustic pressure \(p_1\) is valid for boundary-layer widths \(\delta\) much smaller than both the acoustic length scale \(k_0^{-1}\) and the geometry length cavity \(L_{\text{geom}}\), a condition which is usually satisfied.

From the acoustic pressure \(p_1\) we directly obtain the irrotational acoustic velocity \(v_1\) and the acoustic mass density \(\rho_1\) in Eq. (2) through the relations [28]

\[
v_1 = -\frac{i(1 - I \Gamma_\perp)}{\omega \rho_0} \nabla p_1, \quad \nabla \times v_1 = 0, \tag{5a}
\]

\[
\rho_1 = \rho_0 k_0^2 p_1. \tag{5b}
\]

Further, we define the following relevant time-averaged first-order products: the acoustic energy density \(E_{\text{ac}}\), kinetic energy density \(\frac{1}{2} \rho_0 v_1^2\), potential energy density \(E_{\text{pot}}\), energy flux density vector \(S_{\text{ac}}\), as well as the time-averaged acoustic angular...
momentum density $\mathbf{L}_{\text{ac}}$ with respect to the unperturbed fluid position $\mathbf{r}$,
1
$$
E_{\text{ac}} = E_{\text{kin}} + E_{\text{pot}}, \quad E_{\text{kin}} = \frac{1}{2} \rho |\mathbf{v}_1|^2, \quad (6a)
$$
$$
S_{\text{ac}} = \langle \mathbf{p}_1 \mathbf{v}_1 \rangle, \quad E_{\text{pot}} = \frac{1}{2} \kappa |p_1|^2, \quad (6b)
$$
$$
\mathbf{L}_{\text{ac}} = \langle \mathbf{d}_1 \times (\rho \mathbf{v}_1) \rangle, \quad \mathbf{d}_1 (\mathbf{r}) = \frac{i}{\omega} \mathbf{v}_1 (\mathbf{r}). \quad (6c)
$$
Here, $\mathbf{d}_1 (\mathbf{r})$ is the fluid displacement at position $\mathbf{r}$.

B. Acoustic streaming at fluid resonance

Outside the narrow viscous boundary layers of width $\delta \ll 500$ nm (for water at MHz frequencies), the second-order acoustic streaming velocity $\mathbf{v}_2$ is described as a Stokes flow driven by the acoustic body force $f_{\text{ac}}$ in the domain $\Omega$ [see Eq. (8) in Ref. [40] and Eq. (52b) in Ref. [28]], and by the slip velocity $\mathbf{v}_2^{\text{slip}}$ at the domain boundary $\partial \Omega$ [see Eq. (A19) in Ref. [28]]:

$$
\begin{align}
0 &= \nabla \cdot \mathbf{v}_2, & \mathbf{r} &\in \Omega, \quad (7a) \\
0 &= -\nabla p_2 + \eta \nabla^2 \mathbf{v}_2 + f_{\text{ac}}, & \mathbf{r} &\in \Omega, \quad (7b) \\
\mathbf{v}_2 &= \mathbf{v}_2^{\text{slip}}, & \mathbf{r} &\in \partial \Omega. \quad (7c)
\end{align}
$$

Here, $f_{\text{ac}}$ in $\Omega$ and $\mathbf{v}_2^{\text{slip}}$ at $\partial \Omega$, are given by

$$
f_{\text{ac}} = \frac{\Gamma_{\text{fl}}}{c_{\text{fl}}} S_{\text{ac}}, \quad (7d)
$$
$$
\mathbf{v}_2^{\text{slip}} = \frac{1}{2} \kappa \rho S_{\text{ac}} - \frac{1}{2 \rho \omega} \nabla \left[ E_{\text{kin}} - 2 E_{\text{pot}} \right] + \frac{3}{4 \omega} \text{Re} \left[ i \partial_1 \mathbf{v}_1 \mathbf{v}_1^* \right], \quad (7e)
$$

where we have used the expression for the slip velocity $\mathbf{v}_2^{\text{slip}}$ valid near a fluid resonance, in which case the magnitude $|\mathbf{v}_1|$ of the acoustic velocity $\mathbf{v}_1$ in the bulk is much larger than that of the wall velocity $|\mathbf{v}_0 | \ll |\mathbf{v}_1 |$ [see Eq. (55) in Ref. [28] for the slip velocity including the wall motion and Stokes drift]. We use the name bulk-driven acoustic streaming for the streaming driven by the acoustic body force $f_{\text{ac}}$ in Eq. (7d), and the name boundary-driven streaming for the streaming driven by the slip velocity $\mathbf{v}_2^{\text{slip}}$ in Eq. (7e). Note that $f_{\text{ac}}$ increases with the frequency squared, as $\Gamma_{\text{fl}} \propto \omega$, whereas $\mathbf{v}_2^{\text{slip}}$ is independent of the frequency, as $\nabla \cdot \partial_1 \approx 0 = \frac{1}{\omega^2} \omega$.

One central quantity in the governing equations (7) for the acoustic streaming is the acoustic energy flux density $S_{\text{ac}}$, which enters both in the body force $f_{\text{ac}}$ [Eq. (7d)] and in the slip velocity $\mathbf{v}_2^{\text{slip}}$ [Eq. (7e)]. This quantity is rotating (has a nonzero curl) if the acoustic fields are rotating, as seen by taking the curl of $S_{\text{ac}} = \langle \mathbf{p}_1 \mathbf{v}_1 \rangle$ and using Eq. (5a) with $\mathbf{V} p_1 \approx i \omega \rho \mathbf{v}_1$ and $\nabla \cdot \mathbf{v}_1 = 0$,

$$
\nabla \times S_{\text{ac}} = \omega^2 \mathbf{L}_{\text{ac}}, \quad (8a)
$$

where $\mathbf{L}_{\text{ac}}$ is the time-averaged acoustic angular momentum density defined in Eq. (6c). Using Stokes theorem for any closed loop shows that the energy flux density $S_{\text{ac}}$, and therefore the body force $f_{\text{ac}}$, points in a direction which rotates around areas with high acoustic angular momentum density

$$
\int f_{\text{ac}} \cdot d l = \frac{\Gamma_{\text{fl}}}{c_{\text{fl}}^2} \int S_{\text{ac}} \cdot d l = \frac{\Gamma_{\text{fl}} \omega^3}{c_{\text{fl}}^2} \int \mathbf{L}_{\text{ac}} \cdot \mathbf{n} d A. \quad (8b)
$$

Hence, bulk-driven streaming is a consequence of acoustic fields with a nonzero acoustic angular momentum density $\mathbf{L}_{\text{ac}}$ defined in Eq. (6c). We refer to acoustic fields with nonzero $\mathbf{L}_{\text{ac}}$ as “rotating acoustics.”

III. ACOUSTIC RESONANCE MODES IN A RECTANGULAR BOX WITH VISCOUS BOUNDARY LAYERS

We consider the rectangular-box-shaped cavity of dimensions $L_x \times L_y \times L_z$, where $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, and $0 \leq z \leq L_z$. Below, we give the solutions for the resonance modes of the acoustic pressure $p_1$ satisfying Eq. (3) in this geometry with viscouos boundary layers. We refer the reader to Appendix A for the detailed derivations. The resulting expressions are valid for frequencies close to resonances of a high $Q$ factor $Q \gg 1$ and for narrow boundary layers [see Eq. (A3)]. We then evaluate the second-order quantities $S_{\text{ac}}$, $f_{\text{ac}}$ and $\mathbf{L}_{\text{ac}}$ which vanish for all single-mode resonances but not for double-mode resonances, where two single modes are excited simultaneously by the same frequency.

A. Single-mode resonances

First, neglecting viscosity, the resonance solution for $p_1$ is proportional to the eigenfunctions $R_{lmn}^{\text{in}} (\mathbf{r})$ of the Helmholtz equation (3a), which are the usual hard-wall standing resonance modes with integer number $l$, $m$, and $n$ half-waves in the $x$, $y$, and $z$ directions, respectively,

$$
p_1^{\text{in}} \propto R_{lmn}^{\text{in}} (\mathbf{r}) = \cos (K_{x}^{*} x) \cos (K_{y}^{*} y) \cos (K_{z}^{*} z). \quad (9a)
$$

Here, the wave numbers $K_{x}^{*}$, $K_{y}^{*}$, and $K_{z}^{*}$ are real-valued, purely geometrical quantities that dictate the resonance wave numbers $K_{x}^{\text{in}}$

$$
K_{x}^{*} = \left. \frac{l \pi}{L_x} \right., \quad K_{y}^{*} = \left. \frac{m \pi}{L_y} \right., \quad K_{z}^{*} = \left. \frac{n \pi}{L_z} \right.. \quad (9b)
$$

$$
K_{x}^{\text{in}} = \sqrt{\left( K_{x}^{*} \right)^2 + \left( K_{y}^{*} \right)^2 + \left( K_{z}^{*} \right)^2}. \quad (9c)
$$

In the presence of viscosity and hence viscous boundary layers, the resonant wave numbers $K_{x}^{\text{in}}$ and angular frequencies $\omega_{\text{in}}$ are down-shifted slightly relative to the inviscid values

1The standard definition of the full angular momentum density is $\mathbf{L} = \mathbf{r} \times (\rho \mathbf{v}^2 - \rho \mathbf{v}^2)$, defined in terms of the instantaneous Lagrangian fluid position $\mathbf{r}(t) = \mathbf{r} + \mathbf{d}_1 (t) + \mathbf{c} \mathbf{r}$. Taylor expanding $\rho \mathbf{v}^2(t) \approx \rho \mathbf{v}^2 + \mathbf{d}_1 \cdot \nabla \rho$ and $\rho \mathbf{v}^2 \approx \rho \mathbf{v}^2 + \mathbf{d}_1 \cdot \nabla \rho$, and using the expansion in Eq. (2) we obtain the time-averaged full angular momentum density $\mathbf{L} = \mathbf{r} \times (\rho \mathbf{v}_2^{\text{slip}} + \mathbf{d}_1 \times (\rho \mathbf{v}_1))$. Here, the last term is $\mathbf{L}_{\text{ac}}$ from Eq. (6c) and expresses the angular momentum density of each fluid particle around its equilibrium position $\mathbf{r}$.
The bulk-driven acoustic streaming is driven by the body force $f_{ac}$ defined in Eq. (7d) and restated here solely in terms of the pressure $p_1$, by using $v_1 \approx \frac{1}{\rho_0} \nabla \cdot p_1$ as given in Eq. (5a):

$$f_{ac} = -\frac{1}{\rho_0} \langle \frac{1}{2} |\nabla p_1|^2 \rangle = \frac{1}{2} \Re \{ i p_1 \nabla p_1^* \}. \quad (14)$$

For any single-mode resonance $lnm$ of the form of Eq. (12a), this expression will vanish since $i p_1$ is exactly $\frac{1}{2}$ out of phase with $\nabla p_1^*$. This motivates us to consider double-mode resonances, which occur if two single modes $lnm$ and $l'n'nm$ overlap, such that they can be excited simultaneously at the same frequency $f = \frac{1}{2} k_0 c_0$, with $k_0^2 \approx k_0^2 \approx k_0^2$. In that case, the total pressure is the sum of two modes

$$p_1 = p_{lnm} + p_{l'n'nm}, \quad (15)$$

and the body force $f_{ac}$ in Eq. (14) becomes $f_{lnm}$:

$$f_{lnm} = \frac{1}{2} \Re \{ i p_{lnm} \nabla p_{lnm} + i p_{l'n'nm} \nabla p_{l'n'nm}^* \}.$$

Inserting the resonance mode Eq. (12a) with $p_{lnm}$ in polar form yields

$$f_{lnm} = \frac{1}{2} \Re \{ i |p_{lnm}| \nabla p_{lnm} \} \mathcal{K}_{lnm}^{(r)}(r), \quad (17a)$$

$$\mathcal{K}_{lnm}^{(r)}(r) = R^{lnm} \nabla R^{lnm} - R^{lnm} \nabla R^{lnm}. \quad (17c)$$

Here, $\mathcal{K}_{lnm}^{(r)}(k_0)$ gives the essential frequency dependence (as $\mathcal{K}$ is nearly constant over the width of the double mode), while $\mathcal{K}_{lnm}^{(r)}(r)$ with the dimension of a wave vector gives the spatial dependency and direction of the double-mode body force. Furthermore, we obtain the double-mode energy flux density $S_{lnm}$ and acoustic angular momentum density $L_{lnm}$ directly from Eq. (17a) by use of Eqs. (7d) and (8a):

$$S_{lnm} = \frac{1}{2} \frac{1}{\rho_0 \omega} \frac{|p_{lnm}|^2}{\mathcal{K}_{lnm}^{(r)}(k_0)} \mathcal{K}_{lnm}^{(r)}(r), \quad (18a)$$

$$L_{lnm} = \frac{1}{2} \frac{1}{\rho_0 \omega^2} \frac{|p_{lnm}|^2}{\mathcal{K}_{lnm}^{(r)}(k_0)} \mathcal{K}_{lnm}^{(r)}(r). \quad (18b)$$

Note that $\mathcal{K}_{lnm}^{(r)}(k_0)$ is normalized to be between $-1$ and $1$, whereas $\mathcal{K}_{lnm}^{(r)}(r)$ and $\nabla \times \mathcal{K}_{lnm}^{(r)}(r) = 2 \nu R^{lnm} \times \nabla \mathcal{K}^{(r)}$ have the maximum amplitudes $k_0$ and $2k_0$, respectively.

