On the Rotor to Stator Contact Dynamics with Impacts and Friction - Theoretical and Experimental Study

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On the Rotor to Stator Contact Dynamics with Impacts and Friction - Theoretical and Experimental Study

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Summary

The contact between a rotor and its stator can in some cases be considered as a serious malfunction that may lead to catastrophic failure. This major threat arises normally from full annular dry friction backward whirl and whip motion where the rotor runs and rubs at a high frequency on the inner surface of the stator, and thereby traversing the full extent of the clearance. Normal and friction forces are exerted on the rotor at each impact and rubs. These particular forces can sustain the rotor in a persistent backward dry whirl or whip motion. In that case, the friction force plays a significant role during dry contact since it changes the precession of the rotor motion. This thesis gives a comprehensive experimental and theoretical study on the rotor to stator contact dynamics, under the influence of dry friction. Different backward full annular dry motions are generated and analyzed theoretically and experimentally. In the theoretical part of the work, the strong nonlinear effects included in the modelling stem from dry friction and impacts. The contact deformations and forces are assumed to happen in a continuous manner during impacts. On the basis of this assumption, the piecewise smooth and discontinuous nonlinear equations of motion are formulated. This allows the use of a smoothening method in order to facilitate the numerical integration procedure and prevents numerical instabilities. This method has proven to be successful and efficient. The performance of two different types of backup bearings, i.e. an annular guide and a new unconventional pinned backup bearing, is also thoroughly studied. The motion of the rotor is studied by use of trajectory plots, time series and spectral analyses. The pinned backup bearing prevents the rotor from entering a full annular contact state, which leads to significant reduction of contact forces and eliminates dry whirl and whip motions. Research publication on the study of experimental measurements of contact forces and behavior during rotor to
stator contact are rare. In this regards, the full instrumented test rig designed specially to investigate rotor-stator contact problems, allows comprehensive experimental studies on the contact force behavior. The theoretical results appear to be in good agreement with the experiments. Nevertheless, among one of the future aspects, it is necessary to use comprehensive uncertainty analysis in the investigation of the friction coefficient behavior.
fremfærden. I dette studie er de teoretiske resultater i overensstemmelse med de eksperimentelle målinger. Dog er der behov for videre forskning hvor fx friktionskoefficienterne bestemmes ved hjælp af dybdegående statistiske modeller for at underbygge evidensen.
This thesis is submitted as partial fulfillment of the requirements for awarding the Danish Ph.D. degree. The work has been carried out from September 2009 to November 2012 at the Section of Solid Mechanics (FAM), Department of Mechanical Engineering (MEK), Technical University of Denmark (DTU), and at the engineering consulting company Lloyd’s Register ODS. The project was mainly supervised by Professor, Dr.-Ing, Dr-Tech. Ilmar Ferreira Santos, to whom I would like to express my profound gratefulness for being a constant source of inspiration and assistance. I am also very thankful for his support and dedication during the experimental work and for his guidance in technical and human matters. I will truly miss his great cheerfulness in stressful periods and his support - that this project could come to an end. The project was also supervised by the Lloyd’s Register ODS company supervisor, technical manager Mr. Henning Hartmann, whom I also would like to gratefully acknowledge his support and his willpower for finding the fundings for this project. I would also like to thank the Lloyd’s Register ODS’ Managing Director Dr. Claus Myllerup, head of the Energy Department Mr. Morten Theill Jensen and Machinery Dynamics team leader Mr. Anders Crone for their support and for letting me pursue my dream of starting the Ph.D. project.

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Kgs. Lyngby, November 2012

Said Lahriri
The following publications are a part of the thesis.


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5 Conclusion, Discussion and Future Work 55
1.1 Background

Rotor to stator contact and rubbing occurs regularly in real industrial machinery. In this regard, one of the most fatal phenomena occurs when the rotor traverses to full extent of the inner surface of the stator with a backward precessional orbit to that of the direction of the angular speed rotation. In this full annular contact state the motion of the rotor can settle into two different motional regimes; 1) the full annular backward whirl motion or 2) the full annular backward whip motion. In a full annular backward whirl motion the dry friction contact force steadily accelerates the rotor lateral motion along the stator inner surface. In some cases the sliding motion brings to stop and the rotor starts to roll on the inner surface of the stator. This state is referred to as backward whirling, and the whirling frequency is governed by the driving velocity of the rotor and by the radius to clearance ratio. In a dry friction backward whip motion the rotor rubs and slips continuously on the inner surface of the stator. The rubbing whip motion locks into a frequency component of the combined rotor stator system. This rubbing frequency component is independent on the driving frequency of the rotor. The dry backward whip motion causes extensive vibrations and contact forces and it can lead to a potential dangerous hazard to the system.
Confirmed and reported cases of full annular dry friction backward whirl and whip motions that lead to fatal incidents are rare. In fact, some researchers state that even though full annular dry friction whirl and whip can be produced in experimental test rigs they rarely occur in real industrial machines. However, in 1995 Rosenblum [1] reported a fatal incident that led to the complete destruction of a 600 MW turbo generator. Rosenblum suspected that the final stage associated with the complete destruction was caused by dry backward whip motion. Although, well reported and rarely disclosed in literature incidents associated with full annular dry backward whirl and whip in industrial machinery applications are quite modest, destructions in industrial machinery do occur. Whether the last stage of the destruction process is caused by the full annular backward precessional motion of the rotor is yet unknown. Furthermore, with the growth of active magnetic bearings (AMB) applications in real machinery a great need for investigating the backup bearing performances in terms of impact force performances exists. The backup bearings are designed to capture the rotor and prevent damages in case of an AMB failure. The backup bearing should sustain the severity arising from the normal and friction forces and keep the rotor safely within the bearing. In this regard, Lloyd’s Register ODS has set up this research project to investigate the backup bearing performances during rotor to stator contact. Two different backup bearings are studied in this thesis, a conventional annular guide and a new unconventional pinned backup bearing design, for two different experimental test setup systems. Although, publications reporting experimental studies on the backward whirl and whip motions exists, publications assessing experimental measured contact forces during the different backward motions are rare. Therefore, this thesis theoretically and experimentally studies the backward whirl and whip motions and assesses the contact forces, behavior and magnitudes through measurements during the different backward motions.

1.2 Literature Survey and Previous Research

The dynamics and behavior of the rotor to stator contact dynamics have been studied extensively in the past by many researchers. Figure 1.1 depicts a time chart diagram of some of the researchers contributing to the knowledge and research of the rotor to stator contact dynamics. Their work is discussed in this section. To give an overview of the research work conducted up to 1989, Muszynska [2] presented a comprehensive literary survey on rub-related phenomena. In 1962, Johnson [3] based his studies on the motion of a vertical shaft impacting a bearing with clearance. Johnson considered two study cases; one without damping included and one with damping included in the modelling. However, he did not include the dry friction in his model and the assessment of synchronous whirls was based on whether the solutions are real, whether the
reaction between the clearance bearing and the vertical shaft is positive, and finally whether the equilibrium is stable. In 1965, Billet [4] discussed how the reverse whirl induced by dry friction in a clearance can become a near resonant whirl over a large range of the shaft speeds. Billet showed how slip contributes to the onset and violence of whirl, and how this whirl speed cannot exist above the first natural frequency of the system. Furthermore, he also discussed how viscous damping acting on the system decreases the maximum reverse whirl speed. In 1967, Ehrich and O’Connor [5] also studied the rotor to stator contact dynamics where they included the motion of the bearing in the modelling. Here, they discovered a new class of vibrational behavior. These new phenomena revealed stator whirl wherein large stator amplitudes may be experienced at supercritical speeds. Additionally, new jumps and hysteresis phenomena were noted. One of the most discussed works was carried out by Black [6] in 1968. Black used two degree of freedom models for the rotor and stator to investigate the range of precession frequencies for which dry friction whirl and whip are possible. His investigation resulted in the U-shaped curve that separates regions of whirl and whip. With this figure, Black predicted that precession frequency existed in which dry friction whirl occurred and that dry friction whip will develop and persist at the upper limit of this precession frequency range. In 1979, Childs [7] described rub related parametric excitations in rotors. In 1985, Beatty [8] investigated the rotor response due to radial rubbing based on a combination of analytical and experimental results. These results were calculated in terms of Fourier series expansion of a predicted rub induced shaft motion. In addition, in 1986 Szcygielski [9] studied a gyro pendulum touching a plane rigid body. The mathematical model was piecewise linear and globally strongly nonlinear. The preliminary experiments showing the trajectories of the gyro axis showed a good qualitative agreement with the analytical and experimental results. In 1988, Zhang [10] investigated the response of a multi degree of freedom system due to full annular rub. He applied Black’s model to his system and identified the same whirl regions as Black. Within this period Choy et al. [11]-[13] conducted nonlinear analysis to study the transient and steady state behavior of rub induced vibrations. In their work they also simulated and presented the magnitude of the contact and rub forces. In 1990, Lingener [14] and Crandall [15] reported experimental findings that seem to confirm Black’s theoretical results, where they found a stable whirling motion with a frequency slightly below the system’s coupled eigenfrequency.

In the nineties a great deal of work treated the nonlinear analysis of rotor to stator contact dynamics. Due to the non-smooth behavior in stiffness and rubbing forces, the system can exhibit complicated vibration phenomena. Studies on these rubbing phenomena revealed that the rotating system showed a rich class of nonlinear related dynamics such as sub and super-synchronous responses, quasi-periodic responses and even chaotic motions caused by the non-smooth
system which can exhibit different types of motions. In fact, the possibility of chaotic behavior for rotors upon rubbing was suggested in 1992 by Ehrich [16]. To capture the phenomena, he based his mathematical model representing this motion on a nonlinear spring system. In 1994, Goldman and Muszynska [17] reported that the chaotic motion in a nonlinear study is more likely to occur if a proper impact model is employed. They used a discontinuous piecewise approach with extra stiffness and damping terms included during the contact stage. Their numerical simulations revealed that the system can exhibit orderly harmonic motions together with subharmonic responses, as well as chaotic motions. These nonlinear phenomena were also shown in Goldman and Muszynska’s research work conducted in 1994 [18] and in 1995 [19]. Moreover, in 1994, Li and Païdoussis [20] investigated the dynamics of the system analytically and numerically, simulating the dynamic behavior through phase plane plots, bifurcation diagrams as well as Poincaré maps using the dry friction coefficient and the eccentricity of the rotor unbalance as control parameter. Additionally, in 1994 Isakson [21] simulated the dynamical behavior of a rotor interacting with non-rotating parts. For this, a quasi-static solution was derived for a system where the stator offset was neglected and only valid for constant angular velocity of the rotor. In some cases, discontinuous and multi valued solutions were obtained. He also demonstrated that a steady state solution existed for a system with stator offset and continuous rubbing contact. In 1998 Piccoli and Weber [22] investigated and presented an application for chaotic motion identification from a measured signal obtained through experiment. They assessed the measured system that demonstrated chaotic motion by evaluating Poincaré diagrams, Lyapunov exponents and correlation dimension by state space reconstructed with delayed co-ordinates. The assessment established that the system was chaotic. In 2004, Pavlovskaia et al. [23] conducted nonlinear analysis on a two-degrees-of-freedom model of the Jeffcott rotor with preloaded snubber ring subjected to out of balance excitation. Their study was based upon analyzing bifurcation diagrams and phase portraits. Their nonlinear analysis showed that the effects of the preloading is crucial and should be included in the modelling. In 2006, this theoretical work was verified experimentally by Karpenko et al. [24]. Rotor to stator rub is often more likely to occur for real industrial rotating machines without the cause of any severe damages. For this reason, in 2003 Bachschmid et al. [25] presented an identification method to detect rubs by using experimental data from two different industrial 320 MW turbogenerators. In 2007, Bachschmid et al. [26] presented a theoretical model that included thermal effects to observe spiral vibrations due to rub in real rotors. These special vibrations were induced by contact forces which introduce heat effects and by the resulting thermal bow. The model was validated with experimental results of a 50 MW generator. In addition, Pennacchi et al. [27] continued the research work on rubs in rotating machineries, where they validated their theoretical model with experimental data from a test rig. The theoretical model was suited to reproduce the experimental behavior.
However, in 2000, Bartha’s [28, 29] test results contradicted the findings given in Black’s theoretical model and the experimental results of Lingener and Cran-dall, with a particular view to the onset to dry friction whip. In his findings, Bartha suggested an extended model where the bearing is modelled as nested rings. Furthermore, Bartha also discovered that the periodic whirl motion of different systems is unstable although steady state analysis based on Black’s model predicts stable periodic whirl regions. Bartha assessed the stability by considering the whole system and evaluating the Floquet multipliers rather than using static force equilibrium. In addition, in 2002 Yu et al. [30] investigated the onset to dry friction whip. Their investigation led to the conclusion that the dry friction whip can be generated spontaneously without the presence of a dry friction whirl region as suggested in Black’s theoretical model where dry friction whirl goes ahead of dry friction whip motion. In 2002, Choi [31] investigated theoretically and experimentally the onset to the backward whirling motion. Choi’s findings seem to support Black’s theoretical prediction. Choi was able to traverse through the first whirl and whip regimes to reach a higher whirl regime, thus supporting Black’s model. In 2005, Jiang and Ulbrich [32] also investigated the physical reason for the onset of dry friction whip in rotor-stator systems with imbalance. They discovered that rotor in resonance at a negative (natural) frequency of the coupled nonlinear rotor/stator system is the physical reason for the onset of dry whip with imbalance. However, they also discovered that the onset to dry whip follows different paths. For that reason, if the motion follows the path of high frequency whirl through an outside disturbance at a relative low rotating speed as carried out in the work of Lingener, the dry friction whirl motion goes ahead of dry friction whip motion as predicted in Black’s theoretical model. However, if the motion follows the path of gradual increase in rotational speed under the influence of mass unbalance, the partial rub generally goes ahead of dry friction whip. In 2007, Childs and Bhattacharya [33] revisited the work of Black with inclusion of multiple rotor modes that predicted several possible whirl regions. However, only the first whirl region and its whip frequency could be computed in their simulations. In 2008, in continuation of this work, Wilkes et al. [34] investigated through experimental and numerical methods, the nature of multimode dry friction whirl and whip for a variety of rub materials and clearances. Their experimental results showed multiple whirl and whip regions as the rotor speed was increased or decreased through regions characterized by whip, terminated with jumps to different whirl/whip regions. In 2011, Inoue et al. [35] demonstrated theoretically and experimentally the suppression of the forward rubbing by introducing a directional difference in the support stiffness of the guide or the backup bearing. This was carried out by supporting the backup bearing with rubber attached in one direction in order to have an asymmetric bearing support. The study revealed that the rotating shaft can escape from the forward rubbing condition to a non-contacting state of much lower rotational speed than in cases of symmetric support conditions.
In the test rig, the forward rub was observed at 600 rpm. When the rotational speed was further increased, the backward harmonic component began to occur from about 700 rpm, indicating that the shaft orbit became elliptic. At a slight increase in the rotational speed to 750 rpm, the shaft escaped from the forward rubbing condition to the non contacting condition with small amplitudes. Late in 2012, Yu [36] discusses the reverse full annular rub where he proposes an analytical model to effectively analyze dry friction whirl/whip phenomenon.

In the above mentioned research not much attention has been drawn to the contact and rub forces. However, with the application of AMBs, Schweitzer [37] and Ulbrich et al. [38], research in auxiliary bearing designs and performances started to be of a great importance. The auxiliary bearings should be designed in order to catch the rotor due to a failure in the AMB. Normally, the auxiliary bearing is a pair of ball bearings that serve the purpose of preventing the rotor from impacting the pole pairs and slowly decelerating it. As such, the contact forces play a significant role in the design. In 1992, Schmied and Pradetto [39] investigated the vibrational behavior of a one ton compressor being dropped into the auxiliary bearings in case of a failure of its AMBs. The auxiliary bearing employed in their investigations was a ball bearing. In 1994, Kirk et al. [40] designed and performed a full scale test rig in order to better understand the contact dynamics and to evaluate various auxiliary bearing configurations. Consequently, this investigation would lead to the development of analytical tools to simulate turbomachinery supported by AMBs. A theoretical formulation was proposed in 1996, by Ishii and Kirk [41] and simulated numerically to examine the transient response of a flexible rotor dropped on the backup bearing. The response was simulated from the time prior to the AMB shut down, to the rotor drops onto the backup bearing. In this work they simulated the contact forces which turned out to be huge during the contact. In 1996, Fumagalli [42] initiated an investigation to understand the auxiliary bearing performance due to the dynamics of the rotor sliding and tumbling into it, and to identify the effect of parameters on the motion, force and energy dissipation. In this way different contact models could be verified through experiments. He investigated different proposed contact models, such as Hertz [43] and Hunt and Crossley [44], and took the energy dissipation into account. He estimated the contact forces through experiments. His estimation of the contact force was based on measuring the acceleration in the vertical direction of the auxiliary bearing and thereby calculating the contact force. In 1999, Kirk [45] revisited his work conducted with the full scale test rig in 1994, where he reviewed the analytical techniques to predict the transient response of a rotor landing on the auxiliary bearing and to illustrate some of the important parameters in the design. In this work he simulated the contact forces in all of his study cases. In 1999, Lawen Jr and Flowers [46] investigated the application of synchronous interaction dynamics methodology to the design of auxiliary bearing systems, where a proper design introduces parameters such as clearance and essential nonlinear rotordynamics.
Their results showed good agreement with Black’s theoretical model for identifying synchronous interaction. Furthermore, in 2003, Keogh and Cole [47]-[48] and Hawkins et al. [49] contributed to the ongoing research investigations with a comprehensive study on auxiliary bearing performances. In their work they simulate the contact forces in case of an AMB failure. A method studied by many researchers to prevent damages due to rub and impacts is to make use of active vibration control. Studies have been conducted in order to prevent the rotor from rubbing and impacting the auxiliary bearing by use of active control. In 2009, Ginzinger et al. [51] apply active control to reduce the severity arising from rotor to stator contact. Late in 2012, Keogh [50] outlines and discusses the benefit of making the auxiliary bearing an active element by controlling the contact dynamics.

Most of the research work conducted within the field of rotor to stator contact with a particular view to the backup bearing performances, focuses on the conventional backup bearing design. This backup bearing design consists of an annular guide. The objective with this particular design is to capture the impacting rotor and prevent damages. However, the circular shape of the inner surface facilitates the dry friction force to act in the same direction and manifests itself during impacts. The frictional force causes an increase in the rotor tangential velocity in the direction opposite to that of the angular velocity of the rotor. This act can sustain the shaft in a full annular backward motion. Yet, a few unconventional backup bearing designs have been proposed in recent years in order to improve the backup bearing performances during impacts. The following section outlines these new designs.

### 1.2.1 Different types of Backup Bearing Designs

A number of backup bearing designs have been proposed by different researchers in order to improve the dynamical impact behavior. Some of these designs have been investigated purely theoretical and some with experimental verification. In the theoretical and experimental work of Ginzinger et al. [51] a new approach to control a rubbing rotor by applying an active auxiliary bearing is presented. The auxiliary bearing is depicted in Fig. 1.2(I). Applying this design, a three-phase control strategy was developed which stabilizes the rotor system in case of an impact load and effectively avoids backward whirling. By using the active control system, the contact and rub forces were reduced substantially together with the rotor deflection. Additionally, in the purely theoretical work of Keogh [50], he also outlines and discusses the benefit of making the auxiliary bearing an active element. The auxiliary bearing design is depicted in Fig. 1.2(II). However, these two proposed backup bearings require feedback control systems in their applications. Design of backup bearings which do not require any type...
of feedback control systems have also been proposed in recent years. In the theoretical and experimental work of Simon [52] he proposes a polygon shaped inner surface design of the backup bearing. One of the shapes is depicted in Fig. 1.2(III). In Simon’s theoretical and experimental investigations, the rotor could not enter a full annular whirling state at impacts when the triangle shaped backup bearing was employed. However, the rotor settled to a fixed contact motion touching each side of the triangle since the friction force at impacts acts in the same direction and facilitates the rotor to sustain in that particular motion. In the purely theoretical work of Zülow and Liebich [53] they apply flexible pins with roller elements at the free end to reduce the severity in contact forces arising from friction, as depicted in Fig. 1.2(IV). In their work they propose different configurations of these pin elements. These flexible pin elements are modelled as spring-damper elements in their theoretical modelling, thus introducing additional degrees of freedom to the system.
1.2 Literature Survey and Previous Research

In this work, a new unconventional backup bearing is proposed. The primary goal is to reduce the rub-related severity in friction force behavior and to reduce the rotor’s potential risk of entering a full annular contact state. The idea is to center the rotor during impacts and reduce the influence of the friction force. In a similar manner to the backup bearing design proposed by Zülow and Liebich [53], Fig. 1.2(IV), the proposed backup bearing design utilizes pin connections. The backup bearing is built by four adjustable pins which allows the clearance to be customized. However, the pin connections are made robust and rigid.

Figure 1.2: (I) Design proposed by Ginzinger et al. [51] (theoretical and experimental investigations), (II) design proposed by Keogh [50] (purely theoretical investigations), (III) design proposed by Simon [52] (theoretical and experimental investigations), (IV) design proposed by Zülow and Liebich [53] (purely theoretical investigations)
and the free end of these pins is smoothly curved in order to produce different directions of the force components at impacts. These different force components help the shaft escaping the pins, unlike the application with the annular ring where the friction force manifests itself and acts in the same direction. The new unconventional backup bearing is depicted in Figs. 1.3(I) and (II) for the PUC test setup, and in Figs. 1.3(III) and (IV) for the DTU test setup.
1.3 Objectives and Original Contributions of this Research Project

The objective of this thesis is to experimentally and theoretically study the contact dynamics between a rotor and two types of passive backup bearings. These backup bearings do not require any type of active control in their application. This work is a summary of the study presented in [P1], [P2], [P3] and [P4].

The main original contribution in [P1] is to analyze both experimentally and theoretically the impact motion of a vertical rotor interacting with two types of backup bearings. The rotor is driven by a non-ideal motor with no feedback control system, i.e. the torsional behavior of the rotor is treated as a degree-of-freedom and the motor torque characteristics as a function of the angular velocity is known. The advantages of employing the unconventional pinned backup bearing is outlined and demonstrated. In the theoretical formulation, nonlinear discontinuous equations of motion are derived. The contact forces are modelled taking into account the compliance in the contact. In this regard, the discontinuous system is described by a proper choice of subclass system according to its degree of discontinuity. The subclass systems are defined as; 1) non-smooth continuous systems, 2) systems with a discontinuous right hand side referred to as Filippov systems and 3) impulsive systems. The impact system is assumed to belong to the Filippov system and a smoothening approach is proposed for numerically solving the highly nonlinear discontinuous equations. The effects and feasibilities of employing this method are comprehensively studied and explained. This approach is therefore utilized in rotor to stator contact modelling.

The main original contribution of [P2] relies on experimental investigations of the dry friction behavior that brings the rotor to a full annular contact state. The dry friction coefficient is assessed through a static pin-on-disk experiment. In the paper the design of a fully instrumented test rig to investigate rotor-stator contact dynamics is elucidated in details. From the rotor impact motion and the measurement of forces acting on the housing, the friction coefficients are evaluated and compared to the values coming from the pin-on-disk test.

The main original contribution in [P3] relies on presenting comprehensive experimental research work on a horizontal rotor partially levitated by a passive magnetic bearing, impacting the two types of backup bearings. The rotor is driven by an ideal motor with speed control, i.e. not as in [P1] where the torsion/spin is a degree-of-freedom. The primary aim is to demonstrate the advantages of employing force transducers in order to achieve good descriptions of the contact forces during impacts and show the reduction of contact forces
using the unconventional backup bearing. This allows the hysteresis loops and the force versus penetration depths to be visualized and explained. As such, the magnitude of the contact forces is assessed in an accurate manner.

The main original contribution in [P4] is to describe and validate the theoretical compliant contact force model with the experimental results. The aim is to highlight the advantages of using such a model and demonstrate its simplicity in the numerical integration procedure, where the smoothening method is employed. This is used to assess the magnitude of the impact forces theoretically and validate them experimentally.

1.4 Outline of the Thesis

There are two different ways of presenting a Ph.D. thesis at DTU:


2. Paper-based thesis which is composed as a summary of the journal papers conducted during the Ph.D period. The included papers form the basis of the research work accompanied by a summary thesis. The thesis includes a comprehensive introduction to the research field, thoroughly explains the theoretical methods employed in the work and highlights some of the original contributions.

This thesis is based upon method 2 and summarizes the work done during the Ph.D. study. It is composed of 5 chapters and gives an overview of the main results presented in the four publications [P1]-[P4].

Chapter 2 gives an introduction to nonlinear dynamical systems with emphasis on discontinuous systems and on the theoretical tools to describe and solve them. It presents the nonlinear dynamical systems where the dry friction models, the vibro-impact models, the Poincaré sections and different types of discontinuous systems are described. It presents the two different test setup systems and briefly outlines the mathematical modelling. Furthermore, it proposes a numerical smoothening method inspired by the work of Leine [56] in order to numerically integrate the strong nonlinear discontinuous dynamical systems.

Chapter 3 presents the results conducted on the test rig realized at the Catholic University of Rio de Janeiro, PUC. The main experimental and numerical results are discussed and compared.
Chapter 4 presents the results conducted on the test rig realized at the Technical University of Denmark, DTU. It shortly discusses the experimental work of characterizing the kinetic friction coefficient through two different experimental approaches. Subsequently, the main experimental and numerical results are presented and discussed.

Chapter 5 gives a discussion and conclusion on the subject; rotor to stator contact dynamic, conducted in this frame of work. Additionally, it also suggests additional future work and new experimental approaches to investigate this subject and bring the understanding of new phenomena to light.
Introduction
Chapter 2

Nonlinear Dynamics

The rotor-stator systems considered in this work are modeled by ordinary differential equations. These ordinary differential equations include only derivatives with respect to one independent variable. The ordinary differential equations are written as a system of first order derivatives and formulated in the state space form. The vector field is expressed as:

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n \]  

(2.1)

where \( x \) is a real-valued vector function of time \( t \), \( f \) is a real-valued vector function of \( x \), \( (\cdot) \) denotes differentiation with respect to time \( t \), and \( n \) designates the dimension of the phase space (the system). Equation (2.1) dictates the velocity vector \( \dot{x} \) at each points of \( x \). At points where \( \dot{x} = 0 \), there is no flow, and these points are referred to as fixed points. The solution to the problem of Eq. (2.1) is derived when the differential equation is subjected to the initial conditions of \( x = x_0 \) at \( t = t_0 \). If \( f(x) \) is a linear function in \( x \), the system is referred to as a linear system. In all other cases it is considered as nonlinear. As the rotor-stator systems are subjected to impacts and dry friction, the systems in this work are nonlinear. These nonlinear systems can operate in steady state, fixed points, and evolve towards different states due to impacts and dry friction. These states can be periodic states, quasi-periodic states or even irregular chaotic states. The impact systems are considered as strongly nonlinear with a discontinuous right hand side. For such strong nonlinear systems it is not always possible to find closed form solutions and solve the system analytically.
The nonlinear analyses are therefore based on numerical procedures. The concept is based on analyzing the trajectory of the rotor, spectrum analyses and computing the Poincaré sections. In the following sections the tools and procedure for analyzing the nonlinear dynamical system are outlined. The concepts and procedure are based on the work of Strogatz [54] and Moon [55].

2.1 Piecewise-Smooth and Discontinuous Dynamical Systems

Physical phenomena such as dry friction and impacts for mechanical systems are often described by means of mathematical models with some kind of discontinuity. In these discontinuous systems the continuous in time dynamics are interrupted by discrete jumps in time in the governing equation of motion. The jumps in the vector field happens at well defined boundaries in state space. The systems operate in different modes and the transition from one mode to another can often be idealized as an instantaneous, discrete transition. Each mode is associated with its own differential equations governing the dynamics of that particular mode. The discontinuous systems can be separated into three subclasses of systems according to their degree of discontinuity, as proposed in the work of Leine [56]:

1. Non-smooth continuous systems with a discontinuous Jacobian. Such systems are described by continuous vector fields, these vector fields are however non-smooth. Examples of such systems can be support conditions that are modelled as piecewise linear.

2. Systems with a discontinuous right hand side. These are referred to as Filippov systems. The vector field of such systems is discontinuous. Examples of such systems can be visco-elastic supports and systems with dry friction.

3. Systems with discrete jumps in state like impacting systems where the velocity switches direction instantaneously.

These subclass systems have been utilized to describe the dynamics of different non-smooth dynamical systems carried out during the Ph.D. study period. It is worth mentioning, that in the beginning of the Ph.D. a nonlinear study was carried out to investigate the performance of an arc spring damper. In that work, presented in appendix [P5], the arc spring supporting the bearing was formed as a curved beam. While this beam behaves as a linear spring with
constant stiffness, various clearances and allowances in the assembly cause the support conditions to change as function of deflection. As a result of this, the arc spring force becomes piecewise linear and the stiffness changes as a function of the displacement of the bearing. That system was not an impact problem and could be described by means of method 1). The arc spring damper study presented in appendix [P5], serves only as an illustration of the application of subclass system 1) in this thesis and will not be treated in this frame of work.

The following sections outline the nonlinear parameters such as dry friction and impact forces and elucidate the proper choice of subclass system according to the degree of discontinuity of the rotor-stator system.

