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REMARKS ON THE BOUNDARY CURVE OF A CONSTANT MEAN CURVATURE TOPOLOGICAL DISC

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ABSTRACT. We discuss some consequences of the existence of the holomorphic quadratic Hopf differential on a conformally immersed constant mean curvature topological disc with analytic boundary. In particular, we derive a formula for the mean curvature as a weighted average of the normal curvature of the boundary curve, and a condition for the surface to be totally umbilic in terms of the normal curvature.

1. INTRODUCTION

A constant mean curvature (CMC) surface is a model for a fluid interface. As such, a basic configuration of interest is that of a CMC topological disc with boundary. It is well known that any CMC surface admits a real analytic conformal parameterization. We will also assume that the boundary is a real analytic curve in $\mathbb{R}^3$; under reasonable regularity assumptions [14] one can then show that the surface extends analytically across the boundary. Applying the smooth Riemann mapping theorem, we thus consider a conformal CMC immersion $f : U \supset \overline{\mathbb{D}} \to \mathbb{R}^3$, where $\overline{\mathbb{D}}$ is the closed unit disc, and U an open subset of $\mathbb{C}$.

Given a simple closed curve C in $\mathbb{R}^3$, Hildebrandt [7] showed that there exists at least one (possibly branched) CMC topological disc spanning C, if the mean curvature H is not too large: if C lies in a ball of radius R then solutions exist provided that $|H| \leq 1/R$.

In this note we record some observations concerning the consequences that the existence of the holomorphic quadratic differential of Hopf [8, 9] has for the relationship between the boundary curve $\gamma := f\big|_{\partial\mathbb{D}}$ and the surface. For a general real analytic space curve, the choice of a surface normal N along the curve and a constant H locally determine a unique CMC H surface. However, if the curve is closed and the surface is required to be a regular topological disc, then we will see that the normal curvature κ_n and the geodesic torsion τ_g (obtained from the derivative of N along the curve) are closely related and must satisfy some integral relations, amongst them:

$$\int_0^{2\pi} (H - \kappa_n(t)) \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \mathrm{d}t = 0 \tag{1}$$

$$\int_0^{2\pi} \tau_g(t) \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \mathrm{d}t = 0, \tag{2}$$

where $z = e^{it}$ is the restriction of the conformal parameterization to $\partial\mathbb{D}$ and s is the arc-length with respect to the metric induced from $\mathbb{R}^3$. The first equation gives an expression for H as an average of the normal curvature with respect to the measure $\mu = (\mathrm{d}s/\mathrm{d}t)^2\mathrm{d}t$:

$$H = \frac{1}{M} \int_0^{2\pi} \kappa_n(t) \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \mathrm{d}t, \qquad M := \int_0^{2\pi} \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \mathrm{d}t.$$

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It is important to note that these formulae are only valid for a parameterization t such that $z = e^{it}$ extends to the disc as a conformal coordinate.

It obviously follows from (1) that the constant H lies in the range $[\min(\kappa_n), \max(\kappa_n)]$. We will in fact conclude:

Theorem 1. *Let $f : \mathbb{D} \cup \partial\mathbb{D} \to \mathbb{R}^3$ be a CMC H immersion of a topological disc with regular analytic boundary $C = f(\partial\mathbb{D})$. Let κ_n and τ_g be the normal curvature and geodesic torsion of the curve $f|_{\partial\mathbb{D}}$.*

(1) *The surface is totally umbilic if and only if at least one of κ_n or τ_g is constant. In this case, both are constant and $\kappa_n \equiv H$ and $\tau_g \equiv 0$.*
(2) *Otherwise, the mean curvature H must satisfy*

$$\min_{\partial\mathbb{D}}(\kappa_n) < H < \max_{\partial\mathbb{D}}(\kappa_n).$$

Any homotopically trivial simple closed curve on a CMC surface can be taken as $\partial\mathbb{D}$ in Theorem 1. This places restrictions on the geometry of such curves. For example, there are none with constant non-zero geodesic torsion; and none that are lines of curvature ($\tau_g = 0$) unless the surface is totally umbilic.

