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Infinitely connected subgraphs in graphs of uncountable chromatic number

Carsten Thomassen *
Department of Applied Mathematics and Computer Science,
Technical University of Denmark, DK-2800 Lyngby, Denmark

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Abstract

Erdős and Hajnal conjectured in 1966 that every graph of uncountable chromatic number contains a subgraph of infinite connectivity. We prove that every graph of uncountable chromatic number has a subgraph which has uncountable chromatic number and infinite edge-connectivity. We also prove that, if each orientation of a graph G has a vertex of infinite outdegree, then G contains an uncountable subgraph of infinite edge-connectivity.

Keywords: infinite graphs, chromatic number, edge-connectivity
MSC(2010):05C15,05C40,05C63

1 Introduction.

Hajnal and Komjáth [3] gave a detailed investigation of which types of subgraphs can surely be found in graphs of uncountable chromatic number. One of the basic questions that they left open is the conjecture of Erdős and Hajnal [1] that every graph of uncountable chromatic number contains a subgraph of infinite connectivity. This problem is also discussed in [6] and

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[10]. Komjáth [5] proved that every graph of uncountable chromatic number contains a subgraph of uncountable chromatic number and of any finite connectivity. He has also proved that every graph of uncountable chromatic number contains a subgraph with infinite vertex degrees and of any finite connectivity, see [6]. He proved recently in [7] that it is consistent that there is an uncountable chromatic graph with no infinitely connected uncountable chromatic subgraph. More recently, a ZFC example has been given by Soukup [9]. In this paper we prove that the edge-connectivity version of the conjecture is true. The same holds if "chromatic number" is replaced by "coloring number" in both the assumption and conclusion of the result. We also prove that, if each orientation of a graph G has a vertex of infinite outdegree, then G contains an uncountable subgraph of infinite edge-connectivity. All these results generalize to arbitrary infinite regular cardinals.

2 Notation.

A *multigraph* may contain multiple edges but no loops. A *graph* is a multigraph with no loops or multiple edges. If G is a graph and its vertex set is divided into sets A, B, then all edges between A and B form a *cut* in G. We call A, B the *sides* of the cut. The cut is *minimal* if it contains no other cut as a proper subset. It is easy to see that a cut in a connected graph G with sides A, B is minimal if and only if both graphs $G(A), G(B)$ are connected. Using this observation it is also easy to see that every cut can be decomposed into pairwise disjoint minimal cuts.

The *edge-connectivity* of a graph is the smallest cardinality of a cut.

If D is a cut in G, then $G - D$ is obtained from G by a *cut-deletion*. We also say that G is obtained from $G - D$ by a *cut-addition*. If D is a minimal cut in G, then these two operations are called *minimal-cut-deletion* and *minimal-cut-addition*, respectively. If D is finite, then $G - D$ is obtained from G by a *finite-cut-deletion*, and G is obtained from $G - D$ by a *finite-cut-addition*. Thus every finite-cut-deletion can be regarded as a finite sequence of minimal-cut-deletions, and similarly for finite-cut-addition.

As usual, a *sequence* of elements in a set S can be described as a collection a_n of elements in S where n is a natural number. If the indices n are chosen from a set of ordinals (smaller than some fixed large ordinal), then we speak of a *generalized sequence*.

If the vertices of a graph are labelled by distinct ordinals, then we define the *root* of a component as the vertex in that component with the smallest label.

If G is a graph, then a subgraph H of G is obtained from G by a *generalized sequence of cut-deletions* if there exists a generalized sequence of subgraphs G_α of G such that the following hold:
(i) $G = G_1$,
(ii) $H = G_{\alpha_0}$ for some ordinal α_0, and
(iii) If α is a limit ordinal, $\alpha \leq \alpha_0$, then G_α is the intersection of all G_β with $\beta < \alpha$, and
(iv) If α is an ordinal, $\alpha < \alpha_0$, then $G_{\alpha+1}$ is obtained from G_α by a cut-deletion.

If G is a graph, and H is a subgraph of G, then G is obtained from H by a *generalized sequence of cut-additions* if there exists a generalized sequence of subgraphs H_α of G such that the following hold:
(v) $H = H_1$,
(vi) $G = H_{\alpha_0}$ for some ordinal α_0, and
(vii) If α is a limit ordinal, $\alpha \leq \alpha_0$, then H_α is the union of all H_β with $\beta < \alpha$, and
(viii) If α is an ordinal, $\alpha < \alpha_0$, then $H_{\alpha+1}$ is obtained from H_α by a cut-addition.

