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On Cabibbo angle from theory

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Abstract – We find an expression for the Cabibbo angle from quark flavour generators of the first two generations. The flavour generators operate on the toroidal components in an intrinsic dynamics for colour degrees of freedom. The generators have led to parton distributions for u and d valence quarks of the proton that compare well with those derived from experiment. The present result 0.974996... for the cosine of the Cabibbo angle compares rather well with the experimentally established value for the up-down quark mixing element 0.97420+/−0.00021 of the Cabibbo-Kobayashi-Maskawa matrix.

Introduction. – In 1963 Nicola Cabibbo introduced a connection between strangeness-conserving and strangeness-changing processes expressed in cosine and sine factors of a certain angle [1]. Cabibbo remarked as a consequence that the vector coupling constant in beta decay is not the full Fermi constant $G_{F\mu}$ but contains a factor $\cos\theta_C$ where θ_C has come to be known as the Cabibbo angle. This was a first sign of slight differences in the description of purely leptonic decay and leptonic decays of hadrons. Cabibbo's description was taken up by Glashow, Iliopoulos and Maiani in 1970 with the introduction of the charm quark [2]. A few years later Kobayashi and Maskawa introduced a three-generation quark mixing [3] whose elements have come to be understood as mutual quark coupling strengths. For instance the original cosine factor is now understood to express a coupling between a u and a d quark flavour. We shall introduce in the present work, quark flavour generators from which it is possible to derive an expression for the Cabibbo angle.

In the Standard Model of elementary particles [4], the strong interactions are described by quantum chromodynamics where quarks in strong interaction mass eigenstates can change colour by emission or absorption of gluon interaction quanta. The weak interactions are described by the electroweak theory (quantum flavourdynamics), where quarks in weak interaction eigenstates can change flavour by emission or absorption of intermediate vector bosons, $W^{\pm}$. Figure 1 shows the situation in a schematic form. Note that the colour transformations do not change the flavours. Note also that the flavours come in three generations. The generations can be mixed by flavour transformations but not by colour transformations[1].

The mixing between generations is a consequence of the mixing between flavour mass eigenstates and flavour interaction states. The mass eigenstates u, c, t, d, s, b constitute the base for the strong interactions, the (weak) interaction eigenstates u', c', t', d', s', b' constitute the base for the weak interactions. The connection between the two base sets is usually described by the Cabibbo-Kobayashi-Maskawa matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \tag{1}$$

such that the u-type flavours are chosen to be the same for both interactions, whereas the d-type flavours are transformed, *i.e.*,

$$\begin{pmatrix} u' \\ c' \\ t' \end{pmatrix} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{\mathrm{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{2}$$

[1] See the caption of fig. 2 for a reservation on this when higher-order processes are taken into account.

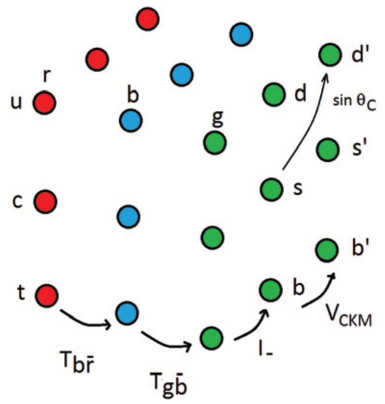

Fig. 1: (Colour online) The *Quark Cube*. The quarks of the Standard Model [4] come in three generations with two flavours in each, (u, d), (c, s) and (t, b), respectively. The six quark flavours, up, down, charm, strange, beauty and top come in three colours, red, blue and green. The gluon generator $T_{b\bar{r}}$ annihilates a red quark of a specific flavour and creates a blue of the same flavour. The gluon generators do not mix the generations. The isospin operator I_- lowers the isospin three-component from $I_3 = \frac{1}{2}$ for the u-family (u, c, t) to $I_3 = -\frac{1}{2}$ for the d-family (d, s, b). The Cabibbo-Kobayashi-Maskawa matrix $V_{\rm CKM}$ transforms mass eigenstates (unprimed) to weak interaction states (primed) and mix all three generations. The Cabibbo mixing angle $\theta_{\rm C}$ describes the mixing between the first two generations. It was originally introduced by Cabibbo [1] before charm, beauty and truth (topness) were discovered. We introduce quark flavour generators T_u, T_d, T_s (5) that act on colour components (21) and find $|\sin\theta_{\rm C}| = |{\rm Tr} T_u^\dagger T_s|$ (24). The result $\frac{2}{9} = 0.222\cdots$ is quite close to the experimentally extracted value [4] for the Wolfenstein parameter $\lambda = 0.22453 \pm 0.00044$ from CKMfitter and $\lambda = 0.22465 \pm 0.00039$ from UTfit.