2.2 Dry Kinetic Friction Models

Dry friction plays an important role in the dynamic behavior for a rotordynamic system with clearance. In terms of tribology, the dry friction is defined as a force that resists relative motion between two contacting surfaces for different objects. The counteracting friction force can bring the relative motion between two contacting bodies to diminish to zero. This motion is referred to as stick. In slip, the bodies slide relative to each other. In a full annular whirling state the two contacting bodies act in stick phase and this is considered as pure rolling without slip. However, dry friction is characterized by different behavior in stick and slip phase. In a stick phase the acting friction force creates equilibrium states with external forces on the bodies. The bodies remain sticking as long as equilibrium is ensured. If the friction force in stick phase exceeds the magnitude of maximum static friction force the two contacting bodies will begin to slip over each other. Therefore, the friction force must not exceed the maximum absolute static friction force when the two bodies stick. In this pure rolling state the kinetic friction coefficient can be reduced substantially, see the work of Kalker [57] and Bartha [29]. In the slip phase the friction force is single valued function of the relative velocity \( v_{rel} \). The friction coefficient during slip is referred to as the kinetic friction coefficient. A correct model for dry friction contains both a description for the stick and slip phase. Different models are proposed for the mathematical description of dry friction which mostly differs in the stick phase modelling. The Coulomb friction model is widely employed despite its simplicity. This model captures the essential features of dry friction and uses a limited amount of parameters. The model however, poses problems when the relative velocity between the two bodies becomes zero. The model is not continuous differentiable at this zero velocity point and problems occur in the numerical integration procedure. To overcome this problem, the function is smoothened with an arctangent function. The smoothened Coulomb model
Nonlinear Dynamics

is depicted in Fig. 2.1(I). Besides the classical Coulomb friction model, more complex tribological models are proposed such as the Strubeck friction model. These models take some of the underlying mechanism of friction into account. The Strubeck model is depicted in Fig. 2.1(II) where \( \mu_s \) is the coefficient of static friction, and \( v_m \) is the velocity corresponding to the minimum coefficient \( \mu_m \) of kinetic friction, \( \mu_m \leq \mu_s \). If \( \mu_m = \mu_s \) the model corresponds simple to Coulomb friction. The Strubeck model suffers from the same jump discontinuity at \( v_{rel} = 0 \) as the Coulomb friction model in numerical integration approaches. Thomsen [58] addresses this problem and presents a procedure for facilitating the numeral integration for the Strubeck friction model. In this present work, the Coulomb model is preferable rather than the Strubeck model. This is due to the efficiency in numerical integration procedure. In a non-zero sliding state

\[
F_\mu = \mu_k F
\]

(2.2)

This law expresses a linear relationship between the absolute value of the normal force \( F \) and the absolute value of the friction force \( F_\mu \). This Coulomb-Amontons law is utilized in this work.

2.3 Vibro-Impact Models

Impact is a complex physical phenomenon, which occurs when two or more bodies collide with each other. The contact implies a continuous process which takes place over a finite time. Gilardi and Sharf [59] give a comprehensive literature survey on contact dynamics and discuss various contact models. Two approaches for impact and contact analysis exist. The first approach is referred
2.3 Vibro-Impact Models

The method is confined primarily to impact between rigid bodies. Hence, assumes that the interaction between the objects happens in a short period of time and that the configuration of the impacting bodies does not change significantly. For such model, the dynamical analysis is divided into two intervals, before and after impacts. In order to assess the process of energy transfer and dissipation between the two bodies, various coefficients are employed. The primary coefficients are the coefficient of restitution and the impulse ratio. However, the application of the impulse-momentum method leads to several problems. The main problem is that energy conservation principles may be violated during frictional impacts, and the discrete approach is not extendible to generic multi-body systems. The second and, in this work, preferable method for impact analysis is by use of compliant, continuous, contact force models. These models overcome the difficulties associated with the impulse-momentum method.

In the continuous contact model the forces and deformations vary and act in a continuous manner during impacts. Figure 2.2 depicts the forces acting upon contact between a rotor and stator. $F$ designates the contact force, $F_\mu$ designates the friction force, $\delta$ designates the local indentation, $\alpha$ designates the contact angle, $x_0$ and $y_0$ designate the offset of the stator, $x$ and $y$ designate the lateral position of the rotor, $A$ designates the center position of the stator and $B$ designates the center position of rotor. Different continuous contact models have been proposed to describe the interaction force at the surfaces of two contacting bodies. The first model was developed by Hertz [43] and based on the theory of elasticity for frictionless contact, to calculate
indentation without the use of damping. The model is written as:

\[ F = k\delta^n \]  

(2.3)

Where \( F \) denotes the contact force, \( k \) denotes the local impact stiffness between the two colliding bodies and depends on the material properties, geometric properties and computed by using elastostatic theory, \( \delta \) denotes the local indentation amount, and \( n \) is an exponent. For circular and elliptical surfaces \( n \) is set to \( 3/2 \). This exponent makes the Hertz's model nonlinear. This model is only applicable for contact between rigid bodies where no energy dissipation takes place. To include damping in the contact modelling a simple spring-damper model was proposed, where the contact force is represented by linear spring-damper element such as in the Kelvin Voigt model. The impact is schematically represented with a linear spring in conjunction with linear damping, accounting for the energy dissipation. The model is written as:

\[ F = b\dot{\delta} + k\delta \]  

(2.4)

Where \( b \) represents the linear damping and related to the coefficient of restitution, \( e \). This model has some weaknesses. The contact force at the beginning of impact is discontinuous due to the damping term. At separation the local indentation tends to zero where the relative velocity between the two bodies becomes zero. This induces a negative force keeping the bodies together. Furthermore, the coefficient of restitution is independent on the impact velocities. To overcome these problems of the spring-damper model and to retain the advantages of the Hertz’s model, Hunt and Crossley [44] proposed a contact model. This model includes a nonlinear damping term defined in terms of local penetration and the corresponding penetration rate. This model satisfies the force boundary conditions during impact and separation and gives a correct description of the contact force behavior. The contact force model is written as:

\[ F = c\dot{\delta}\delta^n + k\delta^n \]  

(2.5)

Where \( c \) is related to the coefficient of restitution. This energy dissipation is related to the ingoing and outgoing velocities. Another important aspect is that the damping depends on the local indentation \( \delta \), entailing that the contact area increases with deformation and a plastic region is more likely to develop for large indentions. This model has successfully been employed in rotor to stator contact studies as in the work of Bartha [29]. A further development of this model was proposed by Lankarani and Nikravesh [60]. This model is employed in this research work.
2.3 Vibro-Impact Models

2.3.1 Compliant Contact Force Model Employed in this Work

In this model a hysteresis damping function is incorporated to represent the energy dissipation during impacts. The model is written as:

\[ F = k\delta^n \left[ 1 + \frac{3(1 - e^2)}{4} \frac{\dot{\delta}}{\delta_-} \right] \tag{2.6} \]

Where \( \dot{\delta}^- \) denotes the initial impact velocity, and \( e \) denotes Newton’s coefficient of restitution. This contact force model has been utilized extensively in the work of Flores et al. [61]-[64] for the study of mechanical systems with revolute joints. This model is utilized for the contact modelling in this thesis. Despite employing a contact force model that gives the correct description of the contact force behavior, problems can occur in the numerical integration approach. The numerical problems arise due to the transition between contact and no contact, to the sustained contact between the colliding bodies and to the presence of dry friction. These conditions can lead to numerical instabilities. A numerical technique is employed in this work which facilitates the numerical integration. This method is outlined later.
2.4 Poincaré Map

The Poincaré map is used to prove the existence of periodic orbits and quasi-periodic attractors as it is often impossible to characterize the motion only by considering the phase portrait. The Poincaré map is interpreted as a discrete dynamical system with the state space reduced by one dimension than the original system. If one considers the $n$-dimensional system $\dot{x} = f(x)$ as depicted in Fig. 2.3, the intersection of the flow designates the surface of section $S$. Thus, $S$ is the $n-1$ dimensional surface of intersection, a lower dimensional subspace. $S$ is required to be transverse to the flow, i.e. all trajectories starting on it intersect this plane. The Poincaré map is obtained from one trajectory intersection with $S$ to the next intersection. If $x_k \in S$ denotes the $k$th intersection, then the Poincaré map is defined as $x_{k+1} = P(x_k)$. Consequently, one point on the Poincaré map represents a periodic response with the same frequency as the driving frequency. Quasi-periodic motion of the trajectories shows up as a closed circle on the map, while chaotic motion results in a fractal shape on the map. Hence, by using the Poincaré map it is possible to distinguish between different types of motions. However, care must be taken when analyzing strange attractors that demonstrate chaotic nature of motion. The assessment of the chaotic motion must be strengthened by evaluating the largest Lyapunov exponent or the fractal dimension of the attractor. The use of Poincaré sections in machinery dynamics applications to establish and investigate the dynamical responses and compute bifurcation diagrams is illustrated in [P5].
2.5 Introduction to the PUC Test Setup System

Figure 2.4(I) depicts the experimental setup. The test rig, in short, is composed of: (a) an AC electrical motor, (b) a velocity sensor, (c) a vertical shaft with attached disk, (d) an annular guide and housing, (e) a disk and (f) proximity sensors. The stator is fixed to a base structure, through four bolts arranged symmetrically. The AC electrical motor is controlled by a frequency inverter that drives the rotor-shaft assembly. A flexible coupling is mounted at the location of the driving motor, connecting the shaft with the driving motor through a ball bearing in order to reduce and prevent lateral motions and vibration of the shaft affecting the driving motor, and reducing the influence of the disk eccentricity and shaft misalignment. Figure 2.4(II) depicts the disk operating within the annular guide, whereas Fig. 2.4(III) depicts the disk operating within the new unconventional backup bearing.

2.5.1 Mathematical Modelling of the Systems

Figure 2.5 depicts the mechanical model of the test system. The system is considered as a lumped mass system where the mass of the thin shaft is negligible compared to that of the disk and disregarded in the modelling and in the analyses. In other words the flexible rotating shaft is considered massless. The disk is considered as a rigid body, the restoring forces acting on it originate from the deflection of the thin shaft, and are represented by linear and angular springs using beam theory. The shaft deflections and definition of the coordinate system are given in Fig. 2.5. The motion of the disk is described with two linear motions, y and z respectively, and three angular motions, β, Γ and φ respectively. The governing equations of motion of disk is derived by use of an energy method. The movements of the stator is also included in this work. The equation of motion of the stator is represented by the linear movements Y and Z. The nonlinear discontinuous equations of motion of the system are thoroughly explained and derived in [P1]. The contact and friction forces acting upon the disk and annular guide are modelled by use of Eqn. 2.6. These contact forces are applied to the right hand side of the equations of motion during contact between the disk and stator. Moreover, the theoretical method of employing the pinned backup bearing is also explained in [P1].
2.6 Introduction to the DTU Test Setup System

The test rig in short consists of the following main components: (a) a shaft mounted with a coupling connection and a removable friction surface, (b) a driving motor, (c) a flexible coupling, (d) a spherical roller bearing, (e) a backup bearing house, (f) the disk and proximity sensor tower and (g) a permanent magnetic housing with drop device. The disk is attached to the shaft, and the shaft is supported by a spherical ball bearing at the drive end and by a permanent magnetic house at the opposite non drive end where the shaft is levitated inside the magnetic housing. The impact house designated with (f) in Fig. 2.6 is the main component of the test rig. It is in this place that the normal and
friction forces exerted on the annular guide are measured. A detailed description of this impact house is given in \[P2\]. Figure 2.7 depicts the measurement setup employed during the experiments. The shaft motion is measured by the two proximity sensors indicated with (\textcircled{a}). The mobility of the inner house, the outer house and the support house is measured by accelerometers indicated with (\textcircled{b}). The shaft is impacted with the pendulum-hammer indicated with (\textcircled{c}). The angular velocity of the shaft is measured at the coupling by a speed sensor indicated (\textcircled{d}). The contact forces are measured with the force-transducers indicated with (\textcircled{e}) and named WEST, EAST, NORTH and SOUTH according to their directions, respectively, and referred to by their name tags in the experimental analyses sections.

### 2.6.1 Mathematical Modelling of the System

The dynamics of the whole system is divided in two sub-systems. One sub-system governing the equations of motion of the shaft and another sub-system governing the equations of motion of the backup bearing house, respectively. The shaft in this study is considered rigid. The restoring forces originate primarily from the restoring support stiffness coming from the permanent magnetic house, $F_{mag}$. The excitation forces originate from three sources; the unbalance mass, $F_{unb}$, the impulse force exerted by the impact pendulum-hammer, $F_{imp}$, and the contact forces between the shaft and the backup bearing, $F_{cont}$. The
The test rig consists of: a shaft with attached disk, a driving motor, a flexible coupling, a spherical roller bearing, a backup bearing house, the disk and proximity sensor tower and a permanent magnetic housing with drop device.

vibration of the backup bearing house is coupled through the interaction forces during impacts. Figure 2.8 depicts the mechanical model of the shaft with the attached disk located at the free end. The shaft is supported at the position $O$ by a ball bearing and coupled to a flexible coupling at this drive end. Therefore, the shaft movement is described around this base point $O$. The movement of the shaft is described and defined with the help of different moving reference frames as depicted in Fig. 2.8. In this way, the Newton-Euler method is utilized for deriving the governing equations of motion and the reaction forces. The contact and friction forces acting upon the shaft and annular guide are derived by use of Eqn. 2.6. The equations of motion of the shaft is represented by the two rotations, $\Gamma$ and $\beta$, respectively, and by the spin $\theta$. The equation of motion of the stator houses are represented by the linear movements $x$ and $y$. The nonlinear equations of motion are thoroughly derived in [P4].
2.6 Introduction to the DTU Test Setup System

Figure 2.7: (I) Test setup with mounted measurement tools, side view where (a) indicates the two proximity sensors measuring the motion of the disk, (b) indicates the accelerometers measuring the mobility of each house, respectively, (c) indicates the pendulum-hammer and (d) indicates the speed sensor. (II) test setup with mounted measurement tools, front view where (a) indicates the two proximity sensors, (b) indicates the accelerometers and (c) indicates the four proximity sensors.

Figure 2.8: Model of the shaft and coordinate frame used to describe the three consecutive rotations $\Gamma$, $\beta$ and $\phi$, and forces acting upon the shaft.
2.7 Numerical Implementation of the Two Systems

The mechanical systems considered in this study are studied by a set of nonlinear ordinary differential equations with a discontinuous right hand side acting at impact events. These systems are strongly nonlinear when viewed from a dynamical perspective. As the contact force and local penetration are assumed to behave in a continuous manner during contact, the systems are assumed to belong to the Filippov system. Mathematical modelling and numerical simulations of such non-smooth systems, where the transition from mode to another often can be idealized as an instantaneous or discrete transition where the time scale of the transition is much smaller than the scale of the individual mode dynamics, make their description often complex. Many different researchers present their contribution to deal with this complex non-smooth system. The main idea is that non-smooth systems can be considered as continuous functions in a finite number of continuous subspaces and the system parameters do not change in an abrupt manner. Among many Wiercigroch et al. [65], Leine [56] and Savi et al. [66] give their contribution by describing non-smooth systems by a smoothed form. The mathematical model uses a smoothened switch model, proposed by Leine [56] on the study of stick-slip vibrations. The switch model treats non-smooth systems by defining different sets of ordinary differential equations. Each of these sets is related to a subspace of the physical system. Leine’s innovative idea is the definition of transition regions that govern the dynamical response during the transitions through these regions. For this matter, each subspace has its own ordinary differential equation. Consequently, each transition region also has its governing equation defined in order to smooth the system dynamics. This approach smooths the discontinuities and facilitates the classical numerical integration procedure. According to Filippov’s theory, the state space of a system $\dot{x} = \tilde{f}(x)$ may be split into two subspaces $\Phi_-$ and $\Phi_+$, separated by a hyper-surface $\Sigma$, and defined by the indicator equation $h(x)$:

$$\dot{x} = \tilde{f}(x, t) = \begin{cases} \tilde{f}_-, x \in \Phi_- \\ \tilde{f}_+, x \in \Phi_+ \end{cases}$$ (2.7)

With the subspaces $\Phi_-$ and $\Phi_+$, and the hyper-surface $\Sigma$ are defined as:

$$\Phi_- = \{ x \in \mathbb{R}^n \mid h(x) < 0 \}$$ (2.8)

$$\Sigma = \{ x \in \mathbb{R}^n \mid h(x) = 0 \}$$ (2.9)

$$\Phi_+ = \{ x \in \mathbb{R}^n \mid h(x) > 0 \}$$ (2.10)

This Filippov theory imposes a correct description of the transition for the impact system. The contact between the rotor and the backup bearing occurs when the rotor radial displacement, $r$, becomes equal to the radial clearance $r_0$. 


and the rotor loses contact with the backup bearing when contact forces vanish i.e. if \( F = K_h \delta^n \left[ 1 + \frac{3(1-e^2)}{4} \frac{\delta}{\delta(-)} \right] = 0 \). Consequently, there are two subspaces with two different transition regions. The two indicator functions define the subspaces:

\[
\begin{align*}
    h_\alpha (x, \dot{x}) &= r - r_0 \\
    h_\beta (x, \dot{x}) &= \delta^n \left[ 1 + \frac{3(1-e^2)}{4} \frac{\delta}{\delta(-)} \right]
\end{align*}
\]  

(2.11)

Therefore, one can state that there is no contact between the rotor and the backup bearing when the state vector \( x \in \Phi_- \), and there is contact between the rotor and the backup bearing if \( x \in \Phi_+ \). The subspaces for this system is depicted in Fig. 2.9(I). These situations are represented by:

\[
\begin{align*}
    \Phi_- &= \{ x \in \mathbb{R}^2 | h_\alpha (x) < 0 \text{ or } h_\beta (x) < 0 \} \\
    \Phi_+ &= \{ x \in \mathbb{R}^2 | h_\alpha (x) > 0 \text{ and } h_\beta (x) > 0 \}
\end{align*}
\]  

(2.12)

However, difficulties arise when trying to numerically integrate a discontinuous system of Filippov type as the impact system presented in this study. The difficulties arise when there are numerous impacts in a very short period of time, and for the transition to sustained contact. For these conditions an ODE integrator will chatter around the hyper-surface, computing points alternating between \( \Phi_- \) and \( \Phi_+ \) and numerical instabilities can occur for the state of sustained contact. A numerical technique is introduced in order to improve the numerical behavior. The methodology introduces a vector field in the transition surface, such that the state vector is forced through the hyper-surface when the transition \( \Phi_- \) to \( \Phi_+ \) takes place and vice-versa. For this matter, the transitions are related to the hyper-surface \( \Sigma \), which consists of two surfaces \( \Sigma_\alpha \) and \( \Sigma_\beta \).
The hyper-surface $\Sigma_\alpha$ defines the transition from $\Phi_-$ to $\Phi_+$, representing the situations where the contact is caused when $r \geq r_0$. The hyper-surface $\Sigma_\beta$ defines the transition from $\Phi_+$ to $\Phi_-$, representing the situation when the contact is lost as the contact forces vanish. Therefore, the non-smooth system must be smoothened in the transition hyper-surfaces (vector field which contains a switching boundary), by assuming that the transition has a linear variation from $\tilde{f}_-$ to $\tilde{f}_+$ and vice-versa in a thin space (boundary layer) defined by a narrow band with thickness $2\eta$. The subspaces and the hyper spaces for this system are depicted in Fig. 2.9(II). The state space can then according to Leine [56] be written as follows:

$$\dot{x} = f(x, t) = \begin{cases} f_-, x \in \Phi_- \\ f_\alpha, x \in \Sigma_\alpha \\ f_\beta, x \in \Sigma_\beta \\ f_+, x \in \Phi_+ \end{cases}$$ (2.13)

If one as an example considers the equation of motion for the PUC system in the $y$-direction described in [P1], the state space is expressed by $\dot{x} = \tilde{f}(x), \ x = (y, \dot{y})$. Letting $f_y$ represents the left hand side of the first order ordinary differential system, the state space is written as:

$$f_-(x, t) = \begin{cases} \dot{y} \\ -f_y - F_{res,y} \end{cases}$$ (2.14)

Where $F_{res,y}$ denotes the restoring forces.

$$f_+(x, t) = \begin{cases} \dot{y} \\ -f_y - F_{res,y} + F_{con,y} \end{cases}$$ (2.15)

Where $F_{con,y}$ denotes the normal and friction forces acting upon contact. The definition of the state spaces associated with the transitions is written by the Filippov’s convex method and defined as:

$$f_\alpha = (1 - q) f_- + qf_+$$
$$f_\beta = (1 - q) f_+ + qf_-, \ q \in [0, 1]$$ (2.16)

Assuming a linear approximation through the transition regions, $\Sigma_\alpha$ and $\Sigma_\beta$ respectively, the transitions are defined as:

$$f_\alpha = \begin{cases} \dot{y} \\ -f_y - F_{res,y} + \cdots \\ K_h\delta^n \left[ \frac{1}{2} (r - r_0 + \eta) + \frac{3}{2} \frac{(1 - \epsilon^2)}{4} \frac{i \beta}{\delta^n} \right] \cdot (-\cos \alpha + \mu_k \sin \alpha) \end{cases}$$ (2.17)
Where $\mu_k$ denotes the kinetic friction coefficient between the two contacting bodies.

\[
\begin{align*}
    f_\beta &= \left\{ \begin{array}{l}
    \dot{y} \\
    -f_y - F_{\text{res},y} + \cdots \\
    K_h \delta_n \left[ \frac{3(1-e^2) - \delta}{4\delta(-)} \right] \cdot (\cos \alpha + \mu_k \sin \alpha)
    \end{array} \right. 
\end{align*}
\]  

(2.18)

Finally, the subspaces and transition hyper-surfaces are defined by:

\[
\begin{align*}
    \Phi_- &= \{ x \in \mathbb{R}^2 | h_\alpha (x) \leq -\eta \text{ or } h_\beta (x) \leq \eta \} 
    \quad \text{(2.19)} \\
    \Phi_+ &= \{ x \in \mathbb{R}^2 | h_\alpha (x) \geq \eta \text{ and } h_\beta (x) \geq \eta \} 
    \quad \text{(2.20)} \\
    \Sigma_\alpha &= \{ x \in \mathbb{R}^2 | -\eta < h_\alpha (x) < \eta \text{ and } h_\beta (x) \geq h_\alpha (x) \} 
    \quad \text{(2.21)} \\
    \Sigma_\beta &= \{ x \in \mathbb{R}^2 | h_\alpha (x) < h_\beta (x) \text{ and } -\eta < h_\beta (x) < \eta \} 
    \quad \text{(2.22)}
\end{align*}
\]

This method allows one to deal with non-smooth systems by employing a smoothening system. This approach is employed for every governing equation of motion in the numerical work. However, the thickness parameter, $\eta$, must be chosen according to the physical problem. In the numerical study carried out in the following the thickness parameter is set to $\eta = 1.5 \cdot 10^{-7}$. The numerical integration procedure is based on the ode15s stiff solver in MATLAB. In the numerical simulation of these piecewise-smooth systems the transition through the discontinuity surface, for example at impacts and at the switching between different vector fields of the system, is detected by use of event functions. These transitions are triggered by zero crossings of scalar valued event functions. The event functions govern the transition to the different sub-spaces and hyper-spaces. The numerical integration procedure is outlined and summarized in the flowchart depicted in Fig. 2.10.
Initialize

Calculating system parameters

Steady state

(1) external excitation
(2) self excited

Contact? yes no

non

Non contact state

yes

hα > 0

Changing equations, entering sub- and hyper-spaces Σα and Φ+

yes

hβ > 0

Changing equations, entering sub- and hyper-spaces Σβ and Φ-

no

Figure 2.10: Flowchart of the numerical integration procedure when simulating the contact events
The chapter summarizes and highlights some of the main original contributions published in [P1].

3.1 The Main Experimental and Theoretical Results

The presentation and comparison of the results are facilitated by presenting the following plots:

1. Time series of the motion of the disk in the two lateral directions, $y$ and $z$ respectively.
2. Trajectory of the lateral motion of the disk within the guide clearance.
3. The angular velocity of the disk plotted against time.
The sampling frequency during the experimental work was taken to be 6.4 kHz.

### 3.1.1 Conventional Annular Guide Setup for ω = 10 Hz (600 rpm), Experimental Results

In this study the behavior of the disk is investigated as it impacts the backup bearing due to an external excitation. Figure 3.1 depicts the impact motion of the center of the disk. The experiment demonstrates the phenomenon where the impact condition is superposed to the rub, where the rotor spin energy is fully transformed into rotor lateral motion. Consequently, the impacts and rubs act rapidly and overcome the capability of the AC-motor to maintain a constant velocity throughout the entire length of the thin shaft due to dry friction. The angular velocity under the rubbing conditions is almost stopped and all the rotational energy is transformed in a kind of "self-excited" rotor lateral vibration with repeated impacts against the annular guide. The full spectrum analysis presented in [P1] confirms that the motion of the disk transforms directly from a normal operational non-contact state to a dry full annular backward whip motion. The disk is locked into this state.

### 3.1.2 Conventional Annular Guide Setup for ω = 10 Hz (600 rpm), Theoretical Results

The findings from the experiments involving the damping factor and the impulse velocities, exerted in the two lateral directions on the disk, causing it to impact the annular guide and the coefficient of restitution are found in [P1] and employed in this numerical study. These velocities are computed as initial conditions given to the center of the disk. The following study is based on dry surface condition. It was not possible to determine the kinetic friction coefficient experimentally at PUC. The analyses are therefore based on the reference value for steel sliding against steel. The reference value is approximately given to $\mu_k = 0.56 \pm 0.01$. In the first approach, the system is considered as a discontinuous impulsive system without the use of smoothening methods in the numerical integration procedure. Figure 3.2 depicts the motion of the center of the disk. Shortly after the collision with the guide, a full annular backward rub state is developed and the angular velocity drops rapidly to zero. This leads to a diverging state and the numerical integration is therefore terminated. This diverging backward precession state for dry friction cases has also been presented in the work of Yanabe et al. [68] for their rotor-stator system illustrating the difficulties of solving these highly nonlinear systems. A study on the impact motion as a function of the dry friction coefficient values was carried out in this
3.1 The Main Experimental and Theoretical Results

The investigations revealed that the numerical results during the impact state were sensitive to the values of the dry friction. All the approaches of slightly varying the friction value led to numerical instability. However, in order to improve the numerical performances, the system is considered to belong to the Filippov system since the deformations and forces are assumed to act in a continuous manner during impacts. The smoothening method outlined in section 2 is employed. This numerical study is thoroughly explained in [P1]. Figure 3.3 depicts the motion of the center of the disk. This procedure overcomes the problems with the numerical instabilities and the result is qualitatively in good agreement with the experiment depicted in Fig. 3.1. This numerical procedure demonstrates its efficiency in solving highly nonlinear discontinuous systems under the influence of dry friction and contact forces. This method is therefore employed throughout the rest of the impact study. The experiments and the

Figure 3.1: Experiment: (I) time series y-motion, (II) time series z-motion, (III) trajectories of the center of the disk within the bearing clearance, (IV) angular velocity of the disk, \( \frac{R}{r_0} = 100 \)
Results of the PUC Test Setup System, Non Ideal Drive, [P1]

Figure 3.2: Numerical: (I) time series y-motion, (II) time series z-motion, (III) trajectories of the center of the disk within the bearing clearance, (IV) angular velocity of the disk

Theoretical findings show that the disk should be prevented from entering a full annular backward contact and whip state. Hence, the unconventional backup bearing design is considered in the following.

3.1.3 Unconventional Disk-Pin Setup for $\omega = 20$ Hz (1200 rpm), Experimental Results

The unconventional backup bearing depicted in Fig. 1.3(II) is employed. It is clearly demonstrated in Fig. 3.4 that this approach has a favorable effect on the impact behavior. The pins are forcing the disk to the center of the backup bearing and preventing a full annular rub condition to be developed. The full
3.1 The Main Experimental and Theoretical Results

Figure 3.3: Numerical: (I) time series y-motion, (II) time series z-motion, (III) trajectories of the center of the disk within the bearing clearance, (IV) angular velocity of the disk, $\frac{R_d}{r_0} = 100$

spectrum analysis presented in [P1] shows that a forward precession of the disk takes place unaffected by the impacts against the pins.

3.1.4 Unconventional Disk-Pin Setup for $\omega = 20$ Hz (1200 rpm), Theoretical Results

Figure 3.5 depicts the dynamics of the disk. In order to facilitate the numerical study, the four pins are positioned as depicted in Fig 3.5(c). Due to the new arrangement, the disk undergoes an impact motion within the imitated square where it impacts the pins. The disk settles to the motion as depicted in Fig 3.5. The disk is prevented from entering a backward contact motion state, and the
Results of the PUC Test Setup System, Non Ideal Drive, [P1]

Figure 3.4: Experiment: (I) time series y-motion, (II) time series z-motion, (III) trajectories of the center of the disk within the bearing clearance, (IV) angular velocity of the disk, $\frac{R_d}{r_0} = 833$

angular velocity is kept constant. Hence the disk is operating with full speed.
3.1 The Main Experimental and Theoretical Results

Figure 3.5: Numerical: (I) time series y-motion, (II) time series z-motion, (III) trajectories of the center of the disk within the bearing clearance, (IV) angular velocity of the disk, $\frac{R_d}{r_0} = 833$
Chapter 4

Results of the DTU Test Setup System, Ideal Drive, [P2]-[P4]

The chapter summarizes some of the main original contributions presented in [P2]-[P4].

4.1 Experimental Characterization of the Kinetic Friction Coefficient, [P2]

In order to achieve good agreements between the experimental and the theoretical results in terms of rotor to stator contact behavior, the kinetic friction coefficient is estimated through two different experiments using the exact same materials as the rotor and stator:

- In the first approach the friction coefficient is measured and studied by use of the pin-on-disk test.
In the second approach, the friction coefficient is evaluated from rotor-stator impacts with uncertainties originating from the measurement setup.

The results for the first method for the case of brass sliding against aluminium are thoroughly described in [P2]. The kinetic friction coefficient behavior was in the initial runs influenced by different wear stages and by stick and slip effects. In all probability, the coefficient value was estimated within the range of 0.17 to 0.76. After a fair amount of sliding distance it settled to the constant value of approximately $\mu_k = 0.25$ for each test setup. In the second method, the dry friction was determined by rotor to stator impact tests. The procedure and results are thoroughly explained in [P2]. The behavior of the kinetic friction coefficient was in good agreement with the pin-on-disk experiment. This method demonstrates that it is feasible to assess the dry friction value through rotor to stator impact test and describe the impact motion on the basis of the friction behavior.

