Experimental characterization of the sound field in a reverberation room

Nolan, Melanie

Publication date: 2019

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Experimental characterization of the sound field in a reverberation room

Mélanie Nolan

Ph.D. Thesis, Acoustic Technology group
Kgs. Lyngby, 2019
This thesis was submitted to the Technical University of Denmark (DTU) in partial fulfilment of the requirements for the degree of Doctor of Philosophy (Ph.D.) in Electronics and Communication. The work presented in this thesis was completed between August 1, 2015 and October 27, 2018 at Acoustic Technology, Department of Electrical Engineering, DTU, under the supervision of Associate Professor Jonas Brunskog and Associate Professor Cheol-Ho Jeong. The project was funded jointly by DTU Electrical Engineering and the Oticon Foundation under Grant No. 14-4015.
Reverberation chambers are an essential laboratory facility for measuring the statistical absorption coefficient of acoustic materials. The validity of the measurements, however, depends critically on the diffusion of the sound field in the test chamber. There is now clear evidence that reverberation-room measurements of absorption coefficients produce systematic errors, generally attributed to a lack of diffuseness of the sound field. In particular, the procedure fails to yield consistent absorption coefficients across chambers, due to differences in the sound fields established in each particular room. The way in which these sound fields depart from an ideal state of diffusion is not yet understood. In fact, the lack of experimental methods to analyze the sound field in reverberation rooms hinders our ability to explain the differences between each sound field, and how they deviate from a diffuse field.

This PhD study investigates sound field analysis techniques for the characterization of reverberation chambers. The directional properties of the stationary and decaying sound field in reverberation chambers are examined experimentally in two ways: (i) by analyzing the distribution of sound energy in the plane wave expansion of the sound field (i.e., in the wavenumber domain); and (ii) by analyzing the distribution of net intensity throughout space. It is shown that the directional properties of the reverberant sound field can be described based on an analysis of the wavenumber spectrum in the spherical harmonics domain, which leads to a characterization of the isotropy of the wave field. Further, a methodology is introduced that can extract the incident intensity field on the measuring sample, enabling to characterize the coupling between the absorptive material and the sound field above it. Additionally, the net flows of acoustic energy are analyzed, and structural differences identified between occupied and empty reverberation rooms, which are accentuated in the decay process.

It is the purpose of this dissertation to examine and discuss the relevant findings and review the contributions of the PhD study in relation to the existing body of knowledge.

**Keywords:** reverberation chambers; measurement of sound absorption; sound field diffusion; plane wave decomposition; intensity; microphone arrays.
Resumé

Efterklangsrum er en væsentlig laboratoriefacilitet til måling af akustiske materiales statistiske absorptionskoefficienter. Målingernes gyldighed er imidlertid kritisk afhængige af diffusionen af lydfeltet i testrummet. Der er nu tydelige beviser for, at efterklangsrumsmålinger af absorptionskoefficienter producerer systematiske fejl, som generelt skyldes, at lydfeltet ikke er tilstrækkeligt diffust. I særdeleshed formår proceduren ikke at give overensstemmende absorptionskoefficienter på tværs af rum på grund af forskelle i lydfelterne, der er etableret i hvert enkelt rum. Den måde, hvor på disse lydfelter afviger fra et ideelt diffust lydfelt, er endnu ikke forstået. Faktisk hindrer manglen på eksperimentelle metoder til at analysere lydfeltet i efterklangssrum vores mulighed for at forklare forskellene mellem hvert lydfelt og hvordan de afviger fra et diffust felt.

Dette ph.d.-studie undersøger teknikker til lydfeltanalyse til karakterisering af efterklangsrum. Retningsegenskaberne for det stationære og det aftagende lydfelt i efterklangsrum undersøges eksperimentelt på to måder: (i) ved at analysere fordelingen af lydenergi i en dekomposition af lydfeltet i plane bølger (det vil sige i bølgetal-domænet); og (ii) ved at analysere fordelingen af aktiv intensitet gennem rummet. Det er vist, at retningsegenskaberne for efterklangsdel af lydfelt kan beskrives ud fra en analyse af bølgetal-spektret i det sfæriske harmoniske domæne, hvilket fører til en karakterisering af isotropien af bølgefeltet. Derudover introduceres en metode, som kan udtrække feltet af indkommende intensitet på måleprøven, hvilket muliggør en karakterisering af koblingen mellem det absorberende materiale og lydfeltet over det. Derudover analyseres strømningen af aktiv akustisk energi, og strukturer forskelle identificeres mellem tomme efterklangsrum og efterklangsrum med en måleprøve, der forstærkes i henfaldsprocessen.

Formålet med denne afhandling er at undersøge og diskutere de relevante resultater og gennemgå ph.d.-studiets bidrag i forhold til den eksisterende viden.

Nøgleord: Efterklangsrum; måling af lydabsorption; lydfelters diffusion; dekomposition i plane bølger; intensitet; mikrofon-arrays
List of publications

The papers included in this dissertation consist of four published journal papers, and a conference proceeding.


In the course of the Ph.D., the following conference contributions were also published. These are not included in the dissertation,

- Due to content overlap with the included publications:
• Because they are relevant, but not essential to the dissertation (note that the PhD Candidate is not the first author of these publications):


List of symbols

The meaning of the symbols encountered in the synopsis is listed below, unless otherwise defined in the running text.

\( a \) Radius of a sphere (in [m])
\( c \) Speed of sound in air (in [m/s])
\( H \) Hermitian operator; i.e., conjugate transpose
\( H \) Sensing matrix
\( h_n \) Spherical Hankel function of the second kind of order \( n \)
\( I \) Identity matrix
\( j \) Imaginary unit, \( j = (-1)^{1/2} \)
\( j_n \) Spherical Bessel function of the first kind of order \( n \)
\( k \) Wavenumber in air, \( k = \omega/c \) (in [m\(^{-1}\)])
\( k \) Wavenumber vector
\( k_x, k_y, k_z \) Wavenumber components in Cartesian coordinates
\( P \) Sound pressure (in [Pa])
\( P \) Sound pressure vector
\( P_n \) Wavenumber spectrum of the sound pressure
\( r \) Position vector
\( r_r \) Measurement position vector
\( r, \theta, \phi \) Spherical coordinates (radius, polar angle, azimuth)
\( t \) Time (in [s])
\( x \) Vector containing the wavenumber coefficients
\( \hat{x} \) Estimate for vector \( x \)
\( x, y, z \) Cartesian coordinates
\( y_n \) Spherical Bessel function of the second kind of order \( n \)
\( Y_n \) Spherical harmonic of order \( n \) and degree \( m \)
\( \alpha \) Absorption coefficient
\( \delta \) Dirac delta function
\( \delta_{ij} \) Kronecker delta
\( \lambda, \mu \) Regularization parameter
\( \varphi \) Phase of a complex amplitude
\( \omega \) Angular frequency (in [rad/s])
\( \|x\|_1 \) \( \ell_1 \)-pseudonorm of a vector \( x \)
\( \|x\|_2 \) \( \ell_2 \)-pseudonorm of a vector \( x \)
Fig. 1: Round Robin test on absorption (ISO 354:2003) conducted on 10.8 m² glass wool (thickness 100 mm) in 7 European laboratories (adapted from Nolan et al., 2014).

Fig. 2: $k$-space diagram. The direction of the plane wave with components $(k_x, k_y, k_z)$ is shown by the vector $\mathbf{k}$, with the spherical angles $\theta_0$ and $\phi_0$. The radius of the sphere is $k$.

Fig. 3: Plane-wave decomposition on a spherical microphone array (2D view). The recorded sound pressure at the array $p$ is a superposition of plane waves from definite directions $\mathbf{k}$, weighted by the complex amplitudes (or wavenumber spectrum) $x$.

Fig. 4: Wavenumber spectra and corresponding spherical harmonics decompositions in the case where (i) the sound field is modeled as four waves propagating in opposite directions; (ii) the sound field is fairly isotropic. Truncation order $N = 7$. The first five orders of spherical harmonics are also displayed (top), with black representing positive values, and white representing negative values.

Fig. 5: Wavenumber spectra due to a single plane wave resulting from the LS (left) and CS (right) solutions (adapted from Fig. 13 in Paper A).
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**Paper A** A wavenumber approach to quantifying the isotropy of the sound field in reverberant spaces
**Paper B** A wavenumber approach to quantifying non-uniform sound incidence in measurements of sound absorption in the reverberation chamber

**Paper C** Experimental characterization of the sound field in a reverberation room

**Paper D** Volumetric reconstruction of acoustic energy flows in a reverberation room

**Paper E** Two definitions of the inner product of modes and their use in calculating non-diffuse reverberant sound fields
1. Introduction

Sound absorption is of decisive influence on the sound field in a room. In room acoustic design, materials are often selected according to their sound absorption properties (i.e., their absorption coefficient) in order to achieve a desired reverberation time, suppress unwanted sound reflections, or reduce the sound pressure level in noisy environments. Knowledge of the acoustic absorption of building materials is therefore indispensable for many tasks related to room acoustical design, i.e., for predicting the reverberation times in the planning phase of ordinary rooms and performance spaces, or for acoustical computer simulations of these environments.

There are two standardized laboratory procedures for measuring the absorption coefficient of a material. In one of them, the test field is a plane wave enclosed in a rigid tube (ISO 10534-2:1998), while the other one requires a reverberation chamber in which a diffuse sound field should be established (ISO 354:2003). A detailed account of these methods is given in (Kuttruff, 2009a). The tube method is restricted to the examination of locally reacting materials with a plane or nearly plane surface, and to normal wave incidence onto the test specimen, far from realistic operational conditions. On the other hand, the reverberation chamber method is in principle well suited for measuring the absorption coefficient of almost any type of wall linings and ceilings. Besides, the measurement is performed under conditions that are much more realistic than those encountered in a one-dimensional waveguide. The test procedure is fairly simple and is therefore the preferred method for the reasons listed above.

In reverberation-room measurements of sound absorption coefficients, a large sample of the material under test is placed in a reverberation room, and the absorption coefficient \( \alpha \) is deduced from the resulting reduction of the reverberation time, once the absorber has been introduced in the room (ISO 354:2003):

\[
\alpha = \frac{55.3V}{cS} \left( \frac{1}{T_S} - \frac{1}{T} \right),
\]

where \( T \) is the reverberation time (in [s]) of the empty reverberation chamber, \( T_S \) is the reverberation time (in [s]) after the test specimen has been introduced, \( S \) is the area (in [m\(^2\)]) of the specimen, \( V \) is the volume (in [m\(^3\)]) of the empty reverberation chamber, and \( c \) is the propagation speed of sound in air (in [m.s\(^{-1}\)]). The procedure is based on the simple reverberation formulae (Sabine, 1922; Morse, 1968), provided that certain assumptions are verified experimentally. In particular, the procedure relies on the underlying notion of a perfectly diffuse sound field. Such sound field can be defined as
the combination of plane waves with random phases coming from all directions. In a perfectly diffuse sound field, the time-averaged energy flux is zero at each point and the average energy density is the same everywhere; that is, the sound field is homogeneous and isotropic (Morse, 1968). The realization of a completely diffuse sound field in any finite space is difficult, if not impossible. Consequently, the reverberation formulae produce systematic errors due to deviations from the diffuse-field assumption in the test chambers. There is in fact considerable uncertainty regarding the reliability and accuracy of the reverberation chamber method. Several round robin tests (Kosten, 1960; Makita et al., 1968; Halliwell, 1983; Nolan et al., 2014), in which the same sample of an absorbing material has been tested in a number of laboratories, have revealed a significant disagreement in the results. This can be attributed to the different sound fields (and thus, to the different degrees of diffusion), established in the various chambers. Figure 1 shows the results of such systematic comparison, where the sound absorption of a given area (10.8 m²) of glass wool has been determined in 7 European laboratories (6 of which are standardized laboratories).

Fig. 1: Round Robin test on absorption (ISO 354:2003) conducted on 10.8 m² glass wool (thickness 100 mm) in 7 European laboratories (adapted from Nolan et al., 2014).

Although the reverberation theory has been subjected to criticism since the early 30’s (Knudsen, 1932; Bolt, 1939), questioning the validity of the reverberation chamber procedure seems too controversial for the standardization bodies. Instead, considerable experimental effort has been spent on methods for promoting a diffuse sound field. Addition of a number of randomly oriented scattering panels or boundary diffusers was expected to build up diffusion and bring reasonable agreement between laboratories (Balachandran et al., 1967; Kuttruff, 2009b). Yet, there is no evidence that installation of such scattering objects results in increased diffusion (Jacobsen, 1979), especially in the presence of non-uniform absorption.
Besides, the question of measuring the degree of diffuseness in reverberant sound fields has been widely investigated, and numerous methods have been proposed (see Section 1.1). Such measurement may be useful for monitoring the sound field in a given reverberation room, examining the diffusing capabilities of various types of diffusing elements (Balachandran et al., 1967; Kuttruff, 1981; Bradley et al., 2014; Nolan et al., 2015), or correcting the absorption coefficients measured in a room lacking diffusion for the case of ideal diffusion (Lautenbach et al., 2013; Nolan et al., 2014).

Yet, the primary problem is to work out an appropriate model of the diffuse sound field in a reverberation chamber. Several quite different models have been described in the literature. Two models, referred to as the perfectly diffuse sound field model and the pure-tone diffuse field model respectively, are described in the following.‡ A detailed account of these models is given in (Jacobsen et al., 2000) and (Jacobsen et al., 2013), and only the main results will be summarized here. Models based on the theory of diffuse reflections and the diffusion equation theory, as well as models making use of the concepts of entropy or mixing, are not dealt with in this report. For further details on these models the reader is referred to (Kuttruff, 2009b), (Picaut et al., 1997), (Lyon, 1974) and (Polack, 1992), respectively.

The perfectly diffuse field model (also sometimes referred to as model of geometrical acoustics [Morse, 1968; Jacobsen, 1979]) is a fairly simple model that describes the perfectly diffuse sound field as a superposition of incoherent plane waves arriving from all directions with equal probability and random phase. The model is based on the assumption that the interference between the various waves can be ignored (the waves are incoherent), which leads to the concept of a completely homogeneous and isotropic sound field. In such a sound field, the energy density is the same at all positions, temporal correlation functions between linear quantities measured at two points depend only on the distance between the two points (an expression for the spatial correlation of the sound pressure amplitude was derived in 1955 by Cook et al. [Cook et al., 1955]), and the net (time-averaged) sound intensity is zero at all positions. An approximation to the perfectly diffuse sound field can be generated by a number of loudspeakers driven with uncorrelated noise in a large anechoic room, as in the experimental investigations described in (Bodlund, 1976) and (Nolan et al., 2018). The attractiveness of the perfectly diffuse sound field model lies in the fact that combined with energy balance considerations, the model leads to useful relations between the sound power emitted by a source of noise in the room, the resulting sound pressure level, and the total absorption of the room.

In reverberation-room measurements of sound absorption, all the sound waves in the room are generated by the same source, and therefore the various waves interfere (due to reflections). This is in obvious disagreement with the perfectly diffuse field model, which assumes that the sound waves that constitute the sound field are uncorrelated. Waterhouse (Waterhouse, 1955) extended the perfectly diffuse sound field model to take account of the interference phenomena that occur near the walls of the room, under

‡ Note: these two models are in fact closely related, and often referred to as the same model in the literature, under the designation random wave theory or statistical theory. Yet, the models are based on very different assumptions: the perfectly diffuse sound field assumes that the waves that compose the sound field are incoherent, while they are coherent in the pure-tone diffuse field model. Therefore, we treat these models as two separate models (as in [Jacobsen et al., 2000]).
the assumption that each incident wave is fully coherent with the corresponding reflected wave. In 1968, he further developed a stochastic pure-tone diffuse-field interference model (Waterhouse, 1968). This model is closely related to the perfectly diffuse field model but describes the sound field as composed of coherent plane waves with random phases arriving from all directions; that is, the sound pressure at a given position \( \mathbf{r} \) can be written as

\[
p(\mathbf{r}) = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |A_n| \exp\left(\omega t + \varphi_n - \mathbf{k}_n \cdot \mathbf{r}\right),
\]

where \( A_n \) is the complex amplitude of the \( n \)’th wave, \( \mathbf{k}_n \) is its wavenumber vector, and each set of random amplitudes and wavenumber vectors corresponds to an outcome of a stochastic process.\(^\dagger\) The phase angles \( \varphi_n \) are uniformly distributed between 0 and \( 2\pi \), and the wavenumber vectors \( \mathbf{k}_n \) are uniformly distributed over all angles of incidence (corresponding to a sinusoidal distribution of the polar angles and a uniform distribution of the azimuth angles). This is a pure-tone model, and therefore the various waves interfere in the entire sound field (not just at the boundaries). As such, the pure-tone diffuse field model is a more realistic model of the sound field in a reverberation chamber driven with a narrow-band signal. Since 1968, the model has been extended considerably: Lubman (Lubman, 1968) examined the spatial variance of the mean square pressure, Schroeder (Schroeder, 1969) studied the effect of frequency averaging, and Jacobsen (Jacobsen, 1979) and Pierce (Pierce, 1981) derived independently the statistics of the power output of a monopole point source. The statistical properties of the particle velocity (Kuno et al., 1974; Jacobsen, 1979) and complex intensity (Jacobsen, 1989; Jacobsen, 1990; Jacobsen, 1991) have also been investigated.

The pure-tone diffuse field model (and, more generally, the statistical theory) is regarded as valid when the modal overlap of the room is high (typically above the Schroeder frequency [Schroeder, 1954]), and above the Schroeder frequency there is no difference between the statistics with respect to position and the full ensemble statistics. At low modal overlap, the source that generates the sound field can no longer be assumed to emit its free-field sound power. Jacobsen et al. (Jacobsen et al., 2009; Jacobsen et al., 2010a; Jacobsen et al., 2010b; Jacobsen et al., 2011) extended the model to lower frequencies by taking into account the variations of the sound power of the source. These random variations modify the ensemble average of all quantities in the sound field. Besides, spatial statistics is no longer the same as ensemble statistics, and it matters whether the frequency coincides with a modal frequency or not (Jacobsen et al., 2013).

Because of the simplicity of the mathematics involved, and because it does not require restrictive assumptions to be utilized analytically, the pure-tone diffuse field model can be very useful, as long as it is recognized that all results should be interpreted as average values (Jacobsen, 1979). Such description of the sound field involves determining statistical properties with respect to an ensemble of reverberation chambers and does not explain the properties of a specific sound field. The ability to analyze the sound field in a specific room is key to understanding the deviations encountered across

\(^\dagger\) When averaging over an ensemble of rooms, the perfectly diffuse sound field model described above is obtained.
standardized laboratories, as well as diagnosing biases in specific laboratory configurations.

An alternative model relies on the mathematical analysis of the wave equation describing the sound field produced by a source in a rectangular room and leads to a general solution (the transmission function) expressed in terms of the normal modes of the room (Morse et al., 1944). Such mode model is deterministic in nature, and generally regarded as valid also outside the limits of applicability of the statistical theory. Besides, it is possible to average the transmission function over an ensemble of rooms with random distributions of the modal frequencies, to derive expressions in closed form for the statistical properties of the sound field in a rectangular room. Such statistical modal theory is essentially due to Lyon (Lyon, 1969), Davy (Davy, 1981; Davy, 2009) and Weaver (Weaver, 1989). Although the mode model is mathematically more complicated than the statistical theory and requires more information about the room, it is probably the only existing analytical model adequate as a description of the sound field in highly damped reverberant rooms. Yet, while the modal theory of the reverberant sound field has been studied extensively for rectangular, parallelepiped rooms with nearly hard walls or walls with small, real and uniform admittance (Morse et al., 1944; Kuttruff, 2009c), rooms with absorption concentrated on one of their boundaries have not received similar attention.

1.1. Assessment of diffusion: state-of-the-art

Considerable experimental effort has been spent on the problem of evaluating diffusion in reverberant sound fields, and numerous methods have been proposed. An account of various descriptors is presented by Schultz (Schultz, 1971), Abdou et al. (Abdou et al., 1994), and more recently by Jeong et al. (Jeong et al., 2018). The following brief review is not intended to be complete.

The general approach found in the existing literature has been to quantify the departure from the state of perfect diffusion. The methods can be divided into three main categories. One approach consists in measuring the acoustic energy at various points across the sound field (Wente, 1935; Nélisse et al., 1997; Lautenbach et al., 2013). The core idea behind this approach is that the acoustic energy should be constant across space in a diffuse sound field. A second approach consists in measuring the acoustic intensity over time, exploiting the fact that in a perfectly diffuse sound field, the average sound intensity is null (Ahonen et al., 2009; Del Galdo et al., 2012; Pulkki et al., 2017). A third method relies on measures based on the cross-correlation between pressures at neighboring positions (Balachandran et al., 1967; Koyasu et al., 1971; Bodlund, 1976), the idea being that, in a diffuse sound field (either perfectly diffuse or “pure-tone diffuse” sound field), the (averaged) cross-correlation function between two omnidirectional microphones follows a sinc function pattern (Cook et al., 1955; Jacobsen et al., 2000). In (Jacobsen et al., 2000), Jacobsen and Roisin presented a method of determining spatial correlation functions in a room, suitable to other quantities than the sound pressure. Noteworthy and perhaps overlooked is the work by Ebeling (Ebeling, 1984), who interpreted the cross-correlation function derived by

In a pure-tone diffuse sound field, the ensemble average (over an ensemble of rooms) of the cross-correlation function follows a sinc pattern.
Cook et al. (Cook et al., 1955) in the spatial frequency domain. Subsequently, he proposed a multipole expansion of the spatial correlation function leading to a measure for spatial diffusivity. Still, other methods exist that are based on measurements of absorption, frequency irregularity, uniformity of decay rate, linearity of decay curves, and higher-order statistics.

The above-mentioned measures can reveal the suitability of diffuse-field assumptions but cannot explain the nature of any deviations from such. Besides, methods based on a comparison with the perfectly diffuse sound field model are incompatible with a realistic description of the sound field in a reverberation room.

An alternative approach consists in measuring the directional distribution of sound energy, the idea being that, in a perfectly diffuse sound field, an equal amount of energy is observed for every direction (Thiele, 1953; Meyer et al., 1956; Schroeder, 1959; Ebeling, 1984; Gover et al., 2002; Gover et al., 2004; Nolan et al., 2016; Berzborn et al., 2018). A more detailed description of these methods will follow later in Section 2.1.

Technical developments in sensing methods provide new possibilities for analyzing sound field diffusion in rooms and, in recent years, different methods have been proposed for the estimation of diffuseness from measurements with an array of microphones. In particular, spherical microphone arrays can provide a panoramic view (4π solid-angle) of the sound field and are therefore particularly well suited for applications where sound waves impinge on the array from multiple directions. Several quite different methods have been proposed for the estimation of diffuseness based on spherical array processing. Epain et al. (Epain et al., 2016) review several methods, meant to quantify the spatial or directional fluctuations of some parameter (Gover et al., 2002; Götz et al., 2015; Pulkki et al., 2017). An alternative approach is suggested in (Epain et al., 2016) and consists in characterizing diffuseness based on the analysis of the spherical harmonic covariance matrix. However, the study is concerned with the estimation of diffuseness arising from the presence of multiple sources. As discussed above, the sound field in a reverberation room driven with noise from a single source is fundamentally different.

1.2. Scope of the thesis

This PhD study aims at characterizing experimentally the sound field in a reverberation chamber used for conventional measurements of sound absorption. The study makes use of microphone array processing techniques to examine the directional properties of the steady and decaying states and discusses the observed behavior against theoretical predictions that follow from assuming a diffuse sound field. The directional properties of the sound field are examined experimentally in two ways: by considering the distribution of sound energy in the plane wave expansion of the sound field (Papers A to C), and by analyzing the distribution of net acoustic intensity (that is, the directions of net energy transport – Papers D and E). The focus of the present work is on experimental techniques for sound field analysis. The question of how the diffusion in a reverberation chamber can be improved is not investigated.

The conception that a given pressure field can be expressed uniquely as a superposition of elementary waves is central to the present investigation. Especially, the use of plane waves as elementary waves is particularly convenient. Such plane wave decomposition
consists in applying a three-dimensional Fourier transform to a measured pressure field (with an array of microphones) to estimate the wavenumber spectrum (Williams, 1999), which is a representation of the sound field in the spatial-frequency domain or k-space (an introduction to k-space is given in Chapter 3).

We propose to examine the directional properties of the stationary sound field in a reverberation chamber, based on an analysis of the wavenumber spectrum in the spherical harmonics domain, and show that the moments from this spherical harmonic expansion can be used to characterize the isotropy of the wave field (Paper A). Based on the wavenumber representation, a method is further introduced to reconstruct the intensity field incident on the measuring sample over a three-dimensional domain in its vicinity. Such reconstruction is used to characterize the directional properties of the sound field incident on the absorbing sample, as well as the coupling between the absorptive material and the sound field above it (Papers B and C).

We then examine the evolution with time of the spatial properties of net energy flow. Successive volumetric reconstructions of the net flow of energy are obtained, so as to capture the three main stages of the decay process: steady state, early decay and late decay (Paper D). The experimental framework is based on a spherical equivalent method (Fernandez-Grande, 2016), which consists of modeling the sound field on a rigid spherical array as the superposition of spherical waves radiated by a combination of monopoles.

Finally, an energy-flow analysis is conducted by means of a numerical model in the steady state (Paper E). The existing modal theory for the reverberant sound field in rectangular rooms with nearly hard walls is extended to the case of rectangular rooms with arbitrary conditions. Essential relations are rederived, and some new results are presented. The orthogonality of the modes is investigated, and it is shown that the modes are not necessarily orthogonal with the usual definition of inner product.

1.3. Thesis structure

This PhD thesis is based on a collection of scientific articles (as recommended by the DTU PhD guidelines), published in peer-reviewed journals and international conference proceedings. The remainder of the thesis develops as follows. Chapter 2 discusses the relevant findings and reviews the contributions of the PhD study to the existing literature. Although the main analytical results are presented in the papers, Chapter 3 introduces background concepts and theory relevant to the study. In particular, we review the formulation of plane-wave sound fields in the spherical harmonics domain, as well as some basic properties of the spherical Fourier transform. In addition, the regularization of ill-posed discrete inverse problems is briefly discussed. The reader familiar with these concepts may skip Chapter 3. Chapter 4 summarizes the main conclusions. Chapter 5 suggests some areas of future work. Six papers are included in the thesis (Papers A-E).
2. Contributions

This chapter reviews the scientific contributions of the conducted research in relation to the existing literature and examines the relevant findings. Section 2.1 provides a literature review where the publications by the author are included. A summary of the papers is provided in Section 2.2.

2.1 Measuring the directional properties of reverberant sound fields: state-of-the-art

Considerable experimental effort has been spent on the problem of measuring the three-dimensional characteristics of reverberant sound fields. In particular, a variety of methods for the measurement of sound field diffusion have been proposed, where diffusion is described on the basis of the angular distribution of sound energy.

One of the earliest attempts to measuring the directional properties of reverberant sound fields is due to Thiele and Meyer (Thiele, 1953; Meyer et al., 1956), who captured the angular distribution of arriving acoustic energy in concert halls, using a concave mirror coupled with a directional microphone. They presented the data in the form of directional “sound hedgehogs”, showing the directions of arrival and amplitudes. Based on such directional distributions, a measure for the isotropy of the sound field, known as directional diffusion (Thiele, 1953), was also derived.

Yamasaki et al. (Yamasaki et al., 1989) developed a more manageable four-microphone probe, with the aim of locating image sources. From measurements of the room impulse response at four closely located microphones, the directions of arriving reflections were derived and assigned to virtual image sources. The technique is however restricted to the early part of the impulse response, which consists of few well-separated reflections. As the density of reflections increases in the reverberant part of the response, individual image sources can no longer be identified.