For an updated Standard Model presentation of the common origin of quark masses and mixing from the Yukawa couplings to the Higgs field, see Ceccucci, Ligeti and Sakai in [4].

The Standard Model has no clue on how to fix the elements of the mixing matrix $V_{\rm CKM}$ except that the unitarity requirement—corresponding to assuming transformations to stay within exactly three generations—reduces the number of independent parameters to three angles, $\theta_{12}, \theta_{23}, \theta_{13}$ and one phase δ. The standard parametrization of the full Cabibbo-Kobayashi-Maskawa-matrix thus reads [4]

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{3}$$

where $c_{12} = \cos\theta_{12}, s_{12} = \sin\theta_{12}$ and so on. The two angles θ_{23} and θ_{13} are considerably smaller[2] than θ_{12} and setting them equal to zero corresponds to having the Cabibbo angle $\theta_{\rm C} = \theta_{12}$ whereby the third generation is decoupled and we have the Cabibbo mixing matrix for the first two generations [5],

$$V_{\rm C} = \begin{pmatrix} \cos\theta_{\rm C} & \sin\theta_{\rm C} \\ -\sin\theta_{\rm C} & \cos\theta_{\rm C} \end{pmatrix}. \tag{4}$$

So far the parameters in $V_{\rm CKM}$ —and thus in the approximate $V_{\rm C}$— have had to be settled by experiment.

Flavour generators. – In the present work we investigate a simple model which combines colour and flavour degrees of freedom to determine the value of the Cabibbo angle from a theoretical point of view. The model is not complete but it does open a road into a possible structure behind the Standard Model. The road opens from an intrinsic relation between flavour and colour degrees of freedom from which it has been possible to derive u and d quark parton distribution functions for a protonic state by the use of flavour generators T_u and T_d acting on the toroidal degrees of freedom in an intrinsic dynamics for baryons like the neutron and the proton [6], see fig. 2[3].

We shall use the same flavour generators here and shall include also a strangeness generator, T_s to have all the three flavours below the charm threshold,

$$T_u = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_d = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_s = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Flavour in colour. – The physical origin of the connection between flavour and colour that we use in the present work and used in [6] to generate the parton distribution functions in fig. 2 is deeply rooted in an underlying model for baryon structure.

We assume baryons to be stationary states of a Hamiltonian on the configuration space $U(3)$,

$$\frac{\hbar c}{a}\left[-\frac{1}{2}\Delta + \frac{1}{2}{\rm Tr}\,\chi^2\right]\Psi(u) = \mathcal{E}\Psi(u), \tag{6}$$

where the length scale a is determined from the classical electron radius $r_{\rm e} = e^2/(4\pi\epsilon_0 m_{\rm e}c^2)$ [4] as $\pi a = r_{\rm e}$ [6] and the configuration variable $u \in U(3)$ contains nine dynamical angular variables $\theta_j, \alpha_j, \beta_j$

$$u = e^{i\chi} = e^{i(\theta_j T_j + \alpha_j S_j/\hbar + \beta_j M_j/\hbar)}, \quad j = 1, 2, 3. \tag{7}$$

[2] From the Wolfenstein parameters $\lambda = 0.22453 \pm 0.00044, A = 0.836 \pm 0.015, \bar{\rho} = 0.122^{+0.018}_{-0.017}, \bar{\eta} = 0.355^{+0.012}_{-0.011}$ in [4] we find a clear hierarchical structure $\theta_{12} = 12.98°, \theta_{23} = 2.42°, \theta_{13} = 0.21°, \delta = 71.03°$ with only a minute angular admixture (but large phase) of the third generation into the first.