### 4.2 Experimental Results of the Rotor-Stator Impact System, [P3]

In the following sections the experimental results are presented and discussed. The presentation and comparison of the results are facilitated by presenting the following plots:

1. Time series of the shaft motion in the two lateral directions, $x$ and $y$ respectively.

2. Trajectory of the lateral motion of the shaft within the ring.

3. The angular velocity of the shaft plotted against time.

4. Time series of the contact forces.

5. Contact force versus penetration rate.

The sampling frequency during the experimental work was taken to be 25 kHz.
4.2 Experimental Results of the Rotor-Stator Impact System, [P3]

4.2.1 Shaft Impacting the Annular Guide During Stable Run, $\Omega = 6.2$ Hz (372rpm)

This experiment is described in details [P3]. In this approach the shaft is suddenly impacted with the pendulum-hammer. Shortly afterwards, a full annular whirling state is developed where the shaft is traversing the full extent of the clearance. Figure 4.1 depicts the motion of the shaft as it impacts the annular guide. The trajectory plot of the center of the shaft together with the displacement plots are presented as the relative motion between the shaft and the annular guide. The full spectrum analysis conducted in [P3] confirms that this backward motion is full annular dry whirling. The whirling frequency is governed by clearance to radius ratio and the driving speed, thus given to; $\Omega_{wh} = \Omega \cdot \frac{R_S}{r_0} \approx 6.2 \text{ Hz} \cdot 13.5 \approx 83.7 \text{ Hz}$.

4.2.2 Impact Forces Between Shaft and Annular Guide, $\Omega = 6.2$ Hz (372rpm)

Figure 4.2 depicts the contact forces measured in each lateral direction for the full annular whirling state. The magnitude of the pre-compression is depicted with the DC-level off-setted from zero in each plot. The oscillating behavior of the contact forces is caused by the compression and relaxation phases of the force transducers, respectively. The hysteresis loops are computed and explained in [P3]. The contact force versus the penetration rate of the shaft into the annular guide is depicted in Fig. 4.3. These figures show the continuous nature of the contact force, which builds up from zero upon impact and smoothly returns to zero upon separation. This plot gives an indication of the energy dissipation as a function of the magnitude of the contact forces. As such, it is demonstrated in [P3] that the energy dissipation increases as the driving speed of the rotor increases. The experimental results of increasing the speed of the shaft and investigating the contact forces and corresponding penetration behavior are in good agreements with the theoretical work of Lankarani and Nikravesh [60]. Moreover, in a full annular whirling state it is experimentally demonstrated in [P3] that the forces acting on the shaft in the radial direction is the centrifugal force $F_C = M_s r_0 \omega_{wh}^2$ and the restoring forces. The centrifugal force $F_C$ is described by the total mass of the shaft, $M_s$, the clearance, $r_0$, and it is primarily influenced by the whirling frequency, $\omega_{wh}$. Therefore, in this state the influence of the unbalance mass is negligible.

Different experiments have been conducted in order to study the full annular backward motion. These motions have been excited both with external impulses (pendulum-hammer), and by self-excited vibrations. It was discovered that the
Results of the DTU Test Setup System, Ideal Drive, [P2]-[P4]

Figure 4.1: Experiment: (I) trajectories of the center of the disk within the bearing clearance $R_S/r_0 = 13.5$, (II) angular velocity of the shaft, (III) time series x-motion, (IV) time series y-motion

The shaft followed different paths for the cases of self-excited vibrations as the shaft velocity was gradually increased and decreased through the first critical speed. In fact, the motion was dry backward whip as the angular velocity was increased, see Fig. 4.4(I), and dry backward whirl, see Fig. 4.4(II), as the angular velocity was decreased. These experiments are explained in details in [P3]. However, these experiments also show that the shaft should be prevented from entering a full annular backward state. Therefore, the unconventional pinned backup bearing presented in Fig. 1.3(III) is employed.
4.2 Experimental Results of the Rotor-Stator Impact System, [P3]

Figure 4.2: Experiment, measured contact forces in: (I) WEST-direction, (II) EAST-direction, (III) NORTH-direction, (IV) SOUTH-direction

4.2.2.1 Shaft Impacting the Pinned Backup Bearing During Stable Run, $\Omega = 6.2$ Hz (372 rpm)

The results of this experiment are explained in detail in [P3]. It is experimentally demonstrated that this type of backup bearing prevents the shaft from entering a full annular contact state. The findings from the different experiments show that the shaft escapes the pins and returns towards the center of the bearing. The different directions of the friction and normal forces help the shaft in escaping the pins. Furthermore, the contact forces are reduced substantially compared to the experiments conducted with the annular guide. Figure 4.5 shows a comparison between the measured contact forces in the EAST direction.
Results of the DTU Test Setup System, Ideal Drive, [P2]-[P4]

Figure 4.3: Experiment, contact force versus penetration in: (I) WEST-direction, (II) EAST-direction, (III) NORTH-direction, (IV) SOUTH-direction

4.3 Theoretical Results of the Rotor-Stator Impact System, [P4]

The presentation and comparison of the results are facilitated by presenting the following plots:

1. Trajectory of the lateral motion of the shaft within the annular guide.

2. Time series of the shaft motion in the two lateral directions, $x$ and $y$ respectively.

3. The angular velocity of the shaft plotted against time.
4.3 Theoretical Results of the Rotor-Stator Impact System, [P4] 47

Figure 4.4: Experiment: (I) waterfall full-spectrum FFT plot for \( \frac{R_s}{r_0} = 13.5 \). The speed is gradually increased through the first critical speed (black arrow), the motion of the shaft transforms from forward whirl to backward whip with partial rub at 8.9 Hz, and finally to backward whirl with heavy rub at 120 Hz. (II) waterfall full-spectrum FFT plot for \( \frac{R_s}{r_0} = 13.5 \). The speed is gradually decreased through the first critical speed (red arrow), the motion of the shaft transforms from forward whirl to backward full annular whirl.
4. Contact forces plotted against time.

The responses of the shaft are transformed to lateral motion in the x and y direction positioned at the displacement proximitior’s location in order to compare the results. Figure 4.6 illustrates forces acting upon contact and how the reaction forces, $R_{\text{horz}}$ and $R_{\text{vert}}$, are influenced by the inertia coming from the support housings. These reaction forces simulate the forces acting upon the force transducers.

### 4.3.1 Shaft Impacting the Annular Guide During Stable Run, $\Omega = 6.2$ Hz (372rpm)

The theoretical result of this simulation case is explained in [P4]. The numerical smoothening method presented in chapter 2 is utilized. The parameters such as the external viscous damping, the restitution coefficient and the dry friction coefficient value are determined by experiments in [P2] and [P3] and employed in the numerical study. As in the experimental case the shaft is suddenly impacted with the pendulum-hammer. Figure 4.7 depicts the motion of the shaft. The
4.3 Theoretical Results of the Rotor-Stator Impact System, [P4]

4.3.2 Impact Forces Between Shaft and Annular Guide, $\Omega = 6.2 \text{ Hz (372 rpm)}$

Figures 4.8(I) and (II) depict the reaction forces, $R_{\text{horz}}$ and $R_{\text{vert}}$, computed in the two lateral directions. The magnitude of the contact forces in the initial state appears quite high. This is caused by the magnitude of the penetration ($\delta$) and the penetration velocity rate ($\dot{\delta}$ and $\dot{\delta}^{-}$). These parameters make the compliant contact force model sensitive to the changes in penetration and corresponding trajectory plot of the center of the shaft together with the displacement plots are presented as the relative motion between the shaft and the annular guide. Shortly after the shaft impacts the annular guide it undergoes to a full annular backward whirl state traversing the full extent of the clearance. The numerical results are in good agreement with the experiments presented presented in Fig 4.1.

Figure 4.6: Subsystem, where contact forces, $F_n$ and $F_\mu$ and reaction forces, $R_{\text{horz}}$ and $R_{\text{vert}}$ act upon contact.
velocity rate. However, in the full annular whirling state, the forces settle to a constant value. Figures 4.9 shows a comparison between the experimental and numerical contact forces, in the horizontal direction. In overall, they show good agreements. Yet, the theoretical computed contact force appears a little higher. This is due to the inertia effect coming from the stator which also causes the oscillating behavior. It was discovered that increasing the stiffness and damping properties of the support houses led to an increase in the bouncing motion of the shaft. Consequently, the contact forces increased substantially in magnitude during a impact and bouncing state. Despite the slight discrepancy in the force magnitude, different simulation cases of varying the driving speed of the shaft show good agreement with the experimental results. This is illustrated in [P4].
4.4 Poincaré Sections for Backward Whirling Motion for $\Omega = 6.2$ Hz (372 rpm), Addition to [P3] and [P4]

The experimental and theoretical analyses for the annular guide application demonstrated that the shaft motion was transformed directly to a backward full annular whirling state for external excitations. For this reason, it is interesting to analyze the behavior and information about the dynamics of the systems at this whirling state through Poincaré maps. The Poincaré maps are obtained by sampling the trajectories in phase plane at a constant interval of the forcing period of $T = 1/\Omega$ and projecting the outcome on the $x(nT) \times y(nT)$ plane. Figure 4.10(I) depicts the trajectory of the shaft together with the Poincaré section as it operates in normal condition. The Poincaré section depicts one point on the trajectory plot indicating a stable period-1 limit cycle motion. A disturbance in the system, external excitation, causes the trajectory to start intersecting the boundary in state space. The shaft motion is directly transformed to a full annular backward motion. The shaft is locked in this motion due to the presence of the radial forces. Figure 4.10(II) depicts the trajectory for the full annular backward motion. The Poincaré section fills up a closed curve which indicates a quasi-periodic motion. This quasi-periodic motion is caused by the non-smooth and discontinuous behavior which is a result of the contact with the annular guide. The dominating frequency component is the rolling/whirling frequency. Figures 4.10(III) and (IV) depict the theoretical results. The theoretical results are very nice related to the experimental results. However, the quasi-periodic
motion is only present due to the annular guide. Removing the annular guide, the trajectory will be attracted to a different cycle/state in space. It means, that the motion of the shaft as it traverses the full extent of the inner surface is purely governed by the annular guide properties.
4.4 Poincaré Sections for Backward Whirling Motion for $\Omega = 6.2$ Hz (372 rpm). Addition to [P3] and [P4]

Figure 4.10: Experiment: (I) phase plane Poincaré section period 1 limit cycle, (II) phase plane Poincaré section quasi-periodic motion, Numerical: (III) phase plane Poincaré section period 1 limit cycle, (IV) phase plane Poincaré section quasi-periodic motion
Results of the DTU Test Setup System, Ideal Drive, [P2]-[P4]
In this thesis, two different rotor to stator contact systems have been thoroughly studied experientially as well as theoretically. In both test systems where the annular guide was employed, the backward whip and whirl motions have been generated during rotor to stator contact due to radial clearance and dry friction. In the theoretical part, the highly nonlinear discontinuous equations of motion were formulated. These high nonlinear systems are difficult to solve due to the effects of dry friction, sustained contact and chatter. Two different theoretical approaches were employed in order to solve the system numerically. In the first approach, the discontinuous system was assumed to behave impulsively at impacts. The investigations showed that the numerical integration procedure failed at the transition to sustained contact. Consequently, numerical instabilities occurred and the integration procedure was terminated. In the second approach, the contact forces and deformations were assumed to happen in a continuous manner during impacts. Hence, the system was referred to the Filippov system where a smoothening method was used. This method showed improved theoretical results and the transition to sustained contact was properly captured. In the experimental part, the two different full annular backward motions were generated and carefully studied. The contact physics that trigger the rotor to enter either a full annular backward whirl or whip motion has also been investigated and outlined in this work. It was discovered that dry backward whip motion are likely to occur in cases of self-excited vibrations where the rotor is crossing the
critical speeds. This process can slowly excite the backward eigenmode of the rotor which facilitates the development of a whip motion. This whip motion is independent on the driving velocity. In contrary, dry backward whirl motions are more easily to trigger by external excitations. It is demonstrated that this motion can be excited for very low rotational speed of the rotor and will sustain that in full annular contact state. The new fully instrumented test rig allows the contact forces to be measured by force transducers. As such, the hysteresis cycles and penetration behavior were studied for full annular contact motions. These experimental findings seemed to confirm the purely theoretical work of Hunt and Crossley [44], and Lankarani and Nikravesh [60]. The theoretical simulated contact forces were also in agreement with the experimental results, thus it is feasible to apply a compliant contact force model in the mathematical formulation of the impact problem. In a contact event, the friction force is the key component of reversing the precession of the rotor. Therefore, two different experimental setups, a pin-on-disk and rotor-stator impact rig, were used in order to describe the coefficient behavior. The results of the pin-on-disk showed that the coefficient value was highly influenced by wear process and stick and slip effects. In all probability, the coefficient value could be described by an interval ranging from 0.17 to 0.76, for the material of brass sliding against steel. The results was unaffected by the changing in relative linear surface velocities. This experiments illustrates the difficulties of estimating a proper dry kinetic coefficient value. The results from the rotor-stator impact test measurements were in good agreement with the pin-on-disk findings. The contact forces and the friction coefficient were evaluated at each impact location. Hence, the transition to a full annular contact state could be described on the basis of the friction behavior. However, it is important to highlight that a further in-depth analyze of the coefficient behavior using thorough statistics models is necessary to assess the value based on statistical evidence. The experimental and theoretical investigations of the rotor impact motion using a conventional annular guide revealed that two components has a significant influence on the rotor full annular motion, namely the dry fiction and the geometric shape of the guide. Therefore, a new pinned unconventional bearing was introduced. It is theoretically and experimentally proven that the geometric shape outlined by the pins eliminates dry full annular whirl and whip motions and reduces the contact forces significantly. The design has been tested for different conditions and demonstrates improved and good performances.

This work has primarily focused on situations where a rotor during normal operation is suddenly subjected to an impulse or out of balance excitation causing a crucial impact situation. The work revealed that the rotor entered directly to a full annular contact state. Future work is proposed to be focused on offsetting the rotor, thereby letting it slightly rub and graze upon the stator. This approach entails that:
• The phase portraits can be computed as a function of velocity or dry friction.

• The co-dimensional one bifurcation diagrams can be visualized and analyzed by changing one parameter. As such, different bifurcation scenarios and shapes can be studied as carried out in [P5]. This is very useful in real machinery application where rotor rub and grazing are likely to occur.

• The DTU test rig is designed and manufactured to mimic different contact scenarios. Hence, different rub cases can be generated.

• Developing advanced statistical tools to assess the dry friction kinetic coefficients and conducting additional experiments to substantiate the statistical evidence.
Bibliography


Appendix A

[P1] On the Nonlinear Dynamics of Two Types of Backup Bearings - Theoretical and Experimental Aspects

On The Nonlinear Dynamics of Two Types of Backup Bearings — Theoretical and Experimental Aspects

The possible contact between rotor and stator can for some cases be considered a serious malfunction that may lead to catastrophic failure. Rotor rub is considered a secondary phenomenon caused by a primary source that leads to a disruption of the normal operational condition. It arises from sudden mass unbalance, instabilities generated by aerodynamic and hydrodynamic forces in seals and bearings among others. The contact event gives rise to normal and friction forces exerted on the rotor at impact events. The friction force plays a significant role by transferring some rotational energy of the rotor to lateral motion, impacting the stator. This event results in persistent coupled lateral vibration of the rotor and stator. This paper proposes a new unconventional backup bearing design in order to reduce the rub related severity in friction. The idea is to utilize pin connections that center the rotor during impacts. In this way, the rotor is forced to the center and the lateral motion is mitigated. The four pins are passively adjustable, which allows the clearance to be customized. A mathematical model has been developed to capture phenomena arising from impact for the conventional backup bearing (annular guide) and for the new disk-pin backup bearing. For the conventional annular guide setup, it is reasonable to superpose an impact condition to the rub, where the rotor spin energy can be fully transformed into rotor lateral movements. Using a nonideal drive, i.e., an electric motor without any kind of velocity feedback control, it is even possible to almost stop the rotor spin under rubbing conditions. All the rotational energy will be transformed in a kind of “self-excited” rotor lateral vibration with repeated impacts against the housing. The vibration of the housing is coupled through the interaction force. The experimental and numerical analysis shows that for the conventional annular guide setup, the rotational energy is fully transformed into lateral motion and the rotor spin is stopped. However, by employing the new disk-pin design the analysis shows that the rotor at impact is forced to the center of the backup bearing and the lateral motion is mitigated. As a result of this, the rotor spin is kept constant. [DOI: 10.1115/1.4007166]

1 Introduction

The dynamics and behavior of the rotating systems have been studied extensively in the past by many researchers. One of the initial studies of a high-speed rotor touching a rigid body was conducted by Szczygelski [1]. The model was based on a gyro pendulum touching a plane rigid body. The mathematical model was piecewise linear and globally strongly nonlinear. The preliminary experiments showing the trajectories of the gyro axis showed a good qualitative agreement with the analytical and experimental results. Among many, Muszynska [2] presented a comprehensive literary survey on rub-related phenomena up to 1989. Additionally, Beatty [3] introduced a mathematical model where the stiffness is given in a piecewise linear form. Due to the nonlinear behavior in stiffness and rubbing forces, the system can exhibit complicated vibration phenomena. Studies on these rubbing phenomena revealed that the rotating system showed a rich class of nonlinear related dynamics such as sub and supersynchronous responses, quasi-periodic responses, and even chaotic motions caused by the nonsmooth system which can exhibit different types of motions. In fact, the possibility of chaotic behavior for rotors upon rubbing was suggested by Ehrich [4]. To capture the phenomena, he based his mathematical model representing this motion on a nonlinear spring system. Moreover, Li and Paidoussis [5] investigated the dynamics of the system analytically and numerically, simulating the dynamic behavior through phase plane plots, bifurcation diagrams, as well as Poincaré maps using the dry friction coefficient and the eccentricity of the rotor imbalance as a control parameter. Additionally, Isakson [6] simulated the dynamical behavior of a rotor interacting with nonrotating parts. For this, a quasi-static solution was derived for a system where the stator offset was neglected and only valid for constant angular velocity of the rotor. In some cases, discontinuous and multivalued solutions were obtained. He also demonstrated that a steady state solution existed for a system with stator offset and continuous rubbing contact.

During rub and impact between the rotor and the stator, a full annular backward rub or partial impact condition can be developed. When the rub is developed into a full rub, the rotor vibration shows backward orbiting due to the present tangential friction force. This feature is normally used to identify rotor-to-stator rub. In fact, the full backward rub and the impact motion can be avoided by lubricating the surfaces and thereby decreasing the surface friction coefficient. Bartha [7,8] investigated theoretically and experimentally the dry friction backward whirl motion. He studied the influence of parameters such as excitation patterns leading to the onset of backward whirl, as well as backup bearing parameters such as coefficient of restitution and modal damping that reduces the rotor’s potential for whirl. In addition, Choi [9]...
also conducted experimental as well as numerical analyses to study the effect of the friction coefficient and the eccentricity between the rotor and stator that can explain the onset of backward rolling and backward slipping. Wilkes et al. [10] investigated through experimental and numerical methods the nature of multimode dry friction whip and whirl for a variety of rub materials and clearances. Their experimental results showed multiple whirl and whip regions as the rotor speed was increased or decreased through regions characterized by whip, terminated with jumps to different whirl/whip regions. In fact, they are the first to experimentally demonstrate these multiple whirl and whip regions. Inoue et al. [11] demonstrated theoretically the suppression of the forward rubbing by introducing a directional difference in the support stiffness of the lubricated surfaces. However, forward rub can still occur even in cases with asymmetric bearing support. The study revealed that the backup bearing with rubber attached in one direction in order to have an asymmetric bearing support. The study showed that the onset of the rubbing phenomena was caused by pin connections that center the disk during the impacts. In this way, the disk is forced to the center and the lateral motion is mitigated. The four pins are adjustable, which allows the clearance to be customized. Using this new design, the disk can encounter one or two pins at the same time. Thus, if the disk is exposed to a sudden drop or impulse, the movement will be restricted in a “kind of square” as the rotor impacts the pins. However, the angular velocity is kept constant and the phenomenon where the rotational energy is transformed in a kind of “self-excited” rotor lateral vibration is prevented as is the diverging full annular rub state.

A test rig is realized in order to validate the mathematical model and capture the impact motion. Figure 2 depicts the experimental setup conducted at PUC-Rio. The test rig, in short, is composed of: (a) an ac electrical motor, (b) a velocity sensor, (c) a shaft, (d) an annular guide and housing, (e) a rotor, and (f) proximity sensors. The stator is fixed to a base structure through four bolts arranged symmetrically. The ac electrical motor is controlled by a frequency inverter that drives the rotor-shaft assembly. Table 1 lists the parameters employed in the experimental and in the numerical work.

2 Mathematical Modeling

Inertial and Moving Referential Frames and Transformation Matrices. The disk is considered rigid, the restoring forces originate from the bending modes of the shaft. The bending mode of the shaft and definition of the coordinate system are given in Fig. 3. The motion of the disk is described with two linear motions, and three angular motions, and , respectively, and three angular motions, , and , respectively. The motion of the disk is described with help from three different moving reference frames obtained using consecutive rotations illustrated below:

\[
\begin{bmatrix}
\cos \beta & -\sin \beta & 0 \\
-\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\]

The angular velocities of the reference frames are written as

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{bmatrix}
= \Gamma
\begin{bmatrix}
0 \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix}
\]

The absolute angular velocity is described with help from the moving frame which is attached to the cross section of the disk, one has

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{bmatrix}
= \Gamma
\begin{bmatrix}
-\sin \beta \\
\sin(\cos \beta / \cos \beta) \\
\cos(\cos \beta / \cos \beta - \sin \beta)
\end{bmatrix}
+ \dot{\beta}
\begin{bmatrix}
0 \\
\cos \phi \\
-\sin \phi
\end{bmatrix}
+ \dot{\phi}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

Solving (3) one gets

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{bmatrix}
= \frac{\dot{\omega}_0}{\begin{bmatrix}
\cos \beta \\
\cos \beta / \cos \beta \\
\cos(\cos \beta / \cos \beta - \sin \beta)
\end{bmatrix}}
+ \dot{\beta}
\begin{bmatrix}
0 \\
\cos \phi \\
-\sin \phi
\end{bmatrix}
+ \dot{\phi}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

Test Facilities—Rotor and Two Types of Backup Bearings. Two types of backup bearings are employed and investigated in this study. The first test setup consists of a conventional annular guide, see Figs. 1(a) and 1(b), catching the impacting rotor and preventing damage. A new backup bearing design is realized in order to improve the behavior with regards to the impact motion and friction. The new bearing model has been developed at PUC-Rio. The objectives of this particular design are to reduce the lateral motion of the rotor and keep a constant angular spin. Figures 1(c) and 1(d) depict the new bearing design. The idea is to utilize bartha [7] and dai et al. [13] have shown that for some cases it is possible to stop the rotor spin under rubbing conditions and repeated impacts against the guide. A method studied by many researchers to prevent damages due to rub and impacts is to make use of active vibration control. In literature there are many publications about active vibration control of flexible rotors. Among these, Ginzingter et al. [14] present a new approach to control a rubbing rotor by applying an active auxiliary bearing. For this auxiliary bearing a three-phase control strategy was developed which stabilizes the rotor system in case of an impact load and effectively avoids backward whirling. However, the work presented in this paper gives an original contribution to the design of passive backup bearings, which does not require any active control system. The new type of backup bearing is therefore employed in this work. The four pins prevent a full annular rub and whirling state and mitigate the lateral motion of the rotor.
Due to unbalance mass in the disk, the center of mass is displayed from the geometric center of gravity. The position vector $B_3 e$ is easily written using the moving referential frame $B_3$, as illustrated in Fig. 4. The position vector from the geometric center to the center of mass (c.m.) is in the inertial frame given to

$$I_r^{CM} = I_r + T_{T_C T_B} T_{B_3} e = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} + T_{T_C T_B} T_{B_3} \begin{bmatrix} 0 \\ y \cos \phi_0 \\ z \sin \phi_0 \end{bmatrix}$$

(5)

The absolute linear velocity of the disk mass center is given to

$$v = \frac{\partial}{\partial t} (I_{r}^{CM}) = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} 0 \\ y \cos \phi_0 \\ z \sin \phi_0 \end{bmatrix}$$

(6)
the disk is found by using Lagrange equation:

\[
\text{deflected shape in the inertial (XZ) plane and (b) deflected shape in the inertial (YZ) plane}
\]

**Dynamics of the Rigid Disk. Energy Consideration.** The total kinetic energy of the disk related to the linear and angular motion is written as

\[
E_{\text{kin}} = \frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v} + \frac{1}{2} \omega^T \mathbf{J} \omega
\]

...

\[
\text{upon introducing the absolute angular velocity given in Eq. (4) and the absolute linear velocity in Eq. (6) into Eq. (7), disregarding higher order terms of the eccentricity such as } O(\epsilon^2) n = 2, 3 \text{ and the linear motion in the } x \text{ direction, the kinetic energy of the disk is expressed in the Appendix. The equations of motion for the disk is found by using Lagrange equation:}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial E_{\text{kin}}}{\partial \dot{q}_i} \right) - \left( \frac{\partial E_{\text{kin}}}{\partial q_i} \right) + \left( \frac{\partial E_{\text{pot}}}{\partial \dot{q}_i} \right) = \mathcal{F}_i, \quad i = 1, 2, \ldots, 5
\]

Here the generalized coordinates are designated \( q_1 = y, q_2 = z, q_3 = \beta, q_4 = \Gamma, \) and \( q_5 = \phi. \) As the disk is considered rigid, there is no potential energy stored on the disk itself; thus \( E_{\text{pot}} = 0. \) Furthermore, the spin of the disk is varying due to the variable torque of the motor and the reduction in spin during rub and impacts. Such forces are included later on as generalized forces \( \mathcal{F}_i, (i = 1, 2, \ldots, 5) \) together with the restoring forces coming from the flexible shaft. The equations of motions are written in the Appendix.

**Equation of Motion of the Stator House.** The movement and vibration of the backup bearing house is coupled through the interaction forces between the disk and backup bearing. At contact and impact events the equation of motion of the 2 degrees of freedom (DOF) system of the house in the two lateral directions \( Y \) and \( Z, \) respectively, is given as

\[
M_Y \ddot{Y} + D_Y \dot{Y} + K_Y Y = \mathcal{F}_Y
\]

\[
M_Z \ddot{Z} + D_Z \dot{Z} + K_Z Z = \mathcal{F}_Z
\]

**Generalized Forces \( \mathcal{F}_i. \)** The generalized forces on the right-hand side of Eqs. (A2)–(A6) are for this particular system written as

\[
\mathcal{F}_Y = \mathcal{F}_{\text{con,y}} - \mathcal{F}_{\text{res,y}}
\]

\[
\mathcal{F}_Z = \mathcal{F}_{\text{con,z}} - \mathcal{F}_{\text{res,z}}
\]

\[
\mathcal{F}_\beta = -\mathcal{F}_{\text{res,}\beta}
\]

\[
\mathcal{F}_\Gamma = -\mathcal{F}_{\text{res,}\Gamma}
\]

\[
\mathcal{F}_\phi = \mathcal{F}_{\text{con,}\phi} - \mathcal{F}_{\text{motor}} - \mathcal{F}_{\text{res,}\phi}
\]

\[
\mathcal{F}_Y = -\mathcal{F}_{\text{con,y}}
\]

\[
\mathcal{F}_Z = -\mathcal{F}_{\text{con,z}}
\]

Where res denotes the restoring forces and con denotes the contact forces. In the following the generalized forces are outlined and derived for the vertically suspended shaft-rotor system.

**Restoring Forces.** The restoring forces originate from the deflected shape of the shaft. The torsional flexibility of the shaft connecting the motor and the disk is also considered in this study. The mass and inertia of the shaft are neglected and only the first bending modes are considered as illustrated in Fig. 3(a). These particular restoring forces are found by applying beam theory; thus for each equation of motion one gets

\[
\mathcal{F}_{\text{res,y}} = c_{yy} \dddot{y} + c_{yz} \ddot{z} + c_{y\beta} \dot{\beta} + c_{y\Gamma} \dot{\Gamma} + k_{yy} y + k_{yz} z + k_{y\beta} \beta + k_{y\Gamma} \Gamma
\]
The stiffness coefficients are utilized in conjunction with the damping coefficient $c$. It is worth mentioning that the damping for this particular system is only applied in the principal direction for the lateral movement of the shaft. This slight damping accounts for the conical lateral movement of the shaft and accounts for the coupling effect due to the mounting procedure. This damping is derived as an equivalent external viscous damping. The order of magnitude of this external damping is determined by experiments. The stiffness coefficients for the depicted modes are given to $k_{yz} = k_{zy} = 12(EL/L^2)$, $k_{yy} = k_{yy} = k_{z} = k_{z} = -6(EL/L^2)$, $k_{yy} = k_{yy} = 4(EL/L)$, $k_{yy} = k_{yy} = GL/L$, and $k_{z} = k_{z} = k_{z} = k_{z} = k_{z}$ = 0. Furthermore, the damping is only applied in the principal direction in each equation given above; thus $c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy} = c_{yy}$ = 0. The torsional damping in the shaft is not considered in this study.

Contact and Rub Forces. In this study the disk is considered infinitely thin for contact and rub properties. Hence, the impact and rubbing between the disk and the annular guide are modeled as contact and friction forces exerted on the disk. The intermittent contact forces are only applied in the lateral directions, $y$ and $z$, respectively. First, the contact and rub forces exerted on the disk. The intermittent contact forces are only applied in the lateral directions, $y$ and $z$, respectively. One has

$$N = K_h \delta^\nu + D_h \cdot \dot{\delta}$$

(17)

The normal contact force is separated into elastic and dissipative force, accounting for the energy dissipation during impacts. This model considers the material and geometric properties of the colliding surfaces as well as information of the impact velocity. The stiffness parameter $K_h$ depends only on the contact stiffness between the colliding surfaces and depends on the material properties and the shape of the contact. For the two spheres in contact, the disk and annular backup bearing, the generalized stiffness coefficient is a function of the radii of the spheres $R_D$ and $R_B$ and the material properties for the disk and back up bearing, respectively. One has

$$K_h = \frac{4}{3(\sigma_D + \sigma_B)} \left( \frac{R_D R_B}{R_D + R_B} \right)^{1/2}$$

(18)

where the material parameters $\sigma_D$ and $\sigma_B$ are given by

$$\sigma_z = \frac{1 - \nu^2}{E_k}$$

(19)

The quantities $\nu$ are the Poisson ratio associated with each sphere. The quantity $D_h$ is a hysteresis coefficient and $\dot{\delta}$ is the relative impact velocity in Eq. (17). The hysteresis coefficient is written as a function of penetration as

$$D_h = \chi \delta^\eta$$

(20)

The hysteresis factor is given by

$$\chi = \frac{3K_h(1 - \nu^2)}{4\delta^\eta}$$

(21)

The normal contact force is finally written as

$$N = K_h \delta^\nu \left[ 1 + \frac{(1 - \nu^2) \dot{\delta}}{4 \delta^\eta} \right]$$

(22)

Where $e$ is the restitution coefficient. The restitution phase starts at this contact point and ends when two bodies separate from each

Fig. 5 Subsystem, impact motion, and forces
other, and the energy loss due to the motion in the normal direction can be expressed in terms of this coefficient. \( \delta \) is the relative penetration velocity, and \( \delta^{-1/2} \) is the initial impact velocity. The exponent \( n \) can, for circular and elliptical contacts, be set to 1.5.