In Fig. 2 we show results for the nine lowest-frequency double modes $lnm \times l'n'nm$ in a square cavity. We plot the body force $f_{lnm} \propto \mathcal{K}_{lnm}^{(r)}(k_0)$ and the z component of the acoustic angular momentum $L_{lnm} \propto \nabla \times \mathcal{K}_{lnm}^{(r)}(k_0)$ in the Supplemental Material [41].
values of $K_l - \phi$ emphasize the total phase difference $lm$. MATLAB code to compute these modes.

IV. CONDITIONS FOR OBTAINING A STRONG ACOUSTIC BODY FORCE

The magnitude of the body force $F_{lm}^{\text{in}}$, the acoustic energy flux density $S_{lm}^{\text{in}}$, and the acoustic angular momentum density $L_{lm}^{\text{in}}(k_0)$ are all proportional to the frequency-dependent factor $F_{lm}^{\text{in}}(k_0)$ in Eq. (17b). To study this factor, we rewrite it to emphasize the total phase difference $\phi_{\text{tot}}^{lm} - \phi_{\text{tot}}^m$ between the single-mode phases given in Eq. (13):

$$F_{lm}^{\text{in}}(k_0) = |F_{lm}^{\text{in}}||F_{lm}^{\text{init}}| \sin (\phi_{\text{tot}}^{lm} - \phi_{\text{tot}}^m).$$  \hspace{1cm} (19)

Consequently, two conditions must be fulfilled to yield a strong body force: (i) the overlap condition, i.e., the two single-mode resonances must have a large overlap in frequency space to ensure a large value of the amplitude $|F_{lm}^{\text{in}}||F_{lm}^{\text{init}}|$, and (ii) the phase condition, i.e., the difference $\phi_{\text{tot}}^{lm} - \phi_{\text{tot}}^m$ in phases must be close to $\pm \pi$ to ensure a large value of the sine factor.

A. Overlap condition: Aspect ratios

The condition that both $|F_{lm}^{\text{in}}|$ and $|F_{lm}^{\text{init}}|$ are large in Eq. (19) is satisfied if the modes overlap. This occurs if the resonance frequencies are nearly identical, $k_{0lm}^{\text{in}} \approx k_{0lm}^{\text{init}}$ or, equivalently,

$$\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2 \approx \left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2,$$  \hspace{1cm} (20)

where we have used that $\Gamma_{l_m}^{lm} \ll 1$ in Eq. (10a) so $k_{0lm}^{\text{in}} \approx k_{0lm}^{\text{init}}$. In the general case, Eq. (20) gives a myriad of possible mode combinations. If we restrict ourselves to horizontal modes, where $n = n' = 0$, we can solve for the aspect ratio $A = \frac{L_x}{L_y}$ in Eq. (20):

$$A = \frac{L_x}{L_y} \approx \sqrt{\frac{l^2 - l'^2}{m^2 - m'^2}}.$$  \hspace{1cm} (21)

Listing the integers $l, m, l', m'$ for which $A$ is real, we find that the square cavity with $A \approx 1$ is the richest case having the highest number of allowed double-mode excitations. In Fig. 2, we show the first nine horizontal double-mode excitations in a square cavity with $A = 1$ and $n = n' = 0$. In this case, the condition in Eq. (20) corresponds to two points ($\frac{l\pi}{L_x}, \frac{m\pi}{L_y}$) and ($\frac{l'\pi}{L_x}, \frac{m'\pi}{L_y}$) both lying on the same quarter circle of radius $K_0$, as illustrated by the black dots in Fig. 2(a).

B. Phase condition: Rotating versus nonrotating actuation

The straightforward way to obtain rotating acoustics is to actuate a given system with two transducers running with a phase difference of $\phi_{\text{act}}^{lm} - \phi_{\text{act}}^m = \pm \pi$, a technique that has been used in many experiments to generate acoustic radiation torques [26,31–35]. Remarkably, even in the standard case of microfluidic ultrasound experiments driven by a single piezoelectric transducer with a nonrotating actuation, $\phi_{\text{act}}^{lm} - \phi_{\text{act}}^m = 0$, a rotating acoustic field may nevertheless be generated because the total phase difference $\phi_{\text{tot}}^{lm} - \phi_{\text{tot}}^m = \phi_{\text{tot}}^{lm} - \phi_{\text{tot}}^m$ may be close to $\pm \pi$ due to the internal phases $\phi_{\text{init}}^{lm}$ of the frequency-dependent factors $F_{lm}^{\text{in}}(k_0)$ [see Eq. (12c) and Fig. 1].
V. NONROTATING ACTUATION OF ROTATING DOUBLE-MODE RESONANCES IN A NEARLY SQUARE CAVITY

We now show how a nonrotating actuation can lead to a rotating double-mode resonance in a nearly square cavity characterized by the average side length $L^* = \frac{1}{2}(L_x + L_y)$ and the small symmetry breaking $d = L_x - L_y$:

$$L_x = L^* + \frac{1}{2}d, \quad L_y = L^* - \frac{1}{2}d, \quad \text{with } |d| \ll L^*. \quad (22)$$

The magnitude of the resulting acoustic body force $f_{ac}$ depends strongly on the symmetry breaking $d$, and for a given double mode we determine how much the aspect ratio $A = L_x/L_y \approx 1 + d/L^*$ should deviate from unity to maximize $f_{ac}$. Specifically, we consider the symmetric double-mode excitation $lm0 + ml0$ in the horizontal $xy$ plane of the type shown in Fig. 2. For such double modes, the analysis simplifies significantly, not only because $z$ drops out, but also because the damping coefficients $\Gamma^{lm0}$ and $\Gamma^{ml0}$ from Eqs. (10b) and (11) are almost identical. In short, we consider the following situation:

- Nearly square cavity: $L_x \approx L^* \approx L_y$, (23a)
- Horizontal $xy$ modes: $n = n' = 0$, (23b)
- Symmetric double modes: $l'm' = ml$, (23c)
- Nonrotating double actuation: $\phi_{act}^{lm0} = \phi_{act}^{ml0}$, (23d)
- Similar damping coefficients: $\Gamma^{lm0} \approx \Gamma^{ml0}$. (23e)

To analyze the strength of the acoustic body force, we examine the factor $F_{lm0}^{ml0}(k_0)$ in Eq. (17b) that describes the frequency dependency of the acoustic body force. For the double mode $lm0 + ml0$, we introduce the short two-subscript notation $F_{lm}(k_0)$ for this quantity:

$$F_{lm}(k_0) = F_{lm0}^{ml0}(k_0) = \text{Re}[i(E_{lm0}||F_{lm0}^{ml0})^*] = |E_{lm0}||F_{lm0}^{ml0}| \sin (\phi_{F_{lm0}}^{ml0} - \phi_{F_{lm0}}^{ml0}). \quad (24)$$

where the functions $F_{lm0}^{ml0}(k_0)$ and $F_{ml0}^{ml0}(k_0)$ behave as shown in Fig. 1. Note that if the modes $lm0$ and $ml0$ have equal resonance frequencies, they have full mode overlap, and the product $|E_{lm0}||F_{lm0}^{ml0}|$ is maximized, while $\sin (\phi_{F_{lm0}}^{ml0} - \phi_{F_{lm0}}^{ml0})$ vanishes, and there is no body force. To maximize $F_{lm}(k_0)$, the two modes must therefore be separated sufficiently to ensure that the phase difference $\phi_{F_{lm0}}^{ml0} - \phi_{F_{lm0}}^{ml0}$ deviates from zero, but not too much, as a decreased mode overlap reduces the modulus product $|E_{lm0}||F_{lm0}^{ml0}|$. Therefore, we cannot in this case perfectly fulfill both the overlap condition, Sec. IV A, and the phase condition, Sec. IV B, so the strongest body force is found at a certain mode separation $\Delta_{lm} = k_{lm0}^0 - k_{ml0}^0$ between the resonance wave numbers, which gives only partial mode overlap and a phase difference that deviates from the ideal $\pm \frac{\pi}{2}$.

The mode separation $\Delta_{lm}$ is induced by the symmetry breaking $d$, and similar to Eq. (22), we introduce the center frequency $k_{lm0}^0 = \frac{1}{2}(k_{lm0}^0 + k_{ml0}^0)$ corresponding to the center frequency between the two modes and write

$$k_{lm0}^0 = k_{lm}^0 + \frac{1}{2} \Delta_{lm}, \quad k_{ml0}^0 = k_{ml}^0 - \frac{1}{2} \Delta_{lm}, \quad (25a)$$

$$k_{lm}^* \approx \left(1 - \frac{1}{2} \Gamma_{lm0}^0 \right) \sqrt{l^2 + m^2} \frac{\pi}{L^*}. \quad (25b)$$

The relation between the mode separation $\Delta_{lm}$, the symmetry breaking $d$, and the aspect ratio $A = L_x/L_y$ is found by inserting Eq. (22) into $k_{lm0}^0 = \sqrt{(l^2 + m^2)/(2L^*)^2 + O(\Gamma_{lm0}^0)}$, expanding in the small ratio $d$, and comparing with Eq. (25a),

$$\Delta_{lm} \approx \frac{m^2 - l^2}{\sqrt{m^2 + l^2}} \frac{\pi}{L^*} \approx \frac{m^2 - l^2}{\sqrt{m^2 + l^2}} \frac{\pi}{L^*}(A - 1). \quad (26)$$

Since the damping coefficients are nearly identical $\Gamma_{lm0}^0 \approx \Gamma_{ml0}^0$, with $\Gamma_{lm0}^0 = \frac{1}{2}(\Gamma_{lm0} + \Gamma_{ml0})$ being the average value, it is convenient to scale all wave-number quantities by their common linewidth $\Gamma_{lm}^* k_{lm0}^0$ (see the overview listed in Table I):

$$\tilde{k}_0 = \frac{k_0}{k_{lm0}^* \Gamma_{lm0}^*}, \quad \tilde{k}_{lm} = \frac{1}{\Gamma_{lm}}, \quad \Delta_{lm} = \frac{\Delta_{lm}}{k_{lm0}^* \Gamma_{lm0}^*}. \quad (27)$$

In terms of these quantities, we write the frequency dependencies $F_{lm0}^{ml0}(k_0)$ and $F_{ml0}^{ml0}(k_0)$ from Eq. (12c) as

$$F_{lm0}^{ml0}(k_0) = \frac{1}{2(k_0 - k_{lm0}^*) + \Delta_{lm} + i}. \quad (28a)$$

$$F_{ml0}^{ml0}(k_0) = \frac{1}{2(k_0 - k_{ml0}^*) + \Delta_{lm} + i}. \quad (28b)$$

In Figs. 3(a) and 3(b) we show as in Fig. 1 the modulus and phase, respectively, of $F_{lm0}^{ml0}$ and $F_{ml0}^{ml0}$ for four different mode separations $\Delta_{lm} = 0.10, 0.58, 1.50, 3.00$. In Fig. 3(c) we show for the same values of $\Delta_{lm}$ the frequency dependency $F_{lm}(k_0)$ from Eq. (24) of the acoustic body force. We note that there exists an optimal mode separation $\Delta_{lm}^\text{opt}$ that yields the largest possible value $F_{lm0}^{ml0} = F_{lm}$ found at the optimal wave number $k_{lm0}^\text{opt}$. In Eq. (B5) of Appendix B we show that

$$k_{lm0}^\text{opt} = k_{lm}^*, \quad (29a)$$

$$\Delta_{lm}^\text{opt} = \pm \frac{1}{\sqrt{3}} \approx \pm 0.58, \quad (29b)$$

$$F_{lm0}^{ml0} = \frac{3\sqrt{3}}{8} = \pm 0.65. \quad (29c)$$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk damping</td>
<td>$\Gamma_{bl}$</td>
<td>Eq. (4)</td>
</tr>
<tr>
<td>Boundary-layer damping</td>
<td>$\Gamma_{bl}$</td>
<td>Eq. (10b)</td>
</tr>
<tr>
<td>Total damping</td>
<td>$\Gamma_{l}$</td>
<td>Eq. (11)</td>
</tr>
<tr>
<td>FWHM of $</td>
<td>F_{lm0}^{ml0}</td>
<td>$</td>
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<tr>
<td>FWHM of $E_{lm0}^{ml0} =</td>
<td>F_{lm0}^{ml0}</td>
<td>^2$</td>
</tr>
<tr>
<td>Optimal mode separation $\Delta_{lm}^\text{opt}$</td>
<td>$\frac{1}{2} \sqrt{3} \Gamma_{lm0}^0 k_{lm0}^*$</td>
<td>Eq. (29b)</td>
</tr>
</tbody>
</table>
The optimal mode separation for rotating acoustics is the center frequency $\pi$ of the optimal mode separation $\tilde{f}$ of $\tilde{F}$, which is the optimal frequency $f_{\text{opt}}$ of the double mode $l m 0 + m 0 l$. We therefore have $f_{\text{opt}} = (L_0 - k_0) \delta_{l m}$, and the corresponding aspect ratio $A_{l m}^{\text{opt}} = (L_0 \delta_{l m})^{\text{opt}}$, where the acoustic body force $f_{\text{ac}}$ of the double mode $l m 0 + m 0 l$ is maximized:

$$d_{l m}^{\text{opt}} = \frac{L_0}{L_0 \delta_{l m}} \frac{l^2 + m^2}{l^2} \frac{1}{\gamma L^*} L^*.$$  

For all the double modes shown in Fig. 2, we have $l > m$, so for a nonrotating actuation, the rotation direction of the body force will therefore be as shown in the case of $d < 0$ (corresponding to $L_0 < L_0$), and opposite for $d > 0$. Note from Eq. (29e) that this asymmetry-induced rotating acoustics is extremely sensitive to $d$. In practice, it may be difficult to control this mechanism, as it requires a fabrication accuracy much smaller than $d/L^* \sim \Gamma_{l m}^{\text{opt}}$, which in microfluidic devices typically is smaller than 1% [42].