[3] The "flavour in colour" relation is *not* an algebraic relation between quantum numbers. Instead it implies flavour degrees of freedom to be considered as specifically generated vector fields on an intrinsic three-dimensional torus.

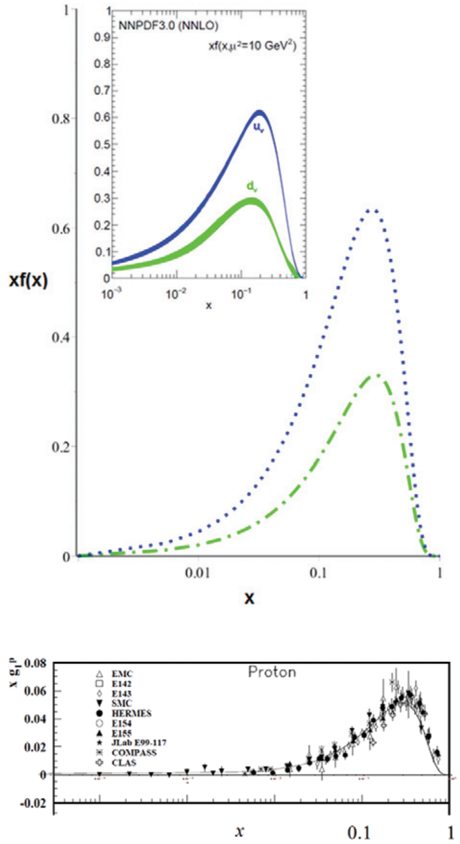

Fig. 2: (Colour online) Top: parton distribution functions for u (dotted, blue) and d (dash-dotted, green) quarks from an intrinsic protonic state [6] to compare with experimental determinations [4,7] (insert). The theoretical distributions are from an approximate state and are generated by T_u and T_d (5), respectively acting on toroidal degrees of freedom in the intrinsic protonic state. The generators T_s (5) and T_c (31) give zero strangeness and charm content in the protonic state at what would correspond to $q^2 \to 0\,\mathrm{GeV}^2$. For finite q^2, sea quarks (and antiquarks) from other generations will evolve via the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi convolutions because of gluon emission leading to pair creation into different generations already in next-to-leading-order processes, see, *e.g.*, pp. 407 in [8]. Bottom: proton spin structure function (solid) [9] based on flavour generators T_u and T_d (5) and overlaid on data from several experimental groups [4,7].

The generators $iT_j = \frac{\partial}{\partial\theta_j}$ are diagonal and we identify them as three colour generators. In topological language, they generate the maximal torus —$U(3)$ is a compact space. The generators S_j commute as intrinsic angular momentum operators. The mixing operators M_j —in combination with the other two sets— generate flavour degrees of freedom and commute like Laplace-Runge-Lenz operators, pp. 236 in [10]. The total set of nine degrees of freedom are excitable kinematically from momentum, angular momentum and Laplace-Runge-Lenz operators in laboratory space as seen in a coordinate representation with projection $x_j = a\theta_j$. For instance $S_3 = a\theta_1 p_2 - a\theta_2 p_1$ and $M_3/\hbar = \theta_1\theta_2 + \frac{a^2}{\hbar^2}p_1 p_2$, where $p_j = -i\hbar\frac{1}{a}\frac{\partial}{\partial\theta_j}$.

The potential $\frac{1}{2}\mathrm{Tr}\chi^2$ is periodic and only depends on the *eigenangles* θ_j (see footnote [4]). With a suitable parametrization of the Laplacian [12][5]

$$\Delta = \sum_{j=1}^{3} \frac{1}{J^2}\frac{\partial}{\partial\theta_j}J^2\frac{\partial}{\partial\theta_j} - \sum_{i<j;k\neq i,j}^{3} \frac{(S_k^2 + M_k^2)/\hbar^2}{8\sin^2\frac{1}{2}(\theta_i - \theta_j)}, \quad (8)$$

the off-diagonal degrees of freedom can therefore be integrated out to get

$$\frac{\hbar c}{a}\left[-\frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2}{\partial\theta_j^2} + W\right] R(\theta_1,\theta_2,\theta_3) = \mathcal{E} R(\theta_1,\theta_2,\theta_3), \quad (9)$$

where R is antisymmetric in the three angles. The total potential $W(\theta_1,\theta_2,\theta_3)$ [6] contains the centrifugal term from the Laplacian (8) and depends on hypercharge and isospin through the value of $S_k^2 + M_k^2$.