The contact angle is taken as the relative contact angle between the two bodies:

\[
\tan(\alpha) = \frac{(y - Y)}{(z - Z)} \tag{23}
\]

The frictional force acting between the disk and annular guide is considered as a Coulomb friction force. In fact, in this study the kinetic friction coefficient around zero velocity is smoothened by use of an arc-tangent function in order to ease the difficulties arising due to the complexity of low velocity. Figure 6 depicts the friction model employed in this study.

The normal and friction forces acting on the disk at each lateral direction with the operational sign according to Fig. 5 can be expressed as

\[
F_{\text{con},y} = N(-\cos z + \mu_k \sin z) \tag{24}
\]

\[
F_{\text{con},z} = N(-\sin z - \mu_k \cos z) \tag{25}
\]

For every impact and rub, these forces act on the disk and ring and are imposed on the right hand-side of Eqs. (A2), (A3), (9), and (10), respectively.

\[
\begin{align*}
\text{Torque Due to Friction.} & \quad \text{The Coulomb friction torque given on the right-hand side of Eq. (11) is stated as} \\
& \quad \mathcal{F}_{\text{con},\phi} = N \cdot R_i \mu_k \text{sign}(\phi) \tag{26}
\end{align*}
\]

However, difficulties arise when modeling with friction and in particular with Coulomb friction. These difficulties arise due to the complexity of low velocity and in particular the stick-slip process. In the slip phase, the friction is simply a constant force opposing the motion. In the stick phase, the relative motion is null and friction appears as a constraint maintaining the zero velocity between the rubbing surfaces. In addition to this, when employing a Coulomb friction model, the model exhibits no friction force at zero velocity and therefore cannot represent the stick to slip transition properly.

Therefore, Piedbœuf et al. [19] present an approach for facilitating the phenomenon of low velocity. Using a simple model, the conditions are written as

\[
\begin{align*}
& \text{if } |\phi| > 0 \Rightarrow \mathcal{F}_{\text{con},\phi} = N \cdot R_i \mu_k \text{sign}(\phi) \\
& \text{if } |\phi| = 0 \Rightarrow \mathcal{F}_{\text{motor}} \leq \mathcal{F}_{\text{con},\phi} \Rightarrow \mathcal{F}_{\text{motor}} \leq N \cdot R_i \mu_k \text{sign}(\mathcal{F}_{\text{motor}})
\end{align*}
\]

3 Implementation

The mechanical system in this work is studied by means of the equations of motion presented previously and in the Appendix with a discontinuous right-hand side acting at impact events. The
system is strongly nonlinear when viewed from a dynamical perspective. Such systems which can be described by a set of first-order ordinary differential equation with a discontinuous right-hand side and discrete jumps in the vector field are known as Filippov systems. Mathematical modeling and numerical simulations of such nonsmooth systems, where the transition from one mode to another mode often can be idealized as an instantaneous or discrete transition where the time scale of the transition is much smaller than the scale of the individual mode dynamics, make their description often complex. Many different researchers present their contribution to deal with this complex nonsmooth system. The main idea is that nonsmooth systems can be considered as continuous functions in a finite number of continuous subspaces where the system parameters do not change in an abrupt manner. Among many, Wiercigroch et al. [20], Leine [21], and Savi et al. [22] contribute to the description of nonsmooth systems and their contribution to deal with this complex nonsmooth system. The difficulties arise when there are numerous impacts in a very short period of time, and for the transition to sustained contact. For these conditions an ODE integrator will chatter around the hyper surface, computing points alternating between \( \Phi_r \) and \( \Phi_s \), and numerical instabilities can occur for the state of sustained contact. A numerical technique is introduced in order to improve the numerical behavior. The methodology introduces a vector field in the transition surface, such that the state vector is forced through the hyper surface when the transition \( \Phi_r \rightarrow \Phi_s \) takes place and vice versa. For this matter, the transitions are related to the hyper surface \( \Sigma \), which consists of two surfaces \( \Sigma_a \) and \( \Sigma_b \). The hyper surface \( \Sigma_a \) defines the transition from \( \Phi_r \) to \( \Phi_s \), representing the situations where the contact takes place, when \( r \geq r_0 \). The hyper surface \( \Sigma_b \) defines the transition from \( \Phi_s \) to \( \Phi_r \), representing the situation when the contact is lost as the contact forces vanish. Therefore, the nonsmooth system must be smoothened in the transition hyper surfaces (vector field which contains a switching boundary), assuming that the transition has a linear variation from \( f_r \) to \( f_s \), and vice versa in a thin space (boundary layer) defined by a narrow band with thickness \( 2\eta \). The state space can then according to Leine [21] be written as follows:

\[
\dot{x} = f(x, t) = \begin{cases} f_r, x \in \Phi_r \\ f_s, x \in \Phi_s \\ f_b, x \in \Sigma_b \\ f_s, x \in \Phi_s \\ \end{cases}
\]

(34)

Letting \( f_r \) represent the left-hand side of Eq. (A2) and rewriting the system to a first order, the state space is written as

\[
f_r(x, t) = \begin{cases} \dot{y} \\ -f_r - \mathcal{F}_{\text{res}, y} \end{cases}
\]

(35)

\[
f_s(x, t) = \begin{cases} \dot{y} \\ -f_s + \mathcal{F}_{\text{con}, y} - \mathcal{F}_{\text{res}, y} \end{cases}
\]

(36)

The definition of the state spaces associated with the transitions are written by the Filippov’s convex method and defined as

\[
f_r = (1 - q)f_r + qf_s
\]

(37)

\[
f_s = (1 - q)f_s + qf_r
\]

Assuming a linear approximation through the transition regions \( \Sigma_a \) and \( \Sigma_b \), respectively, the transitions are defined as

\[
f_r = \begin{cases} \dot{y} \\ -f_r - \mathcal{F}_{\text{res}, y} + \cdots \\ K_d \delta \left[ \frac{(r - r_0 + \eta)}{4} \right] \left( -\cos x + \mu \sin z \right) \end{cases}
\]

(38)

\[
f_s = \begin{cases} \dot{y} \\ -f_s + \mathcal{F}_{\text{res}, y} + \cdots \\ K_d \delta \left[ \frac{(r - r_0 - \eta)}{4} \right] \left( -\cos x - \mu \sin z \right) \end{cases}
\]

(39)

Finally, the subspaces and transition hyper surfaces are defined by

\[
\Phi^- = \{ x \in \mathbb{R}^2 | h_a(x) \leq -\eta \} \\
\phi_s = \{ x \in \mathbb{R}^2 | h_a(x) \geq \eta \}
\]

(40)

\[
\Sigma_a = \{ x \in \mathbb{R}^2 | h_a(x) < \eta \} \\
\Sigma_b = \{ x \in \mathbb{R}^2 | h_a(x) \geq \eta \}
\]

(41)

\[
\bar{\Sigma}_a = \{ x \in \mathbb{R}^2 | h_a(x) > -\eta \} \\
\bar{\Sigma}_b = \{ x \in \mathbb{R}^2 | h_a(x) < -\eta \}
\]

(42)

This method allows one to deal with nonsmooth systems by employing a smoothened system and is used in this impact study. However, the thickness parameter \( \eta \) must be chosen according to

\[
\begin{array}{l}
\Phi_r = \{ x \in \mathbb{R}^2 | h_a(x) < 0 \} \\
\Phi_s = \{ x \in \mathbb{R}^2 | h_a(x) > 0 \}
\end{array}
\]

(33)

However, difficulties arise when trying to numerically integrate a discontinuous system of Filippov type as the impact system presented in this study. The difficulties arise when there are numerical impacts in a very short period of time, and for the transition to sustained contact. For these conditions an ODE integrator will chatter around the hyper surface, computing points alternating between \( \Phi_r \) and \( \Phi_s \), and numerical instabilities can occur for the state of sustained contact. A numerical technique is introduced in order to improve the numerical behavior. The methodology introduces a vector field in the transition surface, such that the state vector is forced through the hyper surface when the transition \( \Phi_r \rightarrow \Phi_s \) takes place and vice versa. For this matter, the transitions are related to the hyper surface \( \Sigma \), which consists of two surfaces \( \Sigma_a \) and \( \Sigma_b \). The hyper surface \( \Sigma_a \) defines the transition from \( \Phi_r \) to \( \Phi_s \), representing the situations where the contact takes place, when \( r \geq r_0 \). The hyper surface \( \Sigma_b \) defines the transition from \( \Phi_s \) to \( \Phi_r \), representing the situation when the contact is lost as the contact forces vanish. Therefore, the nonsmooth system must be smoothened in the transition hyper surfaces (vector field which contains a switching boundary), assuming that the transition has a linear variation from \( f_r \) to \( f_s \), and vice versa in a thin space (boundary layer) defined by a narrow band with thickness \( 2\eta \). The state space can then according to Leine [21] be written as follows:

\[
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\]

(34)

Letting \( f_r \) represent the left-hand side of Eq. (A2) and rewriting the system to a first order, the state space is written as

\[
f_r(x, t) = \begin{cases} \dot{y} \\ -f_r - \mathcal{F}_{\text{res}, y} \end{cases}
\]

(35)

\[
f_s(x, t) = \begin{cases} \dot{y} \\ -f_s + \mathcal{F}_{\text{con}, y} - \mathcal{F}_{\text{res}, y} \end{cases}
\]

(36)

The definition of the state spaces associated with the transitions are written by the Filippov’s convex method and defined as

\[
f_r = (1 - q)f_r + qf_s
\]

(37)

\[
f_s = (1 - q)f_s + qf_r
\]

Assuming a linear approximation through the transition regions \( \Sigma_a \) and \( \Sigma_b \), respectively, the transitions are defined as

\[
f_r = \begin{cases} \dot{y} \\ -f_r - \mathcal{F}_{\text{res}, y} + \cdots \\ K_d \delta \left[ \frac{(r - r_0 + \eta)}{4} \right] \left( -\cos x + \mu \sin z \right) \end{cases}
\]

(38)

\[
f_s = \begin{cases} \dot{y} \\ -f_s + \mathcal{F}_{\text{res}, y} + \cdots \\ K_d \delta \left[ \frac{(r - r_0 - \eta)}{4} \right] \left( -\cos x - \mu \sin z \right) \end{cases}
\]

(39)

Finally, the subspaces and transition hyper surfaces are defined by

\[
\Phi^- = \{ x \in \mathbb{R}^2 | h_a(x) \leq -\eta \} \\
\phi_s = \{ x \in \mathbb{R}^2 | h_a(x) \geq \eta \}
\]

(40)

\[
\Sigma_a = \{ x \in \mathbb{R}^2 | h_a(x) < \eta \} \\
\Sigma_b = \{ x \in \mathbb{R}^2 | h_a(x) \geq \eta \}
\]

(41)

\[
\bar{\Sigma}_a = \{ x \in \mathbb{R}^2 | h_a(x) > -\eta \} \\
\bar{\Sigma}_b = \{ x \in \mathbb{R}^2 | h_a(x) < -\eta \}
\]

(42)

This method allows one to deal with nonsmooth systems by employing a smoothened system and is used in this impact study. However, the thickness parameter \( \eta \) must be chosen according to
4 Theoretical and Experimental Results

The experimental work in this study of characterizing the driving torque from the ac motor was carried out in Alamo [23]. The electric ac motor of 0.12 kW utilized in this frame of work is controlled by a frequency inverter where the frequency band of operation ranges from 0 to 60 Hz. The two-phase induction motor has the maximum rotational speed of the magnitude of 1800 rpm. The rotational speed of the motor \( \Omega \) is controlled by a frequency inverter. The torque is given as a function of the rotating speed of the motor. The torque curves obtained experimentally are computed for different rotational speeds of the motor. The experimental data are obtained by measuring the mechanical friction as given in Crandall et al. [24]. Figure 8(a) depicts three torque curves for different frequencies imposed on the inverter and plotted against the speed of the motor, one for \( 2 \times 10^9 \text{Hz} \), one for \( 2 \times 12 \text{Hz} \), and finally one obtained as an average of the two curves.

An expression of the characteristic torque curve is determined by employing a third order polynomial curve fit to the median curve given in Fig. 8(a):

\[
F_{\text{motor}} = C \left[ -0.002441 \left( \frac{\Omega}{C} \right)^3 + 0.177 \left( \frac{\Omega}{C} \right)^2 - 5.608 \left( \frac{\Omega}{C} \right) + 290.65 \right]
\]  

(44)

\( F_{\text{motor}} \) represents the driven torque multiplied by a constant \( C \). \( C \) is proportional to the driving frequency imposed on the inverter. As an example, if the frequency in the inverter is set to \( 2 \times 100 \text{ rad/s} \) the value of the constant is set to \( C = 100 \). Figure 8(b) shows various curves of the motor torque to different frequencies imposed on the inverter. Equation (44) is imposed on the right-hand side of Eq. (11).

The presentation and comparison of the results are facilitated by presenting the following plots:

1. Time series of the motion in the two lateral directions, \( y \) and \( z \), respectively.
2. Trajectory of the lateral motion of the disk within the ring.
3. The angular velocity of the disk plotted against time.

If not otherwise stated, the parameters listed in Table 1 are employed in the numerical analyses. The damping and stiffness properties of the backup bearing house are found in Alamo [23]. The external viscous damping applied in the principal direction for the lateral movement of the shaft in the mathematical formulation originates from the conical movement and accounts for the mounting procedure where the shaft is connected to the driving motor through a ball bearing and a coupling. The order of magnitude of this equivalent external damping is determined by experiments.

Conventional Annular Guide Setup. As the shaft disk is vertically supported by the shaft, a sudden impulse is given to the disk in order for it to encounter and impact the annular guide. Figure 9 depicts the results obtained experimentally. The trajectory plot of the center of the disk and the displacement plots are presented as the relative motion between the disk and annular guide. The motion of the disk is therefore presented within the clearance, see Fig. 9(c). The experiment conducted here clearly demonstrates the phenomenon where the impact condition is superposed to the rub, where the rotor spin energy is fully transformed into rotor lateral motion. Using the nonideal drive given by the ac motor, the rotor angular velocity under the rubbing conditions is almost stopped and all the rotational energy is transformed in a kind of “self-excited” rotor lateral vibration with repeated impacts against the annular guide. In fact, the disk shortly gains its rotational velocity due to the restoring torsional stiffness of the shaft, as it shortly escapes the annular guide wall before impacting it again.
Figure 10 depicts the velocity of the disk for the impact motion depicted in Fig. 9. Such velocities are obtained by numerical differentiation of the displacements given in Figs. 9(a) and 9(b) and filter the signal from noise. The impulse velocities causing the disk to impact the annular guide are determined by taking the average of the velocities just before and shortly after the impact in the two lateral directions and calculating the radial velocity \( r = \sqrt{x^2 + y^2} \). The velocities are determined to be \( r = 120 \text{ mm/s}, y = 90 \text{ mm/s}, \) and \( z = 80 \text{ mm/s} \), respectively. Furthermore, the restitution coefficient \( e \) is determined by estimating the ratio \( e = v_{\text{out}}/v_{\text{in}} \) between the average impact velocities and the velocities shortly after impact, from the velocity plots in Fig. 10. The average impact velocity is determined to be \( v_{\text{in}} = 55 \text{ mm/s} \), and the average velocity after is determined to be \( v_{\text{out}} = 47 \text{ mm/s} \). This gives a restitution coefficient to \( e = 0.85 \). Figure 11 depicts a contour half-spectrum FFT plot, where frequency is plotted against time. At approximately 20 s the sudden impulse is given to the disk making it impact the annular guide. Regarding this contour plot, \( 1 \times, 2 \times, \) and higher harmonic frequency components are clearly observed before the impact event takes place. As the disk impacts the annular guide, higher frequency components arise. The vibration frequencies are higher during the rub and impacts due to the increase in stiffness when the disk contacts the annular guide. These vibration frequencies are independent on the rotative velocity of the disk.

In order to substantiate the experimental results depicted in Figs. 9 and 11 and verify the precession of the disk, a full-spectrum waterfall plot is depicted in Fig. 12, frequency plotted against time. The full-spectrum plot displays the correlation between the vibration data from the y and z components of the disk. This gives us the opportunity to determine whether the disk orbit motion frequency components are forward or backward in relation to the direction of the disk rotation. Considering this waterfall plot, it can be observed that higher reversed frequency components appear as the disk rubs and impacts the annular guide, approximately after 20 s. This indicates that a backward precession motion takes place, a whip motion. Moreover, Fig. 12 also shows higher harmonic sidebands conveying the backward precession. These reversed frequency components are independent on the rotational velocity of the disk and demonstrate that the rotational energy is transformed into a kind of self-excited rotor lateral vibration. In order to facilitate the presentation of the waterfall plot, a full-spectrum 2D plot and filtered orbit plot are depicted in Figs. 13(a) and 13(b). These plots are computed at 48 s. Considering Fig. 13(a), higher reversed frequency components are clearly observed \( (R_1 = -14 \text{ Hz} \) and \( R_2 = -28 \text{ Hz} \)), indicating a backward whip motion of the disk. Figure 13(b) depicts the filtered orbit showing a reversed elliptic orbit which demonstrates the instantaneous position of the disk on its filtered orbit. This is constructed as the sum of vectors of the forward and reversed orbits \( \sum_i R_i = R_i + R_i, \) \( i = 1, 2, ..., N \), the radii of the forward and backward orbits.

The findings from the experiments involving the damping factor and the impulse velocities, exerted in the two lateral directions on the disk, causing it to impact the annular guide and the coefficient of restitution, are employed in this numerical study. These velocities are computed as initial conditions given to the center of the disk. The following study is based on dry surface condition. The reference value for the kinetic dry friction coefficient for the surface condition of steel sliding against steel is given approximately to \( \mu_k = 0.56 \pm 0.01 \). It is difficult to determine a friction coefficient for the disk sliding against the annular guide for this test setup since the state of the surfaces, contamination, and oxidation of the surfaces have a great effect on the friction. The reference value for the dry friction coefficient seemed a little too high. However, the friction coefficient employed in the numerical study is chosen to be \( \mu_k = 0.3 \). This friction coefficient yielded the best comparable results of the disk motion compared with the experimental results. Two different cases have numerically been studied. In the first case the restoring torsional stiffness of the shaft is disregarded and not included in the numerical analysis. This approach has been conducted in order to study the effect of the restoring torsional stiffness. Figure 14 depicts the motion of the center of the disk. The impact energy is causing the angular velocity to drop to zero as the disk impacts the annular guide. The friction force exerted on the disk acts opposite the driving torque given by the ac motor and reduces the angular velocity at each impact. Shortly after the impacts a backward motion is developed. However, as the disk shortly slips and escapes the annular guide wall the driving torque given by the ac motor fails to reestablish the angular velocity. Consequently, the speed of the disk drops to zero as depicted in Fig. 14(d). In the next case the torsional
stiffness of the shaft is included in the numerical analysis. Figure 15 depicts the motion of the center of the disk. The impact energy is causing the angular velocity to drop rapidly as the disk impacts the annular guide. The friction force exerted on the disk acts opposite the driving torque given by the ac motor and reduces the angular velocity at each impact. The impacts act rapidly and overcome the capability of the ac motor to maintain a constant velocity. However, the restoring torsional stiffness of the shaft has a significant influence on the angular velocity of the disk as it acts opposite of the friction force and facilitates the driving torque. This is depicted in Fig. 15(d) with the oscillating velocity. Shortly after the collision with the guide, a backward rub motion is
developed leading to a self-excited rotor lateral vibration with repeated impacts against the annular guide. Qualitatively, the numerical finding where the restoring stiffness of the shaft is included in the analysis is in good agreement with the experimental results. The analyses demonstrate the phenomenon for the dry surface case where the rotating kinetic energy of the rotor is dissipated quickly through friction, and the angular velocity decreases extensively as a result of this. A backward rub state is developed which leads to a sustained impact state between the disk and annular guide. Moreover, Fig. 16 depicts an enlargement of the displacement plots of the experimental and numerical results in the \(y\) direction, showing the relative motion of the disk as it impacts the annular guide. These figures demonstrate that the disk has small impacts on the surface of the annular guide before it gets impacted away. This impact interaction between the annular guide and disk is due to the low mass and stiffness of the annular guide house. Consequently, this study demonstrates that the disk should be prevented from undergoing a full annular backward contact and whip state since it is difficult for it to escape this state. Hence, the unconventional disk-pin contact design is considered in the following.

Unconventional Disk-Pin Setup. The experiment is conducted by customizing the clearance by adjusting the length of the pins and exerting impulses upon the disk. The clearance for this new setup is set to approximately 0.06 mm yielding a clearance to radius ratio to 833. It is clearly demonstrated in Fig. 17 that this approach has a favorable effect on the impact behavior. The pins are forcing the disk to the center of the backup bearing and thereby preventing the disk to undergo a full rub condition and reducing the displacement of the disk. Furthermore, the angular velocity is kept constant and only slightly affected as the disk impacts the pins due to the impulses exerted on the disk, see Figs. 17(a), 17(b), and 17(d). These four pins secure the disk in the center of the unconventional backup bearing under the impact conditions, and will restrict the motion of the disk due to impacts and unbalance mass, etc. within the square restricted by the four pins as depicted in Fig. 17(c). The impulse velocity exerted on the disk is determined with the same procedure as mentioned in the previous sections. The radial velocity is determined to be \(v_r = 42\) mm/s, \(v_y = 24\) mm/s, and \(v_z = 35\) mm/s, respectively. The restitution coefficient \(e\) is determined to be \(e = 0.85\). The half spectrum contour FFT plot, where frequency is plotted against time, is depicted in Fig. 18 showing the 1X, 2X, and higher harmonic frequency components unaffected by the impulses exerted on the disk. In addition, this setup also induces subfrequency components which are caused by the disk impacting the four pins, respectively. Furthermore, the full-spectrum waterfall FFT plot depicted in Fig. 19 together with the full-spectrum FFT plot and filtered orbit plot in Figs. 20(a) and 20(b) indicate that forward precession of the disk takes place as the disk impacts the pins.

A numerical study has been conducted in order to investigate the behavior of the dynamics of the disk by using this new design. The impulse velocities determined by experiments are employed in the analysis and computed as initial conditions as well as the coefficient of restitution. Moreover, the restoring torsional stiffness of the shaft is included in the numerical analysis. This new bearing design induces that the friction coefficient can be reduced. Figure 21 depicts the dynamics of the disk. In order to facilitate the numerical study, the four pins are positioned as depicted in Fig. 21(c). Due to the pins, the disk undergoes an impact motion within the imitated square where it impacts the pins. The disk settles to the motion as depicted in Fig. 21. For this configuration the disk is prevented from undergoing a backward contact motion state, and the angular velocity is kept constant. Hence the disk is operating with full speed.
allows the clearance to be customized. The four pins are adjustable, which mitigates the lateral motion. In this way, the rotor is forced to the center of annular backward contact state. The rotor spin almost stops under rubbing conditions, and all the theoretical vibration with repeated impacts against the annular guide is transformed in a kind of self-excited rotor lateral vibration with repeated impacts against the annular guide housing. For this state, the precession of the disk vibration is reversed to that of disk angular direction due to the dry friction. This condition can for some cases be very destructive with its orbit shape traversing the full extent of the clearance. The theoretical model demonstrated that the rotor should be forced to the center of the annular guide at impacts and prevent it from undergoing a full annular backward contact state.

The new unconventional backup bearing design was realized and is able to reduce the lateral motion of the rotor and keep a constant angular spin. In this way, the rotor is forced to the center of the annular guide, the backward whip motion is prevented and the lateral motion is mitigated. The four pins are adjustable, which allows the clearance to be customized.

5 Conclusion

Acknowledgment

The authors would like to express great gratitude to Pontifícia Universidade Católica, PUC-Rio de Janeiro, Department of Mechanical Engineering, for hosting and conducting the experimental setup.

Nomenclature

\[
\begin{align*}
\gamma &= \text{lateral displacement of the disk (mm)} \\
z &= \text{lateral displacement of the disk (mm)} \\
\beta &= \text{angular displacement of the disk (rad)} \\
\phi &= \text{angular spin of the disk (rad/s)} \\
\Omega &= \text{rotational speed of the motor (Hz)} \\
I_p &= \text{polar mass moment of inertia of the disk (kg m}^2) \\
I_D &= \text{transverse mass moment of inertia of the disk (kg m}^2) \\
M_D &= \text{mass of the disk (kg)} \\
\varepsilon &= \text{eccentricity (m)} \\
\phi_0 &= \text{phase of the unbalanced mass (rad)} \\
c_u &= \text{damping characteristics, external viscous damping (N s/m)} \\
k_a &= \text{stiffness characteristics originating from the flexible shaft (N/m)} \\
K_1, K_2 &= \text{stiffness characteristics of the support of the stator (N/m)} \\
L &= \text{length of the shaft (m)} \\
R_D &= \text{radius of the disk (m)} \\
r_0 &= \text{clearance (m)} \\
\psi &= \text{clearance to radius ratio} \\
E &= \text{Young modulus (N/m}^2) \\
I &= \text{area moment of inertia of the shaft (m}^4) \\
G &= \text{shear modulus of the shaft (N/m}^2) \\
N &= \text{normal force acting upon contact (N)} \\
\mu_k &= \text{kinetic friction coefficient}
\end{align*}
\]

Appendix: Equation of Motion

The kinetic energy of the disk:

\[
E_{\text{kin}} = \frac{1}{2} M_D \dot{\gamma}^2 - M_D \dot{\gamma} \cos \Gamma \cos \phi - M_D \dot{\gamma} \cos \phi \sin \phi \\
+ M_D \dot{\gamma} \sin \phi \sin \Gamma \cos \phi - M_D \dot{\gamma} \sin \phi \cos \phi \\
+ M_D \dot{\gamma} \sin \phi \cos \Gamma \sin \phi + M_D \dot{\gamma} \cos \phi \cos \phi \\
+ M_D \dot{\gamma} \cos \phi \sin \Gamma \cos \phi + M_D \dot{\gamma} \cos \phi \cos \phi \\
+ M_D \dot{\gamma} \cos \phi \sin \Gamma \cos \phi + M_D \dot{\gamma} \cos \phi \cos \phi \\
+ M_D \dot{\gamma} \cos \phi \sin \Gamma \cos \phi + M_D \dot{\gamma} \cos \phi \cos \phi \\
+ \frac{1}{2} M_D \dot{\phi}^2 - M_D \dot{\phi} \sin \beta \sin \phi + M_D \dot{\phi} \cos \phi \cos \beta \\
+ M_D \dot{\phi} \sin \phi \sin \beta \cos \phi - M_D \dot{\phi} \sin \phi \cos \beta \\
+ \frac{1}{2} I_D \dot{\Gamma}^2 \sin \beta^2 + \frac{1}{2} I_D \dot{\phi}^2 + \frac{1}{2} I_D \dot{\phi} \dot{\Gamma} \sin \beta + \frac{1}{2} I_D \dot{\phi} \dot{\Gamma} \sin \beta \sin \phi \\
+ M_D \dot{\phi} \sin \phi \sin \beta \cos \phi - M_D \dot{\phi} \sin \phi \cos \beta \\
+ \frac{1}{2} I_D \dot{\phi} \dot{\Gamma} \sin \beta + \frac{1}{2} I_D \dot{\phi} \dot{\Gamma} \sin \beta \sin \phi
\]  

(A1)

The equations of motions are written as considering the motion in \( q_1 = y \):

\[
\begin{align*}
M_D \ddot{y} &= -M_D \dot{\gamma} \cos \phi \Gamma + M_D \dot{\gamma} \sin \phi \sin \Gamma \\
+ M_D \dot{\gamma} \sin \phi \cos \phi \Gamma - M_D \dot{\gamma} \sin \phi \cos \phi \\
- M_D \dot{\gamma} \cos \phi \sin \phi \Gamma + M_D \dot{\gamma} \sin \phi \cos \phi \\
+ M_D \dot{\gamma} \cos \phi \cos \phi \Gamma - M_D \dot{\gamma} \sin \phi \cos \phi \\
+ M_D \dot{\gamma} \sin \phi \sin \beta \Gamma + M_D \dot{\gamma} \sin \phi \cos \beta \\
- M_D \dot{\gamma} \cos \phi \cos \beta \Gamma + M_D \dot{\gamma} \cos \phi \cos \beta \\
- M_D \dot{\phi} \sin \phi \sin \beta \cos \beta + M_D \dot{\phi} \sin \phi \cos \beta \\
+ M_D \dot{\phi} \sin \phi \sin \beta \cos \beta - M_D \dot{\phi} \sin \phi \cos \beta \\
+ 2M_D \dot{\gamma} \cos \phi \sin \phi \Gamma + 2M_D \dot{\gamma} \cos \phi \cos \phi \Gamma \\
+ 2M_D \dot{\phi} \sin \phi \sin \phi \Gamma + 2M_D \dot{\phi} \sin \phi \cos \phi \Gamma \\
+ 2M_D \dot{\phi} \sin \phi \sin \beta \cos \phi + 2M_D \dot{\phi} \sin \phi \cos \beta \cos \phi \\
+ 2M_D \dot{\phi} \sin \phi \sin \beta \cos \phi + 2M_D \dot{\phi} \sin \phi \cos \beta \cos \phi \\
- 2M_D \dot{\gamma} \cos \phi \sin \beta \sin \phi + 2M_D \dot{\gamma} \sin \phi \cos \beta \sin \phi
\end{align*}
\]

(A2)

considering the motion in \( q_2 = z \):

\[
\begin{align*}
M_D \ddot{z} &= -M_D \dot{\gamma} \cos \phi \beta - M_D \dot{\gamma} \sin \phi \sin \beta \\
+ M_D \dot{\gamma} \sin \phi \cos \phi \beta - M_D \dot{\gamma} \sin \phi \cos \phi \\
- M_D \dot{\gamma} \cos \phi \sin \phi \beta + M_D \dot{\gamma} \sin \phi \cos \phi \\
+ M_D \dot{\gamma} \cos \phi \cos \phi \beta - M_D \dot{\gamma} \sin \phi \cos \phi \\
+ 2M_D \dot{\gamma} \cos \phi \sin \phi \beta + 2M_D \dot{\gamma} \cos \phi \cos \phi \beta \\
+ 2M_D \dot{\phi} \sin \phi \sin \phi \beta + 2M_D \dot{\phi} \sin \phi \cos \phi \beta \\
+ 2M_D \dot{\phi} \sin \phi \sin \beta \cos \phi + 2M_D \dot{\phi} \sin \phi \cos \beta \cos \phi \\
- 2M_D \dot{\gamma} \cos \phi \sin \beta \sin \phi + 2M_D \dot{\gamma} \sin \phi \cos \beta \sin \phi
\end{align*}
\]

(A3)
considering the motion in $q_3 = \beta$:

$$I_{G} \ddot{\beta} - \left( M_{D} \ddot{\gamma} \cos \phi \sin \beta - \dot{M}_{D} \ddot{\gamma} \cos \phi \dot{\beta} - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} \right) =$$

$$- M_{D} \ddot{\gamma} \cos \phi \sin \beta - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} - M_{D} \ddot{\gamma} \sin \phi \dot{\beta} + M_{D} \dot{\gamma} \sin \phi \dot{\phi} +$$

$$\left[ M_{D} \dot{\gamma} \cos \phi \sin \beta - \dot{M}_{D} \dot{\gamma} \cos \phi \dot{\beta} - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} \right]$$

(A4)

considering the motion in $q_4 = \Gamma$:

$$I_{G} \sin \beta \dot{\beta} + \left( 2 \Gamma \ddot{\gamma} \sin \phi \cos \beta - I_{G} \cos \beta \sin \phi \dot{\beta} - I_{G} \cos \beta \cos \beta \dot{\phi} \right) =$$

$$- M_{D} \ddot{\gamma} \cos \phi \sin \beta - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} - M_{D} \ddot{\gamma} \sin \phi \dot{\beta} + M_{D} \dot{\gamma} \sin \phi \dot{\phi} +$$

$$\left[ M_{D} \dot{\gamma} \cos \phi \sin \beta - \dot{M}_{D} \dot{\gamma} \cos \phi \dot{\beta} - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} \right]$$

(A5)

$$+ \left[ M_{D} \dot{\gamma} \cos \phi \sin \beta - \dot{M}_{D} \dot{\gamma} \cos \phi \dot{\beta} - M_{D} \dot{\gamma} \cos \phi \cos \beta \dot{\phi} \right]$$

References


APPENDIX B

[P2] Experimental Quantification of Contact Forces with Impact, Friction and Uncertainty Analysis

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Experimental Quantification of Contact Forces with Impact, Friction and Uncertainty Analysis

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Abstract
During rotor-stator contact dry friction plays a significant role in terms of reversing the rotor precession. The frictional force causes an increase in the rotor’s tangential velocity in the direction opposite to that of the angular velocity. This effect is crucial for defining ranges of dry whip and whirl motions in rotor-stator contact investigations. Dry friction coefficient is therefore estimated using two different experimental setups: a) standard pin-on-disk tests and b) rotor impact test rig fully instrumented. The findings in both setups indicate that the dry friction coefficient for brass-aluminum configuration significantly varies in a range of 0.17 - 0.76. The rotor enters a full annular contact mode shortly after two impacts with duration of approximately 0.004 seconds.