Over the past few decades, advances in microphone array processing techniques have facilitated the three-dimensional analysis of sound fields. Given the complexity of the sound field in a reverberation room, microphone arrays are a powerful analysis tool. The idea of using an array of microphones to measure the diffusion of reverberation chambers goes back to 1959, when Schroeder (Schroeder, 1959) suggested creating a microphone array by repeated reflections of a planar array at the sidewalls of a reverberation chamber. His paper focuses on the theoretical foundations of the experimental method, and experimental results are not reported.
Rather than an array dependent on the room symmetry, Ebeling (Ebeling, 1984) sampled the sound field in a reverberation chamber at every point of a three-dimensional cubical grid of 16 x 16 x 16 points, with the goal of measuring the cross-correlation functions (Cook et al., 1955) in the spatial frequency domain. He proposed a multipole expansion of the correlation function to measure spatial diffusivity. Measurements in a reverberation room with diffusers indicate that the spatial diffusion does not exceed 70% in the steady state and is negatively influenced by the addition of absorbers.

More recently, Gover et al. (Gover et al., 2002; Gover et al., 2004) have investigated how spherical microphone arrays can be used to characterize directional properties of reverberant sound fields. Instead of manually rotating a directional microphone in many directions as in (Meyer et al., 1956), Gover et al. adapted the method described in (Meyer et al., 1956) to spherical microphone arrays, and measured the directional distribution of acoustic energy by steering directional beams in every direction. Measurements in a rectangular reverberation chamber with diffusers were conducted, where the directional diffusion defined by Thiele (Thiele, 1953) was computed over the entire impulse response. A value of 91% was reported indicating a nearly isotropic sound field. This result is somewhat surprising, as one would not expect such high degree of isotropy in a reverberation chamber driven with a single source (for comparison, the experimental investigation in Paper A shows that even in the case of a sound field generated by 52 loudspeakers driven with random white noise with equal power, isotropy does not exceed 92%). Following a very similar approach, Berzborn et al. (Berzborn et al., 2018) recently introduced the concept of Directional Energy Decay Curves (DEDC) derived from measurements of directional impulse responses using a spherical microphone array.

Epain et al. (Epain et al., 2016) proposed an alternative method for the characterization of diffuseness using spherical microphone arrays, based on the analysis of the spherical harmonic signal covariance matrix. Their work deals with diffusion arising from multiple sources. The sound field in a reverberation room driven with noise from one source is yet, quite different (the waves are coherent).

**Paper A** proposes an experimental method for evaluating isotropy in enclosures, based on an analysis of the wavenumber spectrum (Goodman, 1968) in the spherical harmonics domain. Measurements with a spherical microphone array in a reverberation chamber confirm that the stationary sound field is not isotropic, in agreement with the experimental investigation in (Ebeling, 1984).

Surprisingly, in the literature dealing with the relation between absorption coefficients and directional properties of the sound field in reverberation rooms, no experimental investigation of the distribution of incident acoustic energy on the measuring sample has been reported. Yet, when measuring the statistical absorption coefficient of acoustic materials, a uniform distribution over all directions of incidence is assumed at each moment of the decay process. This seems to be a critical element worthy of examination.

An analytical analysis in terms of normal modes of vibration (wave theory) is found in (Hunt, 1939; Bolt, 1939; Maa, 1939; Hunt et al., 1939) in the case of rectangular rooms with no scattering objects. The results demonstrate that the sample is not exposed to isotropic sound incidence, as each mode will have distinctive attenuation characteristics depending on its orientation with respect to the absorbing surface. A bend in the measured decay curves is reported, corresponding to different decay rates
for waves travelling almost parallel to the absorbing sample (grazing waves), and waves having oblique incidence (non-grazing waves). A related analysis based on the Statistical Energy Analysis (SEA) formalism is found in [Nilsson, 2004a] in the case of rectangular rooms with high absorption at one surface and low absorption at the remaining surfaces.

Diffusing elements in the form of randomly oriented reflecting panels have been found necessary to divert some of the energy in the grazing sound field into the non-grazing sound field, and thereby promote an isotropic sound incidence on the measuring sample. Several authors (Balachandran et al., 1967; Koyasu et al., 1971) have examined the effectiveness of this procedure by measuring the diffuseness of the sound field above an absorbing sample employing the correlation methods described in Chapter 1. Others have studied the effect of scattering elements on the sound decays (Kuttruff, 1981; Nilsson, 2004b; Balint et al., 2018). To date, no experimental method allowing for direct observation of the distribution of sound incidence on the measuring sample is available. As far as the author is aware, the matter has only been addressed numerically, using ray/beam-tracing techniques (Jeong, 2010). Jeong (Jeong, 2010) reviews several of these methods.

As a matter of fact, the incident sound energy on a wall in a reverberation room has rarely been measured, mostly due to a lack of experimental methods to do so. The incident sound intensity cannot be measured using traditional intensity measurement systems, since such measurement gives the net sound intensity (i.e. incident plus reflected). To the author’s knowledge, the only study that reports the measurement of this quantity is due to Jacobsen et al. (Jacobsen et al., 2010). They measured the sound intensity incident on a wall in a reverberation room from pressure measurements at distributed positions on the wall, using Statistically-Optimized Near-Field Acoustic Holography (SONAH) [Hald, 2009].

To cover this knowledge gap, Papers B and C propose an experimental method for characterizing the distribution of sound incidence on the measuring sample. The method relies on a plane wave decomposition (i.e., estimation of the wavenumber spectrum) to determine the magnitude of the sound waves arriving from definite directions onto the absorbing sample. Measurements in a reverberation chamber with diffusers confirm that the distribution of incident acoustic energy on the sample is not uniform in the steady state. Based on the wave components that describe the incident sound field on the sample, the incident intensity field can also be reconstructed over a three-dimensional domain in the vicinity of the absorber, making it possible to visualize and characterize the incident energy flows.

Finally, another relevant path of study addresses the directional measurement problem by analyzing the magnitude and direction of the intensity vector over time. As rightly remarked by Gover et al. (Gover et al., 2002), the intensity vector corresponds to the net flow, and the direction of net energy flow is not always in the direction of a wave front arrival. In other words, when sound waves arrive simultaneously from more than one direction, the sound intensity vector cannot indicate the directions of arrival. Nonetheless, the ability to measure the directions of net energy flux in a reverberation chamber can be key in understanding the acoustic processes therein.

Several studies have used sound intensity measurements from microphone pairs arranged in a variety of ways to calculate three-dimensional intensity vectors over time. Such measurements were conducted in (Yamasaki et al., 1989), (Abdou et al., 1993),
and (Merimaa et al., 2001) for various acoustical spaces. As briefly mentioned in Chapter 1, other studies have measured the acoustic intensity over time with the aim of evaluating sound field diffusion (Kuttruff, 2009d; Ahonen et al., 2009; Del Galdo et al., 2012; Götz et al., 2015; Pulkki et al., 2017). The core idea behind this approach is that, in a perfectly diffuse sound field, the time-averaged intensity is null. Diffuseness is therefore interpreted as the temporal variation of sound intensity and its measure relies on estimating the ratio of active (or propagating) sound intensity with the energy density (Jacobsen, 1989; Schiffrer et al., 1994). Following a quite different approach, Prodi (Prodi, 2018) recently proposed to analyze the energy oscillations due to instantaneous reactive intensity (Stanzial et al., 1996) to examine sound field diffusion in reverberation chambers.

Paper D proposes a volumetric reconstruction of the net energy flow in a reverberation chamber based on an equivalent source model and measurements with a spherical microphone array. Successive reconstructions are conducted to examine the evolution of the intensity vector with time. The results reveal significant structural differences between the energy flows in the steady state and during the decay, and confirm that the time-averaged intensity is non-zero in a reverberation chamber, in clear disagreement with the theoretical predictions that follow from assuming an ideally diffuse sound field. Paper E provides a numerical model (based on the modal theory) for the analysis of sound pressure and energy flow distributions in box-shaped rooms with arbitrary boundary conditions in the steady state.

2.2 Summary of contents and contributions: Papers A-E

Paper A

Paper A proposes an experimental method for evaluating sound field isotropy in steady state, based on an analysis of the wavenumber spectrum (Williams, 1999) in the spherical harmonics domain.

The wavenumber spectrum (or angular spectrum), which results from expanding a measured sound field into a plane-wave basis [see Section 3.1], is used to characterize the magnitudes of the sound waves arriving from definite directions at the observation region. We propose an analysis of the wavenumber spectrum in the spherical harmonics domain, which has suitable mathematical properties when it comes to examining isotropy (Cox, 1973; Ebeling, 1984; Angus, 1998; Baldi et al., 2007; Müller-Trapet et al., 2011). We define the relative monopole moment (zeroth-order term of the spherical harmonics expansion) as the degree of sound field isotropy (isotropy indicator), the underlying hypothesis being that in a perfectly isotropic sound field, the wavenumber spectrum is rotationally symmetric. The measure ranges from 0 to 1 (1 in the case where the flow of energy is equal in all directions; 0 if the incident waves propagate in a single direction), and its definition is based on fundamental properties of the spherical Fourier transform (see Section 3.3). The values of the indicator obtained experimentally are shown to vary from 0% (single loudspeaker in an anechoic chamber) to 92% (52-channel loudspeaker array set in an anechoic chamber), showing that the indicator responds adequately to changes in sound field isotropy. Measurements in a standardized (ISO 354:2003) reverberation room with diffusers indicate that isotropy does not exceed 72%, and that addition of an absorbing sample influences negatively the isotropy of the wave field. These results agree well with the experimental investigation
The robustness to noise is examined numerically; the method is fairly robust to perturbations for SNRs as low as 20 dB. The method described in Paper A does not require a specific array configuration, and the pressure field can be sampled randomly over an arbitrary volume to estimate the wavenumber spectrum. The analysis in the spherical harmonics domain is performed on the wavenumber spectrum, which is defined over a sphere, and not on the recorded sound pressure directly. Therefore, the measured sound pressure does not need to be defined on a spherical surface [as in (Gover et al., 2002)]. Yet, the experimental investigations in Paper A do make use of a rigid sphere of microphones, as such array was readily available. The equations for the plane-wave decomposition on a rigid sphere, not derived in Paper A, are presented in Section 3.2.

Finally, the accuracy of the method is shown to depend heavily on the choice of the regularization scheme. In Paper A, the wavenumber spectrum is calculated via a conventional regularized least-squares inversion. Such inversion promotes smooth estimates (see Section 3.4) and is therefore a sensible choice for estimating the sound field in a reverberant room, where sound waves impinge on the array from multiple directions. Yet, the $\ell_2$–minimization may not be the best regularization choice for non-reverberant sound fields. In Paper A, we show that an alternate estimation of the wavenumber spectrum based on the framework provided by Compressive Sensing (Elad, 2010; Foucart et al., 2013) is better suited to cases where few sound waves impinge on the array. In fact, an advantage of the proposed method is that, because of being formulated as an elementary wave model, the wavenumber spectrum allows for alternative solution strategies (elastic-net [Zou et al., 2005], etc.), conferring a broader application perspective (i.e., the proposed experimental framework is also suited to the analysis of non-reverberant sound fields).

**Paper B**

Paper B proposes an experimental method to characterize the distribution of sound incidence on a measuring sample in a small box-shaped chamber with no diffusers in steady state. The methodology relies on estimating the magnitude of the sound waves arriving from definite directions onto the absorbing sample, to separate the incident from the reflected waves. The measurements are conducted in the empty room, and with the absorber covering one of its walls. A scanning robotic arm is programmed to create a random microphone array within a volume located in the immediate vicinity of the wall. The results confirm that the distribution of sound incidence on the sample is not isotropic. Besides, substantial differences are found between the incident fields in the empty and occupied conditions, demonstrating the influence of the sample on the sound field in its vicinity. This indicates that the sound field impinging onto the sample depends strongly on the properties of the sample itself, not just the room.

Based on the wave expansion, we show that all acoustic quantities (i.e. sound pressure, particle velocity, and active/reactive sound intensity) can be reconstructed in the vicinity of the absorbing specimen. Both the total sound field (incident plus reflected components) as well as the incident field alone can be obtained, which makes it possible to visualize and characterize the power flows incident on the sample. The results show that the incident flow onto the absorptive sample is not constant and uniform throughout space (as would be the case in a perfectly isotropic sound field).
Paper C

Based on the same experimental framework as in Paper B, Paper C examines the case of a standardized (ISO 354:2003) reverberation chamber with diffusers. Measurements with a programmable robotic arm are conducted in the empty room and with an absorber on the floor. In spite of the presence of diffusing elements, the results indicate that the distribution of sound incidence on the absorbing sample is not isotropic in the steady state, and that the incident flow is not constant and uniform. This is attributed to the strong coupling established between the sample and the sound field in its vicinity.

In addition, Paper C shows the quantitative validity of the method via an estimation of the sample’s angle-dependent absorption coefficient. The measured absorption coefficient, obtained by separating the incident from the reflected components in the wavenumber spectrum, agrees well with predictions obtained from a transfer matrix method (Allard, 1993) combined with Miki’s (Miki, 1990) empirical model (the overall deviation is below 10% at all frequencies).

Finally, an analysis of the net flow of energy in the room is presented, and the active and reactive intensity flows (corresponding to flowing and oscillating energy, respectively) are examined.

Paper D

Paper D examines the evolution with time of the spatial properties of net energy flows in a reverberation chamber. The experimental framework is based on a spherical equivalent source model (Fernandez-Grande, 2016), which consists of modeling the sound field on a rigid spherical array as the superposition of waves radiated by a combination of monopoles. This representation is comparable to the plane-wave expansion exploited in Papers A to C yet using a different basis for the wave expansion. Likewise, such representation can yield all acoustic quantities – sound pressure, particle velocity, and sound intensity. The measured instantaneous sound pressures at the microphone positions are windowed in consecutive windows of some interval to obtain successive reconstructions of the energy flows. In this way, it is possible to examine the net flows corresponding to the steady state and the early and late decay times separately, or the evolution of the sound decay by computing the net flows at each time window. Measurements in a standardized reverberation chamber reveal significant structural differences between the intensity field in the steady state and during the decay, and confirm that the time-averaged intensity is non-zero in a reverberation chamber. When a sample of absorbing material is placed on the floor, a large influx of energy directed towards it is detected.

Paper E

Paper E examines numerically the distribution of sound pressure and energy flows (active and reactive) in a box-shaped room with absorption concentrated on the floor. The analysis is conducted by means of a mode model in the steady state and validated against theoretical predictions. The equations describing the response to a point source are derived, and it is shown that the modes are not orthogonal with the usual definition
of inner product. A method for the numerical resolution of the eigenfrequencies is further proposed, which reveals the difficulties related to the treatment of the zeroth-order mode.

The model is deterministic and, as such, provides a powerful means to thoroughly describe the steady-state sound field in rectangular reverberation rooms containing a single concentrated sample of highly absorbing material. As opposed to experimental methods, the proposed numerical model allows for full-scale analysis of the stationary sound field. Although restricted to rectangular rooms with no scattering objects, such model is a fast and flexible way to examine the effect of varying source and receiver positions, varying absorption coefficient of the absorbing surface, and averaging procedures (this is however not examined in Paper E).
3. Background

This chapter provides the reader with the physical and mathematical background, which includes: a brief introduction to the concept of wavenumber spectrum (Section 3.1); the formulation of plane-wave decomposition on a rigid sphere (Section 3.2); an overview of the basic and most important (for this dissertation) properties of spherical harmonics and spherical Fourier transforms (Section 3.3); and an overview of the regularization techniques used in this study (Section 3.4). The reader familiar with the aforementioned topics may skip Chapter 3, and is referred directly to Papers A to E.

In the following, the time convention $e^{j\omega t}$ is chosen. The angular dependency is expressed as $\Omega \equiv (\theta, \phi)$, and $d\Omega = \sin \theta d\theta d\phi$, thus the integration over the sphere is $\iint_\Omega(.)d\Omega = \int_0^{2\pi} \int_0^{\pi} (.) \sin \theta d\theta d\phi$.

3.1 Wavenumber spectrum

Let us consider a steady-state pressure field produced by a pure-tone source in a reverberation chamber. In the far field of the source and any diffracting objects, the pressure at the point characterized by the vector $r = (x, y, z)$ can be expressed by a sum of plane propagating waves with variable amplitudes and propagation directions. This elementary plane wave decomposition consists in applying a three-dimensional Fourier transform to the measured pressure field, to estimate the wavenumber spectrum, or angular spectrum (Goodman, 1968), which is a representation of the sound field in the spatial-frequency domain (or wavenumber domain) [Williams, 1999]:

$$p(k, r) = \iiint_{-\infty}^{+\infty} P(k)e^{jkr}d\mathbf{k}. \quad (3.1)$$

We recognize the exponential term $e^{jkr}$ as a plane wave travelling in a direction specified by the wavenumber vector $\mathbf{k} = (k_x, k_y, k_z)$. The integrals represent a three-dimensional Fourier transform in $k_x$, $k_y$, and $k_z$, respectively, and the quantity $P(\mathbf{k}) = |P(\mathbf{k})|e^{j\varphi(\mathbf{k})}$ is the wavenumber spectrum, with $|P(\mathbf{k})|$ and $\varphi(\mathbf{k})$ its magnitude and phase, respectively. All propagating plane waves satisfy the condition $||\mathbf{k}||^2 = k^2 = k_x^2 + k_y^2 + k_z^2$ with $k^2 \geq k_x^2 + k_y^2$ (indicating that evanescent waves are not present).
Figure 2: $k$-space diagram. The direction of the plane wave with components $(k_x, k_y, k_z)$ is shown by the vector $k$, with the spherical angles $\theta_0$ and $\phi_0$. The radius of the sphere is $k$.

Figure 2 shows the $k$-space (or wavenumber) representation of a plane wave with components $(k_x, k_y, k_z)$. The direction of propagation is shown by the vector $k$ in $k$-space, with the spherical angles $\theta_0$ and $\phi_0$. The radius of the sphere is $k$. It is clear from Eq. (3.1) that the wavenumber spectrum $P(k)$ determines the magnitudes of the plane waves arriving from definite directions at the observation point $r$; hence, plane-wave decompositions are commonly used in acoustics, to analyze the directional properties of a sound field. For instance, such decompositions have been used for sound source identification using NAH (Maynard et al., 1985; Veronesi et al., 1987), for examining spatial correlation functions in arbitrary noise fields (Cox, 1973; Ebeling, 1984), or for converting output data from mesh-based sound field simulations to a format compatible with auralization methods (Støfringsdal et al., 2006). In the present study, the wavenumber representation is employed with the aim of analyzing the isotropy of the sound field in a reverberation chamber (both empty and with an absorber on the floor) [Paper A] and examining the spatial distribution of sound incidence onto the absorber [Papers B and C].

The wavenumber spectrum is typically determined from a set of microphone signals (array of sensors), and its estimation relies on solving a system of linear equations to obtain the amplitudes of the wave expansion used to represent the data captured in the measurement (see Section 3.2). Measurements of the wavenumber spectrum do not require a specific array configuration, and the pressure field can be sampled randomly over an arbitrary volume.

3.2 Plane-wave sound fields in the spherical harmonics domain

The experimental investigations in Paper A make use of a rigid spherical microphone
array. As this is not described in the paper, the following section gives the necessary equations for the plane-wave decomposition on a rigid sphere.

We now consider a rigid spherical array immersed in the sound field described in Section 3.1. The array consists of \( R \) microphones flush-mounted on the surface of the rigid-sphere of radius \( a \). The measured sound pressure at \( \mathbf{r}_p = (a, \Omega_p) \) due to a single plane wave (i.e., \( e^{j\mathbf{k}\cdot\mathbf{r}_p} \)) can be written as a summation of spherical harmonics (Rafaely, 2004; Fernandez-Grande, 2016):

\[
e^{j\mathbf{k}\cdot\mathbf{r}_p} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 4\pi j^n \left( j_n(ka) - \frac{j'_n(ka)}{h_n(ka)} \right) [Y_n^m(\Omega_0)]^* Y_n^m(\Omega_p). \tag{3.2}
\]

The functions \( Y_n^m(\Omega) \) are the spherical harmonics of order \( n \) and degree \( m \), defined as in Eq. (3.7) [see Section 3.3]. The function \( h_n(x) = j_n(x) - j'_n(x) \) is the spherical Hankel function of the second kind, with \( j_n(x) \) and \( y_n(x) \) the spherical Bessel functions of the first and second kind, and \( h'_n(x) \) is the derivative of the Hankel function. Making use of the useful Wronskian relationship \( j_n(ka)h_n(ka) - j'_n(ka)h_n(ka) = -j/(ka)^2 \), Eq. (3.2) becomes:

\[
e^{j\mathbf{k}\cdot\mathbf{r}_p} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi(-j)^{n+1}}{(ka)^2 h_n(ka)} [Y_n^m(\Omega_0)]^* Y_n^m(\Omega_p). \tag{3.3}
\]

We now assume that an infinite number of plane waves arrive at the rigid sphere from all directions \( \Omega_0 = (\theta_0, \phi_0) \), with complex amplitudes \( P(k, \Omega_0) \). The total pressure on the sphere due to all waves can be calculated by integrating Eq. (3.3) over these directions

\[
p(k, \mathbf{r}) = \oint_{\Omega_0} \frac{P(k, \Omega_0)}{(ka)^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi(-j)^{n+1}}{h_n(ka)} [Y_n^m(\Omega_0)]^* Y_n^m(\Omega_p) \, d\Omega_0, \tag{3.4}
\]

where we recognize \( P(k, \Omega_0) \) as the wavenumber spectrum (see Eq. (3.1)). In practice, \( P(k, \Omega_0) \) is obtained using a discrete plane wave expansion, based on a discrete approximation of Eq. (3.4):

\[
p(k, \mathbf{r}) = \sum_{l=1}^{L} \frac{P(k, \Omega_{0,l})}{(ka)^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi(-j)^{n+1}}{h_n(ka)} [Y_n^m(\Omega_{0,l})]^* Y_n^m(\Omega_p) \tag{3.5}
\]

where the directions of propagation of the \( L \) plane waves are uniformly distributed over a spherical domain. Note that for a proper representation of the measured pressure, the number of plane waves should be greater than the number of measurement positions (Williams, 1999; Rafaely, 2004).
By conducting the summation over \( n \) and \( m \) in Eq. (3.5), a transfer matrix \( H \) is obtained that relates the pressure on the sphere and the coefficients of the wave model

\[
p = Hx. \tag{3.6}
\]

The summation is truncated at \( n = N \), given the number of microphones and size of the sphere (Jacobsen et al., 2011). The vector \( p \in \mathbb{C}^R \) consists of the sound pressure measured at a discrete set of \( R \) points on the sphere, and \( H \in \mathbb{C}^{R \times L} \) is the transfer matrix between the amplitudes of the waves (i.e., the wavenumber spectrum) and the measured pressures (in this case, \( H \) includes the scattering introduced by the spherical array). The wavenumber spectrum corresponds to the vector \( x \in \mathbb{C}^L \), i.e., the unknown complex coefficients \( P(k, \Omega_0) \) of the expansion in Eq. (3.5) [see Fig. 3]. This problem is ill-posed, typically underdetermined \((R < L)\), and requires regularization. The question of the regularization of discrete ill-posed problems will be briefly addressed in Section 3.4 (at least to the extent necessary to this work).

### 3.3 Spherical harmonics and spherical Fourier transforms

In this section we define the spherical Fourier transform (Driscoll et al., 1994), or spherical harmonics decomposition, as well as some important properties of the spherical harmonics used in the derivations of Paper A.

**Spherical harmonics**
The spherical harmonic functions \( Y_n^m(\Omega) \), with natural \( n \in \mathbb{N} \) and integer \( m \in \mathbb{Z} \) satisfying \(|m| \leq n\), form an orthogonal basis in the space of \( L^2(\mathbb{S}^2, d\Omega) \) scalar functions on the two-sphere \( \mathbb{S}^2 \), where \( \Omega = (\theta, \phi) \in \mathbb{S}^2 \) are the spherical coordinates with \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), and \( d\Omega = \sin \theta d\theta d\phi \). The spherical harmonics are defined by (Williams, 1999)

\[
Y_n^m(\Omega) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta)e^{im\phi} \tag{3.7}
\]

where

\[
P_n^m(x) = (-1)^m \frac{1}{2^n n!} (1 - x^2)^{n/2} \frac{d^m}{dx^m} \frac{d^m}{dx^m} (x^2 - 1)^n \tag{3.8}
\]

are the associated Legendre functions. Figure 4 shows the first five orders of spherical harmonics. The orthogonality and completeness relations read

\[
\int_{\mathbb{S}^2} Y_n^m(\Omega) \left[ Y_n^m(\Omega) \right]^* d\Omega = \delta_{nn} \delta_{mm} \tag{3.9}
\]

and

\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\Omega) \left[ Y_n^m(\Omega) \right]^* = \delta(\Omega - \Omega), \tag{3.10}
\]

respectively, where the superscript * denotes complex conjugation, \( \delta_{ij} \) is the Kronecker delta, \( \delta(\Omega - \Omega) = \delta(\phi - \phi)\delta(\cos \theta - \cos \theta) \) and \( \delta(x) \) is the Dirac delta function.

**Spherical Fourier transforms**

Since the spherical harmonic functions form a complete, orthogonal basis on the sphere, any square integrable function on the sphere \( f \in L^2(\mathbb{S}^2, d\Omega) \) may be represented by the spherical harmonic expansion

\[
f(\Omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm} Y_n^m(\Omega), \tag{3.11}
\]

where the spherical harmonic coefficients are given by the usual projection onto the spherical harmonic basis functions

\[
f_{nm} = \int_{\mathbb{S}^2} f(\Omega) \left[ Y_n^m(\Omega) \right]^* d\Omega. \tag{3.12}
\]

Equations (3.11) and (3.12) represent the inverse and forward spherical Fourier transforms, respectively (Driscoll et al., 1994). These equations correspond to Eqs (7) and (8) in **Paper A**, in which the spherical Fourier transform is applied on the magnitude of the estimated wavenumber spectrum. For the sake of illustration, Figure 4 shows the spherical harmonics decomposition (up to \( n = 7 \)) of the wavenumber
spectrum in two cases: (i) the sound field is modeled as four waves of equal amplitudes propagating in opposite directions [Fig. 4(a)]; (ii) the sound field is fairly isotropic [Fig. 4(b)]. The directions of propagation of the plane waves used for the plane-wave expansion are uniformly distributed over a spherical domain. The complex coefficients $f_{nm}$ from the spherical harmonics decomposition in Eq. (3.11) [where $f = |P|$] are calculated using a discrete approximation of Eq. (3.12), and displayed in terms of their magnitude $\sum_{m=-n}^{n} |f_{nm}|$. In the four-wave case [Fig. 4(a)], it can be seen that the wavenumber spectrum is best described by the hexadecapole moment (i.e. $n = 4$, as expected given the symmetry of the sound field) and, to a lesser extent, by all axially symmetric moments ($n = 0$, $n = 2$, and $n = 6$). In the case of a fairly isotropic sound field [Fig. 4(b)], the wavenumber spectrum is roughly a constant function over the sphere; thus, its magnitude is (almost) completely determined by the zeroth-order spherical harmonic. The latter result is hereafter shown analytically.

In the following, two useful functions defined on the sphere are presented, along with their spherical Fourier transforms: the constant function and the Dirac delta function. These functions are central to the definition of the isotropy indicator as in Paper A. Additional functions are treated in (Rafaely, 2015).

We first consider the function $f(\Omega) = 1$ constant along both $\theta$ and $\phi$. By writing $f(\Omega) = \sqrt{4\pi} Y_{00}(\Omega)$, inserting in Eq. (3.12), and evaluating the integral using the orthogonality relation defined in Eq. (3.9), one finds

$$f_{nm} = \sqrt{4\pi} \delta_{n0} \delta_{m0},$$

(3.13)

which shows that the constant function can be represented using the zeroth-order spherical harmonic only. In a room and as illustrated in Fig. 4(b), if the acoustic

\[n = 0\]

\[n = 1\]

\[n = 2\]

\[n = 3\]

\[n = 4\]
where (a) the sound field is modeled as four waves propagating in opposite directions; (b) the sound field is fairly isotropic. Truncation order $N = 7$. The first five orders of spherical harmonics are also displayed (top), with black representing positive values, and white representing negative values.

energy is equal in all directions, the resulting magnitude of the wavenumber spectrum will be constant over the entire solid angle. Thus, its energy will reside entirely on the monopole moment of its spherical harmonic expansion. This means that the isotropy indicator, defined as the relative magnitude of the monopole contribution (compared to the total orders), will be unity in the case of an ideally isotropic sound field. As such, Eq. (3.13) determines the upper bound of the isotropy indicator proposed in Paper A.