From the measure-scaled toroidal wave function R, we can generate colour fields c_j by use of the exterior derivative expanded on the torus forms $\mathrm{d}\theta_j$ that are conjugate to the left-invariant coordinate fields $\partial_j = uiT_j$ generated by T_j, *i.e.*, $\mathrm{d}\theta_i(\partial_j) = \delta_{ij}$. Thus, we have

$$\mathrm{d}R = c_j \mathrm{d}\theta_j \quad (10)$$

and can read off colour components

$$c_j(u) = \mathrm{d}R_{u=\exp(\theta i T)}(iT_j) \quad (11)$$

along tracks on the torus, generated by any combination $T = k_1T_1 + k_2T_2 + k_3T_3$ of generators. In particular up and down flavour tracks leading to the parton distribution functions in fig. 2 for the proton correspond to applying $T_u = \frac{2}{3}T_1 - T_3$ and $T_d = -\frac{1}{3}T_1 - T_3$ from (5) on an approximate expression for R. The fractional coefficients —which correspond to the quark electrical charges— enter because $U(3)$ and $SU(3)$ only share off-diagonal generators whereas the diagonal generators are different. Generators X^i_j of $U(3)$ may be defined from creation and annihilation operators $a_i^\dagger$ and a_j, $i,j = 1,2,3$ for the 3-dimensional harmonic oscillator, see pp. 71 and 221 in [13],

$$X^i_j = a_i^\dagger a_j \sim E_{ij}. \quad (12)$$

Here E_{ij} is a 3×3 matrix representation used on (21) with the ij^{th} element equal to 1 and all other elements 0. This set is equivalent to our set of $T_j, S_j/\hbar, M_j/\hbar$, *e.g.*,

[4]This follows from the fact that the eigenvalues of u are not changed by conjugation and that the trace is invariant under conjugation: Any $u = e^{i\chi}$ with eigenvalues $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ can be diagonalized via conjugation with a particularly chosen $v \in U(3)$ such that $v^{-1}uv = e^{i\theta_j T_j} \equiv e^{i\xi}$. Now $e^{i\chi} = 1 + i\chi + \frac{1}{2}(i\chi)^2\cdots$ and we have $v^{-1}e^{i\chi}v = 1 + v^{-1}i\chi v + v^{-1}\frac{1}{2}(i\chi)^2 v + \cdots = 1 + i(v^{-1}\chi v) + \frac{1}{2}v^{-1}i\chi v v^{-1} i\chi v + \cdots = e^{iv^{-1}\chi v} = e^{i\xi}$. From the cyclic property of the trace, $\mathrm{Tr}A^{-1}BA = \mathrm{Tr}B$, it follows that $\mathrm{Tr}\chi^2 = \mathrm{Tr}\ \xi^2 = \sum_{j=1}^{3}\theta_j^2$, with $-\pi \le \theta_j \le \pi$ for the shortest geodesic. See also [11].

[5]Measure-scaling "Jacobian" $J = \prod_{i<j}^{3} 2\sin\frac{1}{2}(\theta_i - \theta_j)$.

$iT_3 = iE_{33}, iS_3/\hbar = E_{12} - E_{21} = i\lambda_2, iM_3/\hbar = i(E_{12} + E_{21}) = i\lambda_1$, where λ_1, λ_2 are two of the six off-diagonal Gell-Mann matrices (see p. 209 in [10]). Generators for $SU(3)$ may be defined as

$$A^i_j = X^i_j - \frac{1}{3}\delta^i_j X, \quad X = X^j_j. \tag{13}$$

It is the factor 1/3 that makes the fractional charges appear in the generators (5) if one assumes the common Gell-Mann-Nakano-Nishijima formula

$$\begin{aligned} Q &= I_3 + \frac{Y}{2}, & Q &= A^1_1, \\ Y &= -A^3_3, & I_3 &= \frac{1}{2}(A^1_1 - A^2_2) \end{aligned} \tag{14}$$

and for $U(3)$ define charge and hypercharge operators as

$$Q' = X^1_1 \sim T_1, \qquad Y' = -X^3_3 \sim -T_3. \tag{15}$$

Thus, eq. (5) mixes quark charges $e_q = \frac{2}{3}, -\frac{1}{3}$ with baryonic hypercharges $y = 1, 0$.