Keywords: contact dynamics, dry friction, pin-on-disk test, impact test

Nomenclature

$x =$ Displacement of the shaft in the horizontal direction [mm]
y = Displacement of the shaft in the vertical direction [mm]
$f_n =$ Natural frequency of the rigid shaft [Hz]
$\sigma =$ Contact angle with the horizontal plane [$^0$]
1. Introduction

The dynamics of rotor to stator contact-dynamics have been studied extensively in the past by many researchers. In 1962, Johnson [1] based his studies on the motion of a vertical shaft impacting a bearing with finite clearance. Johnson considered two case study; one without damping included and one with damping included in the modelling. However, he did not include dry friction in his model and the assessment of synchronous whirls was based on whether the solutions are real, whether the reaction force between the clearance bearing and the vertical shaft is positive, and finally whether the equilibrium is stable. In 1965, Billet [2] discussed how the reverse whirl induced by dry friction in a clearance can become a near resonant whirl over a large range of the shaft speeds. Billet showed how slip contributes to the onset and violence of whirl, and how this whirl speed cannot exist above the first natural frequency of the system. Furthermore, he also discussed how viscous damping acting on the system decreases the maximum reverse whirl speed. In 1967, Ehrich and O’Connor [3] also studied the rotor to stator contact dynamics where they included the motion of the bearing in the modelling. Here, they discovered a new class of vibrational behavior. These new phenomena revealed stator whirl wherein large stator amplitudes may be experienced at supercritical speeds. Additionally, new jumps and hysteresis phenomena were noted. One of the most discussed work was carried out by Black [4] in 1968. Black used two degree of freedom models for the rotor and stator to investigate the range of precession frequencies for which dry friction whirl and whip are possible. His investigation resulted in the U-shaped curve that separates regions of whirl and whip. With this figure, Black predicted that precession frequency existed in which dry friction whirl occurred and that dry friction whip will develop and persist at the upper limit of this precession frequency range. In 1979, Childs [5] described rub related parametric excitations in rotors.

In 1985, Beatty [6] investigated the rotor response due to radial rubbing based on a combination of analytical and experimental results. These re-
sults were calculated in terms of Fourier series expansion of a predicted rub induced shaft motion. In addition, in 1986 Szcygielski [7] studied a gyro pendulum touching a plane rigid body. The mathematical model was piecewise linear and globally strongly nonlinear. The preliminary experiments showing the trajectories of the gyro axis showed a good qualitative agreement with the analytical and experimental results. In 1988, Zhang [8] investigated the response of a multi degree of freedom system due to full annular rub. He applied Black’s model to his system and identified the same whirl regions as Black. Within this period Choy et al. [9]-[11] conducted nonlinear analysis to study the transient and steady state behavior of rub induced vibrations. In their work they also simulated and presented the magnitude of the contact and rub forces. To give an overview of the research work conducted up to 1989, Muszynska [12] presented a comprehensive literary survey on rub-related phenomena. In 1990, Lingener [13] and Crandall [14] reported experimental findings that seem to confirm Black’s theoretical results, where they found a stable whirling motion with a frequency slightly below the system’s coupled eigenfrequency.

In the nineties a great deal of work treated the nonlinear analysis on rotor to stator contact dynamics. Due to the non-smooth behavior in stiffness and rubbing forces, the system can exhibit complicated vibration phenomena. Studies on these rubbing phenomena revealed that the rotating system showed a rich class of nonlinear related dynamics such as sub and supersynchronous responses, quasi-periodic responses and even chaotic motions caused by the non-smooth system which can exhibit different types of motions. In fact, the possibility of chaotic behavior for rotors upon rubbing was suggested in 1992 by Ehrich [15]. To capture the phenomena, he based his mathematical model representing this motion on a nonlinear spring system. In 1994, Goldman and Muszynska [16] reported that the chaotic motion in a nonlinear study is more likely to occur if a proper impact model is employed. They used a discontinuous piecewise approach with extra stiffness and damping terms included during the contact stage. Their numerical simulations revealed that the system can exhibit orderly harmonic motions together with subharmonic responses, as well as chaotic motions. These nonlinear phenomena were also shown in Goldman and Muszynska’s research work conducted in 1994 [17] and in 1995 [18]. Moreover, Li and Païdoussis [19] investigated in 1994 the dynamics of the system analytically and numerically, simulating the dynamic behavior through phase plane plots, bifurcation diagrams as well as
Poincaré maps using the dry friction coefficient and the eccentricity of the rotor imbalance as control parameter. Additionally, in 1994 Isakson [20] simulated the dynamical behavior of a rotor interacting with non-rotating parts. For this, a quasi-static solution was derived for a system where the stator offset was neglected and only valid for constant angular velocity of the rotor. In some cases, discontinuous and multi valued solutions were obtained. He also demonstrated that a steady state solution existed for a system with stator offset and continuous rubbing contact. In 1998 Piccoli and Weber [21] investigated and presented an application for chaotic motion identification from a measured signal obtained through experiment. They assessed the measured system that demonstrated chaotic motion by evaluating Poincaré diagrams, Lyapunov exponents and correlation dimension by state space reconstructed with delayed co-ordinates. The assessment established that the system was chaotic. In 2004, Pavlovskaia et al. [22] conducted nonlinear analysis on a two-degrees-of-freedom model of the Jefcott rotor with preloaded snubber ring subjected to out of balance excitation. Their study was based upon analyzing bifurcation diagrams and phase portraits. Their nonlinear analysis showed that the effects of the preloading is crucial and should be included in the modelling. In 2006, theoretical work of [22] was verified experimentally by Karpenko et al. [23].

However, in 2000, Bartha’s [24, 25] test results contradicted the findings given in Black’s model and with the experimental results of Lingener and Crandall, with a particular view to the onset to dry friction whip. In his findings, Bartha suggested to employ an extended model where the bearing is modelled as nested rings. Furthermore, Bartha also discovered that the periodic whirl motion of different systems is unstable although steady state analysis based on Black’s model predicts stable periodic whirl regions. Bartha assessed the stability by considering the whole system and evaluating the Floquet multipliers rather than using static force equilibrium. In addition, in 2002, Yu et al. [26] investigated the onset to dry friction whip. Their investigation led to the conclusion that, the dry friction whip can be generated spontaneously without the presence of a dry friction whirl region as suggested in Black’s theoretical model where dry friction whirl goes ahead of dry friction whip motion. In 2005, Jiang and Ulbrich [27] also investigated the physical reason for the onset of dry friction whip in rotor-stator systems with imbalance. They discovered that rotor in resonance at a negative (natural) frequency of the coupled nonlinear rotor/stator system is the physical
reason for the onset of dry whip with imbalance. However, they also discov-
ered that the onset to dry whip follows different paths. For that reason, if the
motion follows the path of high frequency whirl through an outside distur-
bance at a relative low rotating speed as carried out in the work of Lingener,
the dry friction whirl motion goes ahead of dry friction whip motion as pre-
dicted in Black’s theoretical model. However, if the motion follows the path
of gradual increase in rotational speed under the influence of mass unbalance,
the partial rub generally goes ahead of dry friction whip. In 2007, Childs
and Bhattacharya [28] revisited the work of Black with inclusion of multiple
rotor modes that predicted several possible whirl regions. However, only the
first whirl region and its whip frequency could be computed in their simula-
tions. In 2008, in continuation of this work, Wilkes et al. [29] investigated
through experimental and numerical methods, the nature of multimode dry
friction whirl and whip for a variety of rub materials and clearances. Their
experimental results showed multiple whirl and whip regions as the rotor
speed was increased or decreased through regions characterized by whip, ter-
minated with jumps to different whirl/whip regions. Late in 2012, Yu [30]
discusses the reverse full annular rub where he proposes an analytical model
to effectively analyze dry friction whirl/whip phenomenon.

A deep investigation in all cited references reveals that all authors agree and
highlight the importance of the parameter dry friction coefficient. Neverthe-
less, only one author, Bartha [25], measures the contact friction by means
of pin-on-disk test setup. The main original contribution of this work relies
on the experimental investigation of dry friction coefficients using two differ-
et experimental setups. The first setup is a pin-on-disk experiment, which
allows the dry friction coefficient to be characterized under the influence of
driving velocity and contact pressure. The second setup, is a special designed
rotor to stator impact test rig fully instrumented. This test rig allows the
shaft impact motion under the influence of dry friction to be investigated.
In this regard, the contact forces such as the normal force and friction forces
together with the friction coefficient are estimated through impacts. This en-
ables the time of contact and penetration to be clearly identified during the
short impact intervals before a complete whirling condition develops. The
results are compared to those obtained by the pin-on-disk experiments.
2. Dry Contact Friction Estimated from Pin-On-Disk Test

The kinetic friction coefficient behavior as a function of velocity, loads and surface coating is a difficult parameter to characterize. These conditions have a significant effect on the friction coefficient and therefore makes it a difficult task to determine an adequate coefficient value. The friction coefficient for the materials of brass against aluminium is investigated. Figure 1 depicts the experimental setup conducted at the Technical University of Denmark, DTU. The test rig is built according to the specifications given in the American Society for Testing and Materials (ASTM) standard G99 and was realized in the work of Jacobsen and Christiansen [31]. The test rig is composed of: (a) a load cell attached on top of the belt stretcher thus increasing the surface pressure of the two contacting surfaces, (b) belt stretcher driving the load cell back and forward in order to increase the contact pressure, (c) the pin, in this particular case made of brass (the exact same material as the shaft in the rotor-stator test rig), (d) the rotating disk, in this case made of aluminium (the exact same material as the annular guide in the rotor-stator test rig). The disk is set to rotate and thereby making the pin sliding against the rotating disk. The pin is ensured orthogonal to the disk in order to achieve a constant surface pressure. The analysis of the kinetic friction behavior of this experiment is based on the work of Sarkar [32]. Different tests are conducted in order to estimate the kinetic friction coefficient and substantiate the findings. In the first test, the disk is set to rotate with the velocity of approximately 0.04 m/s and the applied contact force is set to 78 N. The experiments are repeated for different velocity cases. Figure 2(I) depicts the result for the first test. As the disk starts to rotate, the disk loses enough material to establish geometric conformity with the counter-surface, i.e. the disk loses an amount of mass to attain rapidly a true interfacial area, commensurate with the applied normal load and the tangential stress. The wear debris and shavings of aluminium are clearly depicted in Fig. 1(II) and (III). Afterwards, the transition stage reaches an equilibrium state where the shavings and wear debris process reduces. It appears in Fig. 2(I) that the friction coefficient value is higher in the initial run, due to the wear process before it reduces in magnitude. The friction coefficient $\mu_k$, for the first run oscillates within the interval of $[0.17;0.76]$. The experiment is repeated where the pin runs in the same trace. The result is depicted in Fig. 2(II), where the kinetic friction coefficient seems to oscillate about the value of $\mu_k = 0.25$. The increase in magnitude at 15 m in 2(II)
Figure 1: (I) Pin-on-disk test setup, \( \text{a} \) is a load cell, \( \text{b} \) is a belt stretcher driving the load, \( \text{c} \) is the pin and \( \text{d} \) is the rotating disk. (II) view of the wear debris process, (III) view of brass pin against aluminium disk. This test rig was designed and realized in the work of Jacobsen and Christiansen [31]

can be caused by wear debris and shavings affecting the pin in that short interval and by stick and slip effects. Figures 2(III) and (IV) depicts the results by changing the velocity of the disk to 0.02 m/s. In these two plots, the friction value settles to the constant value of approximately \( \mu_k = 0.25 \). In the last test, the pin location is changed to a new position on the disk. Figure 3 depicts the results. In accordance with the previous results the kinetic friction coefficient seems to be higher at the initial stage before the oscillating behavior settles to the constant value of approximately \( \mu_k = 0.25 \). In fact, the transition stage seems to be reduced. This might be caused by aluminium debris from the previous experiments were adhered onto the brass pin. This could reduce the transition stage. However, this pin-on-disk experiment shows that during the initial contact between two materials, the
kinetic friction coefficient value is given within an interval. For the material of brass against aluminium, the interval is determined to \([0.17;0.76]\). After a fair amount of sliding distance, the coefficient seems to settle to the constant value of approximately \(\mu_k = 0.25\). This value is unaffected by the magnitude of the driving velocity of the disk.

3. Contact Forces Estimated from Rotor-Stator Impact

A new test rig is designed and constructed in order to measure the complex dynamical behavior of a shaft in contact with its stator housing. A detailed design description and application is given in the work of Petersen [33]. This
new test rig enables the contact forces between the rotor and stator to be measured directly in a quite accurate manner, together with the motion of the rotor and the stator housing. Hence, the contact and reaction forces are measured directly by means of force transducers. Figure 4 depicts the new test rig. The test rig in short consists of the following main components; (a) a shaft mounted with a coupling connection and a removable friction surface, (b) a driving motor, (c) a flexible coupling, (d) a spherical roller bearing, (e) a backup bearing house, (f) the disk and proximity sensor tower and (g) a permanent magnetic housing with drop device. The disk is attached to the shaft, and the shaft is supported by a spherical ball bearing at the drive end and by a permanent magnetic house at the opposite non drive end where the shaft is levitated inside the magnetic housing. The impact house designated with (c) in Fig. 4 is the main component of the test rig. It is in this place that the normal and friction forces exerted on the annular guide are measured. It consists of three houses as showed in Fig. 5(I), where (a) is the support house made of stainless steel, (b) is the outer house made of acrylic glass, (c) is the inner house made of acrylic glass and (d) is the changeable annular guide. The main objective with this particular design is to decouple the contact force in the horizontal and vertical plane and make the structure robust to sustain the severity arising from impacts. The contact forces is measured by compression force transducers as depicted in Fig. 5(II). When the shaft impacts the inner house (annular guide) it exerts...
Figure 4: The test rig consists of: (a) a shaft with attached disk, (b) a driving motor, (c) a flexible coupling, (d) a spherical roller bearing, (e) a backup bearing house, (f) a the disk and proximity sensor tower and (g) a permanent magnetic housing with drop device.

A contact force having both a horizontal and vertical component. The vertical force is measured by the force transducer supporting the inner house in the vertical direction. Additionally, the horizontal component is measured by the force transducers supporting the outer house in the horizontal direction. The contact energy is transferred to the force transducers through movements of the two houses. The force transducers are named WEST, EAST, NORTH and SOUTH according to their directions, respectively, and referred to by their name tags in the experimental analyses sections.

Figure 6 depicts the measurement setup employed during the experiments. The shaft motion is measured by the two proximity sensors indicated with (a). The mobility of the the inner house, the outer house and the support house is measured by accelerometers indicated with (b). The shaft is impacted with the pendulum-hammer indicated with (c). The angular velocity
Figure 5: (I): The impact housing, where (a) is the support house, (b) is the outer house and (c) is the inner house and (d) is the backup bearing, (II): alignment of the four pre-compressed force transducers in the lateral directions EAST, WEST, NORTH and SOUTH.

of the shaft is measured at the coupling by a speed sensor indicated (d). The contact forces are measured with the force-transducers indicated with (e). In order to properly capture the impact motion and force behavior, the sampling frequency is taken to 25 kHz. It is important to highlight, that this entails that the impact motion can be captured with the resolution of $\Delta t = 4 \cdot 10^{-5}$ s, allowing the investigation of the time interval of contact and separation. In this setup, the friction coefficient is evaluated from rotor-stator impacts with uncertainties originating from the instrumentation. The procedure for estimating the uncertainties associated with the measurements instruments are based on the concepts in [34] which are based by the methods and concepts given in the International Organization for Standardization (ISO) *Guide to the Expression of Uncertainty in Measurement (GUM)*. The measurement error sources considered in this study, originate solely from the uncertainties associated with the sensors and transducers employed during the experiments. The error specifications given by the force transducer manufacturer and the proximity sensor manufacturer are listed in appendix A.
Figure 6: (I) Test setup with mounted measurement tools, side view where (a) indicates the two proximity sensors measuring the motion of the disk, (b) indicates the accelerometers measuring the mobility of each house, respectively, (c) indicates the pendulum-hammer and (d) indicates the speed sensor. (II) test setup with mounted measurement tools, front view where (a) indicates the two proximity sensors, (b) indicates the accelerometers and (c) indicates the four proximity sensors.

4. Friction Forces Estimated from Rotor-Stator Impacts

The normal and friction force components are evaluated from the rotor-stator impact experimental setup. These force components are computed from the measurements signal in the four lateral directions. The shaft is set to rotate with the angular velocity of approximately 6.2 Hz (372 rpm), yielding a relative linear surface velocity of $v = 0.52$ m/s, and impacted with the pendulum-hammer. Shortly after the shaft impacts the annular guide it undergoes directly to a full annular backward contact state. This is depicted in Fig. 7. As the acquisition frequency is set to 25 kHz, it is possible to capture the first couples of impacts before the full annular motion develops. The measured force components are used to evaluate the kinetic friction coefficient. In order to obtain qualitatively reasonable estimates, the uncertainties associated with the experimental setup are taken into considerations. Figure 8 depicts the shaft trajectory as it impacts the annular guide three times before it settles to a full annular contact state. Figure 8(I) depicts the three contact zones illustrated with the red dots. Figure 8(II) illustrates the forces acting on the annular guide at each impact event. These forces are illus-
Figure 7: Trajectories of the center of the disk within the bearing clearance. The shaft enters a backward whirling state immediately after impact with the annular guide.

notated with black arrows. Additionally, the red arrows illustrate the forces measured by each force transducer at the different impact locations. These contact forces are depicted in Fig. 9 for each lateral location, respectively.

Considering the measured forces for the first impact location depicted in Figs. 9(I) and (II), in the EAST and NORTH directions, it is observed that the contact in the two lateral directions has the same duration of time, $\Delta t = 0.004$ s, before the shaft escapes the annular guide and impacts the lower part. The impact forces for the second impact is depicted in Figs. 9(III) and (IV). The description of the force curves is influenced by the minor deflection of the inner backup bearing house and the penetration of the shaft into the annular guide. This impact is also very impulsive with the contact time of approximately $\Delta t = 0.003$ s. At the third impact location the shaft settles to a full annular contact state. Figures 9(V) and (VI) depicts the contact forces. The force in the NORTH direction does not decrease to
Figure 8: (I) Contact zone for the first 3 impacts, (II) force components. The black dashed line illustrates the movement of the shaft center after pendulum-hammer impact. The relative linear velocity of the shaft surface is approximately $v = 0.52 \text{ m/s}$

zero indicating sustained contact, and the full annular whirling state begins. These measured contact forces are utilized in order to compute the normal and friction forces at each location, respectively, and thereby to evaluate the kinetic friction coefficient. The measured forces employed in the analysis are taken at the instantaneous impact. In this way, the components of the measured forces, the normal and friction forces, can be separated in quite an accurate manner. The measured contact forces in each lateral direction, the position of the shaft (estimated from the center of the annular guide), the contact angle with the horizontal plane $\sigma$, the estimated normal and friction forces and finally the estimated kinetic friction coefficient are listed in Table 1. A combination analysis is conducted in order to combine the uncertainties from the proximity sensors with the force transducers. For the first two impacts the friction coefficient seems to be relatively high, approximately $\mu_k = 0.65 \pm 0.06$. This is attributable to the contact direction of the shaft against the annular guide, and the magnitude of the impact forces in terms of the high normal force. This causes the impact to become more impulsive. Yet, the direction of the friction force component changes the precession direction of the shaft immediately after the first impact. At the second
Table 1: Evaluated Parameters

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<th>3rd</th>
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<td>(-1.09, 0.94)</td>
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<td>$\approx 53^0$</td>
<td>$\approx 41^0$</td>
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<tr>
<td>$F_{SOUTH}$ [N]</td>
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</tbody>
</table>

impact the friction value seems to reduce slightly. At the third impact, the shaft seems to slide on the annular guide surface which causes the friction coefficient to reduce to the value of approximately $\mu_k = 0.17 \pm 0.01$. The kinetic friction coefficient for this impact and sliding state has the same order of magnitude as the coefficient estimated from the pin-on-disk experimental setup. In fact, it is described within the same value interval of [0.17;0.76] for the initial contact between the bodies. Moreover, the reduction in the friction force at the third impact indicates that the relative motion between the two contacting surfaces reduces and the shaft enters a full annular backward whirling state, where it rolls on the inner surface of the annular guide.

5. Conclusion

As the friction coefficient plays a significant role of defining ranges of dry whip and whirl motions in rotor-stator contact theoretical investigations, the friction behavior has been investigated in this work. The pin-on-disk experiments showed how different stages of the wear process have an effect on the kinetic friction coefficient behavior. In the initial tests, the value seemed high and was described within the interval of [0.17;0.76] for a constant value of the relative sliding velocity. After a fair amount of sliding distance the coefficient settled to a more constant value of approximately $\mu_k = 0.25$. This implies that, the oscillating behavior, minimum and maximum values, of the friction coefficient must be taking into consideration when simulating the initial contact in rotor to stator research study unaffected by the relative surface velocities.
In the rotor-stator experiment the shaft motion, the contact forces and friction values were captured and described for the first couples of impacts between the shaft and the annular guide. The friction coefficient behavior and order of magnitude seemed to behave in a similar manner as the results obtained from the pin-on-disk experiments. Moreover, the penetration interval was captured and identified for the first couples of impacts. The contact time at each location was estimated to be of approximately $\Delta t = 0.004$ seconds, indicating a very impulsive behavior before the full annular whirling motion takes place. The analysis also revealed that the first impact between the shaft and the annular guide, changes the precession of the shaft due to the high friction force and friction coefficient value given to approximately $\mu_k = 0.65 \pm 0.06$. At the third impact, the friction force was reduced which indicates that the relative motion between the two contacting surfaces reduces to zero and the transition to a full annular whirling motion develops. In the beginning of this whirling state, the friction coefficient reduced to the value of approximately $\mu_k = 0.17 \pm 0.01$. This experiment demonstrates that the friction coefficient can be evaluated and assessed through dynamical impact test of the rotor-stator system, and describe the impact motion on the basis of the friction behavior, if the motion is measured properly.

References


Figure 9: 1st contact zone, forces measured in: (I) EAST, (II) NORTH. 2nd contact zone, forces measured in: (III) EAST, (IV) SOUTH. 3rd contact zone, forces measured in: (III) WEST, (IV) NORTH
Appendix A

Error specifications given by the force transducer manufacturer and the proximity sensor manufacturer;

Proximity Sensor (prox): Pulso-Tronic KJ4-M12MN50-ANU
- Relative reproducibility error, $\epsilon_{\text{rep}}$, $\leq 1\%$
- Relative linearity, $\epsilon_{\text{lin}}$, $\leq 5\%$

Force Transducer (trans): HBM C9B
- Relative sensitivity error, $\epsilon_{\text{sen}}$, $\leq 1\%$
- Relative reproducibility error, $\epsilon_{\text{rep}}$, $\leq 0.5\%$
- Relative linear error, $\epsilon_{\text{lin}}$, $\leq 0.5\%$
Appendix C

[P3] Experimental Quantification of Dynamic Forces and Shaft Motion in Two Different Types of Backup Bearings under Several Contact Conditions

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Experimental Quantification of Dynamic Forces and Shaft Motion in Two Different Types of Backup Bearings under Several Contact Conditions

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Abstract

This paper treats the experimental study on a shaft impacting its stator for different cases. The paper focuses mainly on the measured contact forces and the shaft motion in two different types of backup bearings. As such, the measured contact forces are thoroughly studied. These measured contact forces enable the hysteresis loops to be computed and analyzed. Consequently, the contact forces are plotted against the local penetration in order to assess the contact force loss during the impacts. The shaft motion during contact with the backup bearing is verified with a two-sided spectrum analyses. The analyses show that by use of a conventional annular guide, the shaft undergoes a direct transition from normal operation to a full annular backward whirling state for the case of external excitation. However, in a self-excited vibration case, where the speed is gradually increased and decreased through the first critical speed, the investigation revealed that different paths initiated the onset of backward whip and whirling motion. In order to improve the whirling and the full annular contact behavior, an unconventional pinned backup bearing is realized. The idea is to utilize pin connections that center the rotor during impacts and prevent the shaft from entering a full annular contact state. The experimental results show that the shaft escapes the pins and returns to a normal operational condition during an impact event.

Keywords: backup bearing, measured contact forces, full annular backward motion, two sided spectrum analysis

Preprint submitted to Elsevier November 23, 2012
Nomenclature

\( x = \) Displacement of the shaft in the horizontal direction \([\text{mm}]\)
\( y = \) Displacement of the shaft in the vertical direction \([\text{mm}]\)
\( R_s = \) Radius of the contacting shaft \([\text{m}]\)
\( r_0 = \) Radial clearance \([\text{mm}]\)
\( e = \) Coefficient of restitution
\( \Omega = \) Angular velocity of the driving motor \([\text{Hz}]\)
\( \omega_{wh} = \) Whirling frequency \([\text{rad/s}]\)
\( \Omega_{wh} = \) Whirling frequency \([\text{Hz}]\)

1. Introduction

Numerous research works conducted on the study of auxiliary bearing performances in AMB applications, simulate the contact forces such as in the research work of Keogh and Cole [1]-[2] and in the work of Hawkins et al. [3]. However, studies on the contact force behavior in terms of contact time and deformations are modest. Yet, Markert and Wegener [4] analyze various contact models and the respective contact forces between a rotor and a backup bearing bearing, the hysteresis cycles and the resulting global behavior. However, they consider the friction force as a minor component since the minimization of friction is a design criteria for the retainer bearings, like for example roller in bearing applications. Therefore, they neglect the friction force component in their modelling and base their findings on the normal force component behavior. Nevertheless, the study conducted here reveals that if dry friction condition is present for a body spinning and contacting a counterpart body, the friction force is of great importance since it changes the motion of the rotor and transforms some of the spinning energy of the rotor to lateral movement. This will increase the contact forces and cause massive impact forces in the case of for example dry whip motions. To the authors’ knowledge, not much experimental work has been conducted in order to measure and study the contact forces in machinery dynamics applications. Yet, Fumagalli [6, 5] experimentally investigated the behavior of auxiliary bearing performances due to the dynamics of a rotor sliding and tumbling into it, and identified the effect of parameters on the motion, force and energy dissipation. In this way, different

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contact models could be verified through experiments. He investigated different proposed contact models, such as proposed by Hertz [7] and Hunt and Crossley [8], and took the energy dissipation into account. However, his estimation of the contact force was based on measuring the acceleration in the vertical direction of the auxiliary bearing and thereby calculated the contact force. Moreover, Ginzinger et al. [9] also present contact forces derived through experiments. As in the work of Fumagalli they used accelerometers to estimate the contact forces. The approach of using accelerometers can be feasible to describe the magnitude of the contact forces if one has a very good description of state parameters such as mass, damping and stiffness of the system. Despite having good descriptions of such parameters, the contact force behavior in terms of impact duration of time, hysteresis cycles and the penetration ratios cannot be evaluated by assessing the contact forces through the mobility of the stator.