Conversely, we now consider the Dirac delta function over the sphere: $f(\Omega) = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$. Inserting in Eq. (3.12) and using the relation $\sin\theta\delta(\phi - \phi')\delta(\cos\theta - \cos\theta') = \delta(\phi - \phi')\delta(\theta - \theta')$ yields

$$f_{nm} = \int_{S^2} \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') [Y_n^m(\theta, \phi)]^* \sin\theta d\theta d\phi$$

$$= \int_{S^2} \delta(\phi - \phi')\delta(\theta - \theta') [Y_n^m(\theta, \phi)]^* d\theta d\phi$$

$$= [Y_n^m(\theta', \phi')]^*.$$  

Equation (3.14) shows that the spherical Fourier coefficients for the Dirac delta function are the spherical harmonics. Since in the case of a single propagating plane
wave (least isotropic case), the magnitude of the wavenumber spectrum resembles a Dirac delta function, Eq. (3.14) defines the lower bound of the isotropy indicator proposed in Paper A.

3.4 Regularization

As mentioned in Section 3.2, the system of linear equations in Eq. (3.6) relates a set of measured pressures to the coefficients (complex amplitudes) of the plane wave expansion (i.e., the wavenumber spectrum). To estimate this wavenumber spectrum, it is necessary to invert the system of equations. In the general case the system is underdetermined \((R < L)\), i.e., more unknown coefficients than measurement points, leading to a non-unique solution, and must be inverted using regularization. This section introduces the standard regularization techniques and regularization parameter choice methods used in this thesis. For a comprehensive treatment and for references to the extensive literature on the numerical aspects of inverse problems and regularization methods one may refer to the books by Hansen (Hansen, 1998; Hansen, 2010). Additionally, an introduction to uncertainty quantification in inverse problems can be found in (Bardsley, 2018).

We recall from Section 3.2 that the wavenumber spectrum estimation problem can be expressed with the linear model \(p = Hx\), where \(p \in \mathbb{C}^R\) is the complex-valued data vector from the measurements at the \(R\) microphones, \(x \in \mathbb{C}^L\) is the unknown vector of the complex amplitudes (i.e., the wavenumber spectrum), and \(H \in \mathbb{C}^{R \times L}\) is the sensing matrix that maps the hypothetical amplitudes \(x\) to the observations \(p\). The Tikhonov regularized solution (or \(\ell_2\)-norm regularized least squares) is defined by (Tikhonov et al., 1977; Golub et al., 1999)

\[
\hat{x}_\lambda(\lambda) = \arg \min_{x \in \mathbb{C}^L} \left\{ \|p - Hx\|_2^2 + \lambda \|x\|_2^2 \right\},
\]

and seeks the solution with the minimum \(\ell_2\)-norm (minimum energy of the coefficients/unknowns) that provides the best data fit. The regularization parameter \(\lambda\) controls the relative importance between the data fit \(\|p - Hx\|_2\) and the \(\ell_2\)-norm of the solution. The solution can be equivalently expressed as \(H^H(HH^H + \lambda I_R)^{-1}p\), where \(I_R\) is the \(R \times R\) identity matrix. Tikhonov regularized solution promotes smooth, minimum-energy estimates that are well suited for applications in rooms, where sound waves impinge on the array from multiple directions. Thus, this is the regularization scheme used throughout this study (Papers A to D).

In the case of few waves impinging on the array, the \(\ell_2\)-norm regularized solution suffers from low resolution and the presence of side lobes (Xenaki et al., 2014; Fernandez-Grande et al., 2016). An alternative estimation of the wavenumber spectrum uses the framework provided by Compressive Sensing (Elad, 2010; Foucart et al., 2013) that promotes a sparse solution to the problem (i.e., an optimal representation of the measured data with as few non-zero coefficients as possible)
\[
\hat{x}_1^{(\mu)} = \arg \min_{x \in C} \left\{ \|p - Hx\|_2^2 + \mu \|x\|_1 \right\}.
\] (3.16)

Equation (3.16) is a least squares optimization method regularized with the \(\ell_1\)-norm of the solution \(x\) and provides the best data fit for the sparsity level determined by the regularization parameter \(\mu\). Equation (3.16) corresponds to the well-known LASSO formulation (Tibshirani, 1996), used in Paper A to estimate the wavenumber spectrum due to a single wave impinging on the array (see Appendix A in Paper A). In this specific case, it is meaningful to assume a sparse representation of the sound field, and we show that the CS solution outperforms the estimation obtained via Tikhonov regularization (the spatial resolution is enhanced and approaches an ideal delta function, as seen in Figure 5). Nonetheless, selecting the \(\ell_1\)-norm is a poor regularization choice when processing the sound field in a reverberant enclosure, which yields non-physical solutions because the actual problem is not sparse (many waves compose the sound field). As such, the choice of the regularization scheme involves imposing prior information on the solution \(x\), to promote either sparsity or smoothness.

![Fig. 5: Wavenumber spectra due to a single plane wave resulting from the LS (left) and CS (right) solutions (adapted from Fig. 13 in Paper A).](image)

The usefulness of regularization methods depends upon there being effective techniques for estimating the regularization parameter. In the following, we introduce the L-curve criterion (Hansen, 1992), a widely used regularization parameter choice method that has been found to be robust on many examples in inverse problems. The L-curve criterion is used throughout Papers A to C (with Tikhonov regularization). Paper D uses an alternative regularization parameter choice method known as Generalized Cross Validation (Golub et al., 1979), also very commonly used in inverse problems.

The L-curve criterion is a heuristic developed in (Hansen, 1992) that we consider here as a method for choosing the Tikhonov regularization parameter \(\lambda\). A brief description of the L-curve method can be found in (Bardsley, 2018), which we
summarize in the following. The *L-curve criterion* is based on the fact that the parameterized curve

\[
\left( \log \| \hat{x}_l(\lambda) \|_2^2 , \log \| H \hat{x}_l(\lambda) - p \|_2^2 \right)
\]

(3.17)

has a distinct L-shape. The corner of the curve roughly corresponds to the transition point between the values of \( \lambda \) for which \( \| \hat{x}_l(\lambda) \|_2 \) is large and \( \| H \hat{x}_l(\lambda) - p \|_2 \) is small (too little regularization), and the values of \( \lambda \) for which \( \| \hat{x}_l(\lambda) \|_2 \) is small and \( \| H \hat{x}_l(\lambda) - p \|_2 \) is large (too much regularization). The *L-curve criterion* chooses \( \lambda \) corresponding to the corner (or point of maximum curvature) along the curve, since this will often correspond to the value of \( \lambda \) at which this transition occurs. The equation for the curvature \( C \) for the parametric curve defined in Eq. (3.17) is given by

\[
C(\lambda) = -\frac{r(\lambda)s(\lambda)[\dot{r}(\lambda) + \lambda^2 s(\lambda)] + [r(\lambda)s(\lambda)]^2 / s'(\lambda)}{[r^2(\lambda) + \lambda^2 s^2(\lambda)]^{3/2}},
\]

(3.18)

where \( s(\lambda) = \| \hat{x}_l(\lambda) \|_2^2 \) and \( r(\lambda) = \| H \hat{x}_l(\lambda) - p \|_2^2 \). The *L-curve* choice for the regularization parameter \( \lambda \) is the value that maximizes the curvature defined in Eq. (3.18).
This PhD study addresses the problem of reverberation-chamber acoustics for the measurement of absorption coefficients, for which a completely diffuse sound field should be established in the test chamber. The shortcomings of the measurement procedure have become increasingly evident as a result of numerous investigations (Kosten, 1960; Makita et al., 1968; Halliwell, 1983; Nolan et al., 2014): several systematic comparisons of the absorption coefficients determined in a number of standardized laboratories resulted in the conclusion that the absorption coefficient of a given area of the same material can take virtually any value. A vast amount of literature has grown around the subject and shows that the major cause of difficulties with the reverberation-chamber measurement of sound absorption can be attributed to a lack of diffuseness of the sound field.

A variety of methods for the measurement of sound field diffusion in reverberation chambers can be found in the literature. However, an exact interpretation of any departure from an ideally diffuse field (or ensemble average) is difficult, and it appears valuable to adopt sound field analysis techniques. In this connection, the purpose of this PhD thesis was twofold: first, to examine the directional properties of the sound field in a reverberation chamber experimentally; second, to discuss the observed behavior against predictions that follow from assuming an ideally diffuse sound field. The directional properties of the sound field in a standardized reverberation chamber (empty and with absorption) have been studied experimentally in two ways: by considering the distribution of sound energy in the plane wave expansion of the sound field; by analyzing the distribution of net acoustic intensity. The following conclusions can be drawn:

- The directional properties of the reverberant sound field can be characterized by analyzing the wavenumber spectrum, which results from expanding the sound field into a plane-wave basis (Papers A to C). Measurements in a standardized (ISO 354:2003) reverberation chamber show that the sound field is not isotropic (that is the wavenumber spectrum is not spherically/hemispherically symmetrical), neither in the room volume, nor in the vicinity of the measuring sample. The accuracy of the estimation depends on (i) the measurement system (the sampling of the pressure field should be sufficient to estimate the wavenumber spectrum correctly; yet, no specific array configuration is required); (ii) the choice of the regularization scheme. It can be remarked that in a reverberation chambers with diffusing elements (such as panel or boundary diffusers), exponentially attenuated waves (evanescent waves) will appear at the observation point, if the latter is located in close proximity.
of the diffusers. Besides, evanescent waves can be caused by diffraction evoked at the sample edges. **Papers A to C** are however concerned with analyzing the sound field away from any source and diffracting element (that is, evanescent waves are not present [Schroeder, 1959]).

- The isotropy of a sound field can be characterized by expanding the wavenumber spectrum in the spherical harmonics domain (**Paper A**). The relative monopole strength determines the degree of isotropy. Various experimental investigations show that this indicator responds adequately to changes in sound field isotropy. Yet, it must be recognized that such quantitative indicator alone is of no help in interpreting the results. For instance, it seems impossible to distinguish between anisotropy attributable to low modal density and anisotropy caused by non-uniform placement of absorbing material. In this connection, the inspection of the wavenumber spectrum is much more informative. Nevertheless, the proposed measure may be useful for comparative investigations in a given room (in examining the influence of source position, diffusers placement, etc.). Note however that the metric is in no way a full measure of the degree of diffusion, since it disregards the correlation between waves incoming from different directions.

- By estimating the wavenumber spectrum in the vicinity of the measuring sample, it is possible to reconstruct and characterize the incident energy flows over a three-dimensional domain near the sample. Measurements in a standardized (ISO 354:2003) reverberation chamber confirm that the incident flow is not constant and uniform throughout space despite the presence of diffusing elements. This is attributed to the strong coupling between the measuring sample and the sound field in its vicinity.

- The complex intensity vectors in a room can be determined and visualized, which provides a valuable tool for understanding the relation between flows of net energy (active intensity) and oscillating energy (reactive intensity) in an empty reverberation chamber or with absorption on the floor.

- The angle-dependent absorption coefficient (averaged over the azimuth angle) can be determined from measurement of the wavenumber spectrum in front of an absorbing plane.

- A method has been presented to analyze experimentally the net flows of acoustic energy as a function of time. Such analysis can be useful in finding the directions of net energy transport throughout the decay process. Although the intensity vector is not directly related to the directions of arrival of the wavefronts, analysis of net intensity sheds light on the overall isotropy and spatial properties of the sound field in a reverberation chamber.

**** In **Paper D**, a different wave basis is used, e.g., the wave expansion is formulated in terms of point sources instead of plane waves. Such expansion is typically suitable in the near field of a source or diffuser, for the decay of the acoustic field is modeled via the spherical spreading of the point sources (Fernandez-Grande, 2016). Yet, the results (not shown in **Paper D** for conciseness) obtained using a plane-wave basis in place of a point-source basis lead to the same observations, which in turn confirms that the wavefronts can be regarded as locally planar in a reverberation chamber, provided that the measurement system is placed at a sufficient distance away from the source and any diffracting element.
- A mode model is provided that can accurately describe the steady-state power flows in rectangular rooms containing a single concentrated sample of highly absorbing material. The mathematical derivations exhibit interesting properties of the modes: in particular, it is shown that the modes are not orthogonal with the usual definition of inner product due to losses at the absorbing boundary.
5. Directions for future work

Some suggestions for future research, and questions deserving attention are listed below:

- It would be of interest to examine the directional properties of the sound field in a reverberation chamber on a temporal basis, based on the experimental methods outlined in Papers A to C. For instance, the evolution of the sound decay could be examined by estimating the wavenumber spectrum in successive windows of some time interval and inspecting the incident energy in each time window. This would lead to a time-dependent analysis of isotropy. A similar procedure is described in Paper D, with the aim of reconstructing the active intensity vector over time. In view of the results obtained in Paper D, an increase in sound field isotropy can be expected during the early part of the decay, due to a less prominent energy transport in the directions of the source and dominant reflections.

- Regarding Paper A, more measurements should be conducted with a larger aperture (to extend the low-frequency limit), and a finer sampling density (to extend the high-frequency limit). In particular, one could examine if the procedure can detect anisotropy attributable to low modal density below Schroeder’s frequency;

- It should be investigated how various averaging procedures (spectral averaging, spatial averaging over source positions, etc.) influence isotropy in a reverberation chamber. This could be done numerically based on the mode model described in Paper E;

- The experimental methods outlined in this report may find important application in measurements in ordinary rooms (especially in rooms with absorbent ceilings, e.g., classrooms, offices, etc.). The ability to characterize the spatial properties of the sound field in these rooms may supplement reverberation time measurements in tasks related to room acoustical design.

- Finally, an alternative to measuring the absorption properties of materials using the ISO 354:2003 standard (that is, by studying the decay process in a reverberation chamber) should be developed. The problem could be approached (among others) in several ways:
• It may be possible to develop an alternative method of determining the absorption coefficient in a reverberation chamber, where the reverberation time is computed based on the early decay only. Such approach has already been suggested in (Kuttruff, 1958) and more recently in (Balint et al., 2018). There is however still need for experiments.

• A steady-state method of determining the absorption coefficient could be developed, by separating the incident from the reflected components in the wavenumber spectrum. Preliminary results in a reverberation chamber are shown in Paper C. It is expected that the proposed framework can successfully be extended to in situ measurements in ordinary rooms.

• The acoustic performance of poroelastic materials could be evaluated based on intrinsic parameters (e.g., porosity, flow resistivity, tortuosity, characteristics lengths, etc.). Yet, such approach requires an understanding of both the areas of materials science and acoustics.


Jacobsen F. and Juhl P. M. (2013), Fundamentals of general linear acoustics (Wiley & Sons), pp. 139-150


A wavenumber approach to quantifying the isotropy of the sound field in reverberant spacesa)

Mélanie Nolan, b) Efren Fernandez-Grande, Jonas Brunskog, and Cheol-Ho Jeong
Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark (DTU),
Building 352, Ørsteds Plads, DK-2800 Kongens Lyngby, Denmark

(Received 1 August 2017; revised 10 March 2018; accepted 29 March 2018; published online 27 April 2018)

This study proposes an experimental method for evaluating isotropy in enclosures, based on an analysis of the wavenumber spectrum in the spherical harmonics domain. The wavenumber spectrum, which results from expanding an arbitrary sound field into a plane-wave basis, is used to characterize the spatial properties of the observed sound field. Subsequently, the obtained wavenumber spectrum is expanded into a series of spherical harmonics, and the moments from this spherical expansion are used to characterize the isotropy of the wave field. The analytical framework is presented. The method is examined numerically and experimentally, based on array measurements in four chambers: two anechoic chambers (one with a single source and another with an array of 52 sources), a reverberation chamber, and the same reverberation chamber with a sample of absorbing material on the floor. The results indicate that the proposed methodology is suitable for assessing the isotropy of a sound field. © 2018 Acoustical Society of America. https://doi.org/10.1121/1.5032194

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I. INTRODUCTION

Many acoustical measurements rely on the assumption that the sound field is diffuse. Examples include standardized measurements of sound absorption and transmission loss in reverberation rooms.1,2 The diffuse sound field is yet an idealized concept, and the sound field in any reverberant space differs in fundamental aspects from the perfectly diffuse sound field.3,4 It is therefore of interest to examine the behavior of sound fields in real rooms, and the concept of acoustic diffusion in a room.

Various models of diffuse sound fields have been described in the literature.3–6 The conception that any complex sound field can be defined as the superposition of a set of plane waves is used as a starting point for a model referred to as the random wave model, which theory is essentially due to Schroeder,7 Waterhouse,8 Lubman,9 Jacobsen,10,11 and Pierce.12 The diffuse sound field is described as composed of plane waves with random phases and equal magnitudes, which directions of propagation are uniformly distributed over all angles of incidence, such that the same amount of energy arrives at the observation point from each element of solid angle. Since infinitely many plane waves are assumed, this model is idealized, but gives a good approximation to the sound field in a reverberation room driven with a pure tone in the frequency range where the modal overlap is high (typically above Schroeder’s frequency). In this study, we associate the concept of diffusion with this theory.

Different methods have been proposed for evaluating the degree of diffusion in a room. Cook et al.,13 (and later Bodlund14), examined the cross-correlation between pressure measurements at neighboring positions. The core idea behind this approach is that, in a perfectly diffuse sound field, the cross-correlation function between two omnidirectional microphones follows a sinc function pattern. In Ref. 15, Jacobsen and Roisin presented a method of determining spatial correlation functions in a room, suitable to other quantities than the sound pressure. Noteworthy and perhaps overlooked is the work by Ebeling,16 who interpreted the cross-correlation function derived by Cook et al.13 in the spatial frequency domain. Subsequently, he proposed a multipole expansion of the spatial correlation function leading to a measure for spatial diffusivity. More recently, other methods have investigated how spherical microphone arrays can be used to characterize diffuseness. Gover et al.17 estimate the directional impulse responses of a room using a spherical array beamformer, to evaluate the distribution of acoustic energy arriving to the array from different directions. Following a different approach, Epain and Jin18 analyze the spherical harmonic covariance matrix to estimate diffuseness arising from the presence of multiple uncorrelated sources. Yet, other measures have been proposed, consisting in measuring the acoustic intensity over time,19–24 or the acoustic energy at various points across space.25

From the standpoint of the random wave theory, sound field diffusion relies on two essential features: (i) the directions of propagation of the plane waves that conform the sound field must be uniformly distributed over all angles of incidence (i.e., isotropic sound field), and (ii) these plane waves must have random relative phases. This publication is strictly concerned with quantifying sound field isotropy [i.e., condition (i)].

An experimental method for evaluating isotropy in enclosures is proposed that is based on an analysis of the wavenumber spectrum in the spherical harmonics domain.

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a) Portions of this work were presented in “A wavenumber approach to characterizing the diffuse field conditions in reverberation rooms,” Proceedings of the 22nd International Congress on Acoustics, Buenos Aires, Argentina, September 2016.
b) Electronic mail: melnola@elektro.dtu.dk
On the one hand, the wavenumber spectrum characterizes the magnitudes of the sound waves arriving from definite directions at the observation point.26 On the other hand, a spherical harmonic basis is best suited to analyze isotropy, as it depends only on direction (polar and azimuth angles), and can provide an unequivocal characterization of the symmetry of a given quantity. Hence, the use of spherical harmonics as a basis for describing isotropy is commonly used in several areas of physics.27–30 In this work, we propose to use a spherical harmonic expansion on the wavenumber spectrum, the underlying hypothesis being that in a perfectly isotropic sound field, the wavenumber spectrum is rotationally symmetric. Because the spherical harmonic expansion is performed on the wavenumber spectrum, and not on the recorded pressure signals directly,18,29,30 the proposed method is not restricted to measurements with a spherical array or other geometry. The method is valid for uniform or random spatial sampling, as opposed to Refs. 17 and 18. Besides, the analytical framework proposed in this paper is far simpler than the theory developed in Ref.16, in that it considers the actual pressure field directly, rather than its ensemble statistics and spatial correlation. Consequently, the proposed methodology is valid even when the random wave theory no longer holds (this would be the case at low frequencies or when absorbing material is spread over one or several surfaces).

The present paper is organized as follows: the theoretical background is presented in Sec. II, and the validity of the method is evaluated in Secs. III and IV, based on a numerical and an experimental study using array measurements.

II. THEORETICAL BACKGROUND

A. Wavenumber spectrum

We consider the steady-state sound field produced by a pure-tone source in a reverberation chamber. The resulting sound field at the point characterized by the vector \( \mathbf{r}_m = (x_m, y_m, z_m) \) can be represented as a superposition of plane waves, each traveling in a direction specified by the wavenumber vector \( \mathbf{k} = (k_x, k_y, k_z) \). Each plane wave may have different amplitudes and phases, which we account for by using a complex coefficient term \( P(k_x, k_y, k_z) = P(\mathbf{k}) \),

\[
p(\mathbf{r}_m) = \int_{-\infty}^{\infty} P(\mathbf{k}) e^{-jk_x x + jk_y y + jk_z z} d\mathbf{k}.
\]  

(1)

The integrals represent a three-dimensional inverse Fourier transform in \( k_x, k_y, \) and \( k_z \), respectively, which guarantees that any pressure distribution may be represented by Eq. (1). The quantity \( P(\mathbf{k}) = |P(\mathbf{k})| e^{i\phi(\mathbf{k})} \) is the wavenumber spectrum, with \( |P(\mathbf{k})| \) and \( \phi(\mathbf{k}) \) its magnitude and phase, respectively. We must keep in mind that all propagating plane waves satisfy the condition \( ||\mathbf{k}||^2 = k_x^2 + k_y^2 + k_z^2 \) with \( k^2 \geq k_x^2 + k_y^2 \) (indicating that evanescent waves are not present). Introducing spherical coordinates, \( x_m = r \sin \theta \cos \phi, y_m = r \sin \theta \sin \phi, z_m = r \cos \theta \) and \( k_x = k \sin \theta \cos \phi, k_y = k \sin \theta \sin \phi, k_z = k \cos \theta \), Eq. (1) becomes

\[
p\left(\mathbf{r}_m\right) = \int_0^{2\pi} \int_0^{\pi} P(\mathbf{k}) e^{-jk_x x + jk_y y + jk_z z} d\mathbf{k}.
\]  

(2)

Since we are interested in the sound field produced by a pure-tone with frequency \( f_0 \), all propagating waves should appertain to the surface of the radiation sphere of radius \( k_0 = 2\pi f_0/c \) in the wavenumber domain. In other words, the wavenumber spectrum \( P(\mathbf{k}) \) must only consist of components that fulfill

\[
P(k, \theta, \phi) = \frac{\delta(k - k_0)}{4\pi k^2} \tilde{P}(\theta, \phi),
\]  

(3)

where \( \delta(k - k_0)/4\pi k^2 \) corresponds to the Dirac delta function in spherical coordinates with symmetry with respect to both \( \theta \) and \( \phi \). \( \tilde{P}(\theta, \phi) \) denotes the two-dimensional wavenumber spectrum expressed in spherical coordinates. Combining Eqs. (2) and (3) yields

\[
p(r, \vartheta, \varphi) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \tilde{P}(\theta, \phi) e^{-jk_0 \sin \theta \sin \phi (\varphi - \varphi_0)} e^{jkr \cos \phi \cos \theta} \sin \theta d\theta d\phi.
\]  

(4)

The pressure distribution measured over a surface associated with \( \mathbf{r}_m = (a, \theta, \phi) \), can now be represented by

\[
p(a, \vartheta, \varphi) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \tilde{P}(\theta, \phi) e^{-jk_0 \sin \theta \sin \phi (\varphi - \varphi_0)} e^{jka \cos \phi \cos \theta} \sin \theta d\theta d\phi.
\]  

(5)

where a spherical measurement area of radius \( a \) is chosen. The two-dimensional inverse Fourier transformation required for explicitly calculating \( \tilde{P}(\theta, \phi) \) reads

\[
\tilde{P}(\theta, \phi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^\pi p(a, \vartheta, \varphi) e^{jk_0 \sin \theta \sin \phi (\varphi - \varphi_0)} e^{jka \cos \phi \cos \theta} \sin \theta d\theta d\phi.
\]  

(6)

In practice, no assumption whatsoever concerning the shape of the measurement area is necessary, since the analysis is done via discrete Fourier transforms, based on a discrete approximation of Eq. (4), see Sec. II.C. In fact, the pressure field can be sampled randomly over an arbitrary volume, as shown in Sec. III.

B. Isotropy

A wave field is termed isotropic if the wavenumber vectors of the incident plane waves are uniformly distributed over all angles of incidence (corresponding to a sinusoidal distribution of the polar angles and a uniform distribution of the azimuth angles).4 In order to evaluate isotropy in an acoustic field, it is necessary to analyze the direction of the waves that comprise the sound field. If the sound field is isotropic, its wavenumber spectrum is spherically symmetric (i.e., the magnitude of the waves is constant with angle). Contrarily, in an anisotropic sound field, the wavenumber
spectrum is asymmetric, as there is variable energy in different directions. Therefore, a spherical harmonic basis is best suited to analyze isotropy, as it depends only on the angles (polar and azimuthal angles).

The magnitude of the wavenumber spectrum \( \hat{P}(\theta, \phi) \) determined in Eq. (6) is thus expanded into a series of spherical harmonics

\[
|\hat{P}(\theta, \phi)| = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(k_0) Y_n^m(\theta, \phi).
\]

(7)

Since the spherical harmonics are orthonormal, the complex coefficients \( A_{mn}(k_0) \) of the expansion can be calculated from

\[
A_{mn}(k_0) = \int_0^{2\pi} \int_0^\pi |\hat{P}(\theta, \phi)| Y_n^m*(\theta, \phi) \sin \theta d\theta d\phi.
\]

(8)

It is interesting to note that Eq. (8) corresponds to a two-dimensional spherical Fourier transform.

In the case of a perfectly isotropic sound field, the magnitude of the wavenumber spectrum is constant over the entire solid angle (i.e., spherically symmetric), which corresponds to a constant function over a sphere. Consequently, the energy of the wavenumber spectrum \( \hat{P}(\theta, \phi) \) resides entirely on the monopole moment of the spherical harmonic expansion in Eq. (7) [i.e., \( A_{00}(k_0) \)]. This will not be the case if the interfering waves cover just a partial section of the solid angle, as all moments of the spherical harmonic expansion in Eq. (7) would characterize the wave field (in the case of a single propagating plane wave, the magnitude of the wavenumber spectrum equals a Dirac delta function, the spherical Fourier coefficients of which are the spherical harmonics \( A_{nm} = [Y_n^m(\theta, \phi)]^* \)). More generally, as soon as there is any degree of asymmetry in the wave field, part of the energy will be represented by the higher-order moments (which are spherically asymmetric).

The magnitude of the \( n \)th order moment is given by \( \sum_{m=-n}^{n} |A_{mn}(k_0)|^2 \), so that the relative magnitude of the monopole contribution (compared to the total orders) can now be expressed as

\[
i(k_0) = \frac{|A_{00}(k_0)|}{\sum_{n=0}^{\infty} \sum_{m=-n}^{n} |A_{mn}(k_0)|}.
\]

(9)

This quantity is here suggested as an isotropy indicator and will be denoted \( i \) in the following. The measure ranges between zero and one and equals unity in the case where the flow of acoustic energy is equal in all directions and is, therefore, perfectly isotropic. Conversely, it approaches zero if the incident waves propagate in a single direction. The measure is independent of the specific choice of coordinate directions. A similar measure was previously proposed in Ref. 32 for characterizing the radiation pattern of monopoles.