To sum up the interpretation: The colour degrees of freedom are contained in the torus of the $U(3)$ configuration space and the flavour structure follows from the Laplacian on this configuration space. The interconnection between the two originates from the assumed Hamiltonian in (6) (see footnote [6]). The model thus contains both the exact colour $su_c(3)$ symmetry and the broken flavour $su_f(3)$; the first reflecting the $U(3)$ Lie group configuration space, the second reflecting the $u(3)$ algebra of the generators from which the Laplacian (8) is constructed [12]. It can be shown [9,16] that the exterior derivative of the wave function generates colour quark fields, the c_j's in (11), and gluon fields that transform under $su(3)$ as the fundamental and adjoint representations, respectively. Thus, left invariance of the coordinate fields used for the intrinsic configuration space leads to local gauge invariance for the fields generated in the space-time frame of the laboratory space.

Cabibbo angle from strange decay. – Let us consider a strangeness-changing, non-leptonic decay

$$\Lambda \to p + \pi^-. \tag{16}$$

At tree level, this decay is described by the Feynman diagram in fig. 3. We follow the notation conventions of [5]. In the standard description for a two-generation model one inserts a factor $\cos\theta_C$ in the upper vertex and $\sin\theta_C$ in the lower vertex to have the decay amplitude

$$\begin{aligned} \mathcal{M} = \; & \frac{g^2_W}{8(M_W c)^2}[\bar{u}(3)\gamma^\mu(1-\gamma^5)(\sin\theta_C)u(1)] \\ & \cdot [\bar{u}(4)\gamma_\mu(1-\gamma^5)(\cos\theta_C)v(2)]. \end{aligned} \tag{17}$$

[6]Historically it grew out of work on lattice gauge theory with Manton's action [14] in a Kogut-Susskind Hamiltonian [15], which is here completely reinterpreted [6].

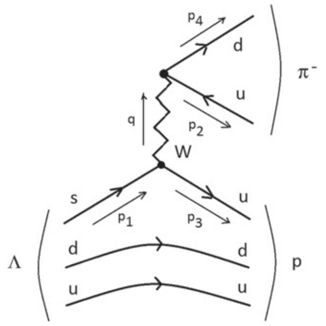

Fig. 3: Feynman diagram for Λ decay. An s quark with u, d as spectators in the Λ baryon transforms to a u quark by emission of an intermediate gauge boson, W^- which creates a $d\bar{u}$ pair. Leaving the scene, one observes a baryon, the proton p and a meson, the negative pion, π^-. We discuss the factor $\sin\theta_C$ in the lower vertex (17).

Here the integration over the W propagator

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu/(M_W c)^2)}{q^2 - (M_W c)^2} \tag{18}$$

has been carried out under the condition that the four-momentum exchange $q = p_1 - p_3$ fulfils $q^2 \ll (M_W c)^2$, where M_W is the W mass. Further, g_W is the electroweak coupling constant $g^2_W = e^2/\sin^2\theta_W = 4\pi\alpha/\sin^2\theta_W$ with θ_W the electroweak mixing angle and u and v are particle and anti-particle spinors, respectively. Unitarity of the Cabibbo matrix (4) is automatically fulfilled by the parametrization via a single angle θ_C,

$$\cos\theta_C = \sqrt{1 - \sin^2\theta_C}. \tag{19}$$

We want to derive $\sin\theta_C$. The lower vertex in fig. 3 involves quarks of strange and up flavours, respectively. We can generate these flavours according to (11) from acting on colour components in the following way:

$$c^f_j = T_f c_j, \qquad f = u, d, s; \qquad j = r, b, g. \tag{20}$$

Here the colour states are [5][7]

$$c_r = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad c_b = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad c_g = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{21}$$