If the surface is totally umbilic, i.e., part of a 2-sphere or a plane, the geodesic torsion along any curve at all is zero, as is the quantity $\kappa_n - H$. But the characterization of total umbilicity as the vanishing of just *one* of these quantities along a closed curve is not valid if the curve does not bound a topological disc - counterexamples are easily given on a round cylinder.

Theorem 1 is proved in Section 2. In Section 3 we briefly discuss applications to CMC surfaces with circular boundary and to capillary surfaces.

2. The Hopf differential and some consequences

2.1. **The Hopf differential along a line.** Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{R}^3$ be a conformally parameterized immersion, i.e.,

$$2\langle f_z, f_{\bar{z}}\rangle = \langle f_x, f_x\rangle = \langle f_y, f_y\rangle = \nu^2 > 0,$$

where $\nu : U \to \mathbb{R}_{>0}$. Given a choice N of unit normal, the mean curvature is

$$H = \frac{1}{2}\nu^{-2}\langle f_{xx} + f_{yy}, N\rangle.$$

As observed by Hopf [9], if we define $Q = \langle N, f_{zz}\rangle$, then the Codazzi equation gives $Q_{\bar{z}} = \nu^2 H_z/2$, so the mean curvature is constant if and only if Q is holomorphic. Assuming this, the *Hopf differential* is the (coordinate independent) holomorphic quadratic differential

$$\mathscr{H} = Q(z)\mathrm{d}z^2, \quad Q = \langle N, f_{zz}\rangle.$$

Suppose that U contains the real line. We want an expression for $\mathscr{H}$ along $\mathbb{R}$, in terms of the geometric data of the curve $\gamma(x) = f(x,0)$. We first assume that x is the arc-length parameter along γ, i.e., that

$$\langle f_x(x,0), f_x(x,0)\rangle = \nu^2(x,0) = 1.$$

This assumption can be achieved on a sufficiently small open set containing the real line by changing to the conformal coordinates $\zeta = \int_0^z \hat{\nu}(z)\mathrm{d}z$, where $\hat{\nu}$ is the holomorphic extension of $\nu(x,0)$. Under this assumption, the Darboux frame along γ is:

$$X(x) = f_x(x,0), \quad Y(x) = f_y(x,0), \quad N(x) = N(x,0).$$

The geodesic and normal curvatures, κ_g and κ_n, and geodesic torsion τ_g of γ are defined by:

$$\begin{aligned} X' &= \kappa_g Y + \kappa_n N, \\ Y' &= -\kappa_g X + \tau_g N, \\ N' &= -\kappa_n X - \tau_g Y. \end{aligned}$$

Now the coefficient Q of $\mathscr{H}$ is:

$$Q = \frac{1}{4}\left(\langle N, f_{xx}\rangle - \langle N, f_{yy}\rangle\right) - \frac{1}{2}i\langle N, f_{xy}\rangle.$$

Along γ, we have $\langle N, f_{xx}\rangle = \kappa_n$ and $\langle N, f_{yy}\rangle = 2H - \langle N, f_{xx}\rangle$. Finally, $f_{xy} = Y'$, so $\langle N, f_{xy}\rangle = \tau_g$. Therefore, along $y = 0$,

$$Q = \frac{1}{2}(\kappa_n - H - i\tau_g).$$

Along the curve, the second fundamental form is $II = \kappa_n \mathrm{d}x^2 + 2\tau_g \mathrm{d}x\mathrm{d}y + (2H - \kappa_n)\mathrm{d}y^2$, and the principal curvatures are $\kappa_\pm = H \pm \sqrt{(H-\kappa_n)^2 + \tau_g^2}$. Thus the geodesic torsion is a measure of how far the curve γ deviates from being a line of curvature, and an umbilic point corresponds to $\kappa_n - H = \tau_g = 0$, that is, to the zeros of the Hopf differential.