C. Implementation of the method

In practice, the two-dimensional wavenumber spectrum \( \hat{P}(\theta, \phi) \) is obtained using a discrete plane wave expansion, based on a discrete approximation of Eq. (4),

\[
p(r_m) = \sum_{l=1}^{L} \tilde{P}(k_l) e^{-jk_l r_m},
\]

(10)

where the directions of propagation of the plane waves are uniformly distributed over a spherical domain. In the limit \( L \to +\infty \) the pressure distribution in Eq. (4) is obtained.

The pressure field is sampled at a discrete number \( M \) of positions, and can be expressed in matrix form as

\[
p = \begin{bmatrix} \psi_1(r_1) & \psi_2(r_1) & \cdots & \psi_N(r_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(r_M) & \psi_2(r_M) & \cdots & \psi_N(r_M) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_L \end{bmatrix}, \quad p = Wc,
\]

(11)

where \( p \) is the measured sound pressure vector, \( c \) is a complex coefficient vector containing the wavenumber spectrum \( \hat{P}(k_l) \) in Eq. (10) and \( W \) is a matrix containing the plane wave functions \( \psi(r) = e^{-jk \cdot r} \). This is an ill-posed (typically underdetermined) problem, which requires regularized inversion. The solution of Eq. (11) can be calculated in a least-squares sense, i.e., via a regularized matrix pseudo-inverse. The problem can be formulated as an unconstrained problem, introducing a regularization parameter \( \lambda \), which determines the penalty weight of the \( \ell_2 \)-norm of the solution vector. Throughout this study, the \( \ell_2 \)-norm of the solution is chosen,

\[
\hat{c} = \arg\min_{c} (||Wc - p||^2 + \lambda ||c||^2_{\ell_2}),
\]

(12)

which has the well-known closed form analytical solution

\[
\hat{c} = W^H (WW^H + \lambda I)^{-1} p,
\]

(13)

where the superscript \( H \) denotes the conjugate transpose and \( I \) is the identity matrix. Equation (13) corresponds to the least-squares solution of the problem with Tikhonov regularization.

Subsequently, a spherical harmonic expansion of the magnitude of each component of \( c \) can be obtained based on a discrete approximation of Eqs. (7) and (8). Note however, that the discrete Fourier inversion required for calculating \( \hat{P}(k_l) \) necessarily extends over a finite surface, limiting the angular resolution.

D. Numerical example

For the sake of illustration, we consider an ideal wavenumber spectrum [by ideal, we mean that the wavenumber spectrum \( \hat{P}(\theta, \phi) \) is estimated perfectly, hence disregarding numerical errors in the inversion of Eq. (11)], discretized into 1000 directions that are solutions to the so-called Thomson problem, which considers equally charged particles on a sphere, hence yielding a uniform sampling over a spherical domain (Fig. 1). Two reference test cases are considered, where the sound field is modeled as (a) a single propagating plane wave; (b) a perfectly isotropic wave field. The complex coefficients \( A_{mn} \) from the spherical harmonic
expansion in Eq. (7) are calculated using a discrete approximation of Eq. (8).

Figure 1 shows the magnitude of the ideal wavenumber spectra, along with the first seven moments (i.e., \(A_{mn}\) computed up to \(n = 7\)) from their respective spherical harmonic expansions. The moments are displayed in terms of their magnitude \(|\sum_{m=-n}^n A_{mn}|\). Figure 1(a) shows the case of the single propagating plane wave, which corresponds to a wavenumber spectrum with a single non-zero coefficient. It is apparent that all moments from the spherical harmonic expansion in Eq. (7) characterize the wave field and that the isotropy indicator in Eq. (9) is zero \((i = 0)\). Figure 1(b) displays the case of an ideal isotropic sound field, with a uniform spatial distribution of the directions of propagation of all waves. It can be seen that the wavenumber spectrum is rotationally symmetrical, and therefore its magnitude resides entirely on the monopole moment of the spherical harmonic expansion, i.e., all \(A_{mn}\) for \((m, n) \neq (0, 0)\) are null. Analytically, the magnitude of such a spectrum corresponds to a constant function over the sphere. Consequently, the magnitude of the wavenumber spectrum is represented using the zeroth-order spherical harmonic only, and the indicator in Eq. (9) equals unity \((i = 1)\), indicating a perfectly isotropic sound field.

### III. NUMERICAL RESULTS

A simulation is conducted to examine the validity of the method. The simulated pressure field is produced by a variable number of pure-tone point sources with equal volume velocity \(Q = 10^{-5} \text{ m}^3 \text{s}^{-1}\). The resulting pressure field due to the monopoles is sampled at 64 randomly distributed points within a cubical volume of side length 20 cm, centred at the origin of coordinates (as shown in Fig. 2). The minimum distance between neighbouring measurement points is set to 5 cm.

All acoustic sources (monopoles) are distributed over a spherical domain of radius 2.4 m around the centre of the array. Thus, the complex pressures generated by the point sources are perfectly in phase at the array centre. Three different source configurations are simulated: seven sources evenly distributed over one-eighth of the spherical domain [case (a), see Fig. 3(a), left]; 26 sources evenly distributed over one-half of the spherical domain [case (b), see Fig. 3(b), left]; 52 sources evenly distributed over the entire spherical domain [case (c), see Fig. 3(c), left]. Additive noise of 30 dB signal-to-noise ratio is included in the simulated measurements.

For the plane-wave expansion described in Eq. (10), a plane-wave basis of 1000 plane waves of unknown amplitudes is considered, whose directions of propagation are distributed uniformly based on a Thomson problem. The number of plane waves should be greater than the number of measurement positions, for a proper representation of the measured pressure. The complex coefficient vector \(c\) corresponding to the wavenumber spectrum (i.e., the amplitudes of the waves) is estimated using Eq. (11). Tikhonov regularization [i.e., a \(\ell_2\) least-squares (LS) solution] is used for the plane-wave expansion described in Eq. (10), a plane-wave basis of 1000 plane waves of unknown amplitudes is considered, whose directions of propagation are distributed uniformly based on a Thomson problem. The number of plane waves should be greater than the number of measurement positions, for a proper representation of the measured pressure. The complex coefficient vector \(c\) corresponding to the wavenumber spectrum (i.e., the amplitudes of the waves) is estimated using Eq. (11). Tikhonov regularization [i.e., a \(\ell_2\) least-squares (LS) solution] is used.
for the regularized inversion, along with the L-curve criterion as a parameter-choice method.\textsuperscript{34} As for the expansion of the wavenumber spectrum in Eq. (7), only a limited number of spherical harmonics orders can be used in practice. The spherical harmonic expansion is truncated at $n_{\text{trunc}} = N = 7$, corresponding to 64 coefficients [so that $(N + 1)^2 = M$, where $M$ is the number of measurement positions]. Although possible, adding more spherical harmonics in the expansion of the wavenumber spectrum does not contain relevant information.\textsuperscript{37,38}

The resulting wavenumber spectrum magnitudes $|P(k)|$ and corresponding spherical harmonic expansions are illustrated in Fig. 3 (centre and right columns, respectively) for the third-octave band centred at 500 Hz (the wavenumber results have been averaged over the third-octave band, and the spherical harmonic expansion conducted on the averaged wavenumber spectra). The moments from the respective spherical harmonic expansions are displayed in terms of their magnitude $\sum_{n=-n}^{n} |A_{mn}|$. In the first configuration [case (a), least isotropic configuration], the contributing waves cover a partial section of the solid angle, and therefore all moments are needed to describe the magnitude of the wavenumber spectrum. In the second scenario [case (b)], the contributing waves cover half of the solid angle, resulting in a wavenumber spectrum that is best described by the monopole and dipole moments of its spherical harmonic expansion. In the last case [case (c), most isotropic case], the wavenumber spectrum is nearly constant over the sphere, and therefore its magnitude resides primarily on the monopole moment of its spherical harmonic expansion, i.e., $A_{mn}$ for $(m, n) \neq (0, 0)$ are (almost) null. These results are well in line with the estimated isotropy indicator values: 0.17, 0.38, and 0.96, for the three cases, respectively.

Figure 4 shows the isotropy indicators of cases (a), (b), and (c) as a function of frequency, for the third-octave bands ranging from 125 Hz to 1 kHz. The results confirm the isotropy of sound field (c) in the entire frequency range (values ranging between 0.93 and 0.97). The indicator is not unity because the sound field is due to 52 sources only (and therefore not perfectly isotropic).

The robustness to noise of the method is examined in Appendix B.

\section*{IV. EXPERIMENTAL RESULTS}

The validity of the proposed methodology is examined experimentally in a large (215 m$^3$) reverberation room at the Technical University of Denmark (DTU), with two different
A. Experimental results in the anechoic chamber

An omnidirectional source (an “Omnisource,” Brüel & Kjær), which radiates approximately like a point source, is placed 4 m away from the surface of the rigid spherical array. The source is driven with random white noise, and a spectral resolution of 1 Hz is used for the analysis. The pressure on the surface of the array is shown in Fig. 6(a) at 500 Hz ($ka = 0.89$). Tikhonov regularization is used for the regularized inversion of Eq. (11), along with the L-curve

![Image](image_url)

FIG. 4. Isotropy indicator as a function of frequency for the sound fields (a), (b), and (c). Truncation order: $N = 7$.

FIG. 5. Sixty-four-channels rigid spherical microphone array.

FIG. 6. Sound pressure level measured in the anechoic chamber at the 64 microphone positions (a); wavenumber spectrum, rotated for display convenience (b); spherical harmonic expansion (c). Frequency: 500 Hz. Truncation order: $N = 7$.

damping conditions. Validation measurements are also conducted in the DTU anechoic chamber (1000 m$^3$) and in the DTU Audio Visual Immersion Lab (AVIL), for they provide tractable environments to examine the methodology. All measurements are performed using a rigid spherical microphone array of radius $a = 9.75$ cm (Brüel & Kjær, Nærum, Denmark, see Fig. 5). The array consists of 64 microphones near-uniformly distributed over its surface, and can sample up to 7 orders of spherical harmonics.

Spherical microphone arrays are widely used for the analysis and reconstruction of complex sound fields and are particularly well suited for applications in enclosures, where the sound waves impinge on the array from multiple directions. Hence, several authors have proposed methods for quantifying diffuseness or isotropy, using spherical array measurements. The approach described in this work does not require a specific array configuration. We use the spherical array here (unlike in Sec. III) for convenience, as this equipment is readily available. Note that the scattering induced by the presence of the rigid sphere in the medium is accounted and compensated for.

As in Sec. III, a plane-wave basis of 1000 plane waves is considered for the plane-wave expansion described in Eq. (10), and the spherical harmonic expansion of the wavenumber spectrum is truncated at $n_{\text{trunc}} = N = 7$. 

$$a = 9.75 \text{ cm}$$
criterion as a parameter-choice method, to estimate the wavenumber spectrum.

Figures 6(b) and 6(c), respectively, show the wavenumber spectrum magnitude, and corresponding spherical harmonic expansion for the third-octave band centred at 500 Hz. It is apparent that all moments from the spherical expansion in Eq. (7) are needed to describe the magnitude of the angular spectrum, which in turn confirms the anisotropy of the sound field. Nevertheless, the angular resolution of the estimated wavenumber spectrum is compromised (the response exhibits a main lobe and concentric side lobes, as in a conventional array output) due to the limited measurement aperture. Hence, the isotropy indicator resulting from the least-squares solution is likely to have higher values than expected throughout the whole frequency range. In the present case of a single wave impinging on the array, the use of the compressive sensing (CS) framework would significantly improve the angular resolution, leading to a wavenumber spectrum closer to that in Fig. 1(a). This is shown in Appendix A. However, in this study, the conventional least-squares solution with Tikhonov regularization is chosen instead, as this choice is more appropriate for rooms and enclosures, where the wave field cannot be assumed to be spatially sparse.

Figure 7 shows the estimated isotropy indicator as a function of frequency for the third-octave bands ranging from 125 Hz to 1 kHz. The corresponding wavenumber spectra are also shown. At low frequencies, the spatial resolution is poor (as the wavelength is large compared to the dimension of the array, $ka < 0.45$), leading to a wide main lobe in the wavenumber spectrum. This in turn results in an isotropy indicator ranging between 0.25 and 0.35. At medium frequencies ($0.45 < ka < 0.89$), the resolution of the array is finer and the isotropy indicator has values around 0.2. At high frequencies ($ka > 0.89$), the resolution is higher (the main lobe and side lobes are therefore narrower), leading to an isotropy indicator below 0.2. Ideally, the isotropy indicator would be zero, as shown in Sec. II D, but it is not zero because of the regularization employed. When using a sparse regularization approach (see Appendix A), the indicator drops to 0.02 in the entire frequency range.

B. Experimental results in a sound field reproduction room

An experimental test is conducted using a 64-channel loudspeaker array, set in an anechoic chamber (Audio Visual Immersion Lab, AVIL, at DTU). The 64 loudspeakers (KEF LS50) are arranged over a spherical domain of radius 2.4 m, thereby surrounding the rigid spherical microphone array (i.e., the microphone array is placed at the centre of the loudspeaker array, so that the distance between any of the loudspeakers and the centre of the microphone array is 2.4 m, see Fig. 8). Only 52 out of the 64 available speakers were used for the experiment, so as to obtain a (quasi) uniform distribution of sources over the spherical domain. It should be noted that the speakers are not only positioned above and around the array, but also below the laboratory’s suspended floor. The speakers, which radiate approximately like point sources, are driven with random white noise signals with equal power, so as to approximate a homogeneous and isotropic sound field. Since the sources are uncorrelated, this experimental arrangement corresponds to an approximation to the perfect diffuse sound field as described in Ref. 4 and in the experimental investigation of Ref. 14. The pressure at the 64 microphone positions is calculated based on the measured autospectra. Although this is sufficient for the purpose of this study, one cannot possibly disregard the phase of the pressure signals when evaluating sound field diffusion in a room.

Figure 9 shows the pressure on the surface of the array at 125 Hz [Fig. 9(a)] and 400 Hz [Fig. 9(b)], the resulting wavenumber spectra (averaged over the respective third-octave bands of frequencies) and the corresponding spherical harmonic expansions ($N = 7$). As in Sec. IV A, Tikhonov regularization is used for the regularized inversion, along
with the L-curve criterion as a parameter-choice method. In both cases, the measured pressure has variations of $\pm 1$ dB, resulting in a wavenumber spectrum that is nearly constant over the spherical domain. Therefore, its magnitude is best described by the zeroth-order moment of its spherical harmonic expansion. At 125 Hz the isotropy indicator is 0.91, and 0.89 at 400 Hz. It does not reach exactly unity, due to errors resulting from differences in the speakers’ frequency responses ($\pm 2$ dB), positioning errors, and transducer mismatch.

C. Experimental results in the reverberation room

Experiments are conducted in a large (215 m$^3$) reverberation room, both empty and with an added sample of absorptive material on the floor. Figure 10(a) shows the absorption coefficient of the 10.8 m$^2$ sample, measured according to ISO 354 (Ref. 1) in one-third octave bands using the interrupted noise method and a Bruel & Kjær sound level meter (type 2250). The room complies with the ISO 354 requirements, and is essentially rectangular although there are 85 built-in concrete boundary diffusers and 12 hanging panel diffusers [see Fig. 10(b)]. The room is driven to steady-state conditions with random white noise using a built-in loudspeaker placed in one of the upper-corners of the room (that is, at a sufficiently large distance away from the surface of the rigid spherical array, so as to maximally excite the room modes and reduce the amount of direct radiation on the surface of the array). A spectral resolution of 0.125 Hz is used for the analysis, corresponding to a time window of 8 s. This corresponds to measuring at 6400 independent discrete frequencies with a frequency span of 800 Hz. The measurements cover the third-octave bands ranging from 125 Hz to 1 kHz. Once again, Tikhonov regularization is used for the regularized inversion, along with the L-curve criterion as a parameter-choice method.

Figures 11(a) and 11(b) compare the resulting wavenumber spectrum at 1 kHz, in the empty and damped room. In the undamped room [free of absorption, Fig. 11(a)], a few dominant incident directions are detected (i.e., a few waves that carry considerably more energy than others, seemingly
corresponding to the direct radiation from the source and a few early reflections), indicating that the field is not perfectly isotropic [as would be the case in the ideal example shown in Fig. 1(b)]. In the damped room [added absorption, Fig. 11(b)], the wavenumber spectrum is less omnidirectional, as there are no waves propagating in the positive z-direction, because no sound is being reflected by the absorbing sample ($\alpha \approx 1$ at 1kHz). This is in good agreement with results described in Ref. 40, which show that there exists a large influx of energy directed towards the absorber. The results are confirmed by the corresponding spherical harmonic expansions displayed in Fig. 11(c) ($N = 7$). In the undamped case, the wavenumber spectrum is best described by the monopole moment of its spherical harmonic expansion, yielding an isotropy indicator value of $i = 0.67$. The sound field is not perfectly isotropic, due to the few dominant directions in the wavenumber spectrum (the stationary sound field in a reverberation chamber driven with a single source is, in fact, not expected to be fully isotropic). In the damped case, the spherical harmonic expansion is no longer dominated by the monopole moment, resulting in a sound field that is less isotropic than in the empty room ($i = 0.35$).

Figure 12(a) compares the magnitude of the moments from the spherical harmonic expansions in the undamped and damped room, for the third-octave bands ranging from 125 Hz to 1 kHz. Figure 12(b) shows the corresponding isotropy indicators as a function of frequency. In the undamped room, the monopole moment dominates the spherical harmonic expansion of the wavenumber spectrum throughout the entire frequency range, yielding values of the isotropy indicator that range from 0.65 to 0.72. In the damped room, higher-order moments are required to describe the wavenumber spectrum. The isotropy indicator in this case ranges from 0.35 to 0.57, indicating that the sound field is less isotropic than in the undamped case. At low frequencies (below 400 Hz), the isotropy indicator is greater than at high frequencies, because the absorption in the room is lower. In fact, it can be observed that the absorption of the material [see Fig. 10(a)] influences the isotropy of the sound field: as the absorption increases, isotropy tends to decrease.

V. DISCUSSION

The current measurement system (64-channels microphone array of radius 9.75 cm) is not expected to provide valid results below 120 Hz, where the circumference of the
sphere corresponds to about 10% of the wavelength in air (\(ka = 0.1\)), nor at high frequencies, where aliasing effects start to appear. In the operational frequency range of the array, the results indicate that the isotropy indicator responds correctly to changes in the isotropy of the sound field. The robustness to noise of the method is examined in Appendix B, which shows that the method is fairly robust to perturbations for SNRs as low as 20 dB, a common range in room acoustic measurements. Nevertheless, its accuracy depends on the wavenumber spectrum estimation, which relies on (a) the measurement system: the sampling of the pressure field should be sufficient to estimate its wavenumber spectrum correctly; (b) the choice of the regularization scheme: in the present study, the wavenumber spectrum has been calculated via a conventional regularized least-squares inversion. This is a sensible choice for estimating the wave field in a reverberant room. However, alternative solution strategies (\(\ell_1\)-norm, elastic-net, etc.) can be further examined. An alternate estimation of the wavenumber spectrum based on the framework provided by CS is presented in Appendix A, with application to sparse problems.

An advantage of the approach described in this work is that it does not require a specific array configuration. As briefly mentioned in the introduction, other methods have been proposed for the estimation of diffuseness from a set of measured microphone signals. Epain and Jin suggested characterizing diffuseness based on the analysis of the spherical harmonic covariance matrix. Yet, the analysis in the spherical harmonic domain is performed on the recorded signals directly (rather than on the wavenumber spectrum), requiring the use of a spherical array of microphones. Moreover, the study is concerned with the estimation of diffuseness arising from the presence of multiple sources. The sound field in a reverberation room driven with noise from one source is, of course, quite different.

This work examines steady-state sound fields in a reverberation chamber. The results evaluate how isotropic the sound field is at a particular location of the room; hence, they do not directly evaluate the compliance of the reverberation room with ISO 354:2003 and ISO 140–10:1991 (which they do not directly evaluate the compliance of the reverberation chamber. The results evaluate how isotropic the sound field in a reverberation room driven with noise from a spherical array, and the pressure field can be sampled arbitrarily (e.g., using regular or random spatial sampling schemes). Furthermore, because of being formulated as an elementary wave model, the wavenumber spectrum can be obtained in a least-square sense using conventional regularization schemes, but also allows for alternative strategies (\(\ell_1\)-norm, elastic-net, etc.), conferring a broader application perspective.

The numerical and experimental results obtained in two anechoic chambers (one with a single source and another with an array of 52 sources uniformly distributed around a spherical microphone array) and in a reverberation chamber (both empty and with absorption on the floor) indicate that the method is suitable for assessing the isotropy of a sound field. The results convey an interesting prospect for characterizing the diffuse field conditions in enclosures.

ACKNOWLEDGMENTS

The authors would like to thank Antoine Richard and Marton Marschall for help with the experimental arrangement, and John L. Davy for comments and discussion. This work is funded by the Oticon Foundation.

APPENDIX A: WAVENUMBER ESTIMATION FOR SPARSE PROBLEMS

In the case of few sound waves impinging on the array, solving the system of linear equations in Eq. (11) via \(\ell_2\)-minimization yields a poor representation of the measured data (i.e., the solution tends to produce many non-zero coefficients), compromising the angular resolution, and consequently the value of the isotropy indicator. An alternate estimation of the wavenumber is based on the framework provided by CS that promotes a sparse solution to the problem (i.e., an optimal representation of the measured data with as few non-zero coefficients as possible) via \(\ell_1\)-minimization. The problem can be formulated in an unconstrained form by introducing a regularization parameter \(\lambda\) which determines the weight of the \(\ell_1\)-norm penalty:

\[
\hat{c} = \arg \min_c \| Wc - p \|^2_2 + \lambda \| c \|_1. \tag{A1}
\]

Equation (A1), which corresponds to the well-known LASSO formulation, is identical to Eq. (12), but using the \(\ell_1\)-norm \(\| \cdot \|_1\) instead.

The method is examined experimentally, based on the anechoic measurements introduced in Sec. IV A. The \(\ell_2\) least-squares (LS) solution obtained with Tikhonov regularization is compared with the CS (LASSO) solution. The CS solution is obtained as in Eq. (A1), and the LS solution is calculated as in Eq. (12).

Figure 13(a) shows the magnitude of the wavenumber spectrum resulting from the LS [Eq. (12)] and the CS [Eq. (A1)] solutions, respectively, for the third-octave band centered at 500 Hz. It is apparent that the obtained complex coefficients are significantly different. In the LS approach all of the wavenumber coefficients are non-zero, whereas the CS solution returns approximately four non-zero coefficients.

VI. CONCLUSION

An experimental method to evaluate sound field isotropy in enclosures is proposed in this study. The method is based on an analysis of the wavenumber spectrum in the spherical harmonics domain, which has suitable mathematical properties when it comes to examine isotropy.

Since the spherical harmonic expansion is performed on the wavenumber spectrum, and contrary to existing methods, the proposed method is not restricted to measurements with a spherical array, and the pressure field can be sampled arbitrarily (e.g., using regular or random spatial sampling schemes). Furthermore, because of being formulated as an elementary wave model, the wavenumber spectrum can be obtained in a least-square sense using conventional regularization schemes, but also allows for alternative strategies (\(\ell_1\)-norm, elastic-net, etc.), conferring a broader application perspective.
yielding a wavenumber spectrum that resembles roughly a Dirac delta function on the sphere. This in turn indicates that the CS solution accurately detects the incoming direction of the waves used in the expansion in Eq. (10) (i.e., the direction of arrival of the waves radiated by the loudspeaker). Figure 13(b) compares the corresponding spherical harmonic expansions of the wavenumber spectra resulting from the LS and CS estimations, respectively ($N = 7$). The CS solution results in less energy in the monopole moment and lower order moments, in good agreement with the results obtained

FIG. 13. Magnitude of the wavenumber spectrum in the anechoic chamber resulting from the LS (top) and CS (middle) solutions, respectively (a); corresponding spherical harmonic expansions (b). Frequency: 500 Hz. Truncation order: $N = 7$.

FIG. 14. Isotropy indicator resulting from the LS and CS solutions, respectively, as a function of frequency in the anechoic chamber. Numerical predictions are superimposed. Truncation order: $N = 7$.

FIG. 15. (Color online) Box plots of the isotropy indicator as a function of frequency, for the numerical study presented in Sec. III. The SNR values vary between 20 dB SNR and 60 dB SNR. Truncation order: $N = 7$. 

...
in the ideal case of a single propagating wave, as illustrated in Fig. 1(a). The LS solution yields an isotropy indicator value of 0.17, whereas the CS solution results in a value of 0.02, much closer to zero, representing accurately the anisotropy of the sound field.

Figure 14 shows the isotropy indicator as a function of frequency, for the third-octave bands ranging from 125 Hz to 1 kHz, resulting from the LS and CS [Eq. (A1)] solutions, respectively. It is apparent that the LS solution overestimates the isotropy indicator, whereas the CS solution yields values close to zero throughout the entire frequency range, in agreement with the theoretical considerations presented in Sec. II. Nonetheless, selecting the $\ell_1$-norm is a poor regularization choice when processing the sound field in reverberant enclosures, which yields non-physical solutions since the problem is not sparse. A $\ell_2$ least-squares solution with Tikhonov regularization is therefore best suited to applications in rooms.

Further numerical results were determined based on simulations using an identical 9.75 cm radius rigid-sphere array with 64 microphones. The predicted isotropy of the sound field.

APPENDIX B: ROBUSTNESS TO NOISE

The influence of the signal-to-noise ratio (SNR) on the evaluation of the isotropy indicator is investigated, based on the numerical study presented in Sec. III. Figure 15 shows box plots of the isotropy indicator as a function of frequency, for the same source configurations as in Sec. III [Figs. 15(a), 15(b), and 15(c), respectively] and SNR values varying between 20 dB SNR and 60 dB SNR. For each SNR, the isotropy indicators have been computed for 25 separate realizations, over the third-octave bands ranging from 125 Hz to 1 kHz. The central marks in the figures are the median of the isotropy indicator, the box represents the first and third quartiles (isotropy indicator between 25% and 75%), and the whiskers are 1.5 times the interquartile distance. Outliers outside this range are removed and correspond to wrong automatic-choice regularization parameters that can be detected from inspection of the L-curve. For the three test cases, the results show that the method is robust to perturbations up to 20 dB SNR, which is sufficient, as noise levels in room acoustic measurements are typically lower.

A wavenumber approach to quantifying non-uniform sound incidence in measurements of sound absorption in the reverberation chamber

Mélanie Nolan, Samuel A. Verburg, Cheol-Ho Jeong, Jonas Brunskog and Efren Fernandez Grande
Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Building 352, DK-2800 Kongens Lyngby, Denmark

Summary
Measured values of acoustic absorption obtained from standard reverberation chamber measurements often differ from theoretical random incidence absorption coefficients. These discrepancies mostly arise due to non-isotropic sound incidence on the absorbing specimen and diffraction at the sample edges. This study examines an experimental method for characterizing the distribution of sound incidence on the specimen under test conditions. The methodology relies on estimating the wavenumber spectrum to determine the magnitude of the sound waves arriving from definite directions onto the absorbing plane. The method is examined experimentally, based on measurements with a programmable robotic arm in a reverberant room in two damping conditions. The corresponding spatial distribution of the active sound intensity field can also be inferred, making it possible to benchmark the proposed methodology with observations of the incident energy flows.

PACS no. 43.55.Nd, 43.55.Br

1. Introduction
The conventional method of measuring sound absorption in a reverberation room relies on the assumption of a perfectly diffuse sound field [1]. From the standpoint of the random wave theory [2-3], this assumption implies that: i) the directions of propagation of the waves incident on the absorber are uniformly distributed over a hemisphere (i.e. isotropic sound incidence); ii) these plane waves have random relative phases. Yet, the sound field in a reverberation chamber driven by a single source does not fulfill these conditions, yielding inaccurate values of absorption coefficient. Besides, a common reason for inconsistent absorption coefficients in different reverberation chambers (especially at low frequencies) is the lack of a uniform sound incidence on the absorbing sample.