As the main tool, which is hopefully of independent interest, we prove that, if G is a graph, and H is a subgraph obtained from G by a generalized sequence of finite-cut-deletions, then G is obtained from H by a generalized sequence of finite-cut-additions. However, some of the added cuts may have to be distinct from all the deleted cuts. Also, it may be worth noting that, if G is obtained from H by a generalized sequence of finite-cut-additions, then we cannot conclude that H can be obtained from G by a generalized sequence of finite-cut-deletions, as shown by a complete graph with a countably infinite vertex set.

3 Cut-deletion and cut-addition.

Proposition 1 *If G is a graph, then there exists a generalized sequence of finite-cut-deletions resulting in a graph H such that each component of H is* *either a single vertex or a subgraph of infinite edge-connectivity.*

Proof of Proposition 1. Using Zorn's lemma it is easy to find a maximal generalized sequence of finite-cut-deletions resulting in a graph H. Consider any component H' of H. If H' has more than one vertex and has a finite cut, then that cut can be used to extend the generalized sequence of finite-cut-deletions resulting in a graph H''. But this contradicts the maximality property of the generalized sequence.

■

Proposition 2 *If G is a graph obtained from a subgraph H of countable chromatic number by a generalized sequence of finite-cut-additions, then G has countable chromatic number.*

Proof of Proposition 2. We label the vertices of H by ordinals. Recall that the vertex of smallest label in a component of H is called the root of that component. We color the vertices of H by the natural numbers. We may assume that each cut we add is a minimal cut in the resulting graph. Consider now a cut-addition where we add the finite cut D between components H_1, H_2, say. Assume that the root of H_1 is smaller than the root of H_2. Then we permute colors of H_2 such that all ends of D in H_2 have colors distinct from the colors of the ends of D in H_1. Note that, when a vertex changes color, it also changes root. As there is no infinite decreasing sequence of ordinals (in other words: the ordinals are well-ordered), a vertex can change root (and hence color) only for finitely many cut-additions. So, for any fixed vertex x, the colors of x tend to an ordinal which will be the final color of x. Thus every vertex has a color which is a natural number, and neighboring vertices have distinct colors. So, G has countable chromatic number.

■

Theorem 1 *If G is a graph, and H is a subgraph obtained from G by a generalized sequence of finite-cut-deletions, then G is obtained from H by a generalized sequence of finite-cut-additions.*

■

Before we prove Theorem 1 we observe that Theorem 1 combined with Propositions 1,2 implies the main result, Theorem 2 below.

Theorem 2 *If G is a graph of uncountable chromatic number, then G has a subgraph which has infinite edge-connectivity and uncountable chromatic number.*

■

We now prove Theorem 1. Let the cuts in the finite-cut-deletions be labelled D_α where α is an ordinal. Let G_β be the graph obtained by deleting all cuts D_α where $\alpha < \beta$. Without loss of generality we may assume that D_β is a minimal cut in G_β. Let E be an edge set in G. We say that a cut D_β is *independent of E* if D_β is disjoint from E, and D_β is a cut (and hence a minimal cut) in $G_\beta \cup E$. Otherwise we say that D_β *depends on E*.

Claim (1): If E is a finite edge set, then only finitely many cuts D_β depend on E.

Proof of Claim (1). Let $G_{E,\beta}$ be the multigraph obtained by from G_β by first contracting each component of G_β into a vertex and then adding the edges of E. If D_β is disjoint from E, and D_β depends on E, then there is a set of edges in E which form a cycle in $G_{E,\beta}$ but not in $G_{E,\beta+1}$. Since E has only finitely many edges and hence only finitely many sets of subsets, there can only be finitely many ordinals β such that $G_{E,\beta+1}$ has (strictly) fewer cycles than $G_{E,\beta}$. This proves Claim (1).

Now let $E'_1 = D_1$. Having defined the increasing finite sequence $E'_1, E'_2, \ldots, E'_n$ of finite edge-sets, we define E'_{n+1} as the the union of E'_n and all edges in those cuts D_α which depend on E'_n. Then each E'_n is finite. Put $E_1 = E'_1 \cup E'_2 \cup \ldots$. Then E_1 is countable, and E_1 is the union of cuts D_α.

Claim (2): No cut D_β disjoint E_1 depends on E_1.

Proof of Claim (2). Suppose (reductio ad absurdum) that D_β is disjoint E_1 and depends on E_1. Then there is a set of edges in E_1 which form a cycle in $G_{E_1,\beta}$ but not in $G_{E_1,\beta+1}$. Since there are only finitely many edges in this cycle, these edges are contained in some E'_n. But then D_β is contained in E'_{n+1}, a contradiction which proves Claim (2).