Under a change of coordinates, the Hopf differential transforms as $\mathscr{H} = \langle N, f_{zz}(\mathrm{d}\zeta/\mathrm{d}z)^2 + f_z \mathrm{d}^2\zeta/\mathrm{d}z^2\rangle \mathrm{d}z^2 = \langle N, f_{zz}\rangle(\mathrm{d}\zeta/\mathrm{d}z)^2\mathrm{d}z^2$. Dropping the assumption that $\nu(x,0) = 1$, we have, or the general case of an arbitrary conformally parameterized CMC immersion of an open set containing the real line,

$$Q(x,0) = \frac{1}{2}(\kappa_n(x) - H - i\tau_g(x))\nu^2(x,0).$$

2.2. **Proof of Theorem 1.** Now let $f : U \supset \overline{\mathbb{D}} \to \mathbb{R}^3$ be a conformally parameterized CMC H immersion, as in the introduction. For some $\varepsilon > 0$, the function $p : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{C}$ given by $p(z) = e^{iz}$ is a holomorphic covering map of an annulus $\mathscr{A} \subset U$, that maps the real line onto the unit circle. Then the function $\tilde{f} := f \circ p : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a conformally parameterized CMC immersion, as in the previous section, with the additional property that the functions $\kappa_n(x)$, $\tau_g(x)$ and $\nu(x,0)$ are periodic with period 2π. Let z denote the coordinate for $\mathbb{R} \times (-\varepsilon, \varepsilon)$ and w the coordinate for U, so $w = e^{iz}$. Since $\mathrm{d}z^2 = -(1/w^2)\mathrm{d}w^2$, we have, in the w coordinate,

$$\mathscr{H} = -\frac{1}{2w^2}\left(\alpha(w) + i\beta(w)\right)\mathrm{d}w^2 = \tilde{Q}(w)\mathrm{d}w^2,$$

where, writing $\hat{\kappa}_n$, $\hat{\tau}_g$ and $\hat{\nu}$ for the holomorphic extensions of $\kappa_n(e^{ix})$, $\tau_g(e^{ix})$ and $\nu(e^{ix})$ respectively,

$$\alpha(w) = (\hat{\kappa}_n(w) - H)\hat{\nu}^2(w) \quad \text{and} \quad \beta(w) = -\hat{\tau}_g(w)\hat{\nu}^2(w)$$

are holomorphic on a neighbourhood of $\mathbb{S}^1$, real-valued along $\mathbb{S}^1$, and $\hat{\nu}$ is non-vanishing along $\mathbb{S}^1$. Because $\mathscr{H} = \tilde{Q}(w)\mathrm{d}w^2$ is holomorphic on $\mathbb{D}$, it follows that the function

$$g(w) = \alpha(w) + i\beta(w) = -2w^2\tilde{Q}(w)$$

vanishes at least to second order at $w = 0$. This means that the integrals along $\mathbb{S}^1$ of the functions $g(w)$, $g(w)/w$ and $g(w)/w^2$ are all zero, which gives us:

$$\int_0^{2\pi} \alpha(e^{it})\mathrm{d}t = 0 = \int_0^{2\pi} \beta(e^{it})\mathrm{d}t,$$
$$\int_0^{2\pi} \alpha(e^{it})e^{\pm it}\mathrm{d}t = 0 = \int_0^{2\pi} \beta(e^{it})e^{\pm it}\mathrm{d}t.$$