Different backup bearing designs which do not require any type of feedback control system have been proposed in the past. Among these, Zülow and Liebich [10] applied flexible pins with roller elements at the free end to reduce the severity arising from friction. In their purely theoretical work they proposed different configurations of these pin elements. These flexible pin elements are modelled as spring-damper elements in their theoretical modelling, thus introducing additional degrees of freedom to the system. They simulated the motion of the rotor together with the contact forces for different rotor drop situations.

The original contribution of this paper is to experimentally study a rotor impacting its backup bearing and demonstrate the advantages of employing force transducers in order to achieve good description of the contact forces. This allows the contact forces and behavior to be thoroughly studied. Consequently, the hysteresis loops are computed and studied together with the contact forces versus penetration depths. These findings seem to be in good agreement with the theoretical work given by Hunt and Crossley [8], and Lankarani and Nikravesh [11]. Moreover, the motion of the shaft is studied both for the conditions of external impulses exerted upon the shaft and for self excited vibrations. For these conditions, the shaft undergoes different backward full annular rub motions. Finally, in this present work the pinned backup bearing as proposed by Lahriri et al. [12] is employed in the experimental study. In a similar manner to the design given by Zülow and Liebich [10], the proposed backup bearing design utilizes pin connections. The feasibility of employing this type backup bearing has been theoretically studied in the work of Zülow and Liebich [10]. In contrary, this present work gives a comprehensive
experimental study on pinned backup bearing designs and experimentally demonstrates the advantages of this particular design.

2. Test Setup, Two different Types of Backup Bearings

A detailed description of the test rig employed in the experimental work is given in [13]. Two different types of backup bearings are employed in the experimental work. In the first test setup the conventional annular guide is employed. Figures 1(I) and (II) depict the annular guide and the application. The experimental analysis in this present work is based on measuring the motion of the shaft with proximity sensors and the contact forces by means of force transducers in the four lateral directions, WEST, EAST, NORTH and SOUTH, respectively. The force transducers support the inner stator house. The contact energy is therefore transferred through linear movements of the stator house. Hence, the contact forces are decoupled in the two lateral directions. The movements of the stator house are measured by accelerometers. The pendulum-hammer depicted in Fig. 1(II) is used to impact the shaft. In order to obtain detailed descriptions of the shaft motion and contact force behavior, the sampling frequency is taken to be 25 kHz. In the second test setup, a new unconventional pinned backup bearing is employed to improve the impact motion of the shaft. Figures 1(III) and (IV) depict the unconventional backup bearing, built by four adjustable pins, and the application of this backup bearing. The aim of this design is to change the motion of the shaft during a contact event. In annular guide applications the circular inner shape facilitates the shaft to sustain in persistent contact with the backup bearing due to the presence of the radial forces. The ideas of using the four adjustable pins are to change the contact force components directions at every impact with the pins, and to prevent the acceleration of the tangential friction force acting opposite to that of the spin direction of the shaft to come into play. This study is carried out to purely investigate the behavior of the shaft motion due to some simple changes of the geometric shape of the backup bearing.

3. Experimental Results

In the following sections the experimental results are presented and discussed. The presentation and comparison of the results are facilitated by presenting the following plots:
Figure 1: (I) The annular guide, (II) application of the guide, the pendulum-hammer is used to impact the shaft, (III) unconventional pinned backup bearing, (IV) application of the pinned backup bearing

1. Time series of the shaft motion in the two lateral directions, $x$ and $y$ respectively.

2. Trajectory of the lateral motion of the shaft within the backup bearing.

3. The angular velocity of the shaft plotted against time.

4. Time series of the contact forces.

5. Hysteresis loop of the contact forces.
6. Contact force versus penetration rate.

In the first experimental case the critical speeds of the shaft are computed and identified. In the following experimental study cases, the attention is drawn to the contact force behavior together with the motion of the shaft under the influence of external excitations and self excited vibrations. The properties of the test setup are listed in Table 1. The mass, $M_s$, accounts for the total mass of the rotor including the shaft and disk.

### Table 1: Properties of the test rig

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<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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<tbody>
<tr>
<td>$M_s$</td>
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<td>kg</td>
</tr>
<tr>
<td>$R_s$</td>
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<td>m</td>
</tr>
<tr>
<td>$r_0$</td>
<td>0.001</td>
<td>m</td>
</tr>
<tr>
<td>$\frac{R}{r_0}$</td>
<td>13.5</td>
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#### 3.1. Critical Speeds of the Shaft-Rotor System

Figure 2(I) depicts a frequency response function of the shaft, where frequency is plotted against the driving angular velocity. The first critical speeds are identified by slightly impacting the shaft with the pendulum-hammer at different driving speeds and computing the transfer functions. The first backward and forward components are identified to approximately 8.4 Hz (504 rpm) and 9 Hz (540 rpm), respectively (this backward precession of the shaft can only be excited for cases of external excitations such as impacts). Furthermore, higher harmonic components such as 2X, 3X, 4X etc. are also identified for this system. The normal operational speed of the system is determined to be 6.2 Hz (372 rpm). The amplification of the response when the higher harmonic components cross the critical speed is evident in the contour plots in Figs. 2(II) and (III), showing a FFT spectrum plotted against the driving speed. The shaft is ramped up in speed Fig. 2(II), whereas it is ramped down in Fig. 2(III).

#### 3.2. Shaft Impacting the Annular Guide During Stable Run, $\Omega = 6.2$ Hz (372rpm)

In this section the contact forces and the behavior of such forces between the shaft and the annular guide are thoroughly studied. The force transducers are pre-compressed, which reduces the mobility of the inner stator housing. This mitigates the unfavourable inertia effect where the inner house is accelerated and amplifies the measured contacted forces. After some time, approximately 11 seconds,
the shaft is suddenly impacted with the pendulum-hammer in the lateral direction. Figures 3(I) and (II) depict the impact force originating from the pendulum-hammer. The magnitude of this impact force is approximately 100 N. Figure 4 depicts the motion of the shaft as it impacts the annular guide. The trajectory plot of the center of the shaft together with the displacement plots are presented as the
relative motion between the shaft and the annular guide. The motion of the disk is therefore presented within the clearance. Shortly after the shaft impacts the annular guide, it undergoes directly to a full annular backward whirling state traversing the full extent of the clearance. The precession of this particular motion will be verified later in this paper, by presenting and studying a two sided spectrum.

3.3. Impact Forces Between Shaft and Annular Guide, $\Omega = 6.2$ Hz (372rpm)

Figure 5 depicts the contact forces measured in each lateral direction during the contact period between the shaft and the annular guide. The magnitude of the pre-compression is depicted with the DC-level off-setted from zero in each plot. The magnitudes of the contact forces reaches rapidly a constant level. This level is dependent on the angular velocity of the shaft and will only change as a function of this. The oscillating behavior of the contact forces is caused by the compression and relaxation phases of the force transducers, respectively. This type of behavior will cause some sort of energy dissipation arising from the friction dissipation and from impacts, related to the restitution phase, at each cycle as the shaft is traversing the full extent of the clearance. Therefore, one turns to the hysteresis loop plots for each lateral direction as depicted in Fig. 6 where the magnitude of the measured forces are plotted against the displacement of the shaft.

These figures depict an ellipse hysteresis loop in each lateral direction for each cycle. In fact, this shape of the ellipse hysteresis loop was first presented and theoretically studied in the work of Hunt and Crossley [8] for the theoretical case of a compressed body. The horizontal line in each plot in Fig. 6 represents
Figure 4: (I) Trajectories of the center of the disk within the bearing clearance for $\frac{R_s}{r_q} = 13.5$, (II) angular velocity of the shaft, (III) time series x-motion, (IV) time series y-motion

the magnitude of the pre-compression. The magnitude of the forces above the horizontal line designates the compression forces whereas the magnitude of the forces below that horizontal line designates the tensile/relaxation forces. There is a very good correlation between the measured forces when the different plots in Fig. 6 are compared. If the shape of the hysteresis loop in the compression zone is considered, the shape as it turns out appears as a half ellipse which entails that the damping in compression is small. In order to study the real shape of the hysteresis loop and the damping related to the restitution phase, the clearance together with the relatively modest movement of the inner house is subtracted. As the shaft and the annular guide are in contact, deformation takes place in the local
contact zone resulting in that compression contact force. The contact force versus the penetration rate of the shaft into the annular guide is depicted in Fig. 7. These figures show the continuous nature of the contact force, which builds up from zero upon impact and smoothly returns to zero upon separation. The area within the hysteresis denotes the energy dissipation which is due to kinematic material damping of the two bodies and the modest damping originating from the hysteresis cycles of the force transducers. However, the damping coming from the force transducers are of minor influence and do not affect the results significantly. This type of impact occurs in two phases; the compression phase and the restitution phase. During the compression phase, the two contacting bodies deform in the normal direction to the impact surface and the relative velocity of the contact
points on these two bodies diminishes to zero. As the shaft penetrates into the annular guide, the compression phase reaches its maximum, the restitution phase starts at this point and lasts until the two bodies separate. Consequently, the shaft runs in a trace formed by the geometric conformity between the two bodies. These experimental studies reveal that there are different parameters affecting the energy dissipation. Thus, the restitution coefficient is dependent on the impact velocity. As such, the energy dissipation and the behavior of the contact forces for different angular velocities of the shaft are studied.
3.4. Impact Forces Between Shaft and Annular Guide, $\Omega = 3.2$ Hz (192 rpm) - 6.2 Hz (372 rpm)

The contact forces and the penetrations are studied for different angular velocity of the shaft. The shaft is set to rotate with a constant angular velocity. Afterwards, the shaft is impacted with the pendulum-hammer with approximately the same magnitude as depicted in Fig. 3. Subsequently, the shaft is set to rest by shutting the driving motor down and the annular ring is changed with a new one. The procedure is repeated for various speeds of the shaft. The various speeds of the shaft are taken to be; $\Omega_1 = 3.2$ Hz, $\Omega_2 = 4.2$ Hz, $\Omega_3 = 5.2$ Hz and finally $\Omega_4 = 6.2$ Hz. Figure 8(I) explains the shape of the penetration forces as a function of the contact time between the shaft and the annular guide in the EAST direction.
The impact responses vary with the angular speed of the shaft. As the speed increases the magnitude of the contact forces increases and the response becomes more impulsive. However, the period of contact between the two bodies reduces as the response becomes more impulsive. This type of behavior has been theoretically studied in the work of Lankarani and Nikravesh [11] showing similar simulated force versus time curves by varying the impact velocity. Figure 8(II) shows the time of initial contact, the time of maximum penetration and the time of separation of the local contact surfaces for each velocity case, respectively. Figure 9 depicts the contact forces plotted against the penetration rates for the different speed cases. The hysteresis loop erects as the angular velocity of the shaft is increased and thereby increasing the contact force. Consequently, the energy dissipation increases equivalently in magnitude. The coefficients of restitution in the EAST-direction are estimated from the average velocities of the shaft as it traverses the full extent of the annular clearance. It is estimated from the ratio $e = \frac{v_{out}}{v_{in}}$ and are given to be; $e_1 = 0.97$, $e_2 = 0.95$, $e_3 = 0.91$ and finally $e_4 = 0.90$. In order to achieve a higher energy dissipation, the penetration of the bodies could either be increased in magnitude, thus increasing the energy stored in the deformation (contact spring), or changing the material properties of the contacting bodies in order to increase the damping properties. The correlation between the magnitude of the contact forces and the angular velocity of the shaft is studied for the cases of sustained contact between the shaft and the annular guide.

3.5. Impact Forces Between Shaft and Annular Guide for Sustained Contact, $\Omega = 2.2$ Hz (132 rpm) - 6.7 Hz (372 rpm)

In this experiment the angular velocity of the shaft is gradually increased and decreased with the magnitude of 0.5 Hz (30 rpm). Figure 10 depicts the development, the growth and the reduction, of the contact forces as the angular velocity is varied. The magnitude of the pre-compression (DC-level) is subtracted in the presentation of the force values in Figs. 10(II) and (IV). In this experiment, the shaft is kept in a full annular backward motion unaffected by the changes in speed. Figure 11 depicts the maximum values of such contact forces plotted against the driving speed. This behavior explains that the growth of the contact forces seem to evolve quadratically. In fact, as discussed in the work of Bartha [14], in a pure rolling state no imbalance force is acting on the shaft as it traverses the full extent of the inner surface of the annular guide. For this matter, the forces that are acting on the shaft in the radial direction is the centrifugal force $F_c = M_t r_0 \omega^2$ and the restoring forces. The centrifugal force $F_c$ is described by the total mass of the shaft, $M_t$, the clearance, $r_0$, and it is primarily influenced by the whirling
frequency, $\omega_{wh}$. Figure 11 shows a very good agreement between the theoretical calculated force $F_C$ with the experimental results.

4. Full-Spectrum Analyses

In order to substantiate the experimental results presented in the previous sections with a particular view to the precession of the shaft, full spectrum analyses are conducted. In the first analysis the motion of the shaft as depicted in Fig. 4 is studied. Figure 12 shows the full spectrum waterfall FFT plot, where frequency is plotted against time. Higher reversed frequency components suddenly appear shortly after the shaft is impacted with the impact pendulum-hammer. This indicates that the shaft undergoes directly to a full annular backward whirling motion. The shaft is traversing the full extent of the clearance with the whirling frequency of approximately $\Omega_{\text{wh}} = \Omega \cdot \frac{R_S}{r_0} \approx 6.2 \text{ Hz} \cdot \frac{13.5}{1} \approx 83.7 \text{ Hz}$.

Figure 13 depicts the full spectrum waterfall plots for the case of increased and decreased angular velocity of the shaft. These full spectrum waterfall plots verify that the shaft sustains in a full annular backward whirling motion as the velocity is varied. This backward whirling motion is dependent on the angular velocity of the shaft as depicted in Fig. 13. Therefore, even for very low angular velocities the backward whirling motion can be engaged due to the dry friction
force acting between the surfaces. The shaft can not escape this whirling motion once initiated.

4.1. Self-Excited Reverse Precession

In this study, the shaft is slowly increased and decreased in angular velocity through the first critical speed. Two different backward precession motions take place when the shaft velocity is gradually increased and decreased, respectively. In the first case, the shaft transits immediately to a backward whip motion. This is illustrated in the full spectrum waterfall plot in Fig. 14 where frequency is plotted against the driving speed. This whip motion is independent on the angular velocity of the shaft. The onset to this whip motion is as follow; in Fig. 2(II) it can be seen that as the 2X component crosses the first critical speed the response of the shaft increases. Consequently, the shaft slowly starts to touch the annular guide and the backward component starts the get excited due to the modest impacts. As the speed gradually increases, the motion of the shaft increases and gets locked into the backward whip motion which was slowly excited during this process. There-
Figure 10: (I) Angular velocity of the shaft, speed up, (II) contact forces in EAST-direction, (III) angular velocity of the shaft, speed down, (IV) contact forces in EAST-direction

Therefore, the response of the shaft jumps from a forward synchronous whirling state to a backward whip motion with partial rub, and then to a full annular backward whip motion with heavy rub. The driving motor is shortly afterwards shut down due to the extensive contact forces. These contact forces are depicted in Fig. 15 for the EAST and SOUTH-directions, respectively. In these plots one can slightly observe the modest contact forces for the first whip motion. Afterwards, the contact forces increase substantially due to the second whip motion. Furthermore, this motion can be very crucial to the system. This backward whip motion, was also discussed in the work of Yu et al. [15] where they also discovered in one of their study cases, that around resonance speed synchronous forward precessional rub
developed into reverse precessional full annular rub when the surface friction was high. The work of Wu and Flowers [16] also substantiate this study case. Wu and Flowers studied experimentally and analytically the development of rub responses both for the case of employing a rigid disk and for the case of employing a flexible disk. They discovered that the rub started from light forward bouncing that developed into mixed bouncing and then into backward heavy rub as the speed was slowly increased. However, the direction of the precession of the disk was not verified with a two sided spectrum analysis at that time. In the latter experimental case in this study, the shaft motion undergoes directly to a full annular backward whirling motion as the driving speed is decreased through the first critical speed. This is illustrated in Fig. 16. This behavior is caused by the 1X component is the first to cross the first critical speed in contrary to the former experimental case. Hence, the motion of the shaft suddenly increases substantially and the shaft impacts the annular guide instead of slowly rubbing the annular surface where the backward mode can be excited. A summary of these two experiments is depicted in Fig. 17, where the maximum amplitudes of the forward and backwards radius

Figure 11: Magnitude of the contact forces for speed up and down in EAST-direction
Figure 12: Waterfall full-spectrum FFT plot, where frequency is plotted against time. After the impact the shaft whirls with the frequency of 83.7 Hz governed by the radius to clearance ratio $\frac{R_s}{r_0} = 13.5$ components are plotted against the driving speed.

5. Unconventional Disk-Pin Design

In order to improve the impact motion of the shaft a new pinned unconventional backup bearing is employed in the following experimental study cases. Figure 18 depicts the impact forces exerted on the shaft.

5.1. Shaft Impacting the Pinned Backup Bearing During Stable Run, $\Omega = 6.2$ Hz (372 rpm)

Figure 19 depicts the motion of the shaft as it impacts the pins. The trajectory plot in Fig. 19(I) illustrates the movements within the clearance. In contrary to the experiments conducted with the annular ring, the shaft shortly impacts the pins before it escapes them, settles and gets attracted to the stable limit circle acting in the center of the pinned backup bearing. This limit circle is induced by the passive magnetic bearing house located at the non-drive end and adds support stiffness to
the shaft. This is observed with the displacements plots depicted in Figs. 19(III) and (IV). In order to verify this behavior the shaft is impacted numerous times as depicted in Fig. 18(I). Shortly after every impact, the shaft returns towards the center of the bearing. The different directions of the friction and normal forces facilitate the shaft in escaping the pins. This particular behavior is also in agreement with the contact force measurements in the lateral directions as depicted in Fig. 20, respectively. These measured force diminish rapidly to zero shortly after the impacts with the pendulum-hammer. In fact, the highest peak in the horizontal direction corresponds to the exerted impact force on the shaft.

Figure 21 shows a comparison between the measured contact forces in the EAST direction for the case with the annular ring and with the new backup bearing employed in the experiments. The contact forces with the new backup bearing employed reduces rapidly to zero as the shaft escapes the pins. The precession of the shaft for the new backup bearing employed is verified by the full spectrum waterfall plot depicted in Fig. 22. As the shaft impacts the pins it undergoes a backward precession opposite to that of the direction of the angular velocity. However, this backward motion is not whirling. Shortly afterwards, the shaft traverses towards the center of the backup bearing where the motion becomes forward again.
Figure 14: Waterfall full-spectrum FFT plot for $\frac{R_S}{r_0} = 13.5$. The speed is gradually increased through the first critical speed, the motion of the shaft transforms from forward whirl to backward whip with partial rub at 8.9 Hz, and finally to backward whirl with heavy rub at 120 Hz.

5.2. Self-Excited Vibrations

5.3. Ramp up in Speed, $\Omega = 6.2$ Hz (372 rpm) - 14.2 Hz (852 rpm)

In this experiment the angular velocity of the shaft is gradually increased with the step size of 0.5 Hz (30 rpm). Figure 23 depicts the motion of the shaft as the speed is gradually increased. The lateral movement of the shaft increases as the speed approaches the first critical speed. After crossing the first critical speed, subsequently, the lateral movement reduces until the shaft suddenly escapes the impacting motion at the driving speed of approximately 12 Hz (720 rpm). Figure 24 depicts the full spectrum waterfall plot. This plot confirms that the precession is backward as the shaft impacts the pins. Figure 25 depicts the contact forces measured during this experiment. This contact forces are reduced with approximately a factor of 10 compared with the results obtained with the annular guide.
Figure 15: (I) Contact forces in EAST-direction showing the partial rub forces and the heavy rub forces, (II) contact forces in SOUTH-direction showing the partial rub forces and the heavy rub forces

5.4. Rotor Drop During Stable Run, $\Omega = 5.2$ Hz ($312$ rpm)

In this study case the motion of the shaft is studied for the case of a sudden rotor drop during normal operation. This study case mimics the cases where an AMB suddenly looses the power or the coupling breaks. For that reason, the passive magnetic bearing house is suddenly removed during the experiment. Figure 26 depicts the motion of the shaft. Figure 26(I) depicts how the stable limit circle suddenly looses its stability and the shaft trajectory drops downwards where it is captured by the two pins. The shaft sustains in the capturing motion. Figure 27 depict the contact forces during this experiment. The magnitude of the contact forces in the horizontal direction and as well as in the SOUTH direction is approximately given to 50 N. Figure 28 depicts the full spectrum plot of this drop motion of the shaft. As the shaft lands between the pins, the lateral movement of the shaft reduces substantially and the spectrum seems broad banded.

6. CONCLUSION

This paper has dealt with the experimental study of a shaft impacting its backup bearing. In this study the following have been investigated and revealed:

1. The contact forces have been thoroughly studied by use of force transducers. The hysteresis cycles together with the penetration ratios have been
Figure 16: Waterfall full-spectrum FFT plot for $\frac{R_s}{r_0} = 13.5$. The speed is gradually decreased through the first critical speed, the motion of the shaft transforms from forward whirl to backward full annular whirl analyzed. It was demonstrated that the magnitude of the contact forces during a full annular whirling motion is dependent on the angular velocity. Moreover, the coefficient of restitution decreased as the angular velocity of the shaft was increased indicating an increase in energy dissipation.

2. The motion of the shaft was studied for the cases of external impulses exerted on it. Shortly after the impact, the shaft went directly to a full annular whirling state. This whirling motion has been studied for different angular velocities of the shaft and shows that the shaft is trapped in this motion unaffected by the changes in speed. In the full annular whirling state the dominant force was demonstrated to be of the order $F_c = M r_0 \omega_{wh}^2$.

3. The precession of the shaft was verified through full spectrum analysis. It was discovered that the shaft followed different paths for the cases of self excited vibrations as the shaft velocity was gradually increased and decreased through the first critical speed. In fact, the motion was dry backward whip as the angular velocity was increased, and dry backward whirl as the angular velocity was decreased.
4. The unconventional backup bearing built by four adjustable pins was employed in the further investigations. It was demonstrated that this type of backup bearing has a favorable effect on the shaft behavior. The shaft can escape the pins and return to normal operational conditions at impact events. The pins prevent the shaft from undergoing a full annular contact motion where the impact forces becomes crucial. Furthermore, the contact forces for this design are reduced substantially compared to those obtained with the use of the conventional annular guide. In the case where an external impulse was exerted upon the shaft at the normal operational speed of 6.2 Hz (372 rpm), the contact forces diminished rapidly to zero. In the case of increasing the speed through the first critical speed, the measured contact forces were reduced by a factor of approximately 10. In the rotor drop experiment, the shaft was sustained at the bottom by the two pins and prevented from entering a full annular contact state. In this case the contact forces were also reduced by a factor of approximately 10.
References


Figure 18: (I) Impact forces exerted on the shaft, (II) zoom of the first impact force
Figure 19: (I) Trajectories of the center of the disk within the bearing clearance for \( \frac{R_s}{e_0} = 27 \), (II) angular velocity of the shaft, (III) time series x-motion, (IV) time series y-motion
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Appendix D

[P4] Theoretical Modelling, Analysis and Validation of the Shaft Motion and Dynamic Forces during Rotor-Stator Contact

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Theoretical Modelling, Analysis and Validation of the Shaft Motion and Dynamic Forces during Rotor-Stator Contact

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Abstract
This paper deals with the theoretical study of a horizontal shaft, partially levitated by a passive magnetic bearing, impacting its stator. Rigid body dynamics are utilized in order to describe the governing nonlinear equations of motion of the shaft interacting with a passive magnetic bearing and stator. Expressions for the restoring magnetic forces are derived using Biot Savart law for uniformed magnetized bar magnets and the contact forces are derived by use of a compliant contact force model. The theoretical mathematical model is verified with experimental results, and show good agreements. However, the simulated contact forces are higher in magnitude compared to the experimental results. The cause of this disagreement is addressed and shows that the formulation of the theoretical contact force model slightly overestimates the forces acting during a full annular backward whirl motion.

Keywords: Nonlinear Analysis, Impact Motion, Contact Forces

Nomenclature

\begin{align*}
x & = \text{Displacement of the shaft in the horizontal direction [ mm]} \\
y & = \text{Displacement of the shaft in the vertical direction [ mm]} \\
\beta & = \text{Angular displacement of the shaft [ rad]} \\
\end{align*}

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\[ \Gamma = \text{Angular displacement of the shaft [rad]} \]
\[ \theta = \text{Angular spin of the shaft [rad/s]} \]
\[ I_{xx} = \text{Transverse mass moment of inertia of the disk [kgm}^2\text{]} \]
\[ I_{yy} = \text{Transverse mass moment of inertia of the disk [kgm}^2\text{]} \]
\[ I_{zz} = \text{Polar mass moment of inertia of the disk [kgm}^2\text{]} \]
\[ M = \text{Mass of the whole shaft [kg]} \]
\[ M_{ih} = \text{Mass of the inner house [kg]} \]
\[ M_{oh} = \text{Mass of the outer house [kg]} \]
\[ \Omega = \text{Angular velocity of the driving motor [Hz]} \]
\[ \omega_{wh} = \text{Whirling frequency of the shaft [rad/s]} \]
\[ r_0 = \text{Clearance [m]} \]
\[ R_s = \text{Radius of the shaft [m]} \]
\[ \epsilon = \text{Eccentricity of the unbalance mass [m]} \]
\[ \varphi = \text{Phase of the unbalanced mass [rad]} \]
\[ F_n = \text{Normal force [N]} \]
\[ \mu_k = \text{Kinetic friction coefficient} \]
\[ \alpha = \text{Contact angle [rad]} \]
\[ \zeta = \text{Damping factor} \]
\[ E_s = \text{Young modulus of the shaft [N/m}^2\text{]} \]
\[ E_B = \text{Young modulus of the backup bearing [N/m}^2\text{]} \]
\[ \nu_s = \text{Poisson ratio of the shaft} \]
\[ \nu_B = \text{Poisson ratio of the backup bearing} \]
\[ I_c = \text{Electric current [A/m]} \]
\[ M_0 = \text{Magnetization constant [A/m]} \]
\[ \mu_0 = \text{Relative permeability [Tm/A]} \]

1. Introduction

Impact is a complex physical phenomenon, which occurs when two or more bodies collide with each other. The contact implies a continuous process which takes place over a finite time. Gilardi and Sharf [1] give a comprehensive literature survey on contact dynamics and discuss various contact models. Two different approaches for impact and contact analysis exist. The first approach is referred to as impulse-momentum method. The method is confined primarily to impact between rigid bodies. Hence, assumes that the interaction between the objects happens in a short period of time and that the configuration of the impacting bodies does not change significantly. For such a model, the dynamical analysis is divided into two
intervals, before and after impacts. In order to assess the process of energy transfer and dissipation between the two bodies, various coefficients are employed. The primary coefficients are the coefficient of restitution and the impulse ratio. However, the application of the impulse-momentum method leads to several problems. The main problem is that energy conservation principles may be violated during frictional impacts, and the discrete approach is not extendible to generic multi-body systems. The second method for impact analysis is by use of compliant, continuous, contact force models. These models overcome the difficulties associated with the impulse-momentum method. In the continuous contact model the forces and deformations vary and act in a continuous manner during impacts. Different continuous contact models have been proposed to describe the interaction force at the surfaces of two contacting bodies. The first model was developed by Hertz [2] and based on the theory of elasticity for frictionless contact, to calculate indentation without the use of damping. The local impact stiffness in this model between the two colliding bodies depends on the material properties, geometric properties and computed by using elastostatic theory. This model is only applicable for contact between rigid bodies where no energy dissipation takes place. To include damping in the contact modelling a simple spring-damper model was proposed, where the contact force is represented by a linear spring-damper element such as given in the Kelvin Voigt model. The impact is schematically represented with a linear spring in conjunction with linear damping, accounting for the energy dissipation. However, this model has some weaknesses. The contact force at the beginning of impact is discontinuous due to the damping term. At separation the local indentation tends to zero where the relative velocity between the two bodies becomes zero. This induces a negative force keeping the bodies together. Furthermore, the coefficient of restitution is independent on the impact velocities. To overcome these problems of the spring-damper model and to retain the advantages of the Hertz’s model, Hunt and Crossley [3] proposed a contact model. This model includes a non-linear damping term defined in terms of local penetration and the corresponding rate. This model satisfies the force boundary conditions during impact and separation and gives a correct description of the contact force behavior. The energy dissipation in this model is related to the ingoing and outgoing velocities. Another important aspect is that the damping depends on the local indentation $\delta$. This has its advantages since a contact area increases with deformation and a plastic region is more likely to develop for large indentions. A further development of this model was proposed by Lankarani and Nikravesh [4]. In this model a hysteresis damping function is incorporated to represent the energy dissipation during impacts.
The different contact force approaches outlined above have been employed by many researchers on the study of rotor to stator contact. Yanabe et al. [5] numerically studied the collision of a rotor with an annular guard during the passage through a critical speed. Their results were based upon two different approaches; the collision theory where the law of conservation of momentum and the coefficient of restitution were used, and the compliant contact force approach where they assumed the contact force to behave as a linear spring with no damping. In the end of the nineties the application of the Hunt and Crossley compliant contact force model started to become employed in the rotor to stator contact force modelling. Both Fumagalli [6] and Bartha [7] made use of this compliant contact force model in their theoretical studies. This contact model has also been employed in recent years in the research work of Childs and Bhattacharya [8], Wilkes et al. [9] and lately in the research work of Childs and Kumar [10]. The extension of the Hunt and Crossley contact force model proposed by Lankarani and Nikravesh is employed in this paper. This contact force model has been utilized extensively in the work of Flores et al. [11]-[14] for the study of mechanical systems with revolute joints and in the work of Lahriri et al. [15] for the rotor to stator contact modelling. Yet, the correct description of the contact stiffness and damping parameter is a difficult task since these are functions of the material properties, wear and damages and surface temperature. These parameters have an effect on the contact force behavior and magnitude.