VI. 3D SIMULATION OF ACOUSTIC STREAMING IN A NEARLY SQUARE CAVITY

In this section we validate by direct numerical simulation the theory for rotating acoustics induced by a nonrotating actuation in a weakly symmetry-broken geometry. We simulate the acoustic pressure $p_1$ and the acoustic streaming $v_2$ in a shallow, nearly square cavity in three dimensions (3D). We choose a cavity similar to that investigated experimentally by Hagsäter et al. [39], and simulated in 3D using the hard-wall approximation by Lei et al. [43], and including the actuator and elastic walls by Skov et al. [44]. The cavity has the mean side length $L^* = 2000 \mu m$, $L_x = L^* + \frac{1}{2} d_1$, $L_y = L^* - \frac{1}{2} d_2$, and height $L_z = 200 \mu m$. We use Eq. (29) to optimize the symmetry breaking $d$ and actuation frequency $f = \frac{1}{\gamma L_0 k_0 c_0}$ to obtain the strongest possible rotating acoustic streaming. The coordinate system is the same as in Sec. III, with $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, and $0 \leq z \leq L_z$. To mimic an actuation from a broad piezoelectric transducer, the boundary is taken to be stationary everywhere, except at the bottom boundary $z = 0$, where we apply the nonrotating Gaussian displacement in the normal direction $z$:

$$u_1 = \begin{cases} u_{1,0}^0 G_1(x,y) e_z & \text{for } z = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_1(x,y) = \exp \left[ -\frac{(x - \frac{l}{2})^2}{\left(\frac{l}{2}\right)^2} - \frac{(y - \frac{m}{2})^2}{\left(\frac{m}{2}\right)^2} \right].$$

This actuation displacement has the maximum amplitude of $u_{1,0}^0$ in the cavity center and the minimum amplitude of $\epsilon^{-2} u_{1,0}^0 \approx 0.14 u_{1,0}^0$ in the corners of the cavity. In Secs. VI C–VI C 4 the fluid is water, and in Sec. VI C 6 it is pyridine. The material parameters for these fluids are listed in Table II.

In the following simulations, we excite the double-mode resonance $l m 0 + l m 0 l$ with $l = 2$, 4, and 6. Using our analytical expressions (17), we have in Fig. 2 shown the spatial dependency of the acoustic body force $f_{\text{ac}} = f_{\text{ac}, 0}$ for these particular double modes. As seen in Eq. (7), this body force is the source of the bulk-driven acoustic streaming, so we predict that the numerical simulation of the bulk-driven streaming will result in a $2 \times 2$ pattern as in Fig. 2(c) for $l = 2$, a $4 \times 4$ pattern as in Fig. 2(h) for $l = 4$, and similarly for $l = 6$ (not shown in Fig. 2).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Water</th>
<th>Pyridine</th>
<th>Unit</th>
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<td>Dynamic viscosity</td>
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<td>Bulk viscosity</td>
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<td>mPa s</td>
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<tr>
<td>Speed of sound</td>
<td>$c_a$</td>
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<td>1417</td>
<td>m/s</td>
</tr>
<tr>
<td>Compressibility</td>
<td>$\kappa_a$</td>
<td>448</td>
<td>507</td>
<td>kPa m$^{-1}$</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho_n$</td>
<td>997.05</td>
<td>982</td>
<td>kg m$^{-3}$</td>
</tr>
</tbody>
</table>

**TABLE II.** Material parameters for water [45] and pyridine (C$_5$H$_5$N) [46] at 25°C used in the numerical simulations.
A. Choosing the parameters

We derived in Sec. III analytical expressions for the acoustic resonance modes including the viscous boundary layers, and we derived in Sec. V the conditions on the geometry of a nearly square cavity for which the body force $f_{\infty}$ and the acoustic rotation are maximized for a nonrotating actuation. Remarkably, these analytic results allow an analytical determination of the parameter values that lead to the strongest rotating acoustic streaming in the $xy$ plane.

For the double mode $l00 + 0l0$, the small damping coefficients $\Gamma_{l00}^0 \approx \Gamma_{00l}^0$ in Eq. (4) and $\Gamma_{l00}^0 \approx \Gamma_{00l}^0$ in Eq. (10b) are computed by approximating the angular frequency by the constant value $\omega = c_{fl}k_{l00}^0 \approx c_{fl}l_{x}^{\star}$ [see Eq. (25b)]:

$$\Gamma_{l00}^0 \approx \left( \frac{4}{3} + \frac{\eta_{fl}}{\eta_{l}} \right) \frac{\eta_{l} \pi}{\rho_{fl} c_{fl}} L_{l} \approx \Gamma_{l00}^0, \quad (31a)$$

$$\Gamma_{l0l}^0 \approx \frac{\delta}{L_{y}} + \frac{\delta}{L_{z}} \approx \Gamma_{l00}^0, \quad \delta \approx \sqrt{\frac{2\eta_{l}L_{y}}{\pi \rho_{fl} c_{fl}}} \quad (31b)$$

Combining Eqs. (10a), (29a), and (25b), we obtain an expression for the boundary-layer shifted, optimal double-mode actuation frequency, which is the center frequency of the two single modes

$$f_{l00}^0 \approx \frac{1}{2} \left( 1 - \frac{1}{2} \Gamma_{l00}^0 \right) \frac{c_{fl}}{L_{l}^{\star}}. \quad (32)$$

The optimal symmetry breaking $d_{l00}^{opt}$ and the corresponding aspect ratio $A_{l00}^{opt}$ then follow from Eqs. (29d) and (29a):

$$d_{l00}^{opt} \approx -\frac{1}{3} (\Gamma_{l00}^0 + \Gamma_{00l}^0)L_{l}^{\star}, \quad (33a)$$

$$A_{l00}^{opt} \approx 1 - \frac{1}{3} (\Gamma_{l00}^0 + \Gamma_{00l}^0). \quad (33b)$$

Finally, we set the pressure amplitudes $P_{l00}^{00} = P_{00l}^{00} = 1$ MPa in Eq. (12b), and compute the corresponding amplitude $u_{l00}^{0}$ of the actuation displacement in Eq. (30) as

$$u_{l00}^{0} = \frac{L_{x} L_{y} L_{z} (\Gamma_{l00}^0 + \Gamma_{00l}^0) \delta_{l} P_{l00}^0}{2 \int_{0}^{L_{x}} \int_{0}^{L_{y}} G_{l}(x,y) \cos \left( \frac{\pi x}{L_{x}} \right) \cos \left( \frac{\pi y}{L_{y}} \right) \sin \left( \frac{\pi z}{L_{z}} \right) \sin \left( \frac{\pi z}{L_{z}} \right) \sin \left( \frac{\pi z}{L_{z}} \right) dx dy}. \quad (34)$$

For water and $l = 2, 4, 6$, the values of the expressions in Eq. (31)–(34) are listed in Table III. For pyridine, we adjust the actuation amplitude and actuation frequency to maintain the resonance amplitudes of $P_{l00}^{00} = P_{00l}^{00} = 1$ MPa. Note that for pyridine, we use the same optimal symmetry breaking $d_{l00}^{opt}$ as used for water. Combining Eqs. (24), (28) and (29e) with the material parameters in Table II leads to the values $F_{l0} = 0.65, 0.64, and 0.63$ for $l = 2, 4, and 6$, respectively, which are nearly the same as the optimal value $F_{l00}^{opt} = 0.65$ given in Eq. (29c).

B. Implementation in COMSOL

As in our previous work [24,44,47], the simulation is performed in two steps using the weak form PDE module in the finite-element-method software COMSOL MULTIPHYSICS [48]. In the first step, we solve for the acoustic pressure $p_{l}$ by implementing Eq. (3) in weak form, and in the second step we solve Eq. (7) for the steady streaming $v_{z}$ and pressure $p_{z}$.

C. Simulation results

1. Acoustic pressure

In Fig. 4, we show the simulation results $p_{l00}^{ana}$ of the acoustic pressure $p_{l}$ in water for $l = 6$ in 3D and line plots for $l = 2, 4, 6$. These three modes are examples of rotating double modes acted on by the wide, single, nonrotating Gaussian actuation $G_{l}$ [Eqs. (30) and (34)] at the bottom boundary $z = 0$. In the Supplemental Material [41] we show the rotating dynamics of the pressure contours similar to Fig. 4(a) for $l = 2, 4, 6$ as gif animations. In Fig. 4 we also show the analytical result $p_{l00}^{ana} = p_{l00}^{ana} + p_{l00}^{ana}$, which is obtained from Eqs. (12) and (15) with the optimal mode separation $k_{00l}^{00} = k_{00l}^{00} - \frac{1}{3} \frac{\pi^{2}}{L_{l}^{2}} \left[ \Gamma_{l00}^{00} \right]^{\star}$ [Eq. (29b)] and mode amplitudes $P_{l00}^{00} = P_{00l}^{00} = 1$ MPa:

$$p_{l00}^{ana} = P_{00l}^{00} = 1 \text{ MPa} \times \left[ \cos \left( \frac{\pi x}{L_{x}} \right) + \cos \left( \frac{\pi y}{L_{y}} \right) \right]. \quad (35)$$
In Fig. 4(b), we compare $\text{Im}(p_1^\text{num})$ and $\text{Im}(p_1^\text{ana})$ and find a quantitative agreement better than 1%. The reason for choosing the imaginary part is to single out the resonant part of the field, which is phase shifted by the factor $\exp(i \frac{\phi}{2}) = i$ relative to the actuation displacement $u_1$ at the boundary Eq. (30). All nonresonant contributions are in phase with $u_1$. In the Supplemental Material [41] is shown an animated gif file of the full cycle in the time domain of the line plots in Fig. 4(b) of $p_1$. These animations reveal that the deviation of the analytical solution from the numerical one is $\sim 1\%$ for $l = 2$ at 0.75 MHz, $\sim 15\%$ for $l = 4$ at 1.5 MHz, and $\sim 30\%$ for $l = 6$ at 2.2 MHz. The deviation increases for increasing mode number because the wide vertical $G_1$ actuation couples strongly to the vertical 001 mode through the amplitude factor $P_1^{010}$ of Eq. (12b), while the frequency-dependency factor $F_0^{01}(k_0)$ of Eq. (12c), which is very small away from resonance, increases significantly as the actuation frequency $f = c_0 k_0$ approaches the vertical-mode resonance $f_0^{010} = 3.7$ MHz. We can reduce the coupling $P_1^{01}$ to the vertical 001 mode by reducing the width of the $G_1$ actuation profile. This is demonstrated in the animated gif file of the Supplemental Material [41], where the width of the $G_1$ actuation has been reduced by a factor of 6 from $\frac{1}{2}L'$, this results in a decreased deviation, now less than $\sim 1\%$, of the analytical result from the numerical one for $p_1$ for all three modes $l = 2, 4, 6$.