The first line amounts to (1) and (2), and the second line gives analogous identities with the integrands multiplied by $\cos(t)$ and by $\sin(t)$. The fact that α and β are real on $\mathbb{S}^1$, together with the vanishing conditions above, gives us the Fourier expansions:

$$\alpha(w) = \sum_{n=2}^{\infty}(\bar{\alpha}_n w^{-n} + \alpha_n w^n) \quad \text{and} \quad \beta(w) = \sum_{n=2}^{\infty}(i\bar{\alpha}_n w^{-n} - i\alpha_n w^n),$$

which precisely encodes the relationship between the quantities ν, τ_g and $\kappa_n - H$ that follow from the holomorphicity of the Hopf differential on $\mathbb{D}$. From this one can read off, for example, that τ_g is constant if and only if $\kappa_n - H$ is constant, and in this case both are zero. Since the vanishing of the Hopf differential characterizes umbilics, this proves the first part of Theorem 1. For the second part, (1) implies that $\min(\kappa_n) \leq H \leq \max(\kappa_n)$. Equality cannot occur, because then the vanishing of the integral (1) of $H - \kappa_n$ would imply that $\kappa_n \equiv H$, i.e., that the surface is totally umbilic.

3. Some applications of Theorem 1

3.1. **CMC discs bounded by a circle.** Given a closed curve in $\mathbb{R}^3$, Brezis and Coron [2] showed that there are at least two geometrically distinct CMC H discs spanning the curve, provided H is small enough (see also [18]). However, it is not known precisely how many there are. The most basic case imaginable is where the boundary is a circle, which we may as well take to be of radius 1. If $0 < |H| < 1$ there are obviously at least two CMC H discs bounded by this circle, a large and a small spherical cap on a sphere of radius $1/|H|$. It is a long standing open question, going back as far as the 1980's [10, 5] whether or not there are more solutions in the case $0 < |H| < 1$. A survey of this problem can be found in [12]. From Theorem 1 we have that the unit sphere is the only solution for $|H| = 1$, and, for solutions that are not totally umbilic,

$$-1 \leq \min_{\partial\mathbb{D}}(\kappa_n) < H < \max_{\partial\mathbb{D}}(\kappa_n) \leq 1.$$

Brito and Sa Earp [3] essentially showed this using the *balancing formula* or *flux formula*, which was found by R. Kusner in his Ph.D. thesis, published in [11], and a proof given for the immersed case in [13]. It can be stated as follows: if $f : M \to \mathbb{R}^3$ is a CMC H immersion of a compact surface with boundary, and $\gamma(s)$ the arc-length parameterization of the boundary, then

$$\int_{\partial\Sigma} \langle Y, a\rangle \mathrm{d}s + H \int_{\partial\Sigma} \langle \gamma \times \gamma', a\rangle \mathrm{d}s = 0,$$

where Y is the unit conormal and γ a unit speed parameterization of the boundary curve, and a is any constant vector in $\mathbb{R}^3$. The flux formula can be proved ([13]) by observing that the 1-form $(Hf + N) \times \mathrm{d}f$ is closed if f has constant mean curvature H. This formula differs from the integral formula (1), as it does not depend on the topology of Σ and does not relate to any conformal structure.

For the special case that $\partial\Sigma$ is a circle $\gamma(s) = (\cos s, \sin s, 0)$, taking $a = (0,0,1)$ allows one to again obtain the above bounds on H: the surface normal along γ is $N = -\kappa_n\gamma + \kappa_g a$ and the conormal is $Y = -\kappa_g\gamma - \kappa_n a$. So the flux formula becomes

$$\int_{\partial\Sigma} (\kappa_n - H)\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = 0.$$

This is not the same as (1) (and note that the integral in (1) has a priori a different value for every choice of conformal coordinate), but it does give the same upper and lower bounds on H for the case of circular boundary.

Note that the holomorphicity of the Hopf differential is used in [1] to prove that a *stable* CMC topological disc with circular boundar must be a spherical cap, and in [16] to give a condition for the instability of a higher genus CMC surface with circular boundary.