Claim (2) proves that all those cuts D_β which are disjoint E_1 form a generalized sequence of minimal cuts. More precisely, $H \cup E_1$ is obtained from $G \cup E_1$ by that sequence of finite-cut-deletions.

We shall now investigate E_1.

Claim (3): Every edge e in E_1 joins two distinct components of H.

Proof of Claim (3). The edge e belongs to some D_β. As D_β is a minimal cut in G_β, it follows that e joins two distinct components of $G_\beta - D_\beta$. Since $G_\beta - D_\beta$ contains H, this proves Claim (3).

Claim (4): For any two components M_1, M_2 in H, all edges in E_1 (if any) joining M_1, M_2 belong to the same D_β.

Proof of Claim (4). Suppose (reductio ad absurdum) that e_1, e_2 join M_1, M_2 and that e_1 belongs to D_α and e_2 belongs to D_β where $\alpha < \beta$. Then $H \cup \{e_1, e_2\}$ has a cycle C containing e_1, e_2. As D_β is a minimal cut in G_β, all edges of D_β belong to the same component of G_β and hence they also belong to the same component of $G_\alpha - D_\alpha$. But $(G_\alpha - D_\alpha) \cup \{e_1\}$ contains $H \cup \{e_1, e_2\}$ and has clearly no cycle containing e_1, a contradiction to the existence of the cycle C.

This proves Claim (4).

Claim (5): $H \cup E_1$ can be obtained from H by a sequence of finite-cut-additions.

Proof of Claim (5). Let $V_1, V_2, \ldots$ be those components of H which are incident with an edge of E_1. By Claim (4), the set of edges in E_1 from V_{n+1} to $V_1 \cup V_2 \cup \ldots \cup V_n$ form a finite cut D'_n. These cuts $D'_1, D'_2, \ldots$ are clearly pairwise disjoint, and their union is E_1 by Claim (3). This proves Claim (5).

Put $H_0 = H$, and $H_1 = H_0 \cup E_1$. Let α_1 be the smallest ordinal such that D_{α_1} is not contained in E_1. Recall that after the proof of Claim (2), we noted that this claim proves that all those cuts D_β which are disjoint from E_1 form a generalized sequence of minimal cuts. This means that what we proved for the generalized sequence of finite-cut-deletions transforming G into H can also be proved for the (smaller) sequence of finite-cut-deletions (starting with D_{α_1}) transforming G into H_1. Now we define E_2 from G, H_1 in the same way as we defined E_1 from G, H_0. More precisely, E_2 contains D_{α_1} and all edges in cuts depending on D_{α_1} and all edges in cuts depending on those cuts which depend on D_{α_1}, etc. By Claim (5), $H_1 \cup E_2$ can be obtained from H_1 by a sequence of finite-cut-additions. So, $H_0 \cup E_1 \cup E_2 = H_1 \cup E_2 = H_2$ can be obtained from H_0 by a generalized sequence of finite-cut-additions. We continue like this using Zorn's lemma or transfinite induction. We have shown how to define $H_{\alpha+1}$ after we have defined H_α. So, it only remains to deal with H_α when α is a limit ordinal and we have already defined all H_γ when $\gamma < \alpha$. We define E_α as the union of all E_γ where $\gamma < \alpha$. In order to proceed we need the following:

Claim (6): No cut D_β disjoint E_α depends on E_α.
The proof of Claim (6) is a repetition of the proof of Claim (2).

This completes the proof of Theorem 1.

■

4 Chromatic number, list-chromatic number, and coloring number.

The *list-chromatic number* $\chi_l(G)$ is the smallest cardinal κ with the following property: If each vertex v has a list $L(v)$ with at least κ colors, then it is possible to color the vertices of G such that each vertex v receives a color from $L(v)$ and such that neighboring vertices receive distinct colors. If all lists are the same we get the usual chromatic number $\chi(G)$. The *coloring number* $col(G)$ is the smallest cardinal κ with the following property: The vertex set of G can be expressed as a generalized sequence such that each vertex has less than κ predecessors. Clearly,

$$\chi(G) \leq \chi_l(G) \leq col(G).$$

Proposition 3 *If G is a graph obtained from a subgraph H of countable coloring number by a generalized sequence of finite-cut-additions, then G has countable coloring number.*

The proof of Proposition 3 is very similar to the proof of Proposition 2. In the proof of Proposition 2 we permute colors in a graph H_2. In the proof of Proposition 3 we let instead all vertices in the generalized sequence used for the coloring number of H_2 succede that of H_1. Thus the vertices in H_2 get larger ordinals. But, for any fixed vertex, that happens only finitely many times. ■

Using Proposition 3 instead of Proposition 2 we get the following analogue of Theorem 2.