In Fig. 5, we validate our analysis in Sec. V of rotating double modes induced by a nonrotating actuation. For the double mode 200 + 020, we plot the body force $f_{\text{act}} = F_{200}^{\text{act}}$ from Eq. (17), which according to Eq. (7) is the source of the bulk-driven acoustic streaming, and the acoustic angular momentum density $L_z = L_{z200}$ from Eq. (18b) in the center plane $z = \frac{1}{2}L_z$. We show results for three different cavity aspect ratios $A = L_y/L_x$ for both the nonrotating wide Gaussian actuation $G_1(x,y)$, given in Eq. (30), and for the externally controlled rotating actuation $G_{16}(x,y)$ consisting of four quadruples of narrow Gaussian actuations [see Eq. (C1) of Appendix C]. As expected, the nonrotating actuation $G_1$ used in Figs. 5(a)–5(c) induces no acoustic rotation for a perfect square cavity $A = 1$ [Fig. 5(b)], while it induces different rotation directions for $A$ slightly larger and smaller than unity [see Figs. 5(a) and 5(c), respectively]. In contrast, the rotating actuation $G_{16}$ used in Figs. 5(d)–5(f) for the same three aspect ratios, induces an acoustic rotation, which is maximized in the perfect square cavity $A = 1$ [see Fig. 5(e)]. We show in Eq. (C4) of Appendix C that this rotating actuation excites the modes 200 and 020 with a phase difference $\phi_{200}^{\text{act}} - \phi_{020}^{\text{act}} \approx 0.5\pi$ such that the factor $F_{200}^{\text{act}}$ in Eq. (19) is positive, and consequently the rotation pattern in Figs. 5(d)–5(f) is in the same direction as shown in Fig. 2(c).

In Fig. 6 we investigate the strength and rotation direction of the body force $F_{020}^{\text{act}}$ as a function of the mode separation.
FIG. 6. The largest possible value (positive and negative) of \( \mathcal{F}_{\text{act}} \) versus varying mode separation \( \Delta_{\text{lm}} \) (lower x axis) or aspect ratio \( A \) (upper x axis), shown for nonrotating actuations (blue) and rotating actuations (light cyan). The numerical values (dots) are found by evaluating \( \mathcal{F}_{\text{act}}^{\text{num}} \) from Eq. (36) in the simulation of the double mode 200 + 020 shown in Figs. 5(a)–5(c) for the nonrotating and Figs. 5(d)–5(f) for the rotating actuation. The analytical curves (solid) are from the expressions derived in Eq. (B4a) (nonrotating act.), Eq. (B7a) (rotating act., corotating), and Eq. (B8a) (rotating act., counter-rotating) of Appendix B. The black-edged circles correspond to those of the same color in Fig. 3(c).

\[ \Delta_{20} \text{ or, equivalently, the aspect ratio } A. \text{ From Eq. (17a), this dependency is quantified through the factor } \mathcal{F}_{200}^{\text{num}} \text{ which we obtain numerically from the simulation of the double mode 200 + 020 in Fig. 5 by calculating the acoustic angular momentum } L_{\text{ac}} \text{ and using Eq. (18b)}: \]

\[ (\mathcal{F}_{200}^{\text{num}}) \approx \frac{L_{\text{ac}} - \frac{1}{2}L_z}{|\rho_1^{200}||\rho_1^{020}|(\omega \rho_0 c_0^2)^{-1}}. \]  

(36)

Here, for pressure amplitudes \( |p_{1}^{200}| = |p_{1}^{020}| = 1 \text{ MPa}, \) the denominator evaluates to \( 0.954 \times 10^{-4} \text{ kg m}^{-1} \text{ s}^{-1} \), which is indeed the maximum value found numerically in Fig. 5(e), such that \( (\mathcal{F}_{200}^{\text{num}}) \) takes the maximum value of unity, as expected. In Fig. 6 we plot the largest possible value of \( (\mathcal{F}_{200}^{\text{num}}) \) both for the nonrotating actuation used in Figs. 5(a)–5(c) and for the rotating actuation used in Figs. 5(d)–5(f). We also plot the analytical expressions for \( (\mathcal{F}_{\text{act}}^{\text{num}}) \) derived in Appendix B and find a quantitative agreement better than 1%. For the nonrotating actuation (the dark blue curve in Fig. 6), the acoustic rotation direction is determined by the aspect ratio \( A \), and it reverses as \( A \) crosses over from \( A < 1 \) to \( A > 1 \) around the perfect square \( A = 1 \). For the rotating actuation (the light blue curves in Fig. 6), the rotation direction can be either corotating or counter-rotating (at different frequencies) with respect to the rotation direction of the actuation. However, for small mode separations \( |\Delta_{\text{lm}}| < 1 \), which for the double mode 200 + 020 corresponds to \( |A - 1| < 0.34 \%), the rotation direction of the acoustic field can only be corotating. For further details, see Appendix B.

3. Acoustic streaming

In Fig. 7, we show the acoustic streaming velocity \( v_2 \) in 3D resulting from the double mode 600 + 060 displayed in Fig. 4(a), which is excited by the nonrotating actuation \( G_1 \). As mentioned above, this situation is similar to that investigated experimentally by Hagsater et al. [39]. Remarkably, in the optimized geometry of Fig. 7, the horizontal \( 6 \times 6 \) streaming-roll pattern at any height \( 0 \leq z \leq L_z \) is strongest in the center of the cavity, \( z = \frac{1}{2}L_z \), thus indicating its bulk-driven origin. In the bottom plane \( z = 0 \) (shown in Fig. 7) and in the top plane \( z = L_z \) (not shown in Fig. 7), the slip velocity \( v_2^{\text{slip}} \) is dominated by the gradient term in the expression (7e) for the slip velocity, and has only a weak rotating component.

In Fig. 8, we compare the results for the acoustic streaming \( v_2 \) (see Eq. (7)) for the double modes 100 + 00 with \( l = 2, 4, 6 \). The streaming is shown in the horizontal \( x-y \) plane at height \( z = \frac{1}{2}L_z \) (dark blue edge in Fig. 7) and in the vertical \( x-z \) center plane at \( y = \frac{1}{2}L_y \) (light green edge in Fig. 7). In the first row [Figs. 8(a)–8(c)] we show the bulk-driven streaming alone \( f_\text{ac on}, v_2^{\text{slip off}} \); in the second row [Figs. 8(d)–8(f)] we show the boundary-driven streaming alone \( f_\text{ac off}, v_2^{\text{slip on}} \); in the third row [Figs. 8(g)–8(i)] we show the total streaming \( f_\text{ac on}, v_2^{\text{slip on}} \). Comparing the bottom-row and top-row panels in Fig. 8 strongly indicate that the respective \( 4 \times 4 \) and \( 6 \times 6 \) streaming-roll patterns in Figs. 8(b) and 8(i) are bulk driven. Furthermore, Figs. 8(a)–8(c) show that the bulk-driven acoustic streaming becomes stronger for higher mode number \( l \) and thus for higher frequency \( f \). In contrast, the boundary-driven acoustic streaming [Figs. 8(d)–8(f)] stays almost constant in amplitude and shows no frequency dependency. Note that the bulk-driven streaming in Figs. 8(a)–8(c) has its maximum velocity \( v_2^{\text{max}} \) in the bulk of the fluid whereas the boundary-driven streaming in Figs. 8(d)–8(f) has its maximum velocity \( v_2^{\text{max}} \) at the boundaries.

4. Comparison with the analytical expression for the bulk-driven streaming

When the no-slip condition on the vertical side walls is ignored, we can derive an approximate analytical expression for the bulk-driven streaming \( (v_2^{\text{slip}})_{0} \) for the double modes \( 100 + 00 \). The details are given in Appendix D, where we find \( (v_2^{\text{slip}})_{0} \) to be proportional to the acoustic energy flux density
Plotted here is the bulk-driven streaming velocity in the center plane at $z = \frac{1}{2}L_z$ (dark blue edge as in Fig. 7) and the vertical ($x-z$) center plane at $y = \frac{1}{2}L_y$ (light green edge as in Fig. 7). The length of the longest arrow represents $v^{\text{max}}_z$ listed in each plot. The used parameters are listed in Tables II and III. In each column the double mode $l = 00$ is excited for $l = 2$, 4, and 6, respectively. The first row (a)–(c) shows the bulk-driven streaming, the second row (d)–(f) shows the boundary-driven streaming, and the third row (g)–(i) shows the total streaming. 3D results corresponding to (i) are shown in Fig. 7.

$S_{00} = S_{00}^{(00)}$ of Eq. (18a):

$$\left. v_{2}^{\text{blk}} \right|_0 = \frac{1}{2} \left[ \frac{4}{3} + \frac{\eta_0}{\eta} \right] \left[ 1 - \frac{\cos \left( \frac{2\pi L_z}{L_y} L^{*}_{00} \right)}{\cos \left( \frac{2\pi L_y}{L_z} L^{*}_{00} \right)} \right] \eta_0 S_{00}. \quad (37)$$

Inserting here $L^{*}_{00} = \frac{2\pi}{L_0}$ and expression (18a) for $S_{00}$ with $F_{00} = \frac{2\pi}{L_0}$ from Eq. (29c), we obtain the maximum value of the bulk-driven streaming velocity in the center plane $z = \frac{1}{2}L_z$:

$$\left. v_{2}^{\text{blk}} \right|_0^{\text{max}} = \frac{3\sqrt{3}}{2} \left[ 1 - \text{sech} \left( \frac{\pi L_z}{\sqrt{2} \eta} \right) \right] \left( \frac{4}{3} + \frac{\eta_0}{\eta} \right) P_{100}^{(00)} |P_{100}^{(0)}| \quad (38)$$

For the three cases shown in Figs. 8(a)–8(c), where we have used $P_{100}^{(00)} = P_{100}^{(0)} = 1$ MPa and $l = 2$, 4, 6, respectively, we calculate from Eq. (38) the maximum values 0.018, 0.060, and 0.102 mm/s, which deviate less than 7% from the corresponding results 0.017, 0.060, and 0.110 mm/s from the numerical simulation, listed in Figs. 8(a)–8(c). For $l = 2$, the analytical prediction of 0.018 mm/s overestimates by 5% the numerical value 0.017 mm/s because it neglects the significant viscous damping from the vertical side walls in this case. For $l = 6$, the analytical prediction of 0.102 mm/s underestimates by 7% the numerical value 0.110 mm/s because the latter includes the coupling to the vertical 001 mode through the wide $\frac{1}{2}L_0$-width vertical $G_1$ actuation as discussed in Sec. VI C 1. When changing to the narrow $\frac{1}{2}L_0$-width vertical $G_1$ actuation, also discussed in Sec. VI C 1, the coupling to the vertical 001 mode diminishes so much that the numerical streaming amplitudes become 0.017, 0.060, and 0.103 mm/s for $l = 2$, 4, 6, respectively, where the latter now deviates less than 1% from the analytical prediction 0.102 mm/s. These numerical results for the acoustic streaming are shown in a pdf file in the Supplemental Material [41].

5. Bulk-driven versus boundary-driven streaming rolls

We address the question on whether the horizontal streaming rolls in the $xy$ plane are due to the bulk-driven streaming or the boundary-driven streaming. We restrict the discussion to horizontal double modes $l00 + 0l0$, for which we have found the analytical expressions for the bulk-driven streaming ($v_{2}^{\text{blk}}$) [Eq. (37)] and the slip velocity $v_{2}^{\text{slip}}$ [Eq. (7e)] at the top ($z = L_z$) and bottom ($z = 0$) boundaries.
According to Eq. (7e), the boundary-driven streaming velocity at the double-mode resonance $f_{00} + 0/0$ is driven by a slip velocity, which contains the term $\frac{1}{2} \kappa_0 S_{ac}$ that rotates in the direction of the acoustic energy flux density $S_{ac}$, just as the bulk-driven streaming (see Eq. (37)). The other terms in the slip velocity have similar magnitude, but they are gradient terms which cannot drive a rotating streaming in the horizontal $xy$ plane. Therefore, as seen in Figs. 8(d)–8(f), the boundary-driven streaming does not lead to horizontal flow rolls as clearly as the bulk-driven streaming does [see Figs. 8(a)–8(c)]. There may, however, exist horizontal planes at a few specific heights $z$, where the horizontal flow-roll pattern appears more clearly as suggested by Lei et al. and analyzed further by Skov et al. [44].