Most of the discrepancies between statistical reverberation theory and measurement results can be explained by a modal approach to reverberation in rooms. An analysis in terms of normal modes of vibration is found in Refs 4 to 6 for rectangular rooms with uniformly absorbing walls. The angular distribution of normal modes is of importance, as each mode is attenuated depending on its orientation with respect to the boundaries and the normal impedance of the bounding surfaces [5]. Hence, the excited modes of vibration may be divided into a number of subgroups having common properties, which led Hunt et al. [7] to the logical conclusion that a group of decay equations must be used in place of the single reverberation formula.

Nonetheless, wave-based methods are most often restricted to consider only rectangular rooms containing no scattering objects, and in which absorption material is uniformly distributed at the six bounding surfaces. The methods are also restricted to small values of the absorption coefficient. Geometrical acoustics can deal with complex geometries and non-uniform absorption distributions [8]. However, these techniques are limited to observe only the behaviour of simulated sound fields and prove most useful in the high-frequency range. Alternatively, Nilsson [9] described the decay process in rectangular rooms with high absorption at one surface and low absorption at the remaining surfaces using the...
Statistical Energy Analysis (SEA) formalism. A two-system SEA model was developed, where the sound field is subdivided into a grazing and a non-grazing part, to distinguish between waves travelling almost parallel to the absorbing wall and waves having oblique incidence. A break in the decay curve was reported, where the slope changes from initial to final value, revealing the influence of the non-grazing and grazing waves respectively. Yet, the method examines the decay process. To date, no experimental method allowing for direct observation of the non-uniform sound incidence on the absorbing sample is available.

This study proposes an experimental method to characterize and observe the sound incidence on the absorbing sample in a reverberant room. The methodology relies on estimating the wavenumber spectrum from a set of measured microphone signals, to determine the magnitude of the sound waves arriving from definite directions onto the absorbing plane. The underlying hypothesis is that the wavenumber spectrum contains the directional properties of the waves that constitute the sound field. As such, it is possible to separate incident from reflected components. The proposed method can cope with complex geometries (including cases when scattering objects are added to the room), non-uniform absorption distributions, and is not restricted to any value of absorption coefficient nor to any frequency range (other than the experimental limits of the acquisition system). The corresponding spatial distribution of the sound intensity field can also be inferred, making it possible to visualize and characterize the incident energy flows. To the author’s knowledge, it is the first time that this is visualized experimentally.

2. Theoretical background

2.1. Wavenumber spectrum

We consider the steady-state sound field produced by a pure-tone source in a reverberation chamber in the immediate vicinity of an absorbing sample. The resulting sound field at the point characterized by the vector \( r_m = (x_m, y_m, z_m) \) can be represented as a superposition of \( L \) plane waves, each traveling in a direction specified by the wavenumber vector \( \mathbf{k}_i = (k_{ix}, k_{iy}, k_{iz}) \). Each plane wave may have different amplitudes and phases, which we include in the complex coefficients \( P(\mathbf{k}_i) \):

\[
p(r_m) = \sum_{l=1}^{L} P(\mathbf{k}_i) e^{-j k_r r_m},
\]

where \( P(\mathbf{k}_i) = \vert P(\mathbf{k}_i) \vert e^{j \phi(\mathbf{k}_i)} \) is the wavenumber spectrum (with \( \vert P(\mathbf{k}_i) \vert \)) and \( \phi(\mathbf{k}_i) \) its magnitude and phase, respectively, and the directions of propagation \( \mathbf{k}_i \) of the plane waves are uniformly distributed over a spherical domain. We must keep in mind that all propagating plane waves satisfy the condition \( \vert \mathbf{k}_i \vert^2 = k_i^2 = k_{ix}^2 + k_{iy}^2 + k_{iz}^2 \) with \( k_i^2 \geq k_{ix}^2 + k_{iy}^2 \), indicating that evanescent waves are not present. Besides, since we are interested in the sound field produced by a pure-tone with frequency \( f_0 \), all propagating waves should appertain to the surface of the radiation sphere of radius \( k_0 = 2\pi f_0/c \) in the wavenumber domain.

The pressure field sampled at a discrete number of positions \( M \) can be expressed in matrix form as

\[
p = Hx,
\]

where \( p \in \mathbb{C}^M \) is the measured sound pressure vector, \( x \in \mathbb{C}^L \) is a complex coefficient vector containing the wavenumber spectrum \( P(\mathbf{k}_i) \) in Eq. (1) and \( H \in \mathbb{C}^{M \times L} \) is the sensing matrix containing the plane wave functions \( e^{-j \mathbf{k}_i \cdot \mathbf{r}_m} \). This is an ill-posed (typically underdetermined) problem, which requires regularized inversion. The solution of Eq. (2) can be calculated in a least-squares sense, i.e. via a regularized matrix pseudo-inverse. The problem can be formulated as an unconstrained problem, [10] introducing a regularization parameter \( \lambda \), which determines the penalty weight of the \( \ell_2 \)-norm of the solution vector. Throughout this study, the \( \ell_2 \)-norm of the solution is chosen,

\[
x = \arg\min_x (\|Hx - p\|_2^2 + \lambda \|x\|_2^2),
\]

which has the well-known closed form analytical solution

\[
x = H^T (HH^T + \lambda I)^{-1} p,
\]

where the superscript \( H \) denotes the conjugate transpose and \( I \) is the identity matrix. Eq. (4) corresponds to the least-squares solution of the problem with Tikhonov regularization. [11] The conventional regularized least-squares inversion is a sensible choice for estimating the wave field in a reverberant room, where sound waves impinge on
the absorbing sample from multiple directions. [12].

2.2. Isotropic sound incidence

As seen in Section 2.1., the estimation of the wave amplitudes $|P(k)|$ in Eq. (2) requires the discretization of the continuum of space directions, resulting in $L$ possible directions of propagation uniformly distributed over a spherical domain. These directions of propagation may be subdivided into two hemispheres of possible directions, representing the incident and reflected wave fields, respectively [13]. From now onwards, we write $P_I(k_i) = x_i$ and $P_R(k_r) = x_r$ respectively, the wavenumber spectra comprising the incident and reflected sound fields. It should be remarked that the wavenumber spectrum represents the direction of propagation and not the direction of arrival, as illustrated in Figure 1, which describes the reflection of a spherical sound wave on a plane rigid surface in k-space.

Fig. 1 – Wavenumber spectrum due to a point source - Subdivision into incident and reflected wave fields.

In order to evaluate the uniformity of sound incidence on the absorber plane, it is necessary to analyze the direction of the waves that comprise the incident sound field. In this regard, the magnitude of the wavenumber spectrum obtained from Eq. (2) provides a powerful means to describe the isotropy of a sound field incident on an absorber. If the sound field incident on the absorber is isotropic, its wavenumber spectrum is hemispherically symmetric (i.e. the magnitude of the waves representing the incident sound field is uniformly distributed over a hemisphere). Contrarily, if the incident sound field is anisotropic, the wavenumber spectrum is asymmetric, as there is variable energy in different directions (see Fig. 2). Figure 2(a) shows the wavenumber spectrum corresponding to a perfectly uniform incident field, where the continuum of space directions is discretized into 1000 directions of propagation. It is apparent that the wavenumber spectrum is constant over the hemisphere comprising the incident wave field. Conversely, Figure 2(b) displays the case of a single propagating wave, which yields a wavenumber with only few non-zero coefficients.

Fig. 2 – Wavenumber spectrum corresponding to: (a) a perfectly uniform incidence; (b) a single wave.

2.3. Incident intensity reconstruction

Once the wavenumber spectrum has been determined, the incident component of the sound pressure $p_i$ can be inferred anywhere in the vicinity of the absorbing sample

$$ p_i = H_R x_i \quad (5) $$

where $p_i \in \mathbb{C}^K$ is the sound pressure vector estimated at a set of $K$ positions $r_k = (x_k, y_k, z_k)$, $H_R \in \mathbb{C}^{K \times N}$ is the reconstruction matrix, $x_i \in \mathbb{C}^N$ is the complex coefficient vector containing the wave components that describe the incident sound field on the plane absorber, and $N$ is the number of wave directions conforming the incident sound field. The particle velocity vector can be calculated
from Euler’s equation of motion, \( u = -\nabla p / j \omega \rho \), thus
\[
u_i = -\frac{1}{j \omega \rho} \nabla H_{R_i} x_i, \quad (6)
\]
where \( \nabla H_{R_i} \in \mathbb{C}^{K \times N} \) contains the gradient of the reconstruction matrix. The incident sound intensity vector at each reconstruction point is now obtained as
\[
I_i = \frac{1}{2} \Re \{ p_i u_i^* \}, \quad (7)
\]
where the superscript * denotes the complex conjugate. The full incident intensity vector can be calculated via the Hadamard product of the reconstructed sound pressure vector and the complex conjugate of the particle velocity. It is apparent that the reconstruction provides a full characterization of the incident acoustic field. The reconstruction could alternatively be based on the matrix \( H \) as in Eq. (2) to obtain the total sound field (net intensity).

3. Results

The validity of the proposed methodology is examined experimentally in a small (40 m³) reverberant rectangular room (reverberation time: 2.5 s; critical distance: 0.25 m) in two damping conditions: undamped and with extra absorption on one of its walls (see Fig. 3, the absorption coefficient of the material is given in Table 1). A programmable robotic arm UR5 (Universal Robots, Odense, Denmark) moves a free field microphone (Brüel & Kjær, Nærum, Denmark) within a rectangular volume of dimensions 0.30 m x 0.75 m x 0.75 m located in the immediate vicinity of the absorbing wall, creating a random array of 296 measurement positions (see Fig. 3). The use of a robotic arm in place of a traditional microphone array makes it possible to circumvent positioning and transducer mismatch errors [14]. The room is excited by a loudspeaker (Dynaudio BM6, Skanderborg, Denmark) driven with a logarithmic sine sweep ranging from 20 Hz to 20 kHz with duration of 10 s. The loudspeaker is placed in a corner of the room opposite to the absorbing wall, so as to maximally excite the room modes and reduce the amount of direct radiation at the measurement positions. The frequency response is measured at the 296 positions with a spectral resolution of 0.2 Hz. The total acquisition time is 2 hours.

For the plane-wave expansion described in Eq. (1), a plane-wave basis of 1000 plane waves of unknown amplitudes is considered, whose directions of propagation are distributed uniformly based on a Thomson problem [15]. The number of plane waves is typically greater than the number of measurement positions, for a proper representation of the measured pressure [16,17]. The complex coefficient vector \( x \) corresponding to the wavenumber spectrum (i.e. the amplitudes of the waves) is estimated using Eq. (2) in front of the absorber plane. Tikhonov regularization (i.e. a \( \ell_2 \) least-squares (LS) solution) is used for the regularized inversion, along with the L-curve criterion as a parameter choice method [11].

Table 1 – Absorption coefficient data for the glass wool in third-octave bands (Miki’s model).

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Absorption Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>125</td>
<td>0.44</td>
</tr>
<tr>
<td>250</td>
<td>0.76</td>
</tr>
<tr>
<td>500</td>
<td>1.0</td>
</tr>
<tr>
<td>1000</td>
<td>1.0</td>
</tr>
</tbody>
</table>

3.1. Wavenumber spectrum distribution

Figure 4 shows the estimated magnitude of the wavenumber spectrum in the undamped room at 87 Hz, at a natural frequency corresponding to the (1, 1, 1) oblique mode. The eight interfering plane waves that compose the mode are clearly identified in the wavenumber results (although only four of them are visible in Figure 4). These plane waves propagate in the directions \( (k_x, k_y, k_z), (k_x, -k_y, k_z), (k_x, -k_y, -k_z), (k_x, k_y, -k_z), (-k_x, k_y, k_z), (-k_x, -k_y, k_z), (-k_x, k_y, -k_z), (-k_x, -k_y, -k_z) \), and may be interpreted as one obliquely incident plane wave and its reflections from the walls of the room.
Figure 5 compares at top view of the magnitude of the wavenumber spectrum at 1 kHz in the undamped and damped room. In both rooms, a few dominant incident directions are detected (i.e. a few waves that carry considerably more energy, seemingly corresponding to a few early reflections), indicating that the field is not perfectly isotropic (as would be the case on the ideal example shown in Fig. 2(a) – the stationary sound field in a rectangular chamber driven with a single source is, in fact, not expected to be fully isotropic). In the damped room (added absorption, Fig 5(b)), no waves are propagating in the negative $x$-direction, as no sound is being reflected by the absorbing plane (the absorption coefficient is almost unity at 1 kHz).

The wavenumber spectrum hemisphere that comprises the incident wave field may be plotted as a function of the angle of incidence $\theta$. The definition of the angles is given in Figure 6. If the sound field incident on the absorber plane is isotropic, the magnitude $|P(k)|$ of the wavenumber spectrum (averaged over the angle $\phi$, see Figure 6) should be a constant function of the angle of incidence $\theta$. Figure 7 shows the averaged magnitude of the wavenumber spectrum as a function of the angle of incidence at 125 Hz, 500 Hz and 1 kHz in the undamped and damped room, respectively. For each frequency in the undamped case, the magnitude of the wavenumber spectrum is not a constant function of the angle of incidence, indicating some dominant incident directions, which in turn confirms the anisotropy of the wave field. In the damped case, the wavenumber spectrum magnitude exhibits low values at normal incidence, indicating that only few waves are propagating in the $x$-direction (note that the magnitude is higher at 125 Hz, because the absorption is lower). At oblique incidence, the magnitude of the incident field increases, as these angles are less affected by the presence of the absorber [9]. The results confirm that the waves that comprise the incident sound field are attenuated depending on their orientation with respect to the bounding surfaces and the absorption of the plane absorber. One would expect the
magnitude of the wavenumber spectrum to continue increasing at grazing incidence. This is however not the case at 500 Hz and 1 kHz, where the magnitude, in fact, decreases. A reason for the observed drop is possibly scattering induced by the measurement device and the edge of the absorber itself. The scattered field may be redirected towards the absorber at oblique angles of incidence (indeed, the drop is not as marked at 125 Hz, where scattering is weaker).

Fig. 7 – Magnitude of the incident wave field as a function of the angle of incidence.

3.2. Intensity distribution

As seen in section 2.3, the corresponding spatial distribution of the sound intensity field can also be inferred (see Eqs 5 to 7), making it possible to visualize the net and incident energy flows.

Figure 8 shows a volumetric reconstruction of the total net active intensity in the undamped and damped room, respectively, for the third-octave band centered at 1 kHz. The intensity vectors are computed over a cubical lattice in the vicinity of the absorbing surface (lattice spacing of 5 cm) using Eqs (5) to (7), where $H_k$ has been replaced by $H$ as the reconstruction matrix. The length of the vector is proportional to the intensity (in W/m$^2$) at the base of the cone, and the cones are color-coded to also show the magnitude of the intensity vector. In the undamped chamber (free of absorption), the intensity field seems to emanate from multiple directions. In the damped room, the intensity is pointing towards the absorber (located in the $yz$-plane at $x = L_x$). This is in agreement with previous findings (see Ref. 18). A dominant direction is detected (red cones in Fig. 8(b)), seemingly corresponding to a stronger reflection from the nearest wall (it is interesting to note that this direction is also detected in the wavenumber spectrum in Figure 5(b)).

Fig. 8 – Volumetric intensity reconstruction of the total net field in the undamped (a) and damped (b) rooms for the third-octave band centered at 1 kHz.

For further investigation, an incident intensity distribution is constructed by estimating the incident intensity field at 254 discrete positions...
distributed over a virtual hemispherical surface centered at a reconstruction point. The perpendicular direction to the absorber plane is assumed to be the normal incidence ($\theta = 0^\circ$). The incident sound intensity as a function of the incidence angle $\theta$ can be computed by averaging the magnitude of the individual intensity vectors over the azimuth angle $\phi$. The incident sound intensity distribution is displayed in Figure 9, for the third-octave bands 200 Hz, 250 Hz, 400 Hz, 500 Hz, 800 Hz and 1 kHz, respectively, for ease of comparison across frequency. The intensity magnitudes are normalized by the maximum value for each frequency band. At high frequencies, it is apparent that the incident intensity decreases with the angle of incidence. The normally incident intensity is the strongest whereas the grazing incident intensity is the weakest. At lower frequencies, the incident intensity distribution is more uniform (especially at 200 Hz) because the absorption is lower. At low frequencies, an increase in the reverberant energy can be expected due to the acoustically harder surface. Hence, the incident intensity is redistributed, yielding a more uniform incidence. These results are in line with general observations made by Jeong, [8] who simulated intensity distributions in various settings using a phased beam tracing.

4. Discussion

The proposed methodology is valid for complex geometries (including cases where scattering objects such as panel or boundary diffusers are added to the room), non-uniform absorption distributions, and is not restricted to any value of absorption coefficient nor to any frequency range (other than the working frequency range of the acquisition device). The experimental validation is here limited to the simple case of a rectangular room, for it provides a tractable environment to examine the methodology. This work is also restricted to consider the steady-state sound field in a reverberation chamber. However, its extension to the decay process is straightforward.

In the operational frequency range of the measurement system, the results indicate that the method is suitable for assessing the isotropy of the incident sound field on a sample. Its accuracy depends on the wavenumber spectrum estimation, which relies on (i) the measurement system: the sampling of the pressure field should be sufficient to estimate its wavenumber spectrum correctly; (ii) the choice of the regularization scheme: in the present study, the wavenumber spectrum has been calculated via a conventional regularized least-squares inversion. This is a sensible choice for estimating the wave field in a reverberant room. [12] However, alternative solution strategies (elastic-net, etc) can be used. It should also be noted that the proposed methodology does not require a specific array configuration, and the pressure field can be sampled arbitrarily (e.g. using regular or random spatial sampling).

5. Conclusions

An experimental method allowing for direct characterization of the non-uniform sound incidence on the absorbing sample in conventional measurements of sound absorption is proposed in this study. The methodology relies on estimating the wavenumber spectrum from a set of measured microphone signals, to determine the magnitude of the sound waves arriving from definite directions onto the absorbing plane. The proposed method is valid for any room geometry and absorption distribution.

The experimental results obtained in a reverberant room in two damping configurations with a programmable robotic arm indicate that the method is suitable for assessing the isotropy of the incident sound field on a sample. The

![Fig. 9 – Incident intensity as a function of the angle of incidence in the damped room for the third-octave bands centered at 200 Hz, 250 Hz, 400 Hz, 500 Hz, 800 Hz and 1 kHz.](image-url)
corresponding spatial distribution of the active sound intensity field was also inferred, enabling to visualize the net and incident energy flows.

Acknowledgement
The authors would like to thank Antoine Richard for comments and discussion. This work is funded by the Oticon Foundation.

References
A method to locate spatial distribution of scattering centers from ultrasonic backscatter signal

Sonic boom propagation around a large building using a combined ray tracing/radiosity method

Spatial power spectral density estimation using a Welch coprime sensor array processor

Passive structural monitoring based on data-driven matched field processing

EXPERIMENTAL CHARACTERIZATION OF THE SOUND FIELD IN A REVERBERATION ROOM
Experimental characterization of the sound field in a reverberation room\textsuperscript{a)}

Mélanie Nolan,\textsuperscript{b)} Samuel A. Verburg, Jonas Brunskog, and Efren Fernandez-Grande

Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Building 352, Ørsted Plads, DK-2800 Kongens Lyngby, Denmark

(Received 31 July 2018; revised 12 March 2019; accepted 18 March 2019; published online 23 April 2019)

Measured values of acoustic absorption obtained from standardized reverberation-chamber measurements often differ across laboratories. These discrepancies arise due to non-isotropic sound incidence on the absorbing specimen, diffraction at the sample edges, and differences in the chambers’ shapes and dimensions. The present study examines an experimental method for characterizing the distribution of sound incidence on the specimen in the steady state. The methodology relies on a plane wave decomposition (i.e., estimation of the wavenumber spectrum) to determine the magnitude of the sound waves arriving from definite directions onto the absorbing sample. Based on this decomposition, the sound pressure, particle velocity, and sound intensity can be reconstructed in the vicinity of the absorbing specimen. One can distinguish between the incident and reflected components of the sound field, making it possible to characterize the incident energy flows. Measurements with a programmable robotic arm are conducted in a reverberation chamber in two damping conditions (empty and with absorption on the floor). The quantitative accuracy of the method is examined via an estimation of the sample’s angle-dependent absorption coefficient, showing good agreement with theoretical predictions. It is anticipated that the proposed method will be of value in explaining the deviations encountered across standardized laboratories.


Pages: 2237–2246

I. INTRODUCTION

Reverberation-room measurements\textsuperscript{1} of sound absorption coefficients are based on the simple reverberation formulas,\textsuperscript{2,3} provided that certain assumptions are verified experimentally. In particular, the procedure relies on the underlying notion of a perfectly diffuse sound field, i.e., a sound field in which energy density is uniformly distributed in space and energy flow is isotropic. The absorption properties of the sample are represented by a (random-incidence) absorption coefficient, which corresponds to the average fraction of power absorbed by the sample when sound is arriving on it equally from all directions; i.e., under the assumption of isotropic sound incidence.\textsuperscript{3} In an attempt to achieve isotropic sound incidence, it is common practice to install a number of randomly oriented scattering panels above the absorptive sample. Yet, the sound field in a reverberation chamber containing a single concentrated sample of absorbing material is often far from diffuse, and, whether the panels work or not, the sample is unlikely to be exposed to isotropic sound incidence.\textsuperscript{4} Consequently, the reverberation formulas produce systematic errors, and yield inconsistent absorption coefficients from one chamber to the other, even under the conditions specified by the standard.\textsuperscript{5,6}

\textsuperscript{a)}Portions of this work were presented in “A wavenumber approach to quantifying non-uniform sound incidence in measurements of sound absorption in the reverberation chamber,” Proceedings of Euronoise 2018, Heraklion, Crete, May 2018.

\textsuperscript{b)}Electronic mail: melnola@elektro.dtu.dk

In the literature dealing with the relation between absorption coefficients and directional properties of the sound field in reverberation rooms, no experimental investigation of the distribution of incident acoustic energy on the measuring sample has been reported. An analysis in terms of normal modes of vibration is found in Ref. 3 and Refs. 7–12, in the case of rectangular rooms with no scattering objects. The results analytically demonstrate that the sample is not exposed to isotropic sound incidence, as each mode will have distinctive attenuation characteristics depending on its orientation with respect to the boundaries and the impedance of the bounding surfaces. A related analysis is suggested by Nilsson,\textsuperscript{13} who describes the decay process in rectangular rooms with high absorption at one surface and low absorption at the remaining surfaces using the Statistical Energy Analysis (SEA) formalism. A two-system SEA model is developed, where the sound field is subdivided into a grazing and a non-grazing part. A break in the decay curves is reported, corresponding to different decay rates for waves travelling almost parallel to the absorbing surface (grazing waves), and waves having oblique incidence (non-grazing waves). Yet, the method is concerned with the decay process and the inspection of measured decay curves.

To date, no experimental method allowing for direct observation of the distribution of sound incidence is available. As far as the authors are aware, the matter has only been addressed numerically, using ray/beam-tracing techniques.\textsuperscript{14,15} (Jeong\textsuperscript{15} reviews several of these methods). As a matter of fact, the incident sound energy on a surface in a reverberation room has rarely been measured, mostly due to
a lack of experimental methods to do so. The incident sound intensity cannot be measured using traditional intensity measurement systems, since such measurement gives the net sound intensity (i.e., incident plus reflected). To the authors’ knowledge, the only study that reports the measurement of this quantity is that by Jacobsen et al.\textsuperscript{16} They measured the sound intensity incident on a wall in a reverberation room from pressure measurements at distributed positions using near-field acoustic holography.\textsuperscript{17}

The aim of the present study is to examine experimentally the distribution of sound incidence in reverberation rooms used for standardized measurements of sound absorption coefficients. An experimental framework is developed to observe and characterize the incidence of sound waves on the sample. The method relies on an elementary plane wave expansion to describe the sound field on a set of microphones (as in a recent investigation examining sound field isotropy in reverberation chambers).\textsuperscript{18} Based on this wave expansion, it is possible to determine the magnitude of the sound waves arriving from definite directions onto the absorbing plane.\textsuperscript{19} Further, one can reconstruct the sound field over a three-dimensional domain in the vicinity of the absorber, inferring all acoustic quantities: sound pressure, particle velocity, and sound intensity. It is possible to reconstruct either the total sound field, or just the incident (or reflected) one,\textsuperscript{20} making it possible to visualize and characterize the power flow incident on the sample. To the authors’ knowledge, it is the first time this is visualized experimentally.

The present paper is organized as follows: in Sec. II, the theoretical background and methodology are presented. In Sec. III, an experimental study is presented based on measurements in a reverberation chamber using a programmable robotic arm. The quantitative accuracy of the method is examined via an estimation of the sample’s angle-dependent absorption coefficient. In this paper, methods for the measurement of acoustic absorption are not investigated. The emphasis is on the experimental technique for sound field analysis.

II. THEORETICAL BACKGROUND

A. Wavenumber (or angular) spectrum

We consider the steady-state sound field produced by a pure-tone source in a reverberation chamber. The resulting sound field in the vicinity of an absorbing sample can be represented as a superposition of plane waves, each traveling in a direction specified by the wavenumber vector $\mathbf{k} = (k_x, k_y, k_z)$. Each plane wave may have a different amplitude and phase, included in the complex coefficient $P(\mathbf{k})$.\textsuperscript{18}

$$p(r_m) = \int_{-\infty}^{+\infty} P(\mathbf{k}) e^{-j(k_x x_m + k_y y_m + k_z z_m)} \, dk,$$

(1)

where $r_m = (x_m, y_m, z_m)$ and the integrals represent a three-dimensional inverse Fourier transform in $k_x, k_y,$ and $k_z$, respectively. The quantity $P(\mathbf{k}) = |P(\mathbf{k})| e^{i\phi(\mathbf{k})}$ is the wavenumber (or angular) spectrum, with $|P(\mathbf{k})|$ and $\phi(\mathbf{k})$ its magnitude and phase, respectively. All plane waves satisfy the condition $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$ with $k_x^2 \geq k_y^2 + k_z^2$ (indicating that they are propagating waves). Time dependency $e^{i\omega t}$ is omitted.

Introducing spherical coordinates, $x_m = r \sin \vartheta \cos \phi, \ y_m = r \sin \vartheta \sin \phi, \ z_m = r \cos \vartheta$ and $k_x = k \sin \vartheta \cos \phi, \ k_y = k \sin \vartheta \sin \phi, \ k_z = k \cos \vartheta$, and assuming a pure-tone sound field with frequency $f_0 = k_0 c / (2\pi)$, Eq. (1) becomes\textsuperscript{18}

$$p(r, \vartheta, \phi) = \int_0^{2\pi} \int_0^\pi \tilde{P}(\theta, \phi) e^{-jkr_0 (\sin \vartheta \sin \theta \cos (\phi - \varphi) + \cos \theta \cos \phi)} \sin \vartheta d\theta d\phi,$$

(2)

where $\tilde{P}(\theta, \phi)$ denotes the two-dimensional wavenumber spectrum expressed in spherical coordinates. In practice, the two-dimensional wavenumber spectrum $P(\theta, \phi)$ is obtained using a discrete plane wave expansion, based on a discrete approximation of Eq. (2)

$$p(r_m) \approx \sum_{l=1}^{L} \tilde{P}(\mathbf{k}_l) e^{-j\mathbf{k}_l \cdot r_m},$$

(3)

where the directions of propagation of the plane waves are uniformly distributed over a spherical domain. In the limit $L \to +\infty$, the pressure distribution in Eq. (2) is obtained.