3.2. **Application to capillary surfaces.** If M is a CMC surface with boundary ∂M, and the boundary curve γ lies in another surface S (the support surface), such that the surfaces intersect with constant contact angle along γ, then M is called a *capillary surface*. If M is a topological disc, then this is a model for the surface of a drop of liquid resting on a solid surface in the absence of gravity. The contact angle depends on the physical properties of the solid (hydrophilic/hydrophobic). Two examples are shown to the right in Figure 1, each constructed by adding a ruled support surface with a prescribed contact angle to the boundary of a CMC surface.

It is known that if the support surface is part of a plane or a sphere then a capillary surface with disc topology is necessarily totally umibilic [15, 6, 4]. This follows directly from Theorem 1, because any curve in a sphere or plane is a line of curvature (zero geodesic torsion) and the Terquem-Joachimsthal theorem [17] states that if the intersection of two surfaces M_1 and M_2 is a line of curvature on M_1, then it is also a line of curvature on M_2 if and only if the surfaces intersect at constant angle.

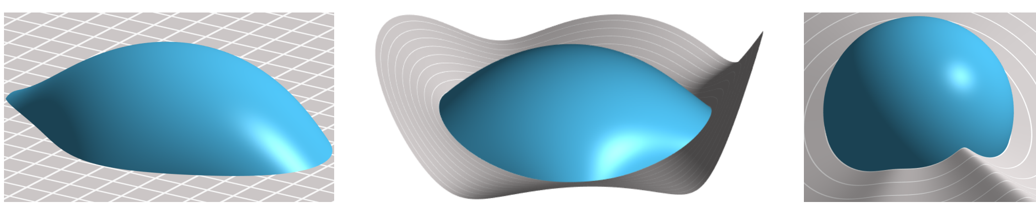

FIGURE 1. Left: this CMC surface cannot meet a plane with constant contact angle because it is not a spherical cap. Middle, right: capillary surfaces with contact angle $\pi/2$. The surface to the right is part of a sphere, so the intersection curve is necessarily a line of curvature in the support surface.

To state the consequence just mentioned in full generality, one can extend the Terquem-Joachimsthal theorem to curves that are not necessarily lines of curvature:

Lemma 2. *Suppose that a surface M_1 meets a surface M_2 along a regular curve γ. Denote by $\tau_{g,j}$ the geodesic torsion of γ with respect to the surface M_j. Then the surfaces meet at constant angle if and only if $\tau_{g,1} = \tau_{g,2}$.*

Proof. Assume γ is parameterized by arc-length. Let N_j and $Y_j = N_j \times \gamma'$ denote the unit normal and co-normal along γ with respect to the surface M_j. Then $\langle Y_1, N_2\rangle = \sin(\theta) =$

$-\langle Y_2, N_1\rangle$, where θ is the angle from N_1 to N_2. Thus,

$$\begin{aligned}\langle N_1, N_2\rangle' &= -\tau_{g,1}\langle Y_1, N_2\rangle - \tau_{g,2}\langle Y_2, N_1\rangle \\ &= (\tau_{g,1} - \tau_{g,2})\langle Y_2, N_1\rangle.\end{aligned}$$

If θ is constant, i.e., $\langle N_1, N_2\rangle' \equiv 0$, then the product $(\tau_{g,1} - \tau_{g,2})\langle Y_2, N_1\rangle$ is zero. Then either $\tau_{g,1} \equiv \tau_{g,2}$ or $\langle Y_2, N_1\rangle$ vanishes at some point; but if the latter occurs then $\langle Y_2, N_1\rangle = -\sin\theta \equiv 0$, because θ is constant, and hence $N_1 \equiv \pm N_2$, and $\tau_{g,1} \equiv \tau_{g,2}$. The converse is also clear. □

Together with Theorem 1, this implies

Theorem 3. *Let M be a closed topological disc, immersed as a capillary surface with support surface S and analytic boundary. Let γ denote the boundary curve of M. Then M is totally umbilic if and only if γ has constant geodesic torsion in the support surface S. In this case, the geodesic torsion is zero.*

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