Thus, we include colour in the incoming and outgoing quark states and sum over colour

$$\begin{aligned} & [\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)] \to \\ & \sum_{j=r,b,g} \left[\bar{u}(3)c^\dagger_j(3)\gamma^\mu(1-\gamma^5)c_j(1)u(1)\right], \end{aligned} \tag{22}$$

[7]Note that $c_j(u)$ in (11) are vector fields on $U(3)$, whereas c_r, c_b, c_g are algebraic state vectors of specific colour. Therefore, $c_1(u), c_2(u), c_3(u)$ are equivalent to, but not identical to c_r, c_b, c_g. However, application of T_f on both sets gives the required connection between the exterior algebraic description and the intrinsic dynamics in the colour degrees of freedom.

with $c_j(1) = T_s c_j$ and $c_j(3) = T_u c_j$. The Dirac matrices γ^μ and γ^5 operate on the spinors independently of the operation of the flavour generators on the colour components. We can therefore factorize the colour algebra from the spinor algebra to get

$$\sum_{j=r,b,g} \left[\bar{u}(3)\,(T_u c_j)^\dagger\,\gamma^\mu(1-\gamma^5)T_s c_j u(1)\right] =$$
$$\left[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right] \sum_{j=r,b,g} c_j^\dagger T_u^\dagger T_s c_j$$
$$= \left[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right] \text{Tr}\ T_u^\dagger T_s. \tag{23}$$

Note that there are no gluon propagators here, so this is not a strong interaction effect as such. Comparing the last expression to the standard expression in (17) we find the Cabibbo angle determined by

$$\sin\theta_\text{C} = \text{Tr}\ T_u^\dagger T_s = -\frac{2}{9}. \tag{24}$$

This yields

$$\cos\theta_\text{C} = \frac{1}{9}\sqrt{77} = 0.974996\cdots$$
$$\approx |V_\text{ud}| = 0.97420 \pm 0.00021\ [4]. \tag{25}$$

The comparison seems quite promising. An exact agreement should not be expected considering that $V_\text{ud} = \cos\theta_{12}\cos\theta_{13}$ contains a mixing into the third generation. This mixing, however, is rather small and therefore $\cos\theta_{13} \approx 1$ and $|V_\text{ud}| \approx \cos\theta_\text{C}$.

Let us now consider the (leptonic) neutron decay

$$n \to p + e + \bar{\nu}_e \tag{26}$$

depicted in fig. 4. At the lower vertex one would traditionally include the Cabibbo factor $\cos\theta_\text{C}$, whereas our model would read

$$\sum_{j=r,b,g} [\bar{u}(3)(T_u c_j)^\dagger\gamma^\mu(1-\gamma^5)T_d c_j u(1)] =$$
$$[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)] \sum_{j=r,b,g} c_j^\dagger T_u^\dagger T_d c_j$$
$$= [\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)]\text{Tr}\ T_u^\dagger T_d. \tag{27}$$

This substitutes $\cos\theta_\text{C}$ by Tr $T_u^\dagger T_d$ which yields $\frac{7}{9}$ —somewhat off the value of V_ud. A radical solution to this discrepancy is to absorb Tr $T_u^\dagger T_d$ in a redefined coupling strength

$$g_0^2 \to g_\text{W}^2 = g_0^2\ \text{Tr}\ T_u^\dagger T_d \tag{28}$$

with $g_0^2 = e^2/(\sin\theta_\text{W}\cos\theta_\text{W})^2$ as an *a priori* coupling strength for weak interactions and then to keep only $\cos\theta_\text{C}$ in the spinor brackets such that V_C remains unitary by virtue of (19). From (17) we would then conclude

$$\cos^2\theta_\text{W} = \text{Tr}\ T_u^\dagger T_d = \frac{7}{9} = 0.777\cdots \tag{29}$$

which —as it should— compares rather well with [4]

$$\frac{m_\text{W}^2}{m_\text{Z}^2} = \left(\frac{80.379(12)\ \text{GeV}}{91.1876(21)\ \text{GeV}}\right)^2 = 0.7771(3). \tag{30}$$

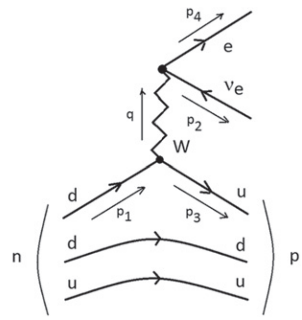

Fig. 4: Feynman diagram for neutron decay (27).