The main original contribution of this work relies on experimental validation of a nonlinear contact force model, taking into account compliance, dry friction, penetration rate and penetration velocity during a contact event. Using 1) a special test rig completely monitored by force transducers, displacement sensors and accelerometers and 2) a theoretical model derived by multi-physics, it is possible to compare theoretical and experimental values of the dynamic contact forces during a full annular contact state. In light of these results, the theoretical employed compliance contact force model is validated.

2. Mathematical Modelling

The equations of motion of the system are derived in the following sections. The dynamics of the whole system is treated in two sub-systems. One sub-system governing the equations of motion of the shaft and another sub-system governing the equations of motion of the backup bearing house, respectively. Figure 1(I)
depicts the test rig. The shaft in this study is considered rigid. The restoring forces originate primarily from the restoring support stiffness coming from the permanent magnetic house. The excitation forces originate from three sources; the unbalance mass, the impulse force exerted by the impact pendulum-hammer and the contact forces between the shaft and the backup bearing. Figure 1(II) depicts the mechanical model of the shaft. The vibration of the backup bearing house is coupled through the interaction forces during impacts.

2.1. Shaft-Disk Dynamics

The shaft is considered as a rigid body. Figure 1(II) depicts the shaft with the attached disk located at the free end. The shaft is supported at the position $O$ by a ball bearing and coupled to a flexible coupling at this drive end. Therefore, the shaft movement is described around this base point $O$. The movement of the shaft is described and defined with the help of different moving reference frames as depicted in Fig. 1(I). In this way, the Newton-Euler method is utilized for deriving the governing equation of motion and the reaction forces. The different moving reference frames are obtained using consecutive rotations illustrated below.

$$
\mathbf{T}_{B_1}^{B_2} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \Gamma & \sin \Gamma \\
0 & -\sin \Gamma & \cos \Gamma
\end{bmatrix},
\mathbf{T}_{B_1}^{B_2} = \begin{bmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{bmatrix},
\mathbf{T}_{B_3}^{B_2} = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

(1)

The angular velocities of the reference frames are written as:

$$
\dot{\mathbf{\Gamma}} = \begin{bmatrix} \dot{\Gamma} \\ 0 \end{bmatrix},
\mathbf{\dot{\beta}} = \begin{bmatrix} \dot{\beta} \\ 0 \end{bmatrix},
\dot{\mathbf{\theta}} = \begin{bmatrix} 0 \\ \dot{\theta} \end{bmatrix}
$$

(2)

The absolute angular velocity, $\Omega$, of the moving reference frame $B_2$, in which the inertia tensor $\mathbf{I}_0$ referred to the contact point $\theta$ remains constant, is determined. The absolute angular velocity is derived in order to employ Euler’s equation with respect to the two angles $\Gamma$ and $\beta$, in which the momentum and moment vectors will be represented.

$$
\mathbf{\Omega} = \mathbf{T}_{B_2}^{B_1} \cdot \mathbf{\dot{\Gamma}} + \mathbf{T}_{B_1}^{B_2} \cdot \mathbf{\dot{\beta}} = \begin{bmatrix} \dot{\Gamma} \cos \beta & \dot{\beta} & \dot{\Gamma} \sin \beta \end{bmatrix}^T
$$

(3)
Figure 1: (I): The test rig consists of: ○ a shaft with attached disk, ◯ a pendulum-hammer, ◊ a flexible coupling, ◜ a spherical roller bearing, ◷ a backup bearing house (the contact forces are measured here by four force transducers attached in each lateral direction), ◱ a the disk and proximity sensor tower, ◲ a permanent magnetic housing with drop device and ▼ a driving motor, (II): model of the shaft and coordinate frame used to describe the three consecutive rotations Γ, β and φ, and forces acting upon the shaft.
The absolute angular velocity of the rotor, $\omega$, represented with help of the moving reference frame $B_2$, the spin happens in this moving reference frame, is expressed as:

$$B_2 \omega = B_2 \Omega + B_2 \dot{\theta} = \begin{bmatrix} \dot{\Gamma} \cos \beta & \dot{\beta} & \dot{\Gamma} \sin \beta + \dot{\theta} \end{bmatrix}^T$$

(4)

The angular angular acceleration of the shaft is expressed as:

$$B_2 \dot{\omega} = \frac{d}{dt} \left( B_2 \omega \right) + B_2 \Omega \times B_2 \omega = \begin{bmatrix} -\dot{\beta} \dot{\Gamma} \sin \beta + \dot{\Gamma} \cos \beta + \dot{\beta} \dot{\theta} \\ \ddot{\beta} - \dot{\beta} \dot{\Gamma} \cos \beta \\ \ddot{\Gamma} \cos \beta + \dot{\Gamma} \sin \beta + \ddot{\theta} \end{bmatrix}$$

(5)

Since the shaft is symmetric revolving around the z-axis the absolute angular velocity, $B_2 \Omega$, is utilized when deriving the linear velocity and linear acceleration of the center of mass of the shaft. The absolute linear velocity of the center of mass of the shaft is given to:

$$B_2 v = B_2 v_0 + B_2 \Omega \times B_2 r_{0-CM} + B_2 v_{rel} = \begin{bmatrix} \beta l_{oc} \\ -\dot{\Gamma} \cos \beta l_{oc} \\ 0 \end{bmatrix}^T$$

(6)

$L_{2}r_{0-CM}$ designates the position vector from 0 to the center of mass given in the moving reference frame $B_2$, thus given to: $B_2 r_{0-CM} = \begin{bmatrix} 0 \\ 0 \\ l_{oc} \end{bmatrix}^T$. The velocity at the base point, $B_2 v_0$, is zero. Since the position vector in the reference frame is constant the relative velocity, $B_2 v_{rel}$, is also zero. The absolute linear acceleration of the center of mass of the shaft is expressed as:

$$B_2 a = B_2 a_0 + B_2 \Omega \times B_2 \Omega \times B_2 r_{0-CM} + B_2 \Omega \times B_2 r_{0-CM}$$

$$+ 2B_2 \Omega \times B_2 v_{rel} + B_2 a_{rel} = \begin{bmatrix} l_{oc} \left( \dot{l}_{oc}^2 \sin \beta \cos \beta + \dot{\beta} \right) \\ l_{oc} \left( 2 \beta \dot{l}_{oc} \sin \beta - \dot{\Gamma} \cos \beta \right) \\ -l_{oc} \left( \dot{l}_{oc}^2 \cos \beta^2 + \dot{\beta}^2 \right) \end{bmatrix}$$

(7)

Fig. 1(II) depicts the forces acting upon the shaft. These forces are expressed in the initial reference frames as:

1. The restoring forces.
   - The gravity force.
     $$F_G = \begin{bmatrix} 0 \\ -Mg \\ 0 \end{bmatrix}^T$$
     (8)
The stiffness forces originating from the permanent magnetic house.

\[
\mathbf{F}_{\text{mag}} = \begin{bmatrix} -F_{\text{mag},x} & -F_{\text{mag},y} & 0 \end{bmatrix}^T = \begin{bmatrix} -K_{m,x} \cdot l_0 \beta & -K_{m,y} \cdot l_0 \Gamma & 0 \end{bmatrix}^T
\]  

(9)

The external viscous damping originating from the conical movement of the shaft and the slightly damping coming from the mounting procedure. Two equivalent viscous damping coefficients are created and included in the modelling in such a way that they affect only the lateral motion of the shaft, i.e. \( D_{m,i} = 2 \zeta \sqrt{K_{m,i} T} \), \( i = x, y \).

\[
\mathbf{F}_{\text{damp}} = \begin{bmatrix} -F_{\text{damp},x} & -F_{\text{damp},y} & 0 \end{bmatrix}^T = \begin{bmatrix} -D_{m,x} \cdot l_0 \beta & -D_{m,y} \cdot l_0 \Gamma & 0 \end{bmatrix}^T
\]  

(10)

2. The excitation forces.

The unbalance forces originating from the unbalanced mass in the disk.

\[
\mathbf{F}_{\text{unb}} = \begin{bmatrix} -F_{\text{unb},x} & -F_{\text{unb},y} & 0 \end{bmatrix} = \begin{bmatrix} M \epsilon \dot{\theta} \cos(\theta t + \varphi) \\
M \epsilon \dot{\theta} \sin(\theta t + \varphi) \\
0 \end{bmatrix}
\]  

(11)

The contact forces between the shaft and the backup bearing.

\[
\mathbf{F}_{\text{cont}} = \begin{bmatrix} -F_{\text{cont},x} & -F_{\text{cont},y} & 0 \end{bmatrix} = \begin{bmatrix} F_n (-\cos \alpha + \mu \sin \alpha \cdot \text{sign} \dot{\theta}) \\
F_n (-\sin \alpha - \mu \cos \alpha \cdot \text{sign} \dot{\theta}) \\
0 \end{bmatrix}
\]  

(12)

The contact angle is taken as the relative contact angle between the shaft and the inner and outer house, respectively, \( x_{ih}, y_{ih} \) and \( x_{oh}, y_{oh} \), thus:

\[
\tan \alpha = \frac{l_{0A} \beta - x_{ih} - x_{oh} - x_0}{l_{0A} \Gamma - y_{ih} - y_{oh} - y_0}
\]  

(13)

Where \( x_0 \) and \( y_0 \) are the offset of the inner house. The shaft encounters the backup bearing when \( r \geq r_0 \) where

\[
r = \sqrt{(l_{0A} \beta - x_{ih} - x_{oh} - x_0)^2 + (l_{0A} \Gamma - y_{ih} - y_{oh} - y_0)^2}
\]
• The impulse force exerted by the impact-pendulum.

\[ \mathbf{F}_{imp} = \begin{bmatrix} -F_{imp,x} & -F_{imp,y} & 0 \end{bmatrix}^T \]  

(14)

The position vectors from the base point 0 to each force components are given to:

- contact forces: \( \mathbf{l}_{a4} = \begin{bmatrix} 0 & 0 & l_{a4} \end{bmatrix}^T \)
- magnetic forces: \( \mathbf{l}_{aB} = \begin{bmatrix} 0 & 0 & l_{aB} \end{bmatrix}^T \)
- gravity force: \( \mathbf{l}_{aC} = \begin{bmatrix} 0 & 0 & l_{aC} \end{bmatrix}^T \)
- disk position: \( \mathbf{l}_0 = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T \)

Euler’s equation is introduced to derive the equation of motion of the shaft. The equation of motion is referred to the base point 0 and represented by the two rotations, \( \Gamma \) and \( \beta \), respectively, and by the spin \( \theta \), thus:

\[
\sum_{b2} M_0 = \mathbf{b}_2 \frac{d}{dt} \left( \mathbf{b}_2 \mathbf{\omega} \right) + \mathbf{b}_2 \mathbf{\Omega} \times \left( \mathbf{b}_2 \mathbf{l}_0 \cdot \mathbf{\omega} \right) + \mathbf{b}_2 \mathbf{r}_{0\cdot CM} \times M_{b2} \mathbf{a}_0 
\]  

(15)

The sum of moments are transformed to the moving reference frame where the tensor of inertia referred to the base point 0 is constant and written as:

\[
\mathbf{b}_2 \mathbf{I}_0 = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} 
\]  

(16)

Finally, the non-linear equations of motion describing the movement of the shaft is derived by solving Eq. (15), thus:

- \( \Gamma \) direction

\[
I_{xx} \left( \ddot{\Gamma} \cos \beta - \dot{\beta} \ddot{\Gamma} \sin \beta \right) + I_{xx} \dot{\beta} \left( \ddot{\Gamma} \sin \beta + \dot{\theta} \right) - I_{y2} \ddot{\beta} \sin \beta - l_{qB} \cos \Gamma F_{mag,y} - l_{qA} \cos \Gamma F_{imp,y} - l_{qA} \cos \Gamma F_{cont,y} - l_{qC} \cos \Gamma F_{unb,y} - l_{qC} \cos \Gamma \cdot M \cdot g 
\]

(17)

- \( \beta \) direction

\[
I_{yy} \ddot{\beta} + I_{yy} \Gamma^2 \sin \beta \cos \beta - I_{zz} \ddot{\Gamma} \cos \beta \left( \ddot{\Gamma} \sin \beta + \dot{\theta} \right) - l_{qB} \left( - \cos \beta F_{mag,x} - \sin \beta \sin \Gamma F_{mag,y} \right) - l_{qA} \left( - \cos \beta \gamma F_{imp,x} - \sin \beta \sin \Gamma F_{imp,y} \right) - l_{qA} \left( - \cos \beta \gamma F_{cont,x} - \sin \beta \sin \Gamma F_{cont,y} \right) - l_{qB} \left( - \cos \beta \gamma F_{damp,x} - \sin \beta \sin \Gamma F_{damp,y} \right) + l_{qC} \sin \beta \sin \Gamma \cdot M \cdot g 
\]

(18)
\[ I_{\varphi} (\ddot{\theta} + \dot{\Gamma} \sin\beta + \dot{\beta} \dot{\Gamma} \cos\beta) + I_{\gamma} \dot{\Gamma} \dot{\beta} \cos\beta - I_{\gamma} \dot{\beta} \dot{\Gamma} \cos\beta + T \] (19)

The reaction forces are derived by employing Newton’s equation and are represented in the inertial reference frame as:

\[ \sum F = M a = N + b_2 F_{mag} + b_2 F_{damp} + b_2 F_{db} + b_2 F_{imp} + b_2 F_{cont} \] (20)

Substituting the above calculated terms into Newton’s equation, one gets:

\[ \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} = T^T \begin{bmatrix} l_{oc} (\dot{\Gamma}^2 \sin\beta \cos\beta + \ddot{\beta}) \\ l_{oc} (2 \dot{\beta} \dot{\Gamma} \sin\beta - \ddot{\Gamma} \cos\beta) \\ -l_{oc} (\dot{\Gamma}^2 \cos\beta^2 + \ddot{\beta}) \end{bmatrix} \] (21)

2.2. The Backup Bearing House Dynamics

Figure 2 depicts the backup bearing houses and the mechanical model. The vibrations of the two houses are coupled through the beams and force transducers supporting the houses in the two lateral directions. The compression forces coming from the force transducers in WEST, EAST, NORT and SOUTH directions are designated \( P_W \), \( P_E \), \( P_N \) and \( P_S \), respectively. Moreover, it is only the damping originating from the bending of the supporting beams that is considered in the mathematical modelling. At contact and impact events the equation of motion of the 2 d.o.f system for each house in the two lateral directions \( x \) and \( y \), respectively are written as:

1. The equation of motion of the inner house.
   - \( x \)-direction.
     \[ M_{ih} \ddot{x}_{ih} + D_{bv} \left( \dot{x}_{ih} - \dot{x}_{oh} \right) + K_{bv} \left( x_{ih} - x_{oh} \right) = -F_{cont,x} \] (22)
     where \( \dot{x}_{ih}, \dot{x}_{oh} \) and \( x_{ih}, x_{oh} \) designate the velocities and displacements of the inner and outer house in the horizontal direction, respectively. \( K_{bv} \) and \( D_{bv} \) designate the bending stiffness and the damping of the vertical beams in the bearing house.
   - \( y \)-direction.
Figure 2: (I): The impact housing, where ⊙ is the support house, ⋄ is the outer house and ⌜ is the inner house and □ is the backup bearing, (II): mechanical model of the backup bearing house and coordinates used to describe the movements of the backup bearing houses

\[ M_{sh} \ddot{y}_{sh} + 2 \cdot K_{ft} (y_{sh} - y_{oh}) - P_N + P_S + M_{sh} \cdot g = -F_{cont,y} \]  
\[ (23) \]

where \( \dot{y}_{sh}, \dot{y}_{oh}, y_{sh}, y_{oh} \) designate the velocities and displacements of the inner and outer house in the vertical direction, respectively. \( K_{ft} \) designates the stiffness of the force transducers.

2. The equation of motion of the outer house.
   - \( x \)-direction.
     \[ (M_{oh} + M_{sh}) \ddot{x}_{oh} + 2 \cdot K_{ft} \dot{x}_{oh} + D_{bh} (\dot{x}_{sh} - \dot{x}_{oh}) + K_{bh} (x_{oh} - x_{sh}) + P_W - P_E = 0 \]
     \[ (24) \]
   - \( y \)-direction.
     \[ (M_{oh} + M_{sh}) \ddot{y}_{oh} + 2 \cdot K_{ft} (y_{oh} - y_{sh}) + D_{bh} \dot{y}_{oh} + K_{bh} y_{oh} + (M_{oh} + M_{sh}) \cdot g = 0 \]
     \[ (25) \]

where \( K_{bh} \) and \( D_{bh} \) designate the bending stiffness and the damping of the horizontal beams in the bearing house.
The magnetic restoring forces are determined by applying magnetism. Afterwards, the remaining forces such as the damping forces, the unbalance forces and the impulse forces are discussed.

3. Passive Magnetic Bearing - Modelling the Magnetic Forces $F_{mag,x}$ and $F_{mag,y}$

The idea of employing the permanent magnetic house, is to use a contact free and obtain a more flexible bearing. In that sense, the shaft is levitated inside the bearing house. The magnetic bearing configuration used in the experimental work is depicted in Fig. 3.

![Figure 3: Picture of the magnetic bearing house](image)

3.1. B-field Acting on a Passive Bar Magnet

The strength of the magnetic field is given as the magnetic flux density, referred to as the B-field. For a permanent magnet with uniform magnetization, the current of the electrons inside the magnet have the same orientation of spin. Consequently, cancels out everywhere and leaving no net current inside the magnet. Hence, the electric current, $I_c$, runs as a loop on the surface and wrapping the bar magnet. An expression of the B-field at any given point, $P$, in space from the steady current
carrying loop as depicted in Fig. 4(I) is derived in the following. The B-field is determined by use of the Biot Savart law. This law relates the magnetic field to the current which is its source and expressed as:

\[
d\vec{B} = \frac{\mu_0}{4\pi} \left( \frac{I_C \cdot dl' \times R_i}{R^2} \right)
\]

Figure 4(I) graphically illustrates the terms employed in the Biot Savart law, where \(d\vec{B}\) is perpendicular to \(dl'\) and designates the magnetic field vector produced by an infinitesimal length of the current carrying loop \(I_C\). \(dl'\) designates a vector representing an infinitesimal length of the current carrying loop in the direction of the current. \(R\) designates the position vector given in spherical coordinates from the section of the loop to the point \(P\). \(R_i\) designates a unit vector given in spherical coordinates and specifies the direction of the vector \(R\) from the current to the field point. \(I_C\) designates the current in the loop. \(\psi\) designates the angle between \(dl'\) and \(R\). Finally, \(\mu_0\) designates the relative permeability of the surrounding space. The Biot Savart law is equivalent to the Amperes circuit law which states that for any closed loop path, the sum of the length elements times the magnetic field in
the direction of the length element is equal to the permeability times the electric current enclosed in the loop. The total B-field can then be written as:

\[
B = \mu_0 I_C \frac{1}{4\pi} \sum \left( \frac{dl' \times R_i}{R^2} \right) = \mu_0 I_C \frac{1}{4\pi} \oint_c \left( \frac{dl' \times R_i}{R^2} \right)
\]  

(27)

According to Cheng [16], Eq. (27) can be written in spherical coordinates as:

\[
B = \mu_0 I_C \frac{1}{4R^3} (a_R 2\cos\psi + a_\phi \sin\psi)
\]

(28)

where \(a_R\) is a unit vector designating the position vector from the center of the loop to the point \(P\). \(a_\phi\) designates the angle between the loop and \(a_R\). The Cartesian coordinates are therefore transformed to spherical coordinates by means of different moving reference frames. After two consecutive rotations the unit vectors are written as:

\[
\begin{align*}
    a_R &= a_x \sin\psi \cos\phi + a_y \sin\psi \sin\phi + a_z \cos\psi \\
    a_\delta &= a_y \cos\phi - a_x \sin\phi \\
    a_\phi &= a_x \cos\psi \cos\phi + a_y \cos\psi \sin\phi - a_z \sin\psi
\end{align*}
\]

(29)

where \(a_x\), \(a_y\) and \(a_z\) are corresponding unit vectors, respectively.

3.2. The Total Current \(I_C\) Acting on a Bar Magnet

The strength of a permanent bar magnet is given by the magnetization vector \(M\), as the sum of the magnetic dipole moment \(m_k\) of each atom per unit volume \(v\), thus;

\[
M = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} m_k}{\Delta v}
\]

(30)

where \(n\) is the atoms per unit volume. The surface current density is written as:

\[
I_{ms} = M \times a_n
\]

(31)

where \(a_n\) is a unit outward normal vector from \(dl'\). Fig. 4(II) depicts a permanent bar magnet with radius \(b\), length \(L\) and with a differential increment in the z-direction, positioned in a Cartesian coordinate system. The axis of the cylinder coincides with the z-axis. The axial magnetization can be expressed as: \(M = \)
\( \mathbf{a}_z \cdot M_0 \), where \( M_0 \) expresses the magnitude of the magnetization vector, it is a material constant. By rewriting Eq. (31) to cylindrical coordinates, one gets:

\[
I_{ms} = (\mathbf{a}_z M_0) \times \mathbf{a}_r = M_0 \mathbf{a}_\phi
\]  

(32)

An expression for the surface current of the differential element \( dz' \) is expressed as:

\[
dI_{ms} = M_0 \mathbf{a}_\phi dz'
\]  

(33)

Hence, the total current density is determined by integrating over the entire length of the bar magnet:

\[
I_{ms} = \int_0^L a_\phi M_0 dz = a_\phi M_0 L
\]  

(34)

The equivalent current in the surface loop is now expressed as \( I_C = M_0 L \). Hence the B-field at any given point from the current carrying loop is expressed as:

\[
B = \mu_0 M_0 L b^2 \frac{(a_R 2 \cos \psi + a_\phi \sin \psi)}{4R^3}
\]  

(35)

Equation (35) is utilized for calculating the magnetic field from all magnets in the bearing house. The permanent bar magnets employed in the experimental work are cylindrical shaped sintered Neodym magnets. These magnets are magnetized in the axial direction. The magnetization constant \( M_0 \) for these types of magnets was determined and verified by experimentally in the work of Lahriri [17] and determined to \( M_0 = 705000 \) A/m.

3.3. Magnetic Forces Acting on a Bar Magnet

The charged particles in the permanent bar magnet run in a plane perpendicular to the B-field, and move in a circular orbit with the magnetic forces playing the role of centripetal forces. If a small element, \( dl' \), of the current carrying loop with cross section, \( S \), is considered, the element consist of \( N \) electrons per unit volume running with the velocity \( \mathbf{v} \) in the direction of \( dl' \). In this way, the magnetic force exerted by this element is given by the Lorentz force law as:

\[
dF_m = -N \cdot q \cdot S \cdot \mathbf{v} \cdot dl' \times \mathbf{B}
\]  

(36)

It appears that \(-N \cdot q \cdot S \cdot \mathbf{v}\) is equivalent to the current \( I_C \), thus Eq. (36) is rewritten to:

\[
dF_m = I_C \cdot dl' \times \mathbf{B}
\]  

(37)
Table 1: Parameters employed in the numerical approach

<table>
<thead>
<tr>
<th>Properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_b ) Radius of bearing magnets</td>
<td>0.006 [m]</td>
</tr>
<tr>
<td>( r_s ) Radius of shaft magnet</td>
<td>0.008 [m]</td>
</tr>
<tr>
<td>( R_b ) Radius of the bearing</td>
<td>0.025 [m]</td>
</tr>
<tr>
<td>( \mu_0 ) Rel. permeability of vacuum</td>
<td>(4 \cdot \pi \cdot 10^{-7} [\text{Tm/A}])</td>
</tr>
<tr>
<td>( M_0 ) Magnetization constant</td>
<td>705000 [A/m]</td>
</tr>
</tbody>
</table>

The resulting force acting on a current carrying loop when interacting with other current carrying loops is given to:

\[
F_m = \begin{bmatrix}
-F_{\text{mag},x} \\
-F_{\text{mag},y} \\
0
\end{bmatrix} = I_C \oint_C dl' \times B = I_C \sum dl' \times B 
\]

(38)

The magnetic force is perpendicular to the B-field. This resultant force equation is employed in the calculation of the restoring magnetic forces acting on the shaft magnet.

3.4. Numerical Implementation

Figure 5 depicts the magnetic configuration at the bearing house and for the shaft magnet. The magnets are positioned and aligned in such way that the interacting current carrying loops are coplanar, generating solely repelling forces with lateral components in the (x,y) plane. The shaft magnet is positioned in the origin of a global coordinate system and local coordinate systems are positioned in the origin of the interacting bearing magnets. The B-field vector is evaluated at different points on the surface loop on the shaft magnet. Consequently, a resulting magnetic force acting on the shaft magnet originating from all the interacting bearing magnet contributions to the B-field is calculated. For every gradual movement of the shaft inside the magnetic field, magnetic restoring force components are calculated. Figure 6 depicts the shaft acting inside the magnetic field. The parameters employed in the numerical derivation of the magnetic forces are given in Table 1. Figure 7 depicts the magnetic forces calculated for the movement of the shaft inside the magnetic field grid.
4. The Remaining Forces

The external viscous damping factors for the shaft damping were determined by experiments, and estimated by the logarithmic decay of the transient responses to; $\zeta_x = 0.016$ and $\zeta_y = 0.035$, respectively. The eccentricity of the unbalance mass in
the disk is estimated approximately to \( \epsilon = 1e - 4 \text{ m} \) and the phase to \( \varphi = 0.21 \text{ rad} \).

The impulse forces are taken from the pendulum-hammer during the experiments.

5. Implementation of the Global Nonlinear Model

The system is assumed to belong to the discontinuous Filippov system. The numerical implementation of such contact systems and the smoothening procedure is inspired by the work of Leine [18]. The numerical integration is based on the ode15s stiff solver in MATLAB. Event functions have been implemented in order to switch between the different cases of contact and no contact. The transition through the discontinuity surfaces, for example at impacts and at the switching between different vector fields of the system, is detected by use of event functions. These transitions are triggered by zero crossings of scalar valued functions. The event functions govern the transition from one state to another. The procedure is outlined and summarized in the flowchart depicted in Fig. 8.

6. Theoretical Results

In this section the numerical modelling of the system is compared to the experimental results. The backup bearing consists of an annular guide in this present work. Presentation and comparison of the results are facilitated by presenting the following plots:

1. Trajectory of the lateral motion of the shaft within the annular guide.
2. Time series of the shaft motion in the two lateral directions, $x$ and $y$ respectively.
3. The angular velocity of the shaft plotted against time.
4. Contact forces plotted against time.

The behavior of the shaft as it impacts the backup bearing is simulated and discussed. The responses of the shaft are transformed to lateral motion in the $x$
Table 2: Parameters employed in the numerical analyses

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Position</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_s$</td>
<td>$7.0 \cdot 10^{10}$ [N/m$^2$]</td>
<td>$l_{0A}$</td>
</tr>
<tr>
<td>$v_s$</td>
<td>0.33 [-]</td>
<td>$l_{0B}$</td>
</tr>
<tr>
<td>$E_B$</td>
<td>$12.5 \cdot 10^{10}$ [N/m$^2$]</td>
<td>$l_{0C}$</td>
</tr>
<tr>
<td>$v_B$</td>
<td>0.33 [-]</td>
<td>$l_{0I}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bearing House</th>
<th>Shaft</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{ih}$</td>
<td>1.70 [kg]</td>
</tr>
<tr>
<td>$M_{oh}$</td>
<td>8.88 [kg]</td>
</tr>
<tr>
<td>$K_{bv}$</td>
<td>$1.23 \cdot 10^7$ [N/m]</td>
</tr>
<tr>
<td>$K_{bh}$</td>
<td>$2.4 \cdot 10^7$ [N/m]</td>
</tr>
<tr>
<td>$K_{ft}$</td>
<td>$8.88 \cdot 10^7$ [N/m]</td>
</tr>
<tr>
<td>$D_{bh}$</td>
<td>1576 [Ns/m]</td>
</tr>
</tbody>
</table>

and $y$ directions positioned at the displacement proximitors’ location in order to compare the numerical responses with the experimental results. The parameters employed in the numerical analyses are listed in Table 2. The stiffness and mass parameters of the housing were determined in the work of Petersen [19]. The equivalent viscous damping coefficients coming from these beams were estimated through experiments, where the inner and outer houses were displayed and the transient responses were measured. In this way, the equivalent viscous damping coefficients are estimated by the logarithmic decay of the transient responses. For the inner house the damping factor in the horizontal direction was estimated approximately to $\zeta_x = 0.026$, thus $D_{bv} \approx 2\zeta_x \sqrt{K_{bv} M_{ih}} \approx 238$ Ns/m. For the outer house the damping factor for the vertical direction was estimated approximately to $\zeta_y = 0.054$, thus $D_{bh} \approx 2\zeta_y \sqrt{K_{bh} M_{oh}} \approx 1576$ Ns/m. The stiffness properties, $K_{ft}$, of the force transducers are given by the force transducer manufacturer.