By using Eqs. (7e) and (37), we compute the ratio of the horizontally rotating bulk-driven streaming $[v_{2}^{blk} (z = \frac{L_z}{2})]_{0}$ in the center of the cavity to the horizontally rotating boundary-driven streaming $[\frac{1}{4} \kappa_0 S_{ac} (z = 0)]$ at the bottom boundary,

$$\left[\frac{[v_{2}^{blk} (z = \frac{L_z}{2})]_{0}}{[\frac{1}{4} \kappa_0 S_{0} (z = 0)]}\right]_{\rho_{w}} = \left(\frac{4}{3} + \frac{\eta_{b}}{\eta_{w}}\right) \left[1 - \text{sech}\left(\frac{\pi L_{c}}{2L^{*}}\right)\right].$$

This velocity ratio takes values from 0 (boundary-driven streaming dominates) for a flat cavity $H_{c} \ll L^{*}$ to $\frac{4}{3} + \frac{\eta_{b}}{\eta_{w}}$ (bulk-driven streaming dominates) for a high cavity $H_{c} \gg L^{*}$. For water, the velocity ratio (39) lies between 0 and 4.13, and in the setup of Figs. 4–8 with $L_{c} = 0.1L^{*}$ and water as the fluid, we obtain the values of the velocity ratio to be 0.38, 1.22, and 2.09 for $l = 2$, 4, and 6, respectively. This result indicates that the $6 \times 6$ streaming roll pattern with $l = 6$ observed by Hagsäter et al. [39] is predominantly bulk-driven, as also seen by comparing Figs. 8(c) and 8(i). In contrast, for the lower mode $l = 2$, the horizontal streaming rolls are predominantly boundary driven, which is seen by comparing Figs. 8(d) and 8(g).

The crossover from boundary- to bulk-driven horizontal streaming rolls may be characterized by the height-to-width ratio $B = H_{c}/L^{*}$: we compute the critical value $B^{crit}$, where the velocity ratio (39) is unity, and thus,

$$\frac{L_{c}}{L^{*}} < B^{crit}, \quad \text{boundary-driven streaming dominates},$$

$$\frac{L_{c}}{L^{*}} > B^{crit}, \quad \text{bulk-driven streaming dominates}. \quad (40)$$

For water, Eq. (39) leads to the ratio $B^{crit}_{water} = 0.35$, and since $L_{c} = 0.1L^{*}$ in the setup of Figs. 4–8, we find that the horizontal streaming rolls in the double modes $00 + 0/0$ with $l \leq 3$ are predominantly boundary driven, while for $l \geq 4$ they are bulk driven.

6. Enhancement of the bulk-driven streaming velocity by increasing the bulk viscosity

For a given resonance mode with a given acoustic pressure amplitude in a given geometry, we obtain from Eqs. (38) and (7e) the following scaling laws for the bulk- and boundary-driven streaming, respectively:

$$v_{2}^{blk} \propto \frac{1}{\rho_{0}^{\frac{1}{3}} c_{0}^{\frac{1}{3}}} \left(4 + \frac{\eta_{b}}{\eta_{w}}\right),$$

$$v_{2}^{slip} \propto \frac{1}{\rho_{0}^{\frac{1}{3}} c_{0}^{\frac{1}{3}}}. \quad (41a)$$

To study the consequences of these scaling laws, we have chosen to compare water with the heterocyclic organic compound pyridine having the chemical formula $C_{5}H_{5}N$. Table II reveals that pyridine has the same material parameters (within 5%) as water, but a bulk viscosity $\eta_{b}$ that is 25.1 times larger than that of water. Using these parameter values in Eq. (41), we find a large enhancement of the bulk-driven streaming, while the boundary-driven streaming is nearly unchanged:

$$v_{2}^{blk}_{pyridine} = 21.3 v_{2}^{blk}_{water},$$

$$v_{2}^{slip}_{pyridine} = 1.2 v_{2}^{slip}_{water}. \quad (42b)$$

We test this prediction in Fig. 9, where we show simulations of the total acoustic streaming in pyridine in the same geometry as in Figs. 8(g)–8(i) but with adjusted actuation amplitude $u_{0}^{l}$ and frequency $f$ given in Table III to maintain the same pressure amplitudes $P_{f_{00}}^{l0} = P_{f_{00}}^{l0} = 1$ MPa. As expected, we find that the total acoustic streaming for the three double modes is enhanced compared to that in water shown in Figs. 8(g)–8(i), and the enhancement factors are 2.8, 8.2, and 13.9, respectively. The increasing factors reflect the increasing weight of bulk-driven relative to boundary-driven streaming as the mode index $l$ increases. The enhancement factors for the bulk-driven streaming alone (not shown in Fig. 9) are 21.4, 21.1, and 21.2, as predicted by Eq. (42a), while they are 1.2,
1.2, and 1.3 for boundary-driven streaming alone (also not shown in Fig. 9) in good agreement with Eq. (42b).

For all three cases \( l = 2, 4, \) and 6, the horizontally rotating bulk-driven streaming in the center \( z = \frac{1}{2} L_z \) is larger than the horizontally rotating boundary-driven streaming at the top-bottom boundaries \( z = 0 \) and \( z = L_z \). This is also predicted by Eq. (40) since \( \frac{L_z}{L} = 0.2, 0.4, 0.6 > B_{\text{pyridine}} = 0.08 \) for all three modes in pyridine.

VII. DISCUSSION

In the following, we raise a few discussion points related to the results of the above analysis, which are based on the single- and double-mode resonances occurring in a closed rectangular-box-shaped cavity with narrow viscous boundary layers. We note that the idealized, hard-wall, numerical model for the acoustic streaming is strikingly sensitive to small differences of only 0.1\% between the cavity side lengths \( L_x \) and \( L_y \). Our results for the resonance frequency in Eq. (10) and the optimal aspect ratio in Eq. (29f) are therefore crucial in the investigation of the largest possible bulk-driven acoustic streaming. This optimization was not taken into account in the experimental study by Hagsäter et al. [39], and the corresponding numerical simulation by Skov et al. [44] of the shallow, nearly square cavity of aspect ratio \( A \approx 0.990 \). As this aspect ratio differs from the optimal value \( A = 0.999 \) for the hard-wall device in Fig. 8(i), it may be possible to increase the bulk-driven acoustic streaming significantly by fine tuning the geometry.

A priori, one may argue that it is difficult to obtain these optimal conditions with a relative precision of the order of 0.1\% in experiments. However, comparing our Fig. 7 with the experimental results in Hagsäter et al. [39], we find good qualitative agreement even though none of the optimal conditions were considered when designing the nearly square fluid cavity. Another example is provided by the straight microchannel with a square cross section and nonrotating actuation that was studied experimentally by Antfolk et al. [11]. There, a single large vertical streaming roll similar to Fig. 2(b) was observed experimentally and analyzed numerically in terms of a double-mode excitation.

There may be several reasons for why the bulk-driven acoustic streaming can be observed experimentally in spite of the low 0.1\%-tolerance level of the aspect ratio suggested by our simplified, hard-wall, microchannel analysis in Fig. 6. First, there are other damping mechanisms in a real experiment [42], aside from the bulk and boundary-layer damping considered here, which may broaden the resonance peaks significantly and thereby increase the change for overlapping resonances. Second, the double-mode body force \( f_{1\text{bl}} \) in Eq. (17a), which is the source of the bulk-driven acoustic streaming, depends on the product \( |P_{1\text{bl}}|^2 |P_{1\text{bl}}|^2 \) of the mode amplitudes, and therefore one can obtain a rotating body force even if the pressure is dominated by one of the modes, say \( lmn \), as long as the other mode \( \text{Inm} \) is just weakly represented. Third, in typical experiments, the driving frequency is scanned over a broad span of frequencies, say 100 kHz around 2 MHz, which increases the possibility for exciting double-mode resonances. Fourth, unintentional asymmetries and imperfections in the experimental setup may cause modes to overlap locally in some regions at some frequencies and in other regions at other frequencies, rather than the idealized homogeneous pattern at a single frequency shown in Fig. 7. This point was also mentioned by Hagsäter et al.: “If the frequency is shifted slightly in the vicinity of 2.17 MHz, the same vortex pattern will still be visible, but the strength distribution between the vortices will be altered.” Moreover, in the 3D simulation by Skov et al. [44], including the surrounding solid and the piezoelectric transducer, this inhomogeneous distribution of the streaming was indeed reproduced in their Fig. 2(b2). Revisiting the experiment by Antfolk et al. [11] of a long straight capillary of square cross section, this effect may also be observed because it is likely that the aspect ratio is slightly less than unity in some regions of the capillary and slightly above unity in other regions, thereby causing the rotation direction to alternate along the channel.

We have in this work mainly considered horizontal acoustics with \( n = 0 \) half-waves in the vertical \( z \) direction. A similar analysis can be done for perpendicular acoustics (with respect to the top and bottom boundaries). In Secs. VI C 1 and VI C 4, we have already briefly discussed the role of the vertical single mode 001 for the pressure \( p_1 \) and the streaming \( v_2 \). It is of course also possible to consider vertical rotating double modes involving, say, a single half-wave in the vertical direction \( lmn \). For relatively shallow cavities, a vertical excitation implies a relatively high frequency and a correspondingly increased body force \( f_{ac} \). Furthermore, for perpendicular acoustics where \( \partial_{\perp} v_{1\perp} \approx \nabla \cdot v_1 = i \omega \kappa_0 p_1 \), the slip velocity \( \nu_{2\perp} \) on the top and bottom boundaries is dominated by the first and last terms in Eq. (7e), \( \nu_2 \approx \frac{1}{2}\kappa_0 S_{ac} + \frac{2}{|\Re[i\partial_{\perp} v_{1\perp} v_{1\perp}^{*}]|} \approx -\kappa_0 S_{ac} \). Remarkably, this slip velocity is in the opposite direction of the body force \( f_{ac} \), so we expect that the bulk- and boundary-driven streaming will be in opposite directions, and thus the cavity could be designed such that these streaming flows tend to cancel. A study of this effect is left for future work.

The bulk-driven acoustic streaming is due to the body force \( f_{ac} \), which we have given in Eq. (7d) for homogeneous fluids. As shown by Karlsen and Bruus [49], this body force contains additional terms if the fluid is inhomogeneous. Furthermore, the analysis by Muller and Bruus [45] showed how thermoviscous effects strongly influence the boundary-driven streaming. The incorporation of inhomogeneities and thermodynamics in future work is therefore expected to reveal a strong effect also on the bulk-driven streaming.

VIII. CONCLUSION

We have derived analytical expressions for the single- and double-mode pressure resonances \( p_{1\text{bl}} = p_{1\text{bl}}^{\text{lmn}} \) and the corresponding boundary-layer damping coefficients \( \Gamma_{1\text{bl}}^{\text{lmn}} \) for a closed rectangular-box-shaped cavity with narrow viscous boundary layers (see Sec. III A and Appendix A). Based on these expressions, we have shown that the body force \( f_{ac} \) can drive a strong bulk-driven acoustic streaming, if two overlapping single modes are excited simultaneously at the same frequency, thereby forming a double-mode resonance \( lmn + \text{Inm} \) (see Secs. III B and IV). In contrast to the conventional wisdom that the bulk-driven streaming can be ignored.
on the scale of a few acoustic wavelengths, these double modes constitute an important example where the bulk-driven streaming is significant.

We have shown in Eq. (8) that the appearance of bulk-driven streaming in closed microcavities is directly related to the amount of rotation (acoustic angular momentum density) of the underlying acoustic fields. Whereas there is no rotation in a single mode, we have demonstrated in Secs. IV B and V that the double modes can be rotating not only due to an externally controlled rotating actuation, but, remarkably, also due to a nonrotating actuation coupled with a weak asymmetry in the geometry of the cavity. Furthermore, in Sec. V we have derived analytical expressions for the optimal aspect ratios that maximize the bulk-driven streaming in a nearly square cavity with a nonrotating actuation. We found that the bulk-driven acoustic streaming is sensitive to small deviations between the cavity side lengths of only 0.1%.

We have validated the theory by direct numerical simulation in Sec. VI of the double modes 200 + 202, 400 + 404, and 600 + 606 in a nearly square cavity with side lengths $L_x \approx 2000$ μm and $L_y \approx 2000$ μm and height $L_z = 200$ μm, similar to the device investigated experimentally by Hagsäter et al. [39]. Using the analytically known optimal condition (29), for which the bulk-driven acoustic streaming is maximized, we have avoided the otherwise required time-consuming parametric sweeping in geometry and frequency to locate these optimal conditions. The numerical results shown in Figs. 4–6 agree with the analytical expressions with a relative accuracy better than 1%.

In Fig. 7, the 3D simulation of the acoustic streaming of the double mode 200 + 202 is seen to reproduce the horizontally rotating $6 \times 6$ streaming-roll pattern observed by Hagsäter et al. In Fig. 8 we showed that this streaming pattern at the higher frequency 2.24 MHz is dominated by the bulk-driven streaming, whereas the streaming of the double mode 200 + 020 at the lower frequency 0.75 MHz is dominated by the boundary-driven streaming. This frequency-dependent crossover occurs because the body force $f_{\text{ac}}$ increases as frequency to the power two, whereas the slip velocity $v_{\text{slip}}$ depends only weakly on frequency [see Eq. (7)]. An analytical expression for computing the crossover for the double mode 100 + 010 is given in Eq. (39).