The pressure field is sampled at a discrete number $M$ of positions, and can be expressed in matrix form as

$$\mathbf{p} = \mathbf{Hx},$$

(4a)

$$\mathbf{p} = \begin{bmatrix}
  e^{-j\mathbf{k}_1 \cdot \mathbf{r}_1} & e^{-j\mathbf{k}_1 \cdot \mathbf{r}_1} & \cdots & e^{-j\mathbf{k}_1 \cdot \mathbf{r}_1} \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{-j\mathbf{k}_m \cdot \mathbf{r}_m} & e^{-j\mathbf{k}_m \cdot \mathbf{r}_m} & \cdots & e^{-j\mathbf{k}_m \cdot \mathbf{r}_m}
\end{bmatrix}
\begin{bmatrix}
  \tilde{P}(\mathbf{k}_1) \\
  \tilde{P}(\mathbf{k}_2) \\
  \vdots \\
  \tilde{P}(\mathbf{k}_M)
\end{bmatrix},$$

(4b)

where $\mathbf{p} \in \mathbb{C}^M$ is the measured sound pressure vector, $\mathbf{x} \in \mathbb{C}^L$ is a complex coefficient vector containing the wavenumber spectrum $\tilde{P}(\mathbf{k}_l)$ in Eq. (3) (i.e., $\mathbf{x}$ is the vector of unknowns that we want to solve for), and $\mathbf{H} \in \mathbb{C}^{M \times L}$ is the transfer (or sensing) matrix containing the plane wave functions $e^{-j\mathbf{k} \cdot \mathbf{r}}$. The problem is typically underdetermined ($L > M$), as there are more unknown coefficients than measurement points. The estimation of $\mathbf{x}$ is classically obtained via a regularized matrix pseudo-inverse (see Sec. II D).\textsuperscript{21} It should be noted that no assumption concerning the shape of the measurement area is necessary. In fact, the pressure field can be sampled randomly over an arbitrary volume.

B. Isotropic sound incidence

In order to evaluate the uniformity of sound incidence on the absorber plane, it is necessary to analyze the direction of the waves that comprise the incident sound field (the sound field due to the waves that propagate towards the absorber). The two-dimensional wavenumber spectrum $\tilde{P}(\theta, \phi)$ obtained in the vicinity of the absorbing sample may be subdivided into two hemispheres of possible directions,
representing the incident and reflected fields, respectively. Figure 1 illustrates $P(\pi - \theta, \phi)$ and $P(\theta, \phi)$ (with $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq 2\pi$), the wavenumber spectra comprising the incident and reflected sound fields, respectively, where it is assumed that the $z$-axis points in the direction perpendicular to the sample, outward from it. It should be recalled that, with the chosen time convention, the wave-number spectrum represents the direction of propagation and not the direction of arrival.

If the sound field incident on the absorber is isotropic, its wavenumber spectrum $P(\pi - \theta, \phi)$ is hemispherically symmetric (i.e., the magnitude of the waves representing the incident sound field is uniformly distributed over a hemisphere). Contrarily, if the incident sound field is anisotropic, the wavenumber spectrum is asymmetric, as there will be variable energy in different directions.

C. Absorption coefficient

Based on the sound field separation defined in Sec. II B and because the problem is formulated as an elementary wave expansion, the angle-dependent absorption coefficient can be determined according to

$$a(\theta) = \frac{P_{\text{abs},\theta}}{P_{\text{inc},\theta}} = 1 - \frac{\int_{0}^{2\pi} |\tilde{P}(\theta, \phi)|^2 d\phi}{\int_{0}^{2\pi} |\tilde{P}(\pi - \theta, \phi)|^2 d\phi},$$

$$0 \leq \theta \leq \frac{\pi}{2},$$

(5)

where $|\tilde{P}(\theta, \phi)|^2$ corresponds to the reflected power, and $|\tilde{P}(\pi - \theta, \phi)|^2$ to the incident power. As such, the estimation of absorption follows from the ratio of absorbed to incident power observed in the measurement aperture, which does not assume any specific waveform incidence. Note that we integrate the individual wavenumber spectra on the azimuth angle $\phi$ to prevent biases at angles with no incidence.

D. Regularized solution to the problem

The solution of Eq. (4) can be calculated in a least-squares sense, i.e., via a regularized matrix pseudo-inverse.

The problem can be formulated as an unconstrained problem,

$$\hat{x} = \text{argmin}\left(\|Hx - p\|^2 + \lambda \|x\|^2\right),$$

(6)

which has the well-known closed form analytical solution

$$\hat{x} = (H^H(HH^H + \lambda I)^{-1}p,$$

(7)

where the superscript $H$ denotes the conjugate transpose and $I$ is the identity matrix. Equation (7) corresponds to the least-squares solution of the problem with Tikhonov regularization. Herein, we use the numerical toolbox in Ref. 25 for analysis and solution of discrete ill-posed problems.

E. Sound field reconstruction

From the estimated solution $\hat{x}$, it is possible to reconstruct the sound field elsewhere in the vicinity of the absorbing sample (pressure, velocity, and sound intensity)

$$p_r = H\hat{x},$$

(8)

where $p_r \in \mathbb{C}^K$ is the reconstructed sound pressure vector estimated at a set of $K$ positions $r = (x_i, y_i, z_i)$ and $H_r \in \mathbb{C}^{K \times L}$ is the reconstruction matrix containing the plane wave functions evaluated at the reconstruction points $r = r_i$.

The particle velocity vector can be calculated from Euler’s equation of motion, $u = -Vp/(j\omega p)$, thus

$$u_r = \frac{-1}{j\omega}VH\hat{x},$$

(9)

where $VH_r \in \mathbb{C}^{K \times L}$ contains the gradient of the reconstruction matrix. The active and reactive intensity vectors are

$$I_r = \frac{1}{2} \text{Re}\{p_ru_r^*\},$$

(10a)

$$J_r = \frac{1}{2} \text{Im}\{p_ru_r^*\},$$

(10b)

where the superscript * denotes the complex conjugate. The full sound intensity vector can be calculated via the element-wise (Hadamard) product of the reconstructed sound pressure vector and the complex conjugate of the particle velocity.

It is possible to reconstruct the incident part of the sound field only. Considering that $\hat{x} = [x_i^{(i)} \ x_r^{(i)}]$, the reconstruction can alternatively be based on the complex coefficient vector $x_i^{(i)} \in \mathbb{C}^N$ and the propagation matrix $H_i^{(i)} \in \mathbb{C}^{K \times N}$ containing the wave components that describe the incident sound field on the plane absorber ($N$ is the number of wave directions conforming the incident sound field). In this case, the reconstruction provides a complete characterization of the incident acoustic field—sound pressure $p_i^{(i)}$, velocity $u_i^{(i)}$, and intensity $I_i^{(i)}$ (or $J_i^{(i)}$), analogous to Eqs. (8) to (10).
\[ p_r^{(i)} = H_r^{(i)} \mathbf{x}^{(i)}, \quad \text{(11a)} \]
\[ u_r^{(i)} = -\frac{1}{j\omega} \mathbf{v} H_r^{(i)} \mathbf{x}^{(i)}, \quad \text{(11b)} \]
\[ I_r^{(i)} = \frac{1}{2} \text{Re} \left\{ p_r^{(i)} \left( u_r^{(i)} \right)^* \right\}, \quad J_r^{(i)} = \frac{1}{2} \text{Im} \left\{ p_r^{(i)} \left( u_r^{(i)} \right)^* \right\}. \quad \text{(11c)} \]

It is naturally possible to calculate the reflected sound field alone, based on the complex coefficient vector \( \mathbf{x}^{(r)} \in \mathbb{C}^{L-N} \) and the propagation matrix \( H_r^{(r)} \in \mathbb{C}^{K \times (L-N)} \) containing the wave components that describe the reflected sound field from the plane absorber.

### III. EXPERIMENTAL RESULTS

The proposed methodology is examined experimentally in a large (215 m\(^3\)) reverberation room, using a programmable robotic arm to scan the acoustic field, as shown in Fig. 2(a). Two damping conditions in the room are examined: the empty (undamped) room and the room with extra absorption (10.8 m\(^2\) glass wool of thickness 100 mm and flow resistivity 12.9 kPa.s/m\(^2\)) on the floor. The room complies with the ISO 354 requirements\(^1\) and is essentially box-shaped, although there are 85 built-in concrete boundary diffusers and 12 panel diffusers [not visible in Fig. 2(a)]. Figure 2(b) shows the absorption coefficient of the 10.8 m\(^2\) sample, measured according to ISO 354\(^1\) in one-third octave bands using the interrupted noise method. For the present experimental study, a scanning robot UR5 (Universal Robots, Odense, Denmark) is programmed to move a free-field microphone (Brüel & Kjær, Nærum, Denmark) within a rectangular volume of dimension 0.6 \times 0.8 \times 0.25 m located in the vicinity of the absorbing sample (the closest microphone is located 28 cm away from the absorber), creating a random array of 290 measurement positions. The measurement positions, the absorptive sample and the room are sketched in Fig. 3. The use of sequential measurements with a robotic arm and a single microphone, in place of a traditional microphone array, makes it possible to circumvent positioning errors (accuracy of approximately 0.1 mm) and transducer mismatch errors,\(^26\) and to achieve greater sampling capability than conventional microphone arrays. The room is excited by a built-in loudspeaker driven with a logarithmic sine sweep ranging from 20 Hz to 20 kHz with duration of 10 s. The loudspeaker is placed in a corner of the room so as to maximally excite the room modes and reduce the amount of direct radiation at the measurement positions [see Fig. 2(a)]. The frequency response is measured at the 290 positions with a spectral resolution of 0.1 Hz. The acquisition time for each damping condition is 50 min.

The complex coefficient vector \( \mathbf{x} \) corresponding to the wavenumber spectrum (i.e., the amplitudes of the waves) in front of the absorber plane is estimated using Eq. (4). A plane-wave basis of 2000 plane waves of unknown amplitudes is considered, whose directions of propagation are distributed uniformly across all propagation angles (based on a Thomson problem).\(^27\) As the number of plane waves is greater than the number of measurement positions for a proper representation of the measured pressure,\(^18,28\) Tikhonov regularization is used for the regularized inversion. The L-curve criterion\(^23\) is used as a parameter choice method.

#### A. Measured wavenumber spectrum

Figure 4 compares the magnitude of the wavenumber spectrum at 125 Hz, 500 Hz, and 1 kHz in the empty room and in the room with absorber (side views). At 125 Hz, the

![FIG. 2. (a) Programmable scanning robot in the damped (absorption on the floor) reverberation chamber; (b) measured absorption coefficient of the absorbing sample as a function of frequency (ISO 354) (Ref. 1).](image-url)
absorptive material has a small influence on the sound field, and the wavenumber spectra in the empty and occupied room are similar. Nonetheless, some changes in magnitude are observed, as the modes of the room are shifted by the presence of the absorber, resulting in sound pressure level changes at a given frequency. At 500 Hz and 1 kHz, which are frequencies at which the material is highly absorptive [see Fig. 2(b)], it becomes apparent that the absorber has a substantial influence on the sound field. In the empty room, the results show that some dominant incident directions are detected (i.e., some waves that carry considerably more energy, seemingly corresponding to a few interfering modes), indicating that the field is not isotropic; the sound field in a room driven with a single source is, in fact, not expected to be isotropic. At both frequencies in the empty room, there are waves travelling in multiple directions, impinging on the floor and being reflected away from it. However, in the damped room (added absorption on the floor), there are no waves propagating away from the absorber, in the $z$-positive direction, as the material is absorptive and no sound is reflected away from it. A significant observation to be drawn from these data is that there are substantial differences between the two incident fields (with and without the sample—lower hemisphere), which demonstrates that the absorbing sample has a strong influence on the incident sound field. This difference between incident fields is particularly marked at high frequencies, where the sample is more absorptive and the empty and occupied conditions are most different.

The lower hemisphere of the wavenumber, which comprises the incident wave field, can be plotted as a function of the angle of incidence $\theta$. If the sound field incident on the floor/absorber is isotropic, the magnitude of the wavenumber spectrum $|\tilde{x}^{(i)}|$ should be a constant function of the angle of incidence. Figure 5 shows the wavenumber spectrum of the incident sound field at 125 Hz, 500 Hz, and 1 kHz in the empty room and in the room with absorption. The right column of Fig. 5 shows the magnitude of the wavenumber spectrum as a function of the angle of incidence $\theta$. The azimuth angle $\phi$ is averaged as

$$|\tilde{x}^{(i)}(\theta)| = \frac{1}{J_I \sum_{j=1}^{J_J}} |\tilde{x}^{(i)}(\theta, \phi_j)|,$$

where a regular grid of $(\theta, \phi)$ points has been defined, and the discrete wavenumber spectrum $|\tilde{x}^{(i)}|$ interpolated over the grid. For each frequency in the empty room, the magnitude of the wavenumber spectrum varies considerably with the angle of incidence, indicating some dominant incident directions, which confirms the anisotropy of the wave field. In the damped room at high frequencies (500 Hz and 1 kHz), the wavenumber spectrum magnitude exhibits lower values at normal incidence compared to the empty case. In fact, in the presence of the absorber, the incoming waves seem to arrive primarily from the direction of the source and the walls (as well as higher-order reflections between the walls and ceiling). However, the acoustic components travelling in the $z$-direction (normal to the sample) are less prominent than in the empty case, because in the presence of the absorptive sample, no waves are redirected from the absorber onto the ceiling (and, in turn, back into the absorber). Contrarily, at oblique incidence, the magnitude of the wavenumber spectrum increases, as these angles are less affected by the presence of the absorber. The results confirm that the waves that comprise the incident sound field are attenuated depending on their orientation with respect to the bounding surfaces and the absorption of the absorber.13

Finally, one would expect the magnitude of the wavenumber spectrum to continue increasing at grazing incidence. This is, however, not the case and the magnitude, in fact, decreases, possibly due to the experimental limits in distinguishing between incident and reflected components near the equator at $\theta = \pi/2$. 

![FIG. 4. Magnitude of the estimated wavenumber spectrum in the empty room and the room with absorber at 125 Hz, 500 Hz, and 1 kHz (side views).](image-url)
B. Intensity distribution

The proposed methodology enables to estimate the spatial distribution of the sound intensity field, as seen in Sec. IIE [Eqs. (8)–(11)], making it possible to visualize the net and incident energy flows onto the absorber. Figures 6(a) and 6(b) show a volumetric reconstruction of the total net active intensity in the empty room [Fig. 6(a)] and in the room with absorption [Fig. 6(b)], respectively, for the third-octave band centered at 1 kHz. Figures 6(c) and 6(d) show the analogous reconstruction of the reactive intensity for the
empty and occupied cases. The intensity vectors are computed over a parallelepiped lattice of 252 points in the vicinity of the absorbing plane (9 × 7 × 4 grid with a lattice spacing of 10 cm) using Eqs. (8)–(10). The length of the vectors is proportional to the intensity (in W/m²) at the base of the cone, and the cones are color-coded to also show the magnitude of the intensity vectors. In the empty room [Fig. 6(a)], the active intensity field seems to flow in multiple arbitrary directions. Contrarily, in the room with absorption [Fig. 6(b)], the active intensity is clearly pointing towards the absorber. This is in agreement with previous findings (see Ref. 29). It is also noticeable that the intensity vectors in this case are slightly angled in the x-direction, corresponding to the position of the source (see Fig. 6—left, for the source position). In fact, the power flow seems to curve from the source towards the absorber. In the empty room [Fig. 6(c)], the reactive intensity has no clear orientation, and its average magnitude is the same as the active intensity. This result agrees with statistical predictions in a reverberant field, for a frequency band above the Schroeder frequency.30

In the damped room [Fig. 6(d)], it is interesting to see that the reactive intensity has a seemingly arbitrary spatial structure, although upon closer inspection, it is noticeable that the reactive flows occur only in the x–y plane (tangential plane) above the absorber, indicating a lack of standing waves between the floor and the ceiling. It is also noticeable that in the damped room, the magnitude of the reactive intensity flow is about one third of the active intensity one, indicating a dominant active intensity flow, as expected due to the presence of the absorptive sample.

It is also of interest to analyze the distribution of the acoustic intensity incident onto the sample. For this purpose, the active incident intensity field is reconstructed in the room with absorption, over the same grid as in Fig. 6, but according to Eq. 11(c). Figure 7 shows such volumetric reconstruction of the incident intensity for selected third-octave bands (250 Hz, 500 Hz, 1 kHz, and 2 kHz). It is observable that at all frequencies, the incident energy is not constant and uniform throughout space, far from the theoretical predictions that follow from assuming an ideally isotropic sound field. At 250 Hz, the distribution is somewhat more uniform, which can be attributed to an increase in the reverberant energy at low frequencies, due to the moderate absorption of the sample. Hence, the incident intensity is redistributed, yielding a somewhat more uniform incidence. It can be remarked that in general, the direction of incident energy flow is not in the direction of a wave front (the direction of the incident intensity vector results from the combination of sound waves arriving simultaneously from more than one direction); as such, the incident energy flows cannot indicate the directional distribution of sound waves arriving onto a sample (as opposed to the wavenumber spectrum representation), but can indicate the departure from the ideal state of diffusion, which is the basis of reverberation-chamber measurements of absorption coefficients.

C. Absorption coefficient

Finally, it is of importance to examine the capability of the method to estimate absorption coefficients, based on the estimated incident and reflected wavenumber spectra. In particular, the proposed experimental methodology makes it

FIG. 7. (Color online) Volumetric intensity reconstruction of the active intensity incident onto the absorbing sample for selected third-octave bands: 250 Hz, 500 Hz, 1 kHz, and 2 kHz.
possible to determine angle-dependent absorption properties of the measuring sample.

Figure 8 shows the angle-dependent absorption coefficient determined from the wavenumber spectrum measurement [Eq. (5)] for the third-octave bands ranging from 200 Hz to 1.25 kHz. The absorption coefficient is averaged over the azimuth angle \( \phi \) and calculated as mean values per third-octave bands (the frequency averaging is performed on the wavenumber spectrum, to avoid biased estimates, as a pure-tone sound field in a reverberation room is not diffuse, and not all frequencies contain energy in every direction, as shown in Sec. III A). The asterisks * represent the experimental data estimated every \( \pi/30 \text{ rad} \) (6°). The solid line is calculated from a transfer matrix model in the case of a single layer of porous material with a rigid backing. The characteristic impedance and the wavenumber for the material are obtained from Miki’s empirical model (a flow resistivity value of 12.9 kPa.s/m² is used and provided by the manufacturer). The experimental values fit reasonably well with the prediction. The agreement is particularly good in the frequency range between 315 Hz and 1.25 kHz and for angles between 0 and \( \pi/3 \text{ rad} \) (60°). Below 315 Hz (200 and 250 Hz), the incident energy is not uniformly distributed with angle, which leads to a bias, and the spatial resolution of the array (determined by its size) is limited (the 3 dB bandwidth at 250 Hz is about 15°). At frequencies where the sample is highly absorptive (above 500 Hz), the calculated absorption at grazing incidence is underestimated because it is not possible to distinguish between incident and reflected components. This is a result of the limited wavenumber resolution of the measurement system: some of the incident energy on the absorber at grazing incidence (in the lower hemisphere of the wavenumber spectrum, close to the equator) “leaks” into the upper half of the wavenumber, which corresponds to directions of reflected components. This is also apparent in Fig. 4 at 1 kHz.

Figure 9 shows the relative deviation \( \epsilon \) (in %) between the experimental results and the prediction from the transfer matrix model, as a function of the angle of incidence

\[
\epsilon(\theta_i) \ [\%] = \left| \frac{x_{th}(\theta_i) - \hat{x}(\theta_i)}{x_{th}(\theta_i)} \right| \times 100, \tag{13}
\]

where \( \hat{x} \) corresponds to the estimated quantity, \( x_{th} \) is given by the transfer matrix method and Miki’s model, and \( \theta_i \) is the discrete angle of incidence. The results corroborate that the estimation is accurate, particularly between 315 Hz and 1.25 kHz (\( \epsilon < 10\% \), which quantitatively is a very close agreement). The deviation is greater at 200 Hz, and towards grazing incidence at high frequencies due to the sensing limitations.

**FIG. 8.** Angle-dependent absorption coefficient determined from wavenumber spectrum measurements for the third-octave bands ranging from 200 Hz to 1.25 kHz. The asterisks * represent experimental data; the solid line is obtained from a transfer matrix method (Ref. 31) along with Miki’s model (Ref. 32).

**FIG. 9.** Relative deviation (in %) between the experimental results and the prediction from the transfer matrix model as a function of the angle of incidence.
IV. DISCUSSION

The current measurement system is expected to provide reliable results between 200 Hz and 2.5 kHz given the size of the measurement aperture and the separation distance between points. The frequency range can be extended by conducting more measurements, either over a larger aperture (to extend the low-frequency limit), or with a finer sampling density (to extend the high-frequency limit). In the present study, the duration of a measurement sequence (one damping condition) was 50 min (including 2 s for the arm to stabilize after each displacement). The sequence runs automatically, based on the programmed robot positioning path.

In the frequency range of validity, the results are found to agree well with theoretical predictions as well as established models. The transfer matrix method, along with Miki’s empirical model, serve here as reference, as they are well-established models to characterize the angle-dependent impedance and absorption of porous absorbers. Yet, the presented experimental method is also valid for other types of materials, as it serves to analyse an arbitrary sound field in the vicinity of an absorber, and does not assume or require a specific absorber type.

When estimating the angle-dependent absorption coefficient, we eventually average the wavenumber spectrum over the azimuth angle, as this angle is of marginal significance in the case of isotropic absorbers.3,3 Besides, as a result of the anisotropy of the sound field on the measuring sample (see Sec. III A), the estimation of the absorption coefficient is singular in directions with no incident energy and therefore biases the final estimate. Averaging the wavenumber spectrum over the azimuth angle guarantees a finite amount of energy at all elevation angles.

A key aspect of the proposed method is that it is concerned with the direct observation of the sound field in the room (as such, it does not aim at evaluating the compliance of the reverberation chamber with ISO 354:2003), and does not employ indirect measures or restrictive assumptions on the nature of the sound field, e.g., an ideally diffuse field. This may be a fundamental reason why the agreement found between the method and the transfer matrix model is better than the agreement (or rather lack of agreement)3,6 typically found in measurements of absorption across standardized laboratories.

V. CONCLUSION

An experimental method to characterize the non-uniform sound incidence on the absorbing sample in a reverberation room is proposed in this study. The methodology relies on using a plane wave decomposition (i.e., estimation of the wavenumber spectrum from a set of measured microphone signals), to determine the magnitude of the sound waves arriving from definite directions onto the absorbing plane. Experiments in a reverberation chamber in two damping conditions have been conducted, using a programmable robotic arm to scan the sound field. The results indicate that the method is suitable for assessing the isotropy of the incident sound field on a sample. Substantial differences between the incident fields in the empty and occupied room have been found, which demonstrate the influence of the sample on the sound field in its vicinity. In other words, the sound field that impinges onto the sample depends strongly on the properties of the sample itself, and not just the room.

Based on the wave expansion, all acoustic quantities (i.e., sound pressure, particle velocity, and complex sound intensity) can be reconstructed in the vicinity of the absorbing specimen. Both the total sound field (incident plus reflected components) as well as the incident field alone can be obtained, which makes it possible to visualize and characterize the power flow incident on the sample. It is the first time that this is visualized experimentally. The results show that the incident flow onto the absorptive sample is not constant and uniform throughout space. Finally, the quantitative validity of the method is shown via an estimation of the sample’s angle-dependent absorption coefficient. The results show good agreement with a transfer matrix model (used with Miki’s empirical model), with an overall deviation (at all frequencies) below 10%.

We foresee that the proposed experimental framework will be of value in explaining the deviations encountered across standardized laboratories in reverberation-chamber measurements of absorption coefficients. To this end, it may be of interest to also examine the evolution of the sound decay by estimating the incident energy in successive time windows of some interval.

ACKNOWLEDGMENTS

The authors would like to thank Cheol-Ho Jeong for advice on background literature, Gerd Marbjerg for discussion on models of porous materials, and Antoine Richard for comments on the manuscript. This work is funded by the Oticon Foundation.

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Paper D
Volumetric reconstruction of acoustic energy flows in a reverberation room

Mélanie Nolan and Efren Fernandez-Grande

Abstract: This study examines the spatial and directional properties of net energy flows in a reverberation chamber. Based on measurements with a spherical array, a method is proposed to estimate the flows of acoustic energy in the volume surrounding the array. The proposed method is used to examine the steady state, early decay, and late decay of the sound field in a reverberation room (both empty and with an absorber on the floor). The results show that the approach is successful in characterizing the spatio-spectral and spatio-temporal properties of power flows in reverberant sound fields, constituting a valuable analysis tool.

1. Introduction

Reverberation rooms have become the primary instrument for standardized measurements of sound absorption. The procedure is based on the reverberation formulas that follow from the assumption of a perfectly diffuse sound field, i.e., a sound field in which energy density is uniformly distributed in space and energy flow is isotropic. In such a sound field, the sound pressure level is the same everywhere, and the time-averaged (net) intensity is zero at all positions. Several round robin tests conducted in different standardized laboratories have revealed a significant disagreement in the results, attributed to different degrees of sound field diffusion established in the various chambers.

The statistical properties of the sound intensity in reverberant stationary sound fields have been studied extensively. The statistical theory has been used for determining the relative ensemble variance of active and reactive sound intensity in reverberation rooms driven with a pure tone and the predictions have been confirmed by experimental results. Such description involves determining statistical properties with respect to an ensemble of reverberation chambers. The theory does not explain the properties of the sound field in a specific room, which is key to understanding the deviations encountered across standardized laboratories. Energy-flow analyses have proven to be a powerful tool for understanding the behaviour of sound in complex environments (e.g., in rooms, based on modeling of the diffusion equation). Experimentally, volumetric energy-flow measurements are valuable for understanding the radiation mechanisms of complex sound sources. Although not explored to date, the methods seem particularly relevant for reverberation chamber acoustics, as the rooms are primarily used to quantify the generation, absorption, and transmission of sound energy.

This study investigates the spatio-temporal properties of acoustic energy flows in a standardized reverberation chamber, both empty and with a sample of absorptive material on the floor. The proposed experimental procedure is based on a spherical equivalent source method, which models an arbitrary sound field (measured with a spherical microphone array) as the superposition of waves radiated by a combination of monopoles. The method can yield all acoustic quantities—sound pressure, particle velocity, and sound intensity in a volume surrounding the array. The measured sound pressure at the microphone positions can be windowed in consecutive time intervals to obtain a temporal reconstruction of the energy flows. Hence, the directional properties of the net flow corresponding to the steady state, early decay time, and late decay time can be examined separately. To the authors’ knowledge, it is the first time that an experimental analysis of the flow of acoustic energy in a reverberation room is presented.

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Date Received: November 16, 2018 Date Accepted: February 13, 2019
2. Theory

This work considers a rigid spherical array immersed in a sound field (see Fig. 1). The array consists of \( M \) microphones, flush-mounted on the surface of a rigid sphere of radius \( a \). The measured sound pressure can be expressed as a discrete combination of \( L \) point sources (equivalent sources) distributed over the positions \( r_{0j} = (r_{0,j}, \Omega_{0,j}) \):}

\[
p_t(a, \Omega_m) = \sum_{j=1}^{L} \frac{-j \rho \omega Q_l}{k a^2} \sum_{m=0}^{\infty} \sum_{n=-m}^{n} h_n(kr_{0,j}) Y_n^m(\Omega_m) Y_n^m(\Omega_{0,j})^*,
\]

where \( k \) is the wavenumber and \( Q_l \) is the volume velocity of each equivalent source. The microphone positions on the sphere are \( r_m = (a, \Omega_m) \) and the origin of coordinates is in the centre of the sphere. The summation over \( n \) and \( m \) corresponds to expanding the sound pressure due to a point source on a rigid sphere into spherical harmonics.