In the numerical modelling constraints are imposed to govern the motion of the shaft as it impacts and whirls on the full extent of the inner surface of the annular guide. The constraints govern whether the shaft performs rubbing or dry full annular whirling. The transition to the backward full annular whirling state must satisfy the condition of rolling, i.e. the relative velocity $v_{rel}$ at the point of contact must diminish to zero. The relative sliding contact velocity can be expressed as:

$$v_{rel,s}(t) = v_{rel,t}(t) + R_s \dot{\theta}(t)$$  \hspace{1cm} (39)
Where the tangential velocity, \( v_{rel,t} \), is expressed as the relative displacement vector and the relative velocity vector between the shaft and the backup bearing. Thus,

\[
v_{rel,t}(t) = \frac{(x_s - x_{ih} - x_{oh}) (\dot{y}_s - \dot{y}_{ih} - \dot{y}_{oh}) - (y_s - y_{ih} - y_{oh}) (\dot{x}_s - \dot{x}_{ih} - \dot{x}_{oh})}{\sqrt{(x_s - x_{ih} - x_{oh})^2 + (y_s - y_{ih} - y_{oh})^2}}
\]  
\( (40) \)

The lateral motion and velocity of the shaft are designated, \( x_s, y_s \) and \( \dot{x}_s, \dot{y}_s \), respectively. When Eq. (39) becomes zero the shaft starts to roll on the inner surface of the annular ring with the whirling frequency given to:

\[
\omega_{wh} = \frac{\dot{\theta} R_y}{r_0}
\]
\( (41) \)

As pointed out by Bartha [7] and verified experimentally in this work, at this whirling state, no imbalance force is acting on the shaft. Consequently, the forces that are acting on the shaft in the radial direction is the centrifugal force \( F_c = Mr_0 \omega_{wh}^2 \), the restoring stiffness force \( F_{mag} \) and the damping force \( F_{damp} \). To maintain this continuous contact condition Bartha [7] states two conditions that must be satisfied. The first condition states that the outgoing centrifugal force must be higher than the ingoing restoring forces, thus:

\[
F_c > F_{mag} + F_{damp}
\]  
\( (42) \)

In addition, the first eigenfrequency of the shaft must be higher than the whirling frequency. The second condition states that as the frictional and damping forces point into the direction opposite to the whirling frequency, the shaft can therefore be slowed down and lose contact with the annular ring. The condition states that:

\[
\omega_{wh} \leq \omega_{wh,crit} = \sqrt{\frac{K_{mag}}{M}} \left( -\frac{D_m}{\mu} - \sqrt{1 + \left( \frac{D_m}{\mu} \right)^2} \right)
\]  
\( (43) \)

The negative sign indicates that the whirling frequency is opposite to that of the driving frequency of the shaft. Figure 9 depicts the forces acting upon contact. The impact location is modelled as a spring-damper element. \( F_n \) and \( F_{\mu} \) designate the normal and friction force respectively, \( R_{horz} \) and \( R_{vert} \) designate the reaction forces in the two lateral directions. The reaction forces simulate the forces acting upon the force transducers during the experiments and are influenced by the inertia
effects coming from the support housings. The contact force is model by use of the compliant model:

$$F = k\delta_n \left[ 1 + \frac{3}{4} \left( 1 - e^2 \right) \frac{\delta}{\delta^-} \right]$$  \hspace{1cm} (44)

$k$ denotes the local impact stiffness between the two colliding bodies and depends on the material properties, geometric properties and computed by using elastostatic theory. $\delta$ denotes the local indentation amount (for circular and elliptical surfaces $n$ is set to 3/2). $\delta^-$ denotes the initial impact velocity, and $e$ denotes Newton’s coefficient of restitution. This compliant contact force model is sensitive to the penetration velocity rate (the ingoing and outgoing velocities).

Figure 9: Subsystem, where contact forces, $F_n$ and $F_\mu$ and reaction forces, $R_{horz}$ and $R_{vert}$ act upon contact

6.1. Rotor Drop Against the Backup Bearing, $\Omega = 0$ Hz
This study case is conducted in order to compare the bouncing forces and the energy dissipation related to the coefficient of restitution. The magnetic support is
removed in this study case. The shaft is released from the topmost position of the bearing clearance. Figure 10 depicts the impact motion of the shaft. Shortly after the shaft is released, it impacts the annular guide where it bounces on the lower part before it settles to rest due to energy dissipation. The numerical results are in good agreement with the experimental results depicted in Fig. 11. Figure 10(III) shows the reaction forces simulated in the SOUTH direction. However, the simulated reaction forces are a little higher than the experimental results. The disagreement is attributable to the mobility of the inner house depicted in Fig. 10(IV) which amplifies the reaction forces. Yet, this study demonstrates that the energy dissipation must be related to the coefficient of restitution which is dependent on the ingoing and outgoing velocities.

![Figure 10: Numerical: (I) trajectory plot of the shaft motion, (II) time series y-motion, (III) reaction force in the SOUTH direction, (IV) acceleration of the inner-house](image)

Figure 10: Numerical: (I) trajectory plot of the shaft motion $\frac{R_0}{m} = 13.5$, (II) time series y-motion, (III) reaction force in the SOUTH direction, (IV) acceleration of the inner-house
6.2. Shaft Impacting the Annular Guide During Stable Run, $\Omega = 6.2$ Hz (372rpm)

In this section the motion of the shaft is studied together with the contact forces. The restitution coefficient is determined experimentally to approximately $e = 0.90$. The dry friction kinetic friction coefficient for brass sliding against aluminium is determined by a pin-on-disk experiment to approximately $\mu_k = 0.25$. The impulse force originating from the pendulum-hammer is also taken from the experimental part to 100 N. This impulse force is applied in the horizontal direction, $F_{imp,x}$. After some time, as the shaft operates in the normal operational condition, it is suddenly impacted with the pendulum-hammer. Figure 12 depicts the motion of the shaft. The trajectory plot of the center of the shaft together with

Figure 11: Experiment: (I) trajectory plot of the shaft motion for $R_S = 13.5$, (II) time series y-motion, (III) measured contact forces in SOUTH-direction, (IV) acceleration of the inner-house in the SOUTH-direction
the displacement plots are presented as the relative motion between the shaft and the annular guide. As in the experimental case, shortly after the shaft impacts the annular guide it undergoes to a full annular backward whirl state traversing the full extent of the clearance. In the initial state, the shaft runs with slipping on the inner surface. However, as the sliding velocity increases the frictional force causes an increase in the rotor’s tangential velocity in the direction opposite to its direction of rotation. At some state the opposite tangential velocity brings the sliding motion to stop and the shaft starts to roll. The numerical results are in good agreement with the experiments. The two-sided full spectrum plot depicted in Fig. 13(I) verifies the backward whirling state and shows good agreements with the experiments.

6.3. Impact Forces Between Shaft and Annular Guide, $\Omega = 6.2$ Hz (372 rpm)

Figures 14(I) and (II) depict the reaction forces computed in the two lateral directions. As the shaft impacts the annular guide before it settles to a full annular contact state, the contact forces appear very high. This is caused by the bouncing motion of the shaft. Since the compliant contact force model is governed by the penetration velocity rate, the shaft impact motion produces high contact forces. This demonstrates that the theoretical compliant models are very sensitive to the penetration ($\delta$) and velocity rate ($\dot{\delta}$ and $\dot{\delta}^-$. In the full annular whirling state, the forces settles to a constant value. Figures 15 shows a comparison between the experimental and numerical contact forces, in the horizontal direction. In overall, they show good agreements. The theoretical result is though little higher. This is due to the inertia effect coming from the stator. The inertia effect also induces the oscillating behavior. By increasing the stiffness and damping properties of the house supports in order to reduce the mobility has a double-sided effect. An increase in these properties causes an increase in the bouncing motion of the shaft. Consequently, the contact becomes more impulsive and massive.

6.4. Impact Forces Between Shaft and Annular Guide, $\Omega = 3.2$ Hz (192 rpm) - $6.2$ Hz (372 rpm)

The angular velocity of the shaft are taken to be; $\Omega = 3.2$ Hz, $\Omega = 4.2$ Hz, $\Omega = 5.2$ Hz and $\Omega = 6.2$ Hz. In each of these velocity cases the shaft is suddenly impacted with the pendulum-hammer. Shortly afterwards, the shaft enters a full annular backward whirling state. Figure 16 shows a comparison between the simulated and measured contact forces during the whirling state. As the speed increases the contact force response becomes more impulsive and the period of contact reduces.
Despite that the theoretical simulated forces appear higher, the model shows good agreements with the experimental results.

7. Conclusion

In this work the discontinuous governing equations of motion of the shaft and stator are derived by use of rigid body dynamics. The restoring forces are derived by use of magnetism and the impact forces are estimated by use of a compliant contact force model. The simulated motion of the shaft during impacts shows good agreements with experimental result. The theoretical compliant contact force model also shows very good agreements. Assuming the contact forces and penetrations to happen in a continuous manner during impacts and addressing the differentiable discontinuous equations of motion to belong to the Filippov system, are feasible in rotor to stator contact modelling. However, the theoretical computed contact forces appear a little higher in magnitude than the measured forces. This is due to the inertia effect coming from the support houses, and the formulation of the force model. The model is governed by the material properties of the two bodies and the penetration rate and velocities. It appears that by reducing the mobility of the support houses has a double-sided effect. This will increase the bouncing motion of the shaft together with the ingoing and outgoing velocities and create massive contact forces. This makes these models sensitive to the penetration velocity rates. Yet, this compliant model is simple to implement in the numerical integration procedure, and the results seem to be in good agreements with the experiments.
References


Figure 12: Numerical: (I) trajectories of the center of the disk within the bearing clearance, (II) angular velocity of the shaft, (III) time series x-motion, (IV) time series y-motion
Figure 13: Waterfall full-spectrum FFT plots, where frequency is plotted against time. After the impact the shaft whirls with the frequency of 83.7 Hz governed by the radius to clearance ratio $\frac{R_s}{r_0} = 13.5$, (I) Numerical, (II) Experimental.

Figure 14: Numerical: (I) reaction forces horizontal direction, (II) reaction forces vertical direction.
Figure 15: Experimental Vs. Numerical results at full annular whirling state: (I) contact forces in the EAST-direction

Figure 16: Contact force versus time for various speeds in EAST-direction for (a) 3.2 Hz, (b) 4.2 Hz, (c) 5.2 Hz and (d) 6.2 Hz, (I) numerical, (II) experiments
Appendix E

[P5] On the nonlinear design of industrial arc spring dampers

On the nonlinear design of industrial arc spring dampers

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Abstract
The objective of this paper is to present a numerical approach for analyzing parameter excited vibrations on a gas compressor, induced by the nonlinear characteristic of the arc spring feature of certain designs of squeeze film dampers, SFDs. The behavior of the journal is studied in preparation for varying the damping characteristics of the SFD as well as the dynamic forces acting on the SFD. Phase plane orbits together with Poincaré maps are given for different arc spring damping and static and dynamic load cases. Besides, bifurcation diagrams as a function of the arc spring damping and forces acting on the SFD are presented. It is worth mentioning, that the maps and diagrams can be used as design guidance.

1 Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_g$</td>
<td>Center of gravity of the compressor</td>
</tr>
<tr>
<td>$D_1$</td>
<td>Damping characteristics at bearing 1 [Ns/m]</td>
</tr>
<tr>
<td>$D_2$</td>
<td>Damping characteristics at bearing 2 [Ns/m]</td>
</tr>
<tr>
<td>$F_x$</td>
<td>Static and dynamic forces acting in the vertical direction [N]</td>
</tr>
<tr>
<td>$F_y$</td>
<td>Static and dynamic forces acting in the horizontal direction [N]</td>
</tr>
<tr>
<td>$f_{dyn1}$</td>
<td>Aerodynamic force acting on bearing 1 in the vertical direction [N]</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Polar mass moment of inertia of the journal [kgm$^2$]</td>
</tr>
<tr>
<td>$I_t$</td>
<td>Transverse mass moment of inertia of the journal [kgm$^2$]</td>
</tr>
<tr>
<td>$K_1$</td>
<td>Stiffness characteristics at bearing 1 [N/m]</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Stiffness characteristics at bearing 2 [N/m]</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of the compressor [m]</td>
</tr>
<tr>
<td>$L_1$</td>
<td>Length from $C_g$ to bearing 1 [m]</td>
</tr>
<tr>
<td>$L_2$</td>
<td>Length from $C_g$ to bearing 2 [m]</td>
</tr>
<tr>
<td>$M_t$</td>
<td>Mass of the compressor [kg]</td>
</tr>
<tr>
<td>$M_x$</td>
<td>Bending moment acting around the vertical direction [Nm]</td>
</tr>
<tr>
<td>$M_y$</td>
<td>Bending moment acting around the horizontal direction [Nm]</td>
</tr>
<tr>
<td>$T$</td>
<td>Time [s]</td>
</tr>
<tr>
<td>$x_1$</td>
<td>Vertical position of the journal at bearing 1 [m]</td>
</tr>
<tr>
<td>$x_2$</td>
<td>Vertical position of the journal at bearing 2 [m]</td>
</tr>
<tr>
<td>$y_1$</td>
<td>Horizontal position of the journal at bearing 1 [m]</td>
</tr>
<tr>
<td>$y_2$</td>
<td>Horizontal position of the journal at bearing 2 [m]</td>
</tr>
<tr>
<td>$\theta_x$</td>
<td>Angular position of the compressor at $C_g$ measured in the x-direction [rad]</td>
</tr>
<tr>
<td>$\theta_y$</td>
<td>Angular position of the compressor at $C_g$ measured in the y-direction [rad]</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Angular velocity of the compressor [Hz]</td>
</tr>
</tbody>
</table>

2 Introduction
Squeeze film damper bearings are widely used in modern turbo machinery of high power density. Pietra and Adiletta [1] give a comprehensive review of the development of the SFD. The SFDs generate their force capa-
bility in reaction to dynamic journal motions, squeezing a thin film where the dominant mechanism is damping from shear dissipation due to the surface squeeze. The hydrodynamic reaction forces are conventionally linearized as equivalent stiffness, damping, and inertia. Diaz and San Andrés [2] provide a comprehensive review of the state of the art in prediction and experimental identification of these linear dynamic coefficients for SFD bearings. San Andrés and Santiago [3] experimentally identified damping force coefficients, for large orbital motion, which agreed well with the prediction based on the short length bearing model only if an effective damper length is used. Furthermore, their measurements of film pressures revealed an early onset of air ingestion. A lot of the theoretical attention has been paid to the centered synchronous, circular rotor motion which only exists when the damper is statically centered. However, preloaded and eccentric damper operation where the journal is statically misaligned from the bearing housing aggravates non-linear effects. These effects result in non-circular orbits, jump-phenomena and increase the likelihood of non-synchronous vibrations. Nikolajsen and Holmes [4] reported observations of non-synchronous vibrations in a test rig consisting of a flexible symmetric rotor supported by journal bearings in series with SFDs with retainer springs. The non-synchronous vibrations occurred at speeds between two and three times the first critical speed. Zhao et al. [5] showed that jump phenomena, subharmonic and quasi-periodic motion are possible in concentric damper motions. They utilized a numerical integration scheme to predict the trajectories, calculate the Poincaré map and power spectra. Different techniques and analyses tools have successfully been employed in the non-linear analyses. Zhao et al. [6] studied how the unbalanced response of a rigid rotor, supported on an eccentric squeeze film damper, was approximated by a harmonic series whose coefficients were determined by the collocation method. Their study revealed that for large values of the unbalance and static misalignment the subharmonic and quasi-periodic motions at above twice the critical speed were bifurcated from the unstable harmonic solution. Furthermore, Sundararajan and Noah [7] exemplify how the shooting and arc-length continuation method can be utilized to locate the periodic responses, determining their stability and describing the bifurcations as a parameter is varied in the SFD system. P. Bonello et al. [8] propose a receptance harmonic balance technique for the determination of the steady state periodic response. This technique showed to be versatile and tractable for large order systems. Inayat Hussain [9] numerically investigates the effects on the bifurcations of a flexible rotor in squeeze film dampers with retainer springs, by varying gravity parameter, mass ratio, and stiffness ratio. One particular design used in large centrifugal compressors comprises a flexible curved beam supporting the journal of the SFD. While this curved beam in itself will behave as a linear spring with constant stiffness, various clearances and allowances in the assembly will cause the support conditions of the beam to change as function of deflection. As a result of this behavior the arc spring force becomes piecewise linear and the stiffness changes as a function of the displacement of the journal. This system is classified as strongly nonlinear. The present paper deals with the nonlinear analyses of the motion of a journal supported by this type of arc spring.

3 SFD Bearing Design, Test and Data

Fig. 1(a) depicts a sketch of the considered SFD bearing design. The oil for the SFD bearing is supplied through a central circumferential groove in the outer cage. The SFD bearing is fitted with two O-rings to prevent side leakage of the oil. This type of SFD bearings designs have been used successfully in series with conventional tilting pad bearings on centrifugal compressors in the oil and gas industry. The O-rings are in radial contact

![Figure 1](image)

(a) Cross-section of the SFD bearing, (b) Arc spring beam, (c) Illustration of the stiffness behavior

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between inner cage and outer housing, and therefore also contribute significantly to the impedance of the SFD bearing. The characteristic of the O-rings was studied by Smalley [10] which revealed that the O-rings contribute with dynamic stiffness $K_{OR}$ and damping $B_{OR}$. These are functions of material, vibration frequency and amplitude, temperature, initial squeeze, stretch and cross-section diameter relative to O-ring groove cross section. Consequently, the impedance of the O-rings may differ in vertical and horizontal direction. For heavy rotors a mechanical spring $K_{CS}$ is sometimes used to center the damper ring in the damper housing. In this study, the inner cage is centered inside the outer housing by means of an arc-spring. The arc-spring is a $\pi$-arc beam supported by "hooks" at both ends, see Fig. 1(b). This arc spring contributes with vertical stiffness. General knowledge about the arc spring behavior and the static push test conducted in this study revealed that the arc-spring has a non-linear behavior. Fig. 1(c) depicts an illustration of the behavior of the arc-spring. For modest loading and thereby modest vertical deflection the arc-spring is not exploited to the full, thus the stiffness tends to be lower and originates both from the bending stiffness of the arc-spring and the squeeze of the O-rings. However, as the loading is increased the radial contraction increases until the hooks encounter the supports and cause the stiffness to increase. Hence, the behavior of the stiffness in this present paper is considered as piecewise linear.

A test rig is constructed in order to measure both the static and dynamic coefficients of the industrial SFD bearings, see Fig. 2. Utilizing this test rig the stiffness of the arc-spring is investigated as a function of eccentricity, the static load level, various O-ring material, etc. This test rig was also employed in the work of Lund et al. [11].

![Figure 2](image-url)

**Figure 2**: (a) Cross-section of test damper bearing, (b) Front view of test damper bearing, (c) SFD test rig

### 3.1 Arc-Spring Stiffness Coefficients-Fem Model and Push Test

The whole raison d’être for the arc spring is to counteract the static weight of the rotating assembly in order to ensure that the SFD is centred during operation of the machine. In the laboratory, the static load is simulated by means of a yoke from the inner cage to the supporting structure. This loading arrangement will in effect act in parallel with the arc spring, so in order to minimize the influence the load is applied via a stack of disc springs. The applied static load is measured by means of a (strain gauge based) load cell and the deflection by means of proximity probes measuring the relative deflection between outer hosing and inner cage. The tests are conducted by static loading a 33 MN/m rated arc-spring mounted together with two Shore 75 O-rings. The configuration for the bearing housing is given in Table 1.

Fig. 3(a) shows the results of the damper bearing vertical displacement of the static push tests. Given that the bearing is preloaded it is not possible to identify the stiffness changes since the hooks of the arc-spring are encountering the support and causing the arc spring to operate with “maximum” stiffness. For this reason, the radial clearance of approximately 100 $\mu$m is inadequate to unload the hooks at the support and thereby reduce the stiffness as the bearing is unloaded. Therefore, it is not possible to identify the drastical change of the stiffness (the fold of the stiffness curve) and capture the nonlinear behavior. An FEM model is conducted in order to identify the nonlinear behavior of the stiffness, investigate the stiffness changes and validate the model.
Figure 3: (a) Static load vs displacement curves obtained experimentally, (b) Deflection vs static load obtained by the FEM model, (c) Piecewise linear stiffness curve

<table>
<thead>
<tr>
<th>Description</th>
<th>[mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of bearing housing</td>
<td>216.16</td>
</tr>
<tr>
<td>Diameter of inner cage</td>
<td>215.91</td>
</tr>
<tr>
<td>Radial bearing clearance</td>
<td>0.127</td>
</tr>
<tr>
<td>Axial length of damper land</td>
<td>23.495</td>
</tr>
<tr>
<td>O-ring hardness</td>
<td>Shore 75</td>
</tr>
<tr>
<td>Oil groove, housing (width $\times$ depth)</td>
<td>28.7 $\times$ 15.3</td>
</tr>
<tr>
<td>Oil groove, inner ring (width $\times$ depth)</td>
<td>19.85 $\times$ 12.7</td>
</tr>
</tbody>
</table>

**Table 1:** Measured test damper dimensions.

An FEM model comprising arc spring and housing is set up and takes various clearances into account by means of simple node to node contact elements. Fig. 3(b) depicts the results from the FEM analysis. In this approach it is possible to identify the changes in the stiffness curve as a function of deflection. Furthermore, good agreement between the experimental results and the FEM model is obtained.

Table 2 shows the findings of the vertical static stiffness of the arc-spring together with the O-rings.

<table>
<thead>
<tr>
<th>Element</th>
<th>Mean static stiffness [N/M]</th>
</tr>
</thead>
<tbody>
<tr>
<td>33MN/m rated arc-spring</td>
<td>30.7</td>
</tr>
<tr>
<td>Shore 75 O-ring at 22 C (stiffness per O-ring)</td>
<td>1.2</td>
</tr>
</tbody>
</table>

**Table 2:** Mean static stiffness for load range 100 kg to 500 kg.

It is noted that the contribution to the static stiffness from the O-rings is considerable. Utilizing the findings of the stiffness behavior a non-linear stiffness characteristic in the range of the operational clearance of the bearing housing is given in Fig. 3(c). The static force acting on the bearing due to gravity will force the journal to operate at an equilibrium position where the arc-spring beam is exploited to the full. In case of exiting an aerodynamic force acting on the bearing the equilibrium point could be change leading to a change in the stiffness. A further increase of the aerodynamic force will cause the rotor to operate connected to the pedestal, where the stiffness changes drastically from 0 to approximately $1 \cdot 10^8 \text{ N/m}$. 

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4 Rigid Rotor on TPJB & Arc-Spring Dampers-Global Model

A rigid rotor, representing a gas compressor, supported by SFD’s at each end is considered, see Fig. 4. The rigid rotor is mounted on two identical journal bearings supported on squeeze film dampers which are supported by arc-springs. The governing equation of motion of the journal related to the the bearings is written as:

\[
T^{-1}MT\ddot{q}_{cg} + T^{-1}GT\dot{q}_{cg} + T^{-1}DT\dot{q}_{cg} + T^{-1}KTq_{cg} = T^{-1}F
\]

or written as,

\[
M\ddot{q}_b + G\dot{q}_b + D\dot{q}_b + Kq_b = F_b
\]

where,

\[
M = \begin{bmatrix}
M & 0 & 0 & 0 \\
0 & M & 0 & 0 \\
0 & 0 & I_t & 0 \\
0 & 0 & 0 & I_t
\end{bmatrix},
G = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -I_p\Omega \\
0 & 0 & 0
\end{bmatrix},
T = \begin{bmatrix}
L_2 & L & L_1 & 0 \\
0 & L_2 & L_1 & L_0 \\
0 & 0 & L_2 & L_1 \\
0 & 0 & L_1 & L_0
\end{bmatrix},
D = \begin{bmatrix}
D_1 + D_2 & 0 & D_1L_1 - D_2L_2 & 0 \\
0 & D_1 + D_2 & 0 & D_1L_1 - D_2L_2 \\
D_1L_1 - D_2L_2 & 0 & D_2L_1 + D_2L_2 & 0 \\
0 & D_1L_1 - D_2L_2 & 0 & D_1L_1^2 + D_2L_2^2
\end{bmatrix},
K = \begin{bmatrix}
K_1 + K_2 & 0 & 0 & 0 \\
0 & K_1 + K_2 & 0 & 0 \\
0 & K_1L_1 - K_2L_2 & K_1L_1^2 + K_2L_2^2 & 0 \\
0 & K_1L_1 - K_2L_2 & 0 & K_1L_1^2 + K_2L_2^2
\end{bmatrix},
F = \begin{bmatrix}
F_x \\
F_y \\
M_x \\
M_y
\end{bmatrix},
q_{cg} = \begin{bmatrix}
x \\
y \\
\theta_x \\
\theta_y
\end{bmatrix},
q_b = \begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_1
\end{bmatrix}
\]

The forces at the left hand side represent both the static forces due to gravity and the harmonic exciting aerodynamic forces. The vertical stiffness at the support is piecewise linear whereas it is linear in the horizontal direction. In the further analysis and study of the system Eq (2) is written in the state space form in order to ease the numerical integration and computing the eigenvalues.

Figure 4: (a) Mechanical model, (b) Cross section at the support
5 Numerical Analyses and Preliminary Experimental Results

A numerical stitching method is employed in the nonlinear analyses of the system where the stiffness matrix changes as a function of the displacements. The numerical integration is based on the one step explicit Runga-Kutta (4,5) formula, the Dormand-Prince pair, in MATLAB. Utilizing this solver, an event function has been implemented in order to switch between the different stiffness cases. The results of the numerical analyses are illustrated in the following plots,

1. Bifurcation diagram depicting the response in the vertical direction, $x_1$ of the journal plotted against the speed of the rotor $\Omega$.
2. Phase plane portraits showing the orbits in the $x_1 \times \dot{x}_1$ plane.
3. Trajectory of the journal showing the $x_1 \times y_1$ motion of the journal.
4. Poincaré map, obtained by sampling the trajectory in phase plane at a constant interval of the forcing period of $T = \frac{1}{\Omega}$ and projecting the outcome on the $x_1(nT) \times \dot{x}_1(nT)$ plane.
5. Fourier spectrum of the response

These plots are in general adequate to provide the necessary behavior and information about the dynamics of the system. However, for a system that demonstrates chaotic nature of motion these plots such as the Poincaré maps and Fourier spectra should be strengthened by either evaluating the largest Lyapunov exponents or the fractal dimension of the attractor in order to pronounce a system chaotic or strange. In this study the fractal dimension of the attractor is calculated to determine whether the journal undergoes a chaotic state of motion. The measure of the fractal has successfully been used among experimentalist. Grassberger and Procaccia [12] proposed an efficient approach for determining the correlation dimension $d_G$. In this approach one discretizes the points $\{x_i, i = 1, ..., n\}$ by letting the system evolve for a long time and calculating the distances between pairs of points $s_{ij} = |x_i - x_j|$ on the attractor. Another approach is to discretize the orbit in a pseudo space where the dimension of the attractor is embedded in an $m$-dimensional Euclidean space from a sample of $N$ points where the orbit is constructed by the lagged variables. Upon choosing the time lag $\tau$ and the dimension of the space $m$ the coordinates of the pseudo space is taking as $x(i), x(i + \tau), i = 1, 2, ..., N - (m - 1)\tau$. Here, the value of the time lag is determined by finding the first zero of the auto-correlation function of the time series $x(i)$, i.e., letting $R(\tau) = 0$, where $R(\tau)$ is the auto-correlation function of the time series $x(i)$. In this study both of the methods are employed for determining the chaotic nature of motion of the journal. A correlation function is then calculated by constructing a sphere of radius $r$ at each point $x_i$ and counting the number of points in each sphere, that is,

$$C(r) = \lim_{n \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} H(r - |x_i - x_j|)$$  \hspace{1cm} (4)

where $H(s)$ is the Heaviside function i.e.,

$$H(s) = \begin{cases} 1, & \text{if } s \geq 0, \\ 0, & \text{if } s \leq 0 \end{cases}$$  \hspace{1cm} (5)

For attractors this function has been found to exhibit a power law dependence on $r$ as $r$ becomes smaller and smaller as pointed by Grassberger and Procaccia [12], that is,

$$\lim_{r \to 0} C(r) = ar^{d_G}$$  \hspace{1cm} (6)

In this way the fractal or correlation dimension is found by taking the slope of the $\ln C$ versus $\ln r$ curve. Besides providing methods for pronouncing system chaotic, the bifurcation scenarios and transition into chaos should also be considered. For non-smooth problems the transition into chaos demonstrate a much broader class of bifurcation phenomena such as border-collision bifurcations than for smooth systems.

The parameters listed in Table 3 are employed throughout the different study cases of the motion of the journal. The stiffness in the horizontal direction is assumed to comprise 10% of the stiffness in the vertical direction at both
Table 3: Parameters used to simulate the gas compressor dynamics

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_t$</td>
<td>504</td>
<td>[kg.m²]</td>
</tr>
<tr>
<td>$I_p$</td>
<td>5.37</td>
<td>[kg.m²]</td>
</tr>
<tr>
<td>$I_t$</td>
<td>221.64</td>
<td>[kg.m²]</td>
</tr>
<tr>
<td>$L$</td>
<td>2.44</td>
<td>[m]</td>
</tr>
<tr>
<td>$L_1$</td>
<td>1.30</td>
<td>[m]</td>
</tr>
<tr>
<td>$L_2$</td>
<td>1.14</td>
<td>[m]</td>
</tr>
<tr>
<td>$D_{1ver} + D_{1hor}$</td>
<td>$10 \times 10^3$</td>
<td>[Ns/m]</td>
</tr>
<tr>
<td>$D_{2ver} + D_{2hor}$</td>
<td>$10 \times 10^3$</td>
<td>[Ns/m]</td>
</tr>
<tr>
<td>$K_{2ver}$</td>
<td>$30.7 \times 10^6$</td>
<td>[N/m]</td>
</tr>
<tr>
<td>$K_{2hor}$</td>
<td>$3.07 \times 10^6$</td>
<td>[N/m]</td>
</tr>
</tbody>
</table>

bearings. With the object of investigating the bifurcations and the dynamic motions of the system, the journal on bearing 1 is excited in the vertical direction by a harmonic driven aerodynamic force with the magnitude of $f_{1dyn} = 2880N$. The frequency of excitation is identical to the speed of the rotor. Furthermore, the arc spring supports the journal at a static equilibrium position of approximately $x_1 = -21.15 \mu m$. The following sections process the different study cases.

Figure 5: (a) One dimensional Bifurcation Diagram, (b) One dimensional Bifurcation Diagram, increased damping

5.1 Constant magnitude of the aerodynamic force and constant damping

Fig. 5(a) depicts the co-dimension one bifurcation diagram which is described on a one dimension manifold; here, it takes only the single parameter $\Omega$ to unfold them. For this figure, the transient is cut off and the motion $x_1$ on bearing 1 in the vertical direction is plotted against the driving frequency $\Omega$. The frequency range is chosen to represent a operational speed range of the compressor. The bifurcation diagram presented in Fig. 5(a) is determined by gradually increasing $\Omega$ from 70 Hz (4200 rpm) to 180 Hz (10800 rpm). In order to capture the different types of cycles within the plane the initial conditions are modified during the simulations. Therefore, after identifying the local attractors within the plane the final position and velocity (post behavior) of the journal of a previous simulation are used as the new set of initial conditions. In that way the journal remains at the attractor. However, at the bifurcation point (D) in Fig. 5(a) the journal is forced back to the center of the damper in order to capture
the dynamics from point (D) to point (F). Furthermore, the system demonstrates that different types of cycles with different dynamic characteristics coexist for the wide range of $\Omega$. Considering the bifurcation