Finally, in agreement with Eckart [29], we have shown in Eq. (37) that the bulk-driven acoustic streaming increases with the ratio $\frac{\eta_x}{\eta_y}$ between the bulk and the dynamic viscosity. This dependency on $\eta_x$ was studied numerically in Sec. V1C6 by exchanging the water in the microcavity with pyridine. As shown in Fig. 9, we found that the bulk-driven streaming for pyridine was enhanced by a factor $\approx 3$ relative to water. This prediction obviously calls for experimental validation.

In this paper we have pointed out the significance of the bulk-driven streaming in resonating acoustic cavities even when the geometry length scale is comparable to a few half-wavelengths. The fundamental requirement for obtaining this effect is the existence of two overlapping resonance modes excited simultaneously at the same frequency. This requirement is easily fulfilled in integrable geometries in 3D, such as rectangular, cylindrical, and spherical cavities, or in 2D, such as rectangular and circular cross sections of long, straight capillary channels, all of which are frequently encountered in experiments reported in the literature. The insight provided by our analysis is therefore relevant and important for acoustofluidics in general, both for fundamental studies and for technological applications.

### APPENDIX A: ACOUSTIC PRESSURE IN A RECTANGULAR BOX WITH BOUNDARY LAYERS

In this Appendix, we derive the solution for the acoustic pressure $p_1$ with boundary layers in a rectangular-box-shaped cavity $\Omega$ for which $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, and $0 \leq z \leq L_z$. The governing equation and boundary conditions for $p_1$ are given in Eq. (3). We consider first an actuation $U_{1,\perp}(x, y)$ which is only nonzero at the bottom boundary $z = 0$. The solution for a general actuation on all six boundaries is then finally constructed by superposition. Using separation of variables to solve the Helmholtz equation (3a) gives $p_1$ in the following form:

$$p_1 = X(x)Y(y)Z(z), \quad k_z^2 = k_x^2 + k_y^2 + k_z^2,$$

$$X(x) = C_x \cos(k_x x) + S_x \sin(k_x x),$$

$$Y(y) = C_y \cos(k_y y) + S_y \sin(k_y y),$$

$$Z(z) = C_z \cos(k_z z) + S_z \sin(k_z z).$$  \hfill (A1)

Inserting this form into boundary condition (3b) gives

$$[\partial_{\perp} + \alpha_{\perp}] p_1 = \frac{k_x^2 U_{1,\perp}}{\kappa_0},$$  \hfill (A2a)

$$\alpha_{\perp} = \alpha_{\perp}(k_z) = \frac{1 + i}{2} \delta (k_z^2 - k_0^2), \quad \perp = x, y, z.$$  \hfill (A2b)

where $\perp$ is the inward direction perpendicular to the boundary, and the quantity $\alpha_{\perp}$ is the boundary-layer correction to the inward-normal derivative of $p_1$. Note that $\partial_{\perp}$ has a sign change at two opposite boundaries, say $x = 0$ and $L_x$, for which $\partial_{\perp} = \partial_x$ and $-\partial_x$, respectively, whereas $\alpha_{\perp}(k_z)$ depends on $k_z^2 = k_z^2$, without a sign change at opposite boundaries. We assume in the following that the length scale introduced by $\alpha_{\perp}$ is much longer than the cavity length scale $L_{\perp}$:

$$|\alpha_{\perp}| L_{\perp} \ll 1, \quad \perp = x, y, z.$$  \hfill (A3)

According to Eq. (A2), the boundary conditions at the stationary boundaries at $x = 0$ and $L_x$ are

$$x = 0 : \quad C_x \alpha_x + S_x k_x = 0,$$  \hfill (A4a)

$$x = L_x : \quad C_x[k_x \sin(k_x L_x) + \alpha_x \cos(k_x L_x)]$$

$$+ S_x[-k_x \cos(k_x L_x) + \alpha_x \sin(k_x L_x)] = 0.$$  \hfill (A4b)

where $\alpha_x = \alpha_x$ in both equations, whereas $\partial_x = -\partial_x$ in Eq. (A4a), while $\partial_x = -\partial_x$ in Eq. (A4b). Nontrivial solutions with $C_x$ and $S_x$ different from zero must satisfy the usual criterion for the equation determinant $D_x(k_z)$:

$$D_x(k_z) = 0, \quad \text{with}$$  \hfill (A5a)

$$D_x(k_z) = (k_z^2 - \alpha_x^2) \sin(k_x L_x) + 2k_x \alpha_x \cos(k_x L_x).$$  \hfill (A5b)
We write the solutions $k_i = k_i^l$ to Eq. (A5a) as perturbations away from the inviscid solutions $K_i^l$,

$$k_i^l = K_i^l + \epsilon_i^l, \quad K_i^l = \frac{l\pi}{L_x}, \quad l = 0, 1, 2, \ldots, \quad (A6)$$

where $\epsilon_i^l L_x \ll 1$. By inserting this into Eq. (A5b), expanding in $\epsilon_i^l$, and writing $\alpha_i^l = \alpha_i(k_i^l)$, we find

$$D_\lambda(k_i^l) = (-1)^l \left\{ (K_i^l + \epsilon_i^l)^2 - (\alpha_i^l)^2 \right\} \epsilon_i^l L_x + 2 (K_i^l + \epsilon_i^l) \alpha_i^l \right\}. \quad (A7)$$

From assumption (A3), $\alpha_i^l \ll \epsilon_i^l L_x = [\alpha_i^l L_x]$, so it can be ignored. We then factorize out $K_i^l + \epsilon_i^l = k_i^l$ and obtain

$$D_\lambda(k_i^l) \approx (-1)^l \left\{ (K_i^l)^2 - (\alpha_i^l)^2 \right\} \epsilon_i^l L_x + 2 \alpha_i^l \right\} \quad (A8)$$

For $l = 0$, we have $K_0^l = 0$ and $k_i^l(0^0 - K_0^l) = (k_i^l)^2$, while for $l > 0$ we have $k_i^l(0^l - K_0^l) \approx \frac{1}{2} (k_i^l + K_0^l)(k_i^l - K_0^l)$ to first order in $\epsilon_i^l$. In either case, by introducing $L_i^l$ as

$$L_i^l = \frac{1}{2} \left( 1 + \delta_0 k_i^l \right) L_x, \quad (A9)$$

the resulting expression for $D_\lambda(k_i^l)$ for all $l$ is written as

$$D_\lambda(k_i^l) \approx (-1)^l k_i^l L_x \left\{ (k_i^l)^2 - (\alpha_i^l)^2 \right\} \frac{2 \alpha_i^l}{L_x}. \quad (A10)$$

The wave numbers $k_i^l$ thus fulfill the zero-determinant criterion (A5a) $D_\lambda(k_i^l) = 0$ if

$$\left( k_i^l \right)^2 = (\alpha_i^l)^2 \frac{2 \alpha_i^l}{L_x}, \quad (A11)$$

and similarly for $(k_i^l)^2$ for all $l$ and $m$. From Eqs. (A1) and (A4a) with $k_i = k_i^l$, we obtain the corresponding eigenfunctions $X_i(x)$,

$$X_i(x) = \cos (k_i^l x) - \frac{\alpha_i}{k_i^l} \sin (k_i^l x), \quad (A12)$$

where the prefactor $C_i^l$ is absorbed into $Z(z)$ in Eq. (A1). We examine the ratio $k_i^l \alpha_i$ for all $l$ using Eq. (A11):

$$\left( \frac{\alpha_i^l}{k_i^l} \right)^2 = \frac{1}{2} L_x \alpha_i^l \ll 1, \quad l = 0, \quad (A13a)$$

$$\frac{\alpha_i^l}{k_i^l} = \frac{1}{L_x} L_x \alpha_i^l \ll 1, \quad l > 0. \quad (A13b)$$

By assumption (A3), we note that even for $l = 0$ this ratio is small, but not as small as for $l > 0$. By inserting $k_i^l$ into Eq. (A11) into Eq. (A12) and Taylor expanding in $\alpha_i^l/k_i^l$, we recover the usual inviscid hard-wall eigenfunctions $\cos (K_i^l x)$ plus a small correction of the order $\alpha_i^l L_x$, due to the boundary layers

$$X_i(x) \approx \cos (K_i^l x) + \alpha_i^l L_x \left\{ \frac{1}{2} L_x (\frac{k_i^l}{L_x} - 1), \quad l = 0, \right. \left. \sin (K_i^l x) \frac{1}{2} (\frac{k_i^l}{L_x} - 1), \quad l > 0. \right\} \quad (A14)$$

Note that $(\partial_x + \alpha_i^l) X_i x = 0$ as required by the boundary condition (A2). Similar expressions are valid for the $y$ eigenfunctions $Y_i(y)$. In general, the pressure is an infinite sum of the eigenfunctions

$$p_1 = \sum_{l,m=0}^{\infty} X_i Y_i Z_{lm}(z), \quad (A15a)$$

$$Z_{lm}(z) = C_{lm}^l \cos (k_{lm}^l z) + S_{lm}^l \sin (k_{lm}^l z), \quad (A15b)$$

$$\left( k_{lm}^l \right)^2 = k_i^2 - (k_i^l)^2 - (k_{lm}^l)^2, \quad (A15c)$$

where $k_{lm}^l$ depends on the angular frequency $\omega$ of the actuation through $k_i$. At $z = L_x$, the sets of coefficients $C_{lm}^l$ and $S_{lm}^l$ satisfy a condition similar to Eq. (A4b), which leads to

$$S_{lm}^l = \frac{k_{lm}^l \sin (k_{lm}^l L_x) + \alpha_i^l \alpha_i^l \sin (k_{lm}^l L_x)}{k_i^l \cos (k_{lm}^l L_x) - \alpha_i^l \sin (k_{lm}^l L_x)} \frac{k_i^l L_x}{k_i}, \quad (A16)$$

where $C_{lm}^l = \alpha_i^l(k_i^l)$. Further, at $z = 0$ we have a condition similar to Eq. (A4a) with $k_i^l = k_i^l U_1(x, y)$ on the right-hand side. Combined with Eqs. (A15) and (A16), the boundary condition at $z = 0$ becomes

$$\sum_{l,m=0}^{\infty} \frac{X_i(x) Y_i(y) D_\lambda(k_i^l) C_{lm}^l}{k_i^l \cos (k_{lm}^l L_x) - \alpha_i^l \sin (k_{lm}^l L_x)} = \frac{k_i^l L_x}{k_i} \frac{k_i^l L_x}{k_i}, \quad (A17)$$

where $D_\lambda$ is defined similar to Eq. (A5b). To find the coefficients $C_{lm}^l$, we write the wall actuation function $U_1(x, y)$ as a generalized Fourier series in the functions $X_i(x)$ and $Y_i(y)$ that form a complete basis set on the interval $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$ to order $\alpha_i^l L_x$ and $\alpha_i^l L_y$, respectively:

$$U_1(x, y) = \sum_{l,m=0}^{\infty} \frac{U_1 L_x X_i(x) Y_i(y)}{D_\lambda(k_i^l) C_{lm}^l} \quad (A18a)$$

$$H_1 \frac{k_i^l}{L_x} \int_0^{L_x} \int_0^{L_y} U_1(x, y) X_i(x) Y_i(y) \, dA \quad (A18b)$$

Inserting the expansion (A18a) into Eq. (A17), we obtain the amplitudes $C_{lm}^l$:

$$C_{lm}^l \approx \frac{k_{lm}^l \cos (k_{lm}^l L_x) - \alpha_i^l \sin (k_{lm}^l L_x)}{D_\lambda(k_i^l) C_{lm}^l} \frac{k_i^l L_x}{k_i} \quad (A19)$$

Finally, using Eq. (A16) for $S_{lm}^l$ and Eq. (A19) for $C_{lm}^l$ in Eq. (A15) for $p_1$, yields the expression for the pressure

$$p_1 = \sum_{l,m=0}^{\infty} \frac{k_i^l L_x \int_0^{L_x} \int_0^{L_y} X_i(x) Y_i(y) Z_{lm}(z)}{k_i^l \cos (k_{lm}^l L_x) - \alpha_i^l \sin (k_{lm}^l L_x)} \quad (A20a)$$

$$Z_{lm}(z) = \cos \left[ k_{lm}^l (L_x - z) \right] - \frac{\alpha_i^l}{k_{lm}^l} \sin \left[ k_{lm}^l (L_x - z) \right]. \quad (A20b)$$

$$\left[ k_{lm}^l (k_0) \right]^2 = (1 + i \Gamma \alpha_0) - (k_i^l)^2 - (k_{lm}^l)^2. \quad (A20c)$$
The infinite sum in Eq. (A20a) is general and applies to all frequencies and actuations at the bottom boundary. For a given actuation \( U_1 \), this solution is largest for frequencies \( \omega = \omega_0 \) where \( D_z(k_{\text{lin}}^z) \) is smallest, which gives the hard-wall resonances \( lmn \) studied below. In real systems, the surrounding solid will have its own resonance properties and, therefore, the actuation \( U_1 \) will depend strongly on the frequency. In this case, the largest value of the prefactor \( \kappa_{\text{lin}}^{lmn} \) is not necessarily found where \( D_z(k_{\text{lin}}^z) \) is smallest.