Charm and GIM-mechanism. – Let us try to enlarge the model to charm quarks. We take the following charm flavour generator to supplement the set in (5)

$$T_c = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

Just like T_s, neither will T_c generate any partons in the protonic state we referred to in fig. 2. We consider the $d \to c$ vertex analogous to the $s \to u$ vertex in fig. 3,

$$d \to c + W^-, \quad \frac{-ig_\text{W}}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)V_\text{cd}, \quad V_\text{cd} \approx -\sin\theta_C. \tag{32}$$

Using the same idea as in (23) we would conclude a Cabibbo factor

$$-\sin\theta_\text{C} = \text{Tr}\ T_d^\dagger T_c = -\frac{2}{9}. \tag{33}$$

It is *the same* value of the trace as in (23) but this time should compare to $-\sin\theta_\text{C}$ which is opposite in sign to the relation $\sin\theta_\text{C} = -2/9$ in (24). Can this immediate inconsistency be remedied? Yes, we can exploit the freedom of choice of an overall phase factor in V_CKM. Instead of the standard V_CKM we may use

$$U_\text{CKM} \equiv -iV_\text{CKM}. \tag{34}$$

As V_CKM is unitary ($V^\dagger = V^{-1}$), so is U_CKM,

$$U^\dagger U = (-iV)^\dagger(-iV) = V^\dagger i(-i)V = 1, \quad U^\dagger = U^{-1}. \tag{35}$$

With U_CKM in stead of V_CKM we get the following vertex Cabibbo factors for the lower vertex in fig. 3 to compare with a similar generation-changing charm decay:

$$\begin{aligned} s &\to u + W^-, \quad U_\text{us} = -iV_\text{us} \approx -i\sin\theta_\text{C}, \\ c &\to d + W^+, \quad U_\text{cd}^* = (-iV_\text{cd})^* \approx -i\sin\theta_\text{C}. \end{aligned} \tag{36}$$

In the second decay we used the vertex factor rule in the right part of fig. 5. Thus, the immediate problem of opposite signs on $\sin\theta_\text{C}$ in (23) and (33) is eliminated, albeit at the price of a common non-zero phase, $i = e^{i\pi/2}$ —a phase which we would like to understand better.

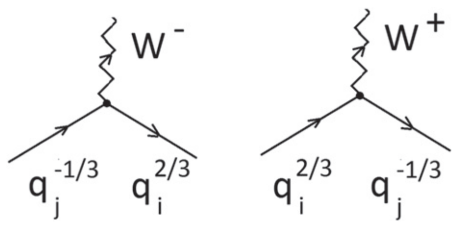

Fig. 5: $\frac{-ig_{\mathrm{W}}}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^5)V_{ij}$ $\frac{-ig_{\mathrm{W}}}{2\sqrt{2}}\gamma^{\mu}(1-\gamma^5)V^*_{ij}$. Vertex factors for quark flavour transformations, $i = u, c, t$ and $j = d, s, b$. The Cabibbo factors V_{ij} refer to the Cabibbo-Kobayashi-Maskawa matrix in (1) —note that the i, j indices are reversed in the left figure. See p. 439 and p. 350 in ref. [5].