Theorem 3 *If G is a graph of uncountable coloring number, then G has a subgraph which has infinite edge-connectivity and uncountable coloring number.*

■

The method used to prove Theorem 2 easily extends Theorem 4 below. Recall that an infinite cardinal κ is *regular* if the union of $< \kappa$ sets of size $< \kappa$ has size $< \kappa$.

Theorem 4 *If κ is an infinite regular cardinal and G is a graph of chromatic number $> \kappa$, then G has a subgraph which has edge-connectivity $\geq \kappa$ and chromatic number $> \kappa$.*

■

Theorem 4 also holds if "chromatic number" is replaced by "coloring number". Komjáth [8] proved that it is consistent that $\chi_l(G) = col(G)$ when $col(G)$ is infinite.

5 Orientations.

An *orientation* of a graph G is obtained from G by giving each edge a direction. Hakimi [4] proved that a finite graph G has an orientation such that each vertex has outdegree $< k$ (where k is a natural number) if and only if G does not contain a subgraph with average degree more that $2(k-1)$. It is natural to ask if a similar characterization is possible when k is an infinite cardinal. We first note that large minimum degree is not the relevant concept. To see this, let T_α denote the tree where every vertex has degree α. Select a root r in T_α. Direct each edge in T_α towards the root r. Then all vertices have outdegree at most 1. The result below involves large edge-connectivity.

Theorem 5 *If G is a graph such that every orientation of G has a vertex of infinite outdegree, then G contains a subgraph which has uncountably many vertices and infinite edge-connectivity.*

Proof of Theorem 5. The proof by Fodor [2] shows that G has uncountable coloring number. Combining this with Theorem 3 gives Theorem 5. We shall here give another argument. Let H be obtained from G as in Proposition 1. If some component of H is uncountable, the proof is complete. So assume (reductio ad absurdum) that every component of H is countable. Then H has an orientation such that each vertex has finite outdegree. (Just enumerate the vertices in each component of H by $v_1, v_2, \ldots$ and direct each edge from the vertex with the large label to the vertex with the small label.) Now G can be obtained from H by a generalized sequence of finite-minimal-cut-additions, by Theorem 1. When we add a minimal cut we direct all edges towards the component with the smallest root. Thus a vertex may get larger outdegree in this process but only finitely many times. So, the final outdegree of each vertex is finite, a contradiction.

■

Theorem 5 can be generalized to the following: If κ is an infinite regular cardinal, and G is a graph such that every orientation of G has a vertex of outdegree at least κ, then G contains a subgraph which has more than κ vertices and edge-connectivity at least κ.

It is easy to minimize the maximum outdegree for complete graphs and complete bipartite graphs. The complete graph with α vertices has an orientation such that all vertices have outdegree $< \alpha$. (Just label the vertices by the ordinals smaller than the smallest ordinal of cardinality α and direct each edge from the vertex with the large label to the vertex with the small label.) On the other hand, if $\beta < \alpha$, then any orientation of the complete graph with α vertices has a vertex of outdegree $\geq \beta$. This is even true for the complete bipartite graph with β vertices in one of the classes, say B, and α vertices in the other class A. For otherwise, we delete all outneighbors of all vertices in B. As we delete at most β vertices, there are still vertices in A left, and they have outdegree β, a contradiction.

However, in general, large edge-connectivity does not guarantee a large minimum outdegree in every orientation.

Proposition 4 *(a) If G is a 2-edge-connected finite or infinite graph, then every orientation of G has a vertex of outdegree at least 2 unless G is a cycle.*

(b) For each infinite cardinal α there exists a graph of edge-connectivity α such that some orientation has maximum outdegree 2.

Proof of Proposition 3. The proof of (a) is an easy exercise. To prove (b), select a root r in the tree T_α. Direct each edge in T_α towards the root r. Now replace each vertex v in T_α by an oriented tree T_v which is a copy of T_α. If u, v are neighbors in T_α, then we add a perfect matching between T_u, T_v

and orient all edges towards T_r. The resulting oriented graph has outdegrees at most 2, and the underlying graph has edge-connectivity α.

■

Fodor [2] proved that an oriented graph with all outdegrees $< \alpha$ has chromatic number at most α. If this is combined with Theorem 5 we obtain a weakening of Theorem 4.

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