### 1. Single-mode resonances in a rectangular box

Resonance occurs when \( k_0 = \frac{1}{k_0} \omega \) equals one of the values \( k_{\text{lin}}^{lmn} \) that minimize \( D_z(k_{\text{lin}}^z) \) in Eq. (A20a). We expect these values of \( k_{\text{lin}}^{lmn} \) to be near the inviscid values \( k_0^{lmn} \), and write, similar to Eq. (A6),

\[
\begin{align*}
    k_{0}^{lmn} &= k_0^{lmn} + \epsilon_0^{lmn}, \\
    k_{l0}^{lmn} &= \sqrt{\left(K_l^0 \right)^2 + \left(K_y^0 \right)^2 + \left(K_n^0 \right)^2}. \\
    K_l^0 &= \frac{\pi}{L_x}, \quad K_y^0 = \frac{m\pi}{L_y}, \quad K_n^0 = \frac{n\pi}{L_z},
\end{align*}
\]

where \( \epsilon_0^{lmn} \) should be chosen to minimize \( D_z(k_{\text{lin}}^z) \). The modes \( l \) and \( m \) in the \( x \) and \( y \) directions together with the resonance condition \( k_0 = k_{\text{lin}}^{lmn} \) fix the wave number \( k_{\text{lin}}^{lmn} = k_{\text{lin}}^{lmn}(k_0^{lmn}) \) in Eq. (A20c) pertaining to the direction \( z \):

\[
(k_{l0}^{lmn})^2 = (1 + i\Gamma_{l0}^{lmn})(k_0^{lmn})^2 - (K_l^0)^2 - (K_y^0)^2 - (K_n^0)^2.
\]

Combining this expression with Eq. (A11) for \( (K_l^0)^2 \) and \( (K_y^0)^2 \), we find

\[
(k_{l0}^{lmn})^2 = (K_l^0)^2 + \epsilon_0^{lmn}(1 + i\Gamma_{l0}^{lmn})\left[ 2\alpha_y^0 + 2\alpha_y^0 + 2\alpha_n^0 \right] + \epsilon_0^{lmn}(1 + i\Gamma_{l0}^{lmn})\left[ 2\alpha_y^0 + 2\alpha_y^0 + 2\alpha_n^0 \right]
\approx (K_l^0)^2 + 2k_0^{lmn}\epsilon_0^{lmn} + i\Gamma_{l0}^{lmn}(k_0^{lmn})^2 + 2\alpha_y^0 + 2\alpha_y^0 + 2\alpha_n^0
\]

\[
(A23)
\]

To evaluate \( D_z(k_{\text{lin}}^z) \) in Eq. (A20a), we note that \( k_{\text{lin}}^{lmn} \) is close to \( K_l^0 \), so we can use Eq. (A10) with \( (K_l^0)^2 - (K_l^0)^2 \to (K_l^0)^2 - (K_l^0)^2 \), which is found from Eq. (A23):

\[
D_z(k_{\text{lin}}^z) \approx (-1)^n\Gamma_{l0}^{lmn}\left[ 2k_0^{lmn}\epsilon_0^{lmn} + i\Gamma_{l0}^{lmn}(k_0^{lmn})^2 + 2\alpha_y^0 + 2\alpha_y^0 + 2\alpha_n^0 \right].
\]

The (real) value of \( \epsilon_0^{lmn} \), which minimizes this expression and leads to the resonance wave number \( k_0^{lmn} \), is then

\[
\epsilon_0^{lmn} = -\text{Re}\left[ \frac{\alpha_y^0}{K_l^0 L_x} + \frac{\alpha_y^0}{K_m^0 L_y} + \frac{\alpha_n^0}{K_n^0 L_z} \right] = -\frac{1}{2}K_0^{lmn}\Gamma_{l0}^{lmn}.
\]

\[
(A25a)
\]
Note that for any $l$, $m$, and $n$, the product term $X^i(x)Y^m(y)Z^n(z)$ only satisfies the boundary condition $A_{2a}$ for a stationary boundary $U_{1\perp} = 0$. The actual boundary condition is satisfied by the infinite sum (A2a) over terms $X^i(x)Y^m(y)Z^n(z)$ because at $z = 0$ and in $Z^n$ at $z = 0$, it must be corrected to deal with frequencies $k_0 \approx k_{lmn}$. This is done by adding $k_0 - k_{lmn}$ to the right-hand side of Eq. (A21).

By Eq. (A24), this procedure is seen to be equivalent to the substitution

$$i(k_0^m)^2 \Gamma_{lmn} \rightarrow 2k_0^m(k_0 - k_{lmn}) + i(k_0^m)^2 \Gamma_{lmn}$$

(A31)
in Eq. (A30).

2. Approximate solutions near resonance for actuation at all boundaries

At resonance, the expression (A30) satisfies all boundary conditions to first order in $\alpha' L_\Omega$, $\alpha'' L_\zeta$, and $\alpha L_\zeta$. Ignoring these first-order corrections as well as the small $Z_{act}^{lmn}$ term in Eq. (A30), the eigenfunctions are approximated equal to the usual hard-wall eigenfunctions $R^{lmn}$:

$$R^{lmn}(x, y, z) = (K_x^I \phi) \cos (K_y^Z \phi) \cos (K_z^Z \phi).$$

(A32)

To generalize our results from an actuation on only the bottom boundary $z = 0$ to all six boundaries, we note first that $L_x^l \approx f_0^l(Z^n(z))$ and $L_y^l \approx f_0^l(Z^n(y))$ and $L_z^l \approx f_0^l(Z^n(z))$, whereby Eq. (A18b) can be rewritten as

$$Q_{lmn}^l \approx \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

(A33)

Using this expression together with the substitution (A31) valid for $k_0 \approx k_{lmn}$, we obtain the following expression for the resonance modes $lmn$ in a rectangular box $\Omega$ with viscous boundary layers and an inward normal displacement $U_{1\perp}(r)$ specified on all six boundaries $\partial \Omega$ [Eq. (3c)]:

$$P_{lmn}^l \approx \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

(A34a)

$$P_{lmn}^l \approx \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

(A34b)

$$F^{lmn}(k_0) = \frac{1}{\Gamma_{lmn}} \frac{(k_0 - k_{lmn}) + i\frac{|k_{lmn}|}{\Gamma_{lmn}}}{(k_0 - k_{lmn}) + i\frac{|k_{lmn}|}{\Gamma_{lmn}}},$$

(A34c)

$$\Gamma_{lmn} \approx \frac{1}{\Gamma_{lmn}},$$

(A35)

where $\Gamma_{lmn}$ is given in Eq. (A27).

APPENDIX B: MAXIMUM ACOUSTIC ROTATION

In the following, we evaluate the frequency-dependent factor $\mathcal{F}^{lmn}_{m0}$ in Eq. (17b) for rotating and nonrotating actuations of the nearly square cavity considered in Sec. V. For convenience, we first introduce the double-mode quantities $\xi_{lm}$ and $\phi_{lm}^{act}$,

$$\xi_{lm} = k_0 - k_{lm}^*, \quad \phi_{lm}^{act} = \phi_{act}^{lm0} - \phi_{act}^{lm}. \quad (B1)$$

where $k_0$ and $k_{lm}^*$ are defined in Eq. (27), and $\phi_{act}^{lm}$ is the difference in the phases by which the modes $lm0$ and $m0$ are actuated [see Eq. (12b)]. By using these quantities and the expressions for $F^{lm0}$ and $F^{m0}$ in Eq. (28), we calculate the frequency-dependency $\mathcal{F}^{lm0}_{m0}$ from Eq. (17b):

$$\mathcal{F}^{lm0}_{m0} = \text{Re}\{i\phi_{act}^{lm} F^{lm0}(F^{m0})^*\}$$

$$= \cos(\phi_{act}^{lm}) 2\Delta_m + \sin(\phi_{act}^{lm}) [1 + (2\xi_{lm})^2 - \Delta_m^2]$$

$$[(2\xi_{lm} - \Delta_m)^2 + 1][2(\xi_{lm} + \Delta_m)^2 + 1].$$

(B2)

This equation expresses the magnitude of the acoustic rotation as a function of the actuation phase difference $\phi_{act}^{lm}$, the actuation frequency $\xi_{lm}$, and the mode separation $\Delta_m$, which by Eq. (26) is related to the aspect ratio $A$. In the following, we study the nonrotating actuation with $\phi_{act}^{lm} = 0$ and the rotating actuation with $\phi_{act}^{lm} = \pm \pi$, and we determine the optimal mode separation $\Delta_m^{opt}$ and frequency $\xi_{lm}^{opt}$, for which $\mathcal{F}^{lm0}_{m0}$ takes its largest positive or negative value ($\mathcal{F}^{lm0}_{m0}$).

1. Extremum values of $\mathcal{F}^{lm0}_{m0}$ for a nonrotating actuation $\phi_{act}^{lm} = 0$

For a nonrotating actuation, we have $\phi_{act}^{lm} = 0$, and Eq. (B2) becomes

$$\mathcal{F}^{lm0}_{m0} = \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

(B3)

The extremum values of this expressions are

$$\mathcal{F}^{lm0}_{m0}^{extr} = \begin{cases} 2\Delta_m & \text{if } |\Delta_m| < 1, \\ \frac{2\Delta_m}{\Omega} & \text{if } |\Delta_m| > 1, \end{cases}$$

(B4a)

which are obtained at the frequencies $\xi_{lm}^{extr}$ given by

$$\xi_{lm}^{extr} = \begin{cases} 0, & \text{if } |\Delta_m| < 1, \\ \pm \frac{1}{2\sqrt{\Delta_m}} & \text{if } |\Delta_m| > 1. \end{cases}$$

(B4b)

In Fig. 6 ("nonrotating act.") we plot the value $(\mathcal{F}^{lm0}_{m0})^{extr}$ from this expression as a function of $\Delta_m$. The optimal value $(\mathcal{F}^{lm0}_{m0})^{opt} = (\mathcal{F}^{lm0}_{m0})^{extr}$ is found at the optimal mode separation $\Delta_m^{opt}$, and optimal scaled frequency $\xi_{lm}^{opt}$.

$$\xi_{lm}^{opt} = 0, \quad \Delta_m^{opt} = \pm \frac{1}{3\sqrt{2}}, \quad (\mathcal{F}^{lm0}_{m0})^{opt} = \pm \frac{3\sqrt{3}}{8}.$$  

(B5)

2. Extremum values of $\mathcal{F}^{lm0}_{m0}$ for a rotating actuation $\phi_{act}^{lm} = \pm \pi$

For an externally controlled rotating actuation, we have $\phi_{act}^{lm} = \pm \pi$ and Eq. (B2) becomes

$$\mathcal{F}^{lm0}_{m0} = \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

$$= \frac{2\Delta_m}{\Omega} \int_0^L U_{1\perp}(z) X^l(z) Y^m(y) Z^n(z) dA$$

(B6a)

$$\text{sgn}(\phi_{act}^{lm}) = \text{sign}(\phi_{act}^{lm}).$$

(B6b)
This expression is more complicated than Eq. (B3) since it only has one extremum for a sufficiently small mode separation $\Delta \lambda_m$, namely, where the angular momentum and the actuation are corotating, $\text{sign}(\lambda_{m0}) = \text{sign} \varphi$. For larger values of $\Delta \lambda_m$, the frequency can be tuned to two different extremum values, namely, both the corotating and counter-rotating cases, $\text{sign}(\lambda_{m0}) = \pm \text{sign} \varphi$. We therefore identify two branches of extremum values. The corotating branch ($\lambda_{m0}^{\text{cor}}$) with subscript “$\text{co}$” and the counter-rotating branch ($\lambda_{m0}^{\text{cnt}}$) with subscript “$\text{cnt}$.”

For the corotating branch, $\lambda_{m0}^{\text{cor}}$ is

$$\begin{align*}
\left( \lambda_{m0}^{\text{cor}} \right)_{\text{co}} &= \text{sgn} \varphi \left[ \frac{-\Delta \lambda_m}{(\Delta \lambda_m + 1)^2}, \left| \Delta \lambda_m \right| \leq \sqrt{2} - 1, \right. \\
&\left. \frac{1}{2 \Delta \lambda_m}, \left| \Delta \lambda_m \right| \geq \sqrt{2} - 1, \right]
\end{align*}$$

(B7a)

found at the frequencies $\tilde{\xi}_{lm}^{\text{co}}$.

$$\tilde{\xi}_{lm}^{\text{co}} = \begin{cases} 
0, & \left| \Delta \lambda_m \right| \leq \sqrt{2} - 1, \\
\pm \frac{1}{2} \sqrt{\Delta \lambda_m^2 + 2 \Delta \lambda_m - 1}, & \left| \Delta \lambda_m \right| \geq \sqrt{2} - 1.
\end{cases}$$

(B7b)

For the counter-rotating branch, $\lambda_{m0}^{\text{cnt}}$ is

$$\begin{align*}
\left( \lambda_{m0}^{\text{cnt}} \right)_{\text{cnt}} &= -\text{sgn} \varphi \left[ \frac{-\Delta \lambda_m}{(\Delta \lambda_m + 1)^2}, 1 \leq \left| \Delta \lambda_m \right| \leq \sqrt{2} + 1, \\
\frac{1}{2 \Delta \lambda_m}, \left| \Delta \lambda_m \right| \geq \sqrt{2} + 1, \right]
\end{align*}$$

(B8a)

found at the frequencies $\tilde{\xi}_{lm}^{\text{cnt}}$.

$$\tilde{\xi}_{lm}^{\text{cnt}} = \begin{cases} 
0, & 1 \leq \left| \Delta \lambda_m \right| \leq \sqrt{2} + 1, \\
\pm \frac{1}{2} \sqrt{\Delta \lambda_m^2 - 2 \Delta \lambda_m - 1}, & \left| \Delta \lambda_m \right| \geq \sqrt{2} + 1.
\end{cases}$$

(B8b)

In Fig. 6 (rotating act.), we plot for $\phi_{\text{act}} = +\frac{1}{2} \pi$ both the corotating and counter-rotating branches ($\lambda_{m0}^{\text{cor}}$) and ($\lambda_{m0}^{\text{cnt}}$).

From Eqs. (B7a) and (B8a), we find that the largest value ($\lambda_{m0}^{\text{opt}}$ of ($\lambda_{m0}^{\text{cnt}}$)) is the corotating double mode found at the optimal mode separation $\Delta \lambda_m$, and the rescaled frequency $\tilde{\xi}_{lm}^{\text{opt}}$.

$$\tilde{\xi}_{lm}^{\text{opt}} = 0, \quad \Delta \lambda_m^{\text{opt}} = 0, \quad (\lambda_{m0}^{\text{cnt}})_{\text{opt}} = \text{sgn} \varphi. \quad (B9)$$

We see that in contrast to the nonrotating actuation (B5), the rotating actuation optimizes the acoustic rotation for zero-mode separation $\Delta \lambda_m = 0$.

**APPENDIX C: ROTATING ACTUATION G_{16} OF THE DOUBLE MODE 200 + 020**

In the simulation shown in Figs. 5(d)–5(f) of Sec. VI C 2 we excite the double mode 200 + 020 by using the rotating actuation

$$u_{1z} = u_{1z}^{0} G_{16}(x, y), \quad 0 < x < L_x, \quad 0 < y < L_y. \quad (C1a)$$

Here, $G_{16}(x, y)$ is constructed as the sum of 16 narrow Gaussians $G^0_{1}$ as follows. The primary narrow Gaussian $G^0_{1}(x, y)$ is centered around $(\frac{1}{2} L_x, \frac{1}{2} L_y)$ in the lower left corner of the domain:

$$G^0_{1}(x, y) = \exp \left[ -\left( \frac{x - \frac{L_x}{2}}{\frac{L_x}{2}} \right)^2 - \left( \frac{y - \frac{L_y}{2}}{\frac{L_y}{2}} \right)^2 \right]. \quad (C1b)$$

Four versions of $G^0_{1}$, centered at $(\frac{2k+1}{4} L_x, \frac{2k+1}{4} L_y)$ and multiplied by specific phase factors in order to create a positive rotation direction in the x-y plane, are added to form a quadrupole $G_{4}$ centered around $(\frac{1}{4} L_x, \frac{1}{4} L_y)$:

$$G_{4}(x, y) = e^{-i \pi} G^0_{1}(x, y) + e^{-i \pi} G^0_{1}(x, y - \frac{1}{2} L_y) \quad + e^{-i \pi} G^0_{1}(x, y) \quad + e^{-i \pi} G^0_{1}(x, y - \frac{1}{2} L_y)$$

(C1c)

Finally, by mirroring the quadrupole $G_{4}(x, y)$ across the center lines $x = \frac{1}{2} L_x$ and $y = \frac{1}{2} L_y$, $G_{16}$ is formed by adding the four resulting quadrupoles

$$G_{16}(x, y) = G_{4}(x, y) + G_{4}(L_x - x, y) + G_{4}(L_x - x, L_y - y) + G_{4}(L_y - x, L_y - y). \quad (C1d)$$

To obtain the complex pressure amplitudes $p_{100}$ and $p_{020}$ resulting from $G_{16}$, we first evaluate the integral in the denominator of Eq. (34), where $G_1$ is substituted by $G_{16}$:

$$\int_0^{L_x} \int_0^{L_y} G_{16}(x, y) \kappa_{200} dy dx L_x L_y = 0.0942 \times e^{0.250 \pi}, \quad (C2a)$$

$$\int_0^{L_x} \int_0^{L_y} G_{16}(x, y) \kappa_{020} dy dx L_x L_y = 0.0942 \times e^{-0.250 \pi}. \quad (C2b)$$

We then use Eq. (34) to compute the desired amplitudes

$$p_{100}^{200} = \frac{2 \mu_c}{L_c (\Gamma_{200}^{200} + \Gamma_{020}^{020})} 0.0942 \times e^{0.250 \pi}, \quad (C3a)$$

$$p_{020}^{200} = \frac{2 \mu_c}{L_c (\Gamma_{200}^{200} + \Gamma_{020}^{020})} 0.0942 \times e^{-0.250 \pi}. \quad (C3b)$$

We see that the actuation in Eq. (C1a) excites the two modes 200 and 020 with a phase difference of $\phi_{\text{act}} - \phi_{\text{act}}^{020} = 0.500 \pi$.

Finally, we use Eq. (C3) to choose the amplitude $u_{1z}^{0}$ of the actuation such that $|p_{100}^{200}| = |p_{020}^{200}| = 1$ MPa,

$$u_{1z}^{0} = L_c (\Gamma_{200}^{200} + \Gamma_{020}^{020}) \kappa_{200}^{020} 2 \times 0.0942 = 1.614 \text{ nm}, \quad (C5)$$

where we used the parameters for water given in Table II.

**APPENDIX D: APPROXIMATE SOLUTION FOR THE BULK-DRIVEN STREAMING FROM DOUBLE MODES 00 + 00**

In this Appendix, we calculate an approximate expression for the bulk-driven acoustic streaming velocity ($\{v_{100}^{00}, v_{00}^{00}, v_{020}^{00}\}_0$) driven by the acoustic body force $f_{\text{ch}} = f_{10}^{00} = f_{10}$ resulting from the combination of the two perpendicular, horizontal modes 00 and 00 in a square cavity. As in Eq. (24), the double subscript 0 refers to the double mode 00 + 00. The
bulk-driven streaming velocity \( v_{\text{blk}}^{(0)} \) satisfies an equation similar to Eq. (7) except for having no-slip at the boundaries:

\[
0 = \nabla \cdot \left( v_{\text{blk}}^{(0)} \right), \quad r \in \Omega, \tag{D1a}
\]

\[
\nabla \cdot \left( \rho_2^{(0)} \right) = \eta_2 \nabla^2 \left( v_{\text{blk}}^{(0)} \right) + \frac{\Gamma_{\text{res}}}{c_0^2} S_{\text{f0}}, \quad r \in \Omega, \tag{D1b}
\]

\[
\left( v_{\text{blk}}^{(0)} \right)_r = 0, \quad r \in \partial \Omega. \tag{D1c}
\]

Here, \( S_{\text{f0}} \) is defined in Eq. (18a) combined with Eq. (24) for \( F_{\text{r0}}(k_0) \), Eq. (17c) for \( K_{\text{r0}}(r) \), and Eq. (9a) for \( R_{\text{r0}}(r) \):

\[
S_{\text{f0}}(r) = \frac{1}{2} \left[ \frac{\rho_1^{(0)}}{\rho_2^{(0)}} \right]^{1/2} F_{\text{r0}}(k_0) K_{\text{r0}}(r), \tag{D2a}
\]

\[
F_{\text{r0}}(k_0) = \text{Re}\{i \mathcal{F}^{(0)}(k_0) \mathcal{F}^{(0)}(k_0) \}, \tag{D2b}
\]

\[
K_{\text{r0}}(r) = k_0^s \left[ \sin(k_0^s x) \cos(k_0^s y) - \cos(k_0^s x) \sin(k_0^s y) \right]. \tag{D2c}
\]

The impact on the bulk-driven streaming from the no-slip condition at the side walls will decrease exponentially with the length scale \( w \) away from the side walls, where \( w \) is the minimum shear length scale, i.e., either the vertical length scale \( L_z \) or the horizontal length scale \( (k_0^s)^{-1} \):

\[
w = \min\{L_z, (k_0^s)^{-1}\}. \tag{D3}
\]

For the narrow channel considered in Sec. VI where \( L_z/L^* = 0.1 \), we have \( w/L^* = 0.1 \) for \( l = 2 \), \( w/L^* = 0.08 \) for \( l = 4 \), and \( w/L^* = 0.05 \) for \( l = 6 \), and therefore the vertical side walls can be ignored in the majority of the cavity for these three double modes.

To compute the acoustic streaming velocity, we ignore the side walls and assume that \( v_{\text{blk}}^{(0)} \) is proportional to the \( xy \)-dependent acoustic energy flux density \( S_{\text{f0}}(x, y) \) and some \( z \)-dependent function \( \zeta(z) \) to be found:

\[
(v_{\text{blk}}^{(0)})_z = \kappa_2 \zeta(z) S_{\text{f0}}(x, y). \tag{D4}
\]

By using \( S_{\text{f0}} \) from Eq. (D2a), we find the three identities

\[
\nabla \cdot S_{\text{f0}} = 0, \tag{D5a}
\]

\[
\nabla^2 S_{\text{f0}} = -2(k_0^s)^2 S_{\text{f0}}, \tag{D5b}
\]

\[
\nabla \times S_{\text{f0}} = \frac{\left| \mathcal{F}^{(0)} \right|}{\rho_0 \omega} |\mathcal{F}^{(0)}| \mathcal{F}^{(0)} (k_0^s)^2 \sin(k_0^s x) \sin(k_0^s y) \mathbf{e}_z. \tag{D5c}
\]

Consequently, by Eq. (D5a), the ansatz (D4) satisfies the continuity equation (D1a). Moreover, inserting Eq. (D4) into Eq. (D1b) using \( \Gamma_{\text{res}} = (\frac{4}{3} + \frac{\eta_2}{\eta}) \eta_2 \kappa_2 \omega \) from Eqs. (4) and (D5b) we find

\[
\nabla \cdot (v_{\text{blk}}^{(0)}) = \eta_2 \kappa_2 \left[ \zeta^\prime(z) - \frac{1}{2} \left( \frac{4}{3} + \frac{\eta_2}{\eta} \right) \right] S_{\text{f0}}. \tag{D6}
\]

Since \( S_{\text{f0}}(x, y) \) has no \( z \) component, \( (v_{\text{blk}}^{(0)})_z \) on the left-hand side is independent of \( z \). Therefore, the square bracket on the right-hand side must be a constant. This constant must be zero since otherwise taking the curl of Eq. (D6) would lead to \( 0 = \nabla \times S_{\text{f0}} \) in conflict with Eq. (D5c). The condition of a vanishing square bracket in Eq. (D6) and the no-slip boundary conditions \( \zeta(0) = \zeta(L_z) = 0 \) lead to the following expression for \( \zeta(z) \):

\[
\zeta(z) = \frac{1}{2} \left( \frac{4}{3} + \frac{\eta_2}{\eta} \right) \left[ 1 - \frac{\cos \left( \sqrt{2k_0^s} (z - \frac{L_z}{2}) \right)}{\cos \left( \sqrt{2k_0^s} \frac{L_z}{2} \right)} \right]. \tag{D7}
\]

The solution for \( v_{\text{blk}}^{(0)} \) is then obtained by inserting Eqs. (D2) and (D7) in Eq. (D4).


[29] C. Eckart, Vortices and streamers caused by sound waves, Phys. Rev. 73, 68 (1948).


[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.100.023104 for extensions of the results of the following three figures: (1) Fig. 2 turned into one pdf file with 55 subfigures of the acoustic body force $f_{ac}$ of the double modes $lm0 + l'm'0$ with $l, m, l', m' = 0, 1, 2, 3, 4$ (the MATLAB source code is also provided); (2) Fig. 4 turned into five animated gif files of the pressure $p_i$ in 3D and as line plots of the double modes $200 + 020, 400 + 040$, and $600 + 060$ both for the wide $\frac{1}{2}L^*$-width and the narrow $\frac{1}{4}L^*$-width $G_1$ actuation; and (3) Fig. 8 recalculated using the narrow $\frac{1}{4}L^*$-width $G_1$ actuation.