The functions \( Y_n^m(\Omega) \) are the spherical harmonics of order \( m \) and degree \( n \), defined as in Ref. 11, \( h_n(x) \) is the spherical Hankel function of the second kind, and \( h_n' \) is the derivative of the Hankel function (the chosen sign convention is \( e^{i\omega t} \)).

The angular dependency is expressed as \( \Omega = (\theta, \phi) \). By conducting the summation over \( n \) and \( m \) in Eq. (1), truncated at \( n = N \), a transfer matrix is obtained that relates the pressure on the sphere and the coefficients of the point source model

\[
p_t = G_N q.
\]

The vector \( p_t \in C^M \) consists of the sound pressure measured at the \( M \) microphone positions (incident plus scattered fields), \( G_N \in C^{M \times L} \) is the transfer matrix containing the Neumann Green’s function, and the vector \( q \in C^L \) consists of the unknown complex coefficients [i.e., the mass acceleration \( \dot{q} = \rho \omega^2 q \)] in Eq. (1). This problem is ill posed, typically underdetermined \((M < L)\), and requires regularization. Its solution can be found via the regularized pseudo-inverse \( \tilde{q} = G_N^H (G_N G_N^H + \lambda I)^{-1} p_t \), where the superscript \( H \) denotes the conjugate transpose, \( \lambda \) is the regularization parameter, and \( I \) is the identity matrix (\( \tilde{q} \) is the least-squares solution of the problem with Tikhonov regularization).

Once the coefficients \( \tilde{q} \) of the equivalent sources have been determined, the sound pressure \( \tilde{p} \in C^K \) can be predicted anywhere in the domain:

\[
\tilde{p} = G \tilde{q}.
\]

where the reconstruction matrix \( G \in C^{K \times L} \) contains the free-field Green’s function between the equivalent sources \( r_0 \) and the reconstruction points \( r \) (showing that the scattering introduced by the rigid sphere is compensated for in the reconstruction). The particle velocity vector can be calculated from Euler’s equation of motion as \( \tilde{u} = (-1/j\omega p) \nabla \tilde{G} q \), where \( \nabla \tilde{G} \in C^{K \times L} \) contains the gradient of the reconstruction matrix, with respect to the normal to the surface \( S \) over which the point sources are distributed. The active intensity vector at each reconstruction point is \( \tilde{I} = \frac{1}{2} \Re(\tilde{p} \tilde{u}^*) \), where the superscript \( * \) denotes the complex conjugate.

3. Experimental results

The sound intensity field inside a large (215 m\(^3\)) reverberation chamber is reconstructed using the experimental framework presented in Sec. 2. Both the steady state and the decay process are analyzed. The room contains 85 built-in concrete boundary diffusers and 12 panel diffusers. Two damping conditions are examined: the empty (undamped)
room and the room with added absorption on the floor (10.8 m² glass wool; thickness 50 mm). Figure 1 shows the absorption coefficient data for the glass wool ($\alpha_p$ values provided by Saint-Gobain Ecophon, Hyllinge, Sweden). A rigid spherical array of radius $a = 9.75$ cm (B&K, Nærum, Denmark) is used to measure the sound pressure 40 cm away from the absorbing sample (40 cm is the distance between the sample and the array’s centre). The array consists of 64 B&K microphones, uniformly distributed over its surface, and can sample up to 7 orders of spherical harmonics. The room is excited to a steady-state condition by a built-in loudspeaker driven with pink noise. The loudspeaker is placed in a corner of the room so as to maximally excite the room modes and reduce the amount of direct radiation at the measurement positions. After 60 s, the excitation is interrupted and the decay process recorded. In this way, the three main stages of the decay process are captured: steady state, early decay, and late decay. Hanning windows of length 1 s and 75% overlap are used for the analysis of the steady-state response. During the decay process, the frequency response is calculated from single consecutive time windows. The complex coefficient vector $\mathbf{q}$ corresponding to the source strengths is estimated using Eq. (2). A point-source basis of 1000 sources is considered, whose positions are distributed uniformly based on a Thomson problem over a spherical surface placed 30 cm away from the surface of the array. Equation (2) is inverted with Tikhonov regularization, and generalized cross-validation is used as regularization parameter selection criterion. The reconstruction takes place close to the array, at 128 points uniformly distributed over a sphere of radius 10 cm. Note that the current measurement system is not expected to provide valid results below 120 Hz, where the circumference of the sphere corresponds to about 10% of the wavelength in air ($ka = 0.1$), nor above 2.5 kHz, where aliasing effects start to appear.

Figure 2 shows the reconstructed active intensity vector at 2 kHz in the undamped [Figs. 2(a) to 2(c)] and damped [Figs. 2(d) to 2(f)] room, respectively, for the three stages of the decay process: steady state [Figs. 2(a) and 2(d)], early decay [Figs. 2(b) and 2(e)], and late decay [Figs. 2(c) and 2(f)]. The length of the vectors is proportional to the magnitude in [W m$^{-1}$] (note that the magnitude is not comparable across sub-figures due to different scale lengths; a systematic quantitative comparison is presented in Fig. 3). The absorbing sample is placed in the $xz$-plane in the negative $y$-direction (see Fig. 1 for definition of the Cartesian axes). In the undamped room during steady state [Fig. 2(a)], the net acoustic energy is flowing in seemingly arbitrary
directions, although some dominant directions are identified, corresponding to the direct sound and a few strong reflections. This is also seen in the magnitude of the source strength distribution \( q \) (not shown for conciseness). Contrarily, in the room with absorption [Fig. 2(d)], the energy is predominantly flowing toward the absorber. On closer inspection of Fig. 2(d), it is noticeable that the intensity vectors are slightly angled in the \( x \)- and \( z \)-directions, indicating that the flow is bending from the source toward the absorber. Additionally, there exist tangential flows of energy due to the effect of boundary diffusers near the walls and the interaction between waves propagating from the source and other reflections of considerable amplitude. During the early and late decays in the undamped room [Figs. 2(b) and 2(c)], the energy flows become less structured, flowing in various arbitrary directions. This indicates that the sound field is composed of multiple waves of similar amplitude, leading to a complex pattern of energy flows. The source is not radiating and there are less prominent reflections than in the steady state (this is also confirmed by inspecting the estimated coefficients \( \tilde{q} \)). Still, it is apparent that the sound field is not isotropic. In the damped room, during the early [Fig. 2(e)] and late [Fig. 2(f)] decays, the intensity vectors are pointing toward the glass wool sample, as energy is being absorbed. It is noticeable that the energy flows become slightly more unstructured (more spatial fluctuations) in comparison to the steady state, due to the absence of direct sound. Both the early and late decays in the damped room show a similar spatial structure, where the flow is mostly directed toward the absorptive sample. Studies in rectangular rooms have reported tangential sound field components in the late decay, corresponding to standing waves in the \( xz \)-plane.\(^{14,15}\) The measurement results show no evidence of such flows here, suggesting that the boundary and panel diffusers re-direct the sound waves successfully.

Figure 3 compares the spatially averaged magnitude (in [dB ref. 1pW m\(^{-2}\)]) of the active intensity vector components (\( x \)-, \( y \)-, and \( z \)-directions) as a function of time in the steady state and decay process, in the undamped room and in the room with added absorption. The results are displayed for the third-octave bands centred at 500 Hz, 800 Hz, 1 kHz, and 2 kHz, respectively. The spatially averaged magnitude of the \( x \)-component (over the 128 reconstruction points) is given by

\[
|I_x| = \left[ \frac{1}{W} \sum_{w=1}^{W} (I_x^{(w)})^2 \right]^{1/2},
\]

where \( W \) is the number of reconstruction points, and \( I_x^{(w)} \) is the \( x \)-component at position \( w \). Equivalent formulas apply for the \( y \)- and \( z \)-components, respectively. The recorded time data are processed every 0.2 s in successive windows of length 0.5 s. The intensity field is reconstructed for each time interval and the averaged magnitude of each component calculated. The results are displayed from 59 to 64 s. Note that the time axes are labeled relatively: 60 s, time at which the source is switched off, is labeled as 0 s. In addition, Multimedia files Mm. 1 and Mm. 2 provide short video files showing the time evolution of the three-dimensional (3D) intensity distributions at 1 kHz, for the empty room and the room with absorption, respectively.
Mm. 1. 3D time-dependent intensity distribution in the undamped room at 1 kHz 
\((-0.1 < t < 1.1 \text{ in relative scale}). \) Reconstruction every 0.05 s from successive windows of 
length 0.5 s. This is a file of type “avi” (0.86 Mb).

Mm. 2. 3D time-dependent intensity distribution in the damped room at 1 kHz 
\((-0.1 < t < 1.1 \text{ in relative scale}). \) Reconstruction every 0.05 s from successive windows of 
length 0.5 s. This is a file of type “avi” (0.88 Mb).

In both damping conditions, the results show that neither the time-averaged 
intensity in the steady state nor the net intensity in the decaying sound field is null (in 
clear disagreement with the theoretical predictions that follow from assuming an ide-
ally diffuse field). In the undamped room [Fig. 3 (top row)], the vector components are 
of comparable magnitude, indicating no dominant orientation of the intensity flow. 
Also, the directional components during the decay seem to contain almost identical 
energy, in agreement with the qualitative results shown in, e.g., Fig. 2(b). In the 
damped room [Fig. 3 (bottom row)], the results indicate an overall lower magnitude of 
the intensity field than in the undamped room, as expected due to the presence of the 
sample. This is particularly marked at high frequencies, where the sample is more 
absorptive. Besides, the decay exhibits a steeper slope, relating to a shorter reverbera-
tion time than in the undamped room. It is noticeable that, in the damped room, the 
\(y\)-component of the intensity vector (normal to the sample) presents a higher averaged 
magnitude than the \(x\)- and \(z\)-components, both in the steady state and during the 
decay process. This is due to the absorption by the sample, and in clear agreement 
with Figs. 2(d)-2(f), where a directional structure of the energy flow can be appreci-
ated. This effect increases toward higher frequencies, revealing a larger influx of energy 
onto the absorbing surface. In the late decay, there is no evidence of tangential flows 
in the \(xz\)-plane. Furthermore, it is interesting to note that the magnitude of the \(y\)-com-
ponent remains higher after the sound field has decayed (i.e., after 2 s in relative scale), 
indicating that acoustic background noise is also being absorbed.

Notice that the equivalent source model in Eq. (1) is well suited for sound field 
reconstructions, as the decay of the acoustic field is modeled via the spherical spread-
ing of the point sources.\(^\text{10}\) Yet, a different basis can be used, e.g., the wave expansion 
can be formulated in terms of plane waves instead of point sources. The results (not 
shown) obtained using a plane wave basis instead of a point source basis lead to the 
same observations (indicating that the wavefronts are locally planar in the test-
region—the measurement system is, in fact, placed at a sufficient distance away from 
the source and diffracting elements).

4. Conclusion

A method has been presented to analyse experimentally the flow of acoustic energy in 
a reverberation chamber, both in steady state and throughout the decay process. A 
spherical equivalent source model\(^\text{10}\) is used to obtain a volumetric reconstruction of 
the intensity field in the test-region and examine the acoustic power flows in the room. 
Two damping conditions are tested: the empty room and the room with absorption on 
the floor. In the undamped room, the results confirm that the time-averaged intensity 
in the stationary sound field and the net intensity in the decaying sound field are non-
zero. Significant differences are found between the sound field in the steady state and 
during the decay process. In steady state, there is a clear influence of the direct sound 
and reflections from nearby surfaces (the configuration of diffusers seems to create 
strong first order reflections onto the sample), whereas in the decay process the inten-
sity field is more unstructured and seemingly arbitrary (which could be interpreted as 
more diffuse, although yet far from ideally diffuse). As expected, when an absorber is 
placed in the room, there is a large influx of energy directed toward it. Significant dif-
ferences are found between the steady state and the decay in the damped room (tan-
gential flows may appear in steady state which are not observed during the decay pro-
cess). On a broader sense, the proposed framework is valuable for analyzing 
experimentally the energy flows in reverberant sound fields, and for examining the 
acoustic processes occurring therein (the analysis of the decaying sound field is of par-
ticular interest for reverberation-chamber measurements of absorption coefficients). It 
is anticipated that the proposed framework will be of value in analyzing the disagree-
ment found across standardized laboratories,\(^\text{3,4}\) as well as diagnosing biases in specific 
laboratory configurations.

Acknowledgments

This work is funded by the Oticon Foundation.
References and links

Paper E
Two definitions of the inner product of modes and their use in calculating non-diffuse reverberant sound fields

Mélanie Nolan1,a) and John L. Davy2,b)

1Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Building 352, Ørsted Plads, DK-2800 Kongens Lyngby, Denmark
2School of Science, Royal Melbourne Institute of Technology University, Melbourne, Victoria, Australia

(Received 7 November 2018; revised 6 May 2019; accepted 9 May 2019; published online 7 June 2019)

There are two definitions of the inner product of modal spatial functions used in the literature. Both definitions integrate the product of the modal spatial functions over a line, area, or volume. The only difference is that one of the definitions takes the complex conjugate of one of the modal spatial functions before multiplying the modes together. The definitions are the same if the modal spatial functions are real. If the modal spatial functions are complex, only the definition which takes the complex conjugate is an inner product. If the specific acoustic impedance of the boundaries has a real part, then the modes are only orthogonal with the definition which does not take the complex conjugate, although this definition is not strictly an inner product because the modal spatial functions are complex in this situation. However, this definition of “inner product” can be used to calculate the coefficients in the modal expansion of the system response. On the other hand, when it comes to calculating the mean pressure squared and the mean sound intensity, the modal spatial functions cross-products cannot be ignored because the modes are not orthogonal for the definition which takes the complex conjugate. © 2019 Acoustical Society of America.

I. INTRODUCTION

It is well-known that the statistical random-wave theory1 is a good approximation to the sound field in a reverberant room in the frequency range where the modal overlap is high; that is, above the Schroeder frequency.2 In rooms where absorption is concentrated on certain surfaces (as in conventional measurements of sound absorption coefficients),3 the resulting sound field responds to non-uniform boundary conditions, and the conditions for the statistical theory are violated. To describe such sound field, a formal solution to the wave equation with the appropriate boundary conditions must be found, resulting in a description in terms of the modes of the room. Each mode corresponds to a frequency (the eigenfrequency or resonance frequency) and is solution to the homogeneous wave equation with the appropriate boundary conditions.

The modal theory of the stationary reverberant sound field has been studied extensively for rooms with nearly hard walls: Lyon4 derived formulas for the spatial variance of the squared sound pressure and the input impedance of a pure-tone reverberant sound field assuming that the modal frequencies were locally spaced according to the Poisson or nearest-neighbor distributions; Davy5 corrected errors in Lyon’s formulas; Weaver6 pointed out that theoretically the Gaussian Orthogonal Ensemble (GOE) distribution should be used for the modal frequency spacings; Davy7 confirmed Weaver’s results experimentally. Rooms with absorption concentrated on one of their walls have not received similar attention. Early derivations of the velocity potential for non-rigid walls are described by Morse and Bolt.8 The case of non-rigid walls is partially treated in Ref. 9, although the walls are assumed to have the same acoustic properties, thus excluding rooms with absorption concentrated on one of their walls. Kergomard et al.10 treat the one-dimensional case in great detail and give a number of references. They do allow the normalized specific acoustic impedances to be different at each end of the one-dimensional duct and show that the modal spatial functions are non-orthogonal if the normalized specific impedances have a non-zero real part indicating wall losses.

Interestingly enough, the question of the orthogonality of the spatial modal functions in rectangular rooms with arbitrary boundary conditions has hardly been addressed. As a matter of fact, two different definitions of the inner product of modal spatial functions are used in the above-mentioned literature. Both definitions integrate the product of the modal spatial functions over a line, area, or volume; the only difference is that one of them takes the complex conjugate of one of the modal spatial functions before multiplying the modal spatial functions together. Essentially, they are the same if the modal spatial functions are real. These two definitions cause great confusion in the literature. For instance, Eq. (1) in a recent paper by Leader and Pan11 on the modes of open cavities gives the modal expansion formula for a closed cavity using the complex conjugate operator in both the numerator and the denominator, while citing Eqs. (9.4.1) and (9.4.2) of Ref. 12, which do not use the complex conjugate operator.

a)Electronic mail: melnola@elektro.dtu.dk
b)Current address: CSIRO Infrastructure Technologies, Private Bag 10, Clayton South, Victoria 3169, Australia.
Equations 7–10 of Ref. 11 also use the complex conjugate operator, while Herman13 is cited for the orthogonality of the modal spatial functions of closed systems. However, Eq. (6.11) of Ref. 13 defines the inner product without using the complex conjugate operator. It appears that Herman13 was assuming that his modal spatial functions were real because his “inner product” is not an inner product unless the modal spatial functions are real. (Note: To be fair to Leader and Pan,11 the authors would like to point out that the same error was made in a paper by Davy.)5 If the modal spatial functions are complex, only the definition that takes the complex conjugate is an inner product. If the specific acoustic impedance of the boundaries has a real part (indicating wall losses), then the modes are only orthogonal with the definition that does not take the complex conjugate, although this definition is not strictly an inner product because the modal spatial functions are complex in this situation. However, this definition of “inner product” can be used to calculate the coefficients in the modal expansion of the room response. On the other hand, when it comes to calculating the mean pressure squared and the mean sound intensity, the modal spatial function cross-products cannot be ignored because the modes are not orthogonal for the definition which takes the complex conjugate. The subsequent extra calculations are the reason why most authors prefer to assume (incorrectly) that the modal spatial functions are orthogonal with the (correct) definition of inner product which takes the complex conjugate of one of the modal spatial functions. The primary aim of this paper is to make researchers aware of these two definitions and to show that the modal spatial functions are orthogonal under the correct definition of inner product only if the real part of the specific acoustic impedance of the boundaries is zero (that is, when the boundaries are free of absorption).

Further, because the boundary conditions depend on the wavenumber of the sound waves, these boundary conditions will vary with frequency even if the normalized specific acoustic impedances of the boundaries are independent of frequency (the only exceptions—not considered in this paper—are the two extreme cases where the normalized specific acoustic impedance of the boundary is infinite or zero). Thus, the eigenfrequencies have to be calculated for each driving frequency of the sound source. There are multiple solutions to the equation for the eigenfrequencies of the modes, hence the selection of an adequate starting value for the numerical calculation is a crucial step. Starting the numerical solution with an analytical approximation to the eigenfrequencies works well at very low frequencies. At higher frequencies, the algorithm fails to converge to the correct solution. This is shown by a sudden jump in the calculated eigenfrequencies as a function of the exciting frequency. The paper shows that this problem can be overcome by means of a warm-start strategy, which implies that the result obtained for a given frequency is used as the starting point for the numerical solution of the eigenfrequency for the next higher driving frequency. For the lowest frequency at which calculations are made, the analytical approximation can be used as the starting point for the numerical solution. The paper also points out that a different analytic approximation is needed for the starting point of the numerical solution for the zeroth-order eigenfrequency at the lowest driving frequency.

This paper is organized as follows: the mode model is presented in Sec. II. The orthogonality relation of the modes is investigated and a method of calculating suitable starting values for the numerical solution of the eigenfrequencies is given. In Sec. III, the derived model is used to examine properties of the sound pressure and complex sound intensity in a rectangular room with absorption concentrated on the floor. The numerical results are compared with theoretical predictions and a finite element model (FEM).

II. THE MODE MODEL

A three-dimensional rectangular room containing a highly absorptive surface and no scattering objects is the basis for the analysis in this study. The assumptions necessary for the calculations may be summarized as follows: (i) The walls are assumed to be locally reacting, which means that their acoustic properties can be described by their respective normalized specific acoustic impedances “ξ.” Although the absorption shall be uniformly distributed on each wall, each of them may have different acoustic properties. No assumptions concerning the value of the normalized impedances are necessary; however, only walls with normalized impedances larger than unity are considered in this paper. For convenience, we will also consider real values of the normalized impedances (although the model is not limited to real values). The assumption that ξ is real is not very restrictive, since an imaginary part simply displaces the resonant frequencies by a small amount corresponding to a small change of room dimensions; (ii) the analysis is confined to pure-tone sound fields, although some pure-tone results are averaged to obtain third-octave band results, and only stationary sound fields (and, accordingly, only time-averaged quantities) are considered; (iii) the sound field is produced by a single monopole source with infinite acoustic impedance; (iv) air absorption is assumed to be negligible.

The sound field produced by a monopole point source of volume velocity Q oscillating at angular frequency ω in a rectangular parallelepiped room of dimensions \( L_x \), \( L_y \), and \( L_z \) is calculated using the well-known modal expansion method.12 The steady-state pressure response as a function of position is a solution to the wave equation

\[
\nabla^2 p(x) + k^2 p(x) = -i\omega \rho Q \delta(x - x_0),
\]

where \( x = (x, y, z) \) and \( x_0 = (x_0, y_0, z_0) \) are the spatial coordinates of the receiver and source, respectively, \( \rho \) is the air density, and \( \delta(x - x_0) \) is the three-dimensional Dirac delta function. The time dependency \( e^{i\omega t} \) is disregarded. The solution to Eq. (1) can be expressed in terms of the modal functions of the room if these modal spatial functions form a complete basis for the space of solutions satisfying the given boundary conditions; i.e.,

\[
p(x) = \sum_n A_n p_n(x)
\]
and

\[ q(x) = Q\delta(x - x_0) = \sum_n B_n p_n(x), \quad (2b) \]

where \( p_n \) is the \( n \)th modal spatial function. The coefficients \( A_n \) and \( B_n \) can be easily determined if the modal spatial functions are orthogonal for a suitable definition of inner product. The modal functions \( p_n \) are solutions to the homogeneous wave equation

\[ \nabla^2 p_n(x) + k_n^2 p_n(x) = 0, \quad (3) \]

with the boundary conditions

\[ \frac{\xi}{\partial N} = -ik p_n(x), \quad (4) \]

where \( \xi \) is the normalized specific acoustic impedance and \( \partial / \partial N \) denotes partial differentiation in the direction of the normal to the surface, pointing outward. Equation (3) has solutions only for certain discrete complex values of \( k \), the modal frequencies (or eigenvalues) \( k_n \).

**A. Modal frequencies and modal functions**

To apply the modal decomposition given in Eqs. (2a) and (2b), one should first solve for the eigenvalues and eigenfunctions of Eq. (3) with the boundary conditions defined in Eq. (4). The calculation details are given in the one-dimensional case and extended to three dimensions using separation of variables. The one-dimensional medium extends from \( x = 0 \) to \( x = L_x \), and a source of volume velocity per unit area \( Q_S \) is placed at \( x = x_0 \). The sound pressure in the one-dimensional case is solution of the inhomogeneous equation \( d^2 p(x)/dx^2 + k^2 p(x) = -i\omega \rho Q_S \delta(x - x_0) \), and the one-dimensional modal functions in the \( x \)-direction can conveniently be written as

\[ p_n(x) = C_n e^{-i k_n x} + D_n e^{i k_n x}. \quad (5) \]

By inserting Eq. (5) into the boundary conditions defined in Eq. (4), a system of two linear homogeneous equations for the constants \( C_n \) and \( D_n \) is obtained,

\[ \begin{cases} -C_n(k_n \xi_{x,0} + k) + D_n(k_n \xi_{x,0} - k) = 0, \\ C_n(-k_n \xi_{x,L_x} + k)e^{-i k_n L_x} + D_n(k_n \xi_{x,L_x} + k)e^{i k_n L_x} = 0, \end{cases} \quad (6) \]

where \( \xi_{x,0} \) denotes the normalized impedance of the wall at \( x=0 \), and \( \xi_{x,L_x} \) that of the wall at \( x=L_x \). This system of equations has a non-vanishing solution only if the determinant of their coefficients is zero, which leads to the following condition:

\[ e^{2ik_n L_x} = \frac{(k_n \xi_{x,0} - k)(k_n \xi_{x,L_x} - k)}{(k_n \xi_{x,0} + k)(k_n \xi_{x,L_x} + k)}, \quad (7) \]

Solving Eq. (7) for \( k_n L_x \) yields the transcendent equation

\[ \tan(k_n L_x) = \frac{ik L_x(\xi_{x,0} + \xi_{x,L_x})}{k_n L_x(\xi_{x,0} + \xi_{x,L_x} + (k L_x)^2)}, \quad (8) \]

which must be solved numerically to determine the modal frequencies \( k_n \) (details concerning the numerical resolution will be given in Sec. II D). Analogous equations can be derived in the \( y \)- and \( z \)-directions. Using separation of variables, the three-dimensional modal functions may take the form \( p_n(x) = \prod_{j=x,y,z} \cos(k_n j + \phi_{jn}) \), where the phase terms \( \phi_{jn} \) are derived from the boundary conditions and given in Eq. (18) [one can also use the equivalent complex exponential formulation as in Eq. (5)]. The allowed values of \( k_{nx}, k_{ny}, \) and \( k_{nz} \) are calculated separately using Eq. (8) (and its equivalents in the \( y \)- and \( z \)-directions), and determine the value of \( k_n^2 = k_{nx}^2 + k_{ny}^2 + k_{nz}^2 \).

**B. Orthogonality relations of the room modes and completeness**

This section derives the orthogonality relations for the modal spatial functions commonly used to calculate the coefficients \( A_n \) and \( B_n \) in Eqs. (2a) and (2b).

1. **Classic inner product**

First, the correct inner product for complex-valued functions of real variables

\[ \iint_V p_n(x)p_m^*(x) dx \quad (9) \]

where * denotes the complex conjugate, is considered. Calculation details are given for the one-dimensional medium defined in Sec. II A. The extension to the three-dimensional case is straightforward. The one-dimensional eigenfunctions \( p_n(x) \) and \( p_m(x) \) satisfy Eq. (3) with the boundary condition given in Eq. (4). Analogous equations for \( p_n^*(x) \) are obtained by taking the complex conjugate on both sides of Eqs. (3) and (4). Integrating the product \( p_n(x)p_n^*(x) \) over the one-dimensional domain yields

\[ \int_0^{L_x} p_n(x)p_n^*(x) dx = \int_0^{L_x} \left( \frac{d^2 p_n(x)}{dx^2} - \frac{d^2 p_n^*(x)}{dx^2} \right) dx. \quad (10) \]

Using integration by parts on the right-hand side and inserting the boundary condition defined in Eq. (4),

\[ \int_0^{L_x} p_n(x)p_n^*(x) dx = \frac{-i k}{(k_n^2)^2} \left( \frac{1}{\xi_{x,0}^2} + \frac{1}{\xi_{x,L_x}^2} \right) \left( p_n(L_x)p_n^*(L_x) + \frac{1}{\xi_{x,0}^2} + \frac{1}{\xi_{x,L_x}^2} \right) p_n(0)p_n^*(0). \quad (11) \]
Equation (11) can be extended to the three-dimensional case using separation of variables, yielding

\[ \iiint_V |p_n(x)p_m(x)|dV = i \prod_{j=x,y,z} \left( \frac{k}{\left( k_m^2 - k_n^2 \right)} \right) \left( \frac{1}{\xi_j L_j} + \frac{1}{(\xi_j L_j)^2} \right) \times p_n(L_j)\bar{p}_m(L_j) + \left( \frac{1}{\xi_j 0} + \frac{1}{(\xi_j 0)^2} \right) p_n(0)\bar{p}_m(0), \]

(12)

It is apparent from Eq. (12) that the scalar product defined in Eq. (9) does not equal zero, except when the walls are free of absorption. This would be the case if the walls are rigid \( (\xi = \infty) \), or more generally if they are purely reflective (then their normalized impedances are purely imaginary). Whenever the normalized impedances have a non-zero real part indicating wall losses, the modes are found to be non-orthogonal for the scalar product defined in Eq. (9).