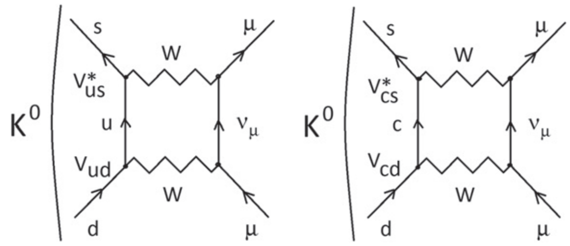

Fig. 6: GIM-mechanism in leptonic kaon decay, $V^*_{\mathrm{cs}}V_{\mathrm{cd}} = -V^*_{\mathrm{us}}V_{\mathrm{ud}}$ for any phase convention in the Cabibbo-Kobayashi-Maskawa matrix V_{CKM} (1) at $\theta_{13} = 0$ and $\theta_{23} = 0$. The two diagrams therefore cancel (except for effects due to $m_{\mathrm{c}} \neq m_{\mathrm{u}}$). As a consequence, the decay $K^0 \to \mu^+ + \mu^-$ is heavily suppressed.

The reader may remember in the back of his or her mind that the opposite signs on the $\sin\theta_{\mathrm{C}}$ in the Cabibbo matrix (4) are the basis of the standard exposition of the Glashow-Iliopoulos-Maiani mechanism [2] that supported the charm quark hypothesis before charm degrees of freedom were observed, namely by explaining the suppression of the leptonic kaon decay

$$K^0 \to \mu^+ + \mu^-. \tag{37}$$

Will the GIM-mechanism survive our phase change in the CKM-matrix? Figure 6 shows the Feynman diagrams to explain the mechanism. We will check that they still cancel using U_{CKM} from (34) instead of V_{CKM} in the standard parametrization (3) which is approximated by the Cabibbo matrix (4). Using fig. 5, we list the Cabibbo factors in the four W-quark-vertex processes needed

$$\begin{aligned}
d &\to u + W^-, \quad U_{\mathrm{ud}} = -iV_{\mathrm{ud}} \approx -i\cos\theta_{\mathrm{C}},\\
u &\to s + W^+, \quad U^*_{\mathrm{us}} \approx (-i\sin\theta_{\mathrm{C}})^* = i\sin\theta_{\mathrm{C}},\\
d &\to c + W^-, \quad U_{\mathrm{cd}} \approx -i(-\sin\theta_{\mathrm{C}}) = i\sin\theta_{\mathrm{C}},\\
c &\to s + W^+, \quad U^*_{\mathrm{cs}} \approx (-i\cos\theta_{\mathrm{C}})^* = i\cos\theta_{\mathrm{C}}.
\end{aligned} \tag{38}$$

We see that the opposite signs in the GIM-mechanism have been shifted to the cosines. In other words the GIM-mechanism still works: the product of the two upper Cabibbo factors in the list (38) refers to the left diagram in fig. 6 and cancels the product of the two lower Cabibbo factors referring to the right part of fig. 6. This is as it should be because U_{CKM} is related to V_{CKM} by a simple phase factor.

Discussion. – To see clearly the hierarchy, we expose the underlying factorization of V_{CKM} in three pairwise rotations among the three generations [4],

$$V = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}. \tag{39}$$

For $\theta_{23} = \theta_{13} = 0$ this expression reduces to the Cabibbo matrix (4) which has been our concern here. Further development of the present model would require a suggestion on the origin of the mixing into the third generation, *i.e.*, $\theta_{23} \neq 0$ and $\theta_{13} \neq 0$. Attempts in the literature tend to combine the quark and lepton sectors [17–21] in search for a common origin of the hierarchical mass and mixing structure observed in both sectors. de Medeiros Varzielas, Rasmussen and Talbert [21] suggest that corrections to fermionic mixings should be sought via other mechanisms than finite group strategies. We note that Kobayashi and Maskawa [3] actually consider a coupling via the Higgs field to strong interactions to explain CP-violation before they suggest their 6-plet model which has now developed into the standard description used with bottom and top quarks in the third generation. Our strategy is to relate electroweak and strong interaction structure where quark degrees of freedom are at play. We see quarks as degrees of freedom excited in hadrons both by electroweak and by strong interactions —but askew with respect to each other. In this sense the Cabibbo angle expresses a mixing between interactions.

Conclusion. – We have relaxed on the independent treatment of the electroweak and strong interactions. We used a relation between flavour generators and colour components to find a possible physical origin of the Cabibbo angle. The relation is supported by previously derived parton distribution functions from the same set of flavour generators that are here related to the Cabibbo angle. The model needs improvement to include mixing into the third generation.

* * *

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