2. Modified “inner product”

In the following, it is shown that the modified “inner product”

\[ \iiint_V |p_n(x)p_m(x)|dV = \sum_{j=x,y,z} \left( \frac{k}{\left( k_m^2 - k_n^2 \right)} \right) \left( \frac{1}{\xi_j L_j} + \frac{1}{(\xi_j L_j)^2} \right) \times p_n(L_j)\bar{p}_m(L_j) + \left( \frac{1}{\xi_j 0} + \frac{1}{(\xi_j 0)^2} \right) p_n(0)\bar{p}_m(0), \]

(13)

leads to the orthogonality of the modes. Following a similar derivation as for Eq. (11) gives

\[ \int_0^{L_x} p_n(x)p_m(x)dx = \frac{1}{(k_m^2 - k_n^2)} \left( -\frac{ik}{\xi x L_x} p_m(L_x)p_n(L_x) + \frac{ik}{\xi x L_x} p_n(L_x)p_m(L_x) \right.
\]

\[ - \frac{ik}{\xi x 0} p_n(0)p_m(0) + \frac{ik}{\xi x 0} p_m(0)p_n(0) \right), \]

(14)

which equals zero for any value of the normalized impedances provided that \( m \neq n \). Note that freely decaying modes are not necessarily orthogonal because the value of the normalized wall impedances is a function of the eigenfrequency of each mode. The driven modes are orthogonal because the normalized wall impedances in this case are a function of the driving frequency. Further, a relation for \( m = n \) can be found by calculating the product \( \int_0^{L_x} p_n^2(x)dx \). For this purpose, it is convenient to write the modal functions in the form \( p_n(x) = \cosh(ik_n x + \phi_n) \), which is equivalent to the exponential form in Eq. (5):

\[ \int_0^{L_x} p_n^2(x)dx = \frac{L_x}{2} \left( 1 + \frac{1}{ik_n L_x} \sinh(ik_n L_x) \right. \]

\[ \left. \times \cosh(ik_n L_x + 2\phi_n) \right), \]

(15)

where use is made of the change of variable \( u = ik_n x + \phi_n \). The value of \( \phi_n \) can be determined from the boundary condition \( \xi_{x,0} \partial p_n(0)/\partial x - ik p_n(0) = 0 \), which gives \( \phi_n = \coth^{-1}(\xi_{x,0} k_n/k) \). Using separation of variables, the foregoing derivation leads to the orthogonality of the modes in the three-dimensional case

\[ \iiint_V |p_n(x)p_m(x)|dV = \sum_{j=x,y,z} \left( \frac{k}{\left( k_m^2 - k_n^2 \right)} \right) \left( \frac{1}{\xi_j L_j} + \frac{1}{(\xi_j L_j)^2} \right) \times p_n(L_j)\bar{p}_m(L_j) + \left( \frac{1}{\xi_j 0} + \frac{1}{(\xi_j 0)^2} \right) p_n(0)\bar{p}_m(0), \]

(16)

with

\[ K_n = \prod_{j=x,y,z} \left( \frac{L_j}{2} \left( 1 + \frac{1}{ik_n L_j} \sinh(ik_n L_j) \cosh(ik_n L_j + 2\phi_n) \right) \right), \]

(17)

and

\[ \phi_{n=x,y,z} = \coth^{-1}(\xi_{x,0} k_n/k). \]

(18)

This shows that the modal functions are orthogonal for the “inner product” defined in Eq. (13). However, the complex-valued modal functions are not orthogonal in the strict sense, as this would require that the product defined in Eq. (13) is an inner product on the space of possible solutions that satisfy the boundary conditions defined in Eq. (4). However, the product defined in Eq. (13) does not satisfy the definition15 of an inner product on the aforementioned space. Nevertheless, the orthogonality relation derived in Eqs. (16)–(18) is sufficient for solving the wave equation, but not sufficient to eliminate the need to evaluate the average values over the room space of cross-products of modal spatial functions. Besides, it is worth mentioning that for rigid walls, the calculation of the quantity defined in Eq. (13) is equivalent to the calculation of the quantity defined when replacing \( p_m \) by its conjugate. It follows that the modal functions (in this case real-valued) are strictly orthogonal. This property has been extensively used in the literature.4–9

3. Completeness

The completeness of the real-valued modal functions in the case of rigid walls is known.4–9 The question of the completeness of the complex-valued modal functions for the set of possible solutions satisfying the boundary conditions defined in Eq. (4) is outside the scope of this investigation. The solution to Eq. (1) obtained under the assumption of completeness of the modal functions and the orthogonality relation derived in Eqs. (16)–(18) shows perfect agreement with the simple closed-form solution in the one-dimensional case10 (not shown for conciseness). This means that the complex-valued modal functions must be close to being a complete basis of elements, which becomes orthogonal for the product defined in Eq. (13). As a consequence, we may say that the definition of orthogonality proposed in Eq. (13) enables the rigorous solution of the wave equation if the modes form a complete basis for the set of possible solutions satisfying the boundary conditions defined in Eq. (4).

C. Steady-state response

In the following, as discussed above, it is assumed that the complex-valued modal functions for the set of possible
solutions satisfying the boundary conditions defined in Eq. (4) is complete. Once the modal spatial functions and modal frequencies have been determined, one may expand the solution to Eq. (1) in terms of the modal functions \( p_n \) as shown in Eq. (2a). The orthogonality of the modes, necessary to calculate the coefficients \( A_n \) and \( B_n \), is derived in Sec. II B, where it is shown that the modes are not orthogonal with the usual definition of inner product if the normalized wall specific acoustic impedances have a non-zero real part [see Eq. (12)]. A modified definition of orthogonality is necessary [see Eqs. (16)–(18)], which gives

\[
B_n = \frac{1}{k_n} \int \int q(x)p_n(x)dV = \frac{Qp_n(x_0)}{k_n}.
\]

(19)

Expanding both sides of Eq. (1) in terms of the modal functions and combining with Eqs. (3) and (19), we have

\[
A_n = \frac{i \omega p B_n}{k_n - k_n^2} = \frac{i \omega p Q p_n(x_0)}{k_n (k_n^2 - k_n^2)},
\]

(20)

and, using the expansion defined in Eq. (2a),

\[
p(x) = \sum_n \frac{i \omega p Q p_n(x_0) p_n(x)}{k_n (k_n^2 - k_n^2)}.
\]

(21)

The particle velocity components can now be derived from Euler’s equation of motion

\[
u_{j=x,y,z}(x) = -Q \sum_n \frac{p_n(x_0) \partial p_n(x)}{k_n (k_n^2 - k_n^2)},
\]

(22)

and the active and reactive intensity vectors are calculated as

\[
I(x) = \text{Re}\{p(x)u^*(x)\},
\]

(23a)

\[
J(x) = \text{Im}\{p(x)u^*(x)\}.
\]

(23b)

It can be remarked that to calculate the sound pressure or the real and imaginary parts of the sound intensity averaged across the position in the room, the modified definition of “inner product” given by Eq. (13) needs to be used. However, to calculate the mean squared sound pressure or the real and imaginary parts of the sound intensity averaged across the whole of the room volume using \( N \) modes, it is necessary to evaluate \( N^2 \) integrals of the form of Eq. (9). If the modes were orthogonal for the definition of inner product given by Eq. (9), all but the \( N \) diagonal products would be zero, which would substantially reduce the amount of calculation.

In addition, a theoretical expression for the spatial standard deviation of the sound pressure level of third-octave bands of random noise in a rectangular room with arbitrary boundary conditions is given in the Appendix. The theoretical expression is based on the theory presented by Davy. An expression for the damping rate \( \gamma_n \) of the \( n \)th modal amplitude which is used in the theoretical expressions is derived in the Appendix for the case of arbitrary boundary conditions. The modal damping rate is a term that appears in the expansion of the \( n \)th modal wavenumber \( k_n = (\omega_n + i \gamma_n)/c \), where the resonant angular frequency of the \( n \)th mode is \( \omega_n \).

D. Numerical solution

The solution of Eq. (8) is computed for each of the three axes using the trust-region\textsuperscript{17} \emph{fseolve} algorithm, available under the Optimization Toolbox in the \textsc{matlab} environment (MathWorks, 2018).\textsuperscript{18} The requirement of an initial guess as a starting point is a critical step. In this section, the determination of an adequate approximate starting value for the numerical calculation of the modal frequencies \( k_n \) for \( i = x, y, \) and \( z \) is addressed.

Considering a one-dimensional medium with rigid or nearly hard terminations (i.e., \( |\xi_{x,0}| \gg 1 \) and \( |\xi_{x,L_x}| \gg 1 \)), an approximate solution to Eq. (8) can be found by applying the series expansion of the exponential function truncated after the second term to the right-hand side of Eq. (7),

\[
e^{2ik_nL_x} \approx e^{-2k_n/(1/\xi_{x,0} + 1/\xi_{x,L_x})}.
\]

(24)

Since the exponential function with imaginary argument is 2\( \pi \)-periodic, \( e^{2ik_nL_x} = e^{2ik_nL_x + 2\pi n} \approx e^{-2k_n/(1/\xi_{x,0} + 1/\xi_{x,L_x})} \), where \( n \) is an arbitrary integer. Equating the exponents yields

\[
k_n L_x = n\pi + ik_n \left( \frac{1}{\xi_{x,0}} + \frac{1}{\xi_{x,L_x}} \right).
\]

(25)

Since \( |\xi_{x,0}| \gg 1 \) and \( |\xi_{x,L_x}| \gg 1 \), the second term on the right-hand side is much smaller than the first one, so that we may replace \( k_n \) in the denominator by its expression in the hard-wall case \((n\pi/L_x)\), which gives

\[
k_n \approx \frac{n\pi}{L_x} + ik_n \left( \frac{1}{\xi_{x,0}} + \frac{1}{\xi_{x,L_x}} \right).
\]

(26)

It is worth mentioning that in the case of rigid or nearly rigid terminations, the well-known expression \( k_n = n\pi/L_x \) is found. Besides, the modal frequencies are complex unless the terminations are absorption free. For infinite or purely imaginary normalized impedances, the modal frequencies are real. This corresponds to the case where the boundaries are rigid or purely reflective, respectively. In the latter case, the modal frequencies will simply be slightly shifted compared to the rigid termination case.

The frequency-dependent approximate solution given in Eq. (26) is a tempting initial point for the solution of Eq. (8). The algorithm, however, sometimes fails to converge to the correct solution when using the approximate solution as an initial guess for the solution of Eq. (8). This is shown by a sudden jump in the modal frequency for a given mode as a function of exciting frequency. Therefore, a \textit{warm-start} strategy is used instead, which implies that the result obtained for a given frequency is used as the starting point for the calculation of the eigenfrequency for the next higher driving
frequency. The approximate solution given in Eq. (26) is used for the computation at 1 Hz.

For each frequency, the zeroth-order modal frequency must be calculated separately, as it is apparent that Eq. (26) is not valid for the zeroth-order mode where \( n = 0 \). An alternative approximate value for \( k_{0,x} \) can be found by using the series expansion of the tangent function truncated after the first term. Since \( k_{0,L_x} \ll 1 \), this expansion can be applied to the left-hand side of Eq. (8), which yields

\[
k_{0,x} \approx \frac{1}{L_x} \sqrt{\left( \frac{kL_x}{\sin \frac{\pi}{L_x}} \right)^2 + i kL_x \left( \frac{1}{\sin \frac{\pi}{L_x}} + \frac{1}{\sin \frac{\pi}{L_z}} \right)}. \tag{27}
\]

In the particular case of a rectangular room with absorption concentrated on the floor, and two pairs of opposite walls which are rigid or nearly rigid, one may use the approximate solutions defined in Eqs. (26) and (27) in the corresponding horizontal directions. In this study, however, the numerically exact values in all directions were always calculated. Also note that the cases when the real part of the normalized impedance is less than unity or when the imaginary part is very different from zero are not dealt with in this paper. These two cases probably need different starting values for the eigenfrequencies, since the assumption of nearly-rigid walls used in Eqs. (24)–(27) will become less correct.

III. NUMERICAL RESULTS

The mode model described in Sec. II is used to examine properties of the sound pressure and complex sound intensity in an existing small (40 m\(^3\)) rectangular room with dimensions 4.38 \times 3.29 \times 2.97 m. A point source is placed in the upper corner at (0.6; 0.5; \( L_z - 0.4 \)) m (that is, at least 0.4 m from any boundary). This position was chosen after calculations with a point source, placed closer to the corner and spaced at the equal distance of 0.25 m from each of the three walls forming the corner, unsurprisingly produced atypical results. Two configurations of the room are considered: (i) all surfaces are nearly rigid (i.e., \( |\xi| \gg 1 \)); (ii) the floor is covered with absorptive material of varying absorption while the remaining surfaces are nearly rigid. For simplicity, the normalized impedances have the same values across frequency, although this is not necessary for the calculation method. The pressure, particle velocity, and complex intensity fields are calculated at 378 points, over a regular grid of receivers placed 0.4 m away from any wall. Receivers closer than 1 m to the source are not used. The steady-state pressure response is calculated according to Eq. (21). The allowed values of \( k_{nx} \), \( k_{ny} \), and \( k_{nz} \) are calculated separately using Eq. (8). The algorithm described in Sec. IID is carried out from 1 to 450 Hz with a frequency step of 1 Hz.

Figure 1 shows the computed sound pressure level in the undamped (left) and damped (right) room (in this case the floor is covered with absorptive material with specific acoustic impedance \( \xi_{z=0} = 2 \), corresponding to an absorption coefficient \( \alpha_{z=0} = 0.9 \)). The results are displayed at 70 Hz, a natural frequency corresponding to the (1,0,1) mode (this natural frequency is estimated in the undamped case). The results (interpolated for clarity) are displayed as orthogonal slice planes normal to the \( x-, y-\), and \( z\)-direction, respectively. The mode is clearly identified in the results from the undamped room (left). Yet, in the damped room (right), wave motion solely occurs in the plane parallel to the absorbing surface. This is in agreement with the general behavior of sound propagating in a rectangular room containing a highly absorptive surface and no scattering objects, as described in, e.g., Refs. 19 and 20: the non-grazing part of the sound field is greatly damped by the absorbing surface, while the grazing part (propagating parallel to the absorbing wall) is much less affected by the absorption.

Additionally in Fig. 1, the sound pressure level computed from the analytical model described in Sec. II is systematically compared with FEM calculations. All calculations were made using the FEM software package\(^{21}\) COMSOL 5.3a, where the longest dimension of the elements was adjusted to guarantee that \( kh/p < 1 \) (\( p = 2 \) is the element order).\(^{22}\) There is fair, if not perfect, agreement between the FEM and analytical results. The spatially averaged relative difference \( \langle \delta \rangle \% = 100 \frac{\| x - x_{FEM} \|_2}{\| x \|_2} \) (where the vector \( x \) corresponds to the analytical sound pressure, \( x_{FEM} \) to the FEM model values, and \( \| \cdot \|_2 \) indicates the \( l_2\)-norm of the pressure vectors) yields values ranging between 4% and 6% in the entire frequency range (mean values for third-octave bands). This indicates that the proposed numerical strategy for solving Eq. (8) is robust.

The fluctuations in sound pressure level across the volume of the room as a function of frequency are shown in Fig. 2. The spatial standard deviation of the sound pressure level (in dB) with respect to the 378 receiver positions is computed using the analytical model described in Sec. II and validated against the theoretical predictions presented in the Appendix. The results are displayed as mean values per third-octave band (ranging from 50 to 400 Hz) for three different absorptions of the floor surface (\( \alpha_{x=0} = 0.9 \), \( \alpha_{x=0} = 0.3 \) and rigid floor, respectively). There is fair agreement between numerical and theoretical predictions. It is confirmed that the spatial standard deviation of the sound pressure level decreases with increasing frequency as the modal overlap increases and exhibits lower values in the undamped case at high frequencies. In fact, the narrower modal bandwidth in the undamped case means that there are more modal frequencies in each third-octave band at which the sound pressure is uncorrelated with the sound pressure at the other modal frequencies. The smaller modal overlap produces the opposite result at low frequencies. The increase of the modal overlap and the third-octave bandwidth with frequency is responsible for the decrease as the frequency increases. At low frequencies, the computed spatial standard deviation fluctuates with frequency. This is expected since the spatial standard deviation at low modal overlap depends strongly on whether one single mode or several modes contribute to the sound field. The damped case has a smaller standard deviation because of the larger modal overlap. At higher frequencies, the opposite occurs because the wider modal bandwidth means less statistically independent modal frequencies in each third-octave band.

Figure 3 shows a volumetric representation of the active intensity field in the undamped and damped room
\( \alpha_z = 0.9 \), respectively, for the third-octave band centered at 125 Hz. The intensity vectors are computed at the 378 receiver points, although positions closer than 1 m from the source are not used. The length of the vectors is proportional to the intensity [in \( \text{W/m}^2 \)] at the base of the cones, and the cones are color-coded to also show the magnitude of the intensity vector. For both damping conditions, it is apparent that the intensity field emanates from the direction of the source placed at \((0.6; 0.5; L_z/L_0)\) m (black dot in Fig. 3). In the undamped room, the intensity field seems to flow in multiple directions. When absorption is added to the floor, the magnitude of the active intensity is less, and the flow is pointing towards the absorbing surface (curving from the source to the absorbing surface). This is in agreement with experimental findings.  

FIG. 1. (Color online) Sound pressure level [dB sound pressure level (SPL)] at 70 Hz (1,0,1 mode) displayed as orthogonal slice planes in the \(x\)-, \(y\)-, and \(z\)-direction, respectively, when (left) all boundaries are nearly rigid; (right) the floor is covered with highly absorptive material (\(\alpha_z = 0.9\)). Maximum order of modes included in the expansion: 15. Results are compared with FEM calculations.

\( \alpha_z = 0.3 \), respectively from left to right). Numerical results (solid lines) are compared with theoretical predictions (dashed lines). Maximum order of modes included in the expansion: 15.

**FIG. 2.** Spatial standard deviation \( s \) of the sound pressure level (in dB) for different floor absorptions (\( \alpha_z = 0.9 \), \( \alpha_z = 0.3 \), and a nearly rigid floor, respectively) for the third-octave band centered at 125 Hz.
Figure 4 shows a volumetric representation of the active (top) and reactive (bottom) intensity fields at 58 Hz, a natural frequency corresponding to the \((0,0,1)\) mode (natural frequency estimated in the rigid wall case). The intensity fields are represented in the undamped room (left) and the damped room (right), respectively. In the undamped room, the magnitude of the active intensity is about one-third of that of the reactive intensity. It is apparent that the sound field is strongly reactive, as indicated by the presence of a standing wave in the \(z\)-direction. These results agree with theoretical predictions.\(^{24–27}\) In the damped room, the magnitudes of the active and reactive intensities are lower than in the undamped case. The active intensity is clearly pointing towards the absorbing surface, although slightly angled in the \(x\)- and \(y\)-directions, corresponding to the position of the source. The magnitude of the active intensity field is also larger than that of the reactive intensity, indicating a more dominant active intensity flow, as expected due to the presence of the absorbing floor. It is also noticeable that the reactive flows occur only in the \(xy\)-plane (plane parallel to the absorbing surface), indicating a lack of a standing wave between the floor and the ceiling.

Figure 5 compares the spatial average of the active and reactive intensity magnitudes in the undamped [Fig. 5(a)] and damped [Fig. 5(b)] room as a function of frequency (the regions of low and high modal overlap are represented separately). The modes are clearly identified in the undamped room at low modal overlap. The sound field is strongly reactive in all modes, in agreement with the theoretical considerations presented in Ref.\(^{27}\). The modal overlap is higher in the damped room, and the magnitude of the active intensity field is larger than that of the reactive intensity, as expected due to the presence of the absorbing floor. At higher modal overlap in the undamped room, the sound field remains strongly reactive in all modes, and active and reactive intensities are of comparable magnitude. In the damped room, it is noticeable that oblique modes are affected by the presence of the absorbing floor (the reactive intensity at the
corresponding frequencies is lower than in the undamped room), while this is not necessarily the case for tangential and axial modes, depending on their orientation with respect to the absorbing surface.

**IV. CONCLUSION**

A mode model has been used to examine the steady-state sound field in a rectangular room with sound absorption concentrated on the floor. The existing modal theory for the reverberant sound field in rectangular rooms with nearly hard walls has been extended to the case of rectangular rooms whose walls have normalized specific acoustic impedances which can differ between walls, and the question of the orthogonality of the spatial modal functions in this case has been examined. The mathematical derivations exhibit interesting properties of the modes: in particular, it is shown that the spatial modal functions are not necessarily orthogonal with the usual definition of inner product, which leads to a modified definition of orthogonality. This modified definition enables the calculation of the modal expansion coefficients but does not guarantee that the average of the cross-products of modal spatial functions is zero when averaged over the room volume.

A method for finding starting values for the numerical calculation of the eigenfrequencies has been proposed, which guarantees that the numerical algorithm converges to the correct solution. A different approximate starting value for the numerical calculation of the eigenfrequency of the zeroth-order mode at the lowest frequency is also given. These numerical considerations are confirmed with sound pressure calculations using a finite element method (FEM).

The model makes it possible to analytically derive expressions characterizing the sound pressure and the intensity flows corresponding to varying positions throughout the volume of the room. A theoretical expression for the spatial standard deviation of the sound pressure level in rooms with arbitrary boundary conditions has been presented. Good agreement was found between this theoretical prediction and a direct computation of the spatial fluctuations of the sound pressure level using the modal expansion method given in this paper. The properties of the active and reactive intensity flows have been examined and found to compare well with the predicted behaviours in the literature. The model provides a powerful means to accurately describe fundamental properties (e.g., sound pressure, particle velocity, complex intensity) of rooms with non-diffuse sound fields and, as such, may find application in the study of, e.g., reverberation chambers used for conventional measurements of sound absorption, or ordinary rooms (classrooms, clinical rooms, etc.).

**ACKNOWLEDGMENTS**

This work is funded by the Oticon Foundation. The authors would like to thank Peter Risby Andersen for advice with the FEM simulations.

**APPENDIX: SPATIAL STANDARD DEVIATION OF THE SOUND PRESSURE LEVEL IN A ROOM WITH ARBITRARY BOUNDARY CONDITIONS**

This appendix shows the derivation of the theoretical expression for the spatial standard deviation of the sound pressure level. The theoretical considerations in Ref. 16 are extended to the case of a room with arbitrary boundary conditions. An expression for the damping rate of the modes is derived. In Sec. III, the derived expression is compared with the computed values.
The theoretical spatial standard deviation \( s \) of the sound pressure level in a reverberant space excited by a frequency band of random noise is given by\(^\text{16} \)

\[
s = \sqrt{(10 \log_{10} P)^2 r}, \tag{A1}
\]

with\(^\text{16} \)

\[
r = \frac{1}{Z} \left( 1 + \frac{K}{M_t} \right) F(\pi Z), \tag{A2}
\]

and \( Z = B_d / \gamma \), where \( B_d \) is the statistical bandwidth in Hz, and \( \gamma \) is the damping rate of the modal amplitudes. \( M_t = n \gamma \) is the statistical modal overlap, which is the product of the modal density \( n \) with the statistical modal bandwidth (here in Hz). The function \( F \) reads\(^\text{16} \)

\[
F(\pi Z) = \frac{2}{\pi} \arctan(\pi Z) - \frac{1}{\pi Z} \ln \left( 1 + (\pi Z)^2 \right), \tag{A3}
\]

and

\[
K = \left( \left\langle \frac{p_{xj}^2(x)}{p_{xj}^2(x)} \right\rangle \right)^2 - 3C(M_s), \tag{A4}
\]

where \( p_{xj}(x) \) is a modal spatial function, and the angular brackets \( \left\langle \cdot \right\rangle \) denote the average value over position in the room and over modes in the frequency range of interest. \( C \) is a function of the distribution of the modal frequency spacings and reads\(^\text{16} \)

\[
3C(M_s) = \left( M_s + 2 - \frac{5}{2} M_s \right) e^{-2M_s} \left( M_s + 1 \right) - E_1(M_s) e^{-M_s} \left( M_s + 1 \right) - E_2(M_s) e^{M_s} \left( M_s + 1 \right), \tag{A5}
\]

where \( E_1 \) is the exponential integral. The modal density \( n \) may be calculated using the formulas for a rectangular parallelepiped room with rigid walls\(^3 \) and reads

\[
n = \frac{4 \pi f^2 L_x L_y L_z}{c^3} + \frac{n f}{c^3} \left( L_x L_y + L_x L_z + L_y L_z \right) + \frac{1}{2c} \left( L_x + L_y + L_z \right). \tag{A6}
\]

Additionally, for all modes \( \left\langle p_{xj}^2(x) \right\rangle = 1 \), and \( \left\langle p_{xj}^3(x) \right\rangle = (3/2)^3 \), \( \left\langle p_{xj}^4(x) \right\rangle = (3/2)^2 \), \( \left\langle p_{xj}^5(x) \right\rangle = 3/2 \) for oblique, tangential, and axial modes, respectively.\(^5 \) Thus

\[
\frac{\left\langle p_{xj}^4(x) \right\rangle}{\left\langle p_{xj}^2(x) \right\rangle^2} = \frac{1}{n} \left( \frac{274 \pi f^2 L_x L_y L_z}{8 c^4} + \frac{9 \pi f}{8 c^2} \left( L_x L_y + L_x L_z + L_y L_z \right) + \frac{3}{8} L_x L_y + L_x L_z \right). \tag{A7}
\]

In order to estimate the modal damping rate \( \gamma \), the square of the wave number is expressed in the form

\[
k^2_n = \sum_{j=x,y,z} k^2_n = \frac{\omega_n^2 + 2i \omega_n \gamma_n}{c^2}, \tag{A8}
\]

where \( \omega_n \) and \( \gamma_n \) are real. It follows that

\[
c^2 k^2_n = \sum_{j=x,y,z} \omega_n^2 + 2i \omega_n \gamma_n = \omega_n^2 \left( 1 + \frac{2i}{\omega_n} \sum_{j=x,y,z} \omega_n \gamma_n \right), \tag{A9}
\]

with \( \omega_n^2 = \sum_{j=x,y,z} \omega_n^2 \). Since \( 2i / \omega_n \sum_{j=x,y,z} \omega_n \gamma_n \ll 1 \), an approximate expression for \( c k_n \) can be found by applying the series expansion of \( (1 + x)^{1/2} \) truncated after the first term, which gives

\[
k_n \approx \omega_n \left( 1 + \frac{i}{\omega_n} \sum_{j=x,y,z} \omega_n \gamma_n \right). \tag{A10}
\]

Besides, Eq. (26) yields

\[
k^2_n = \left( \frac{n_j}{L_j} \right)^2 + \frac{2 i n_j \pi k}{k_n L_j} \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right) - \frac{k^2}{k^2_n L_j^2} \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right)^2, \tag{A11}
\]

for \( j = x, y, z \), respectively. In the special case of nearly rigid boundaries (i.e., \( |\xi_j| \gg 1 \)) and \( |\xi_j L_j| \gg 1 \), one may write

\[
k^2_n \approx \left( \frac{n_j}{L_j} \right)^2 + \frac{2i}{L_j} \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right). \tag{A12}
\]

Recalling that \( c^2 k^2_n = \omega_n^2 + 2i \omega_n \gamma_n \), and equating the imaginary part with that of Eq. (A12), we get

\[
\omega_n \gamma_n \ll 2k \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right), \tag{A13}
\]

for \( j = x, y, z \), respectively. It follows that Eq. (A10) yields

\[
k_n \approx \omega_n \left( 1 + \frac{i c}{\omega_n} \sum_{j=x,y,z} \frac{1}{L_j} \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right) \right), \tag{A14}
\]

where we have assumed that \( \omega = \omega_n \), under the assumption that only modes whose resonant frequencies are close to the actual frequency will contribute significantly to the modal sum (provided that we are not calculating the sound pressure very near the sound source). Finally, by equating the imaginary parts in Eq. (A14) and \( k_n = (\omega_n + i \gamma_n) / c \), we get

\[
\gamma_n \approx c \sum_{j=x,y,z} \frac{1}{L_j} \left( \frac{1}{\xi_j} + \frac{1}{\xi_j L_j} \right). \tag{A15}
\]

This equation can also be derived from equations already published in the literature.\(^8,9,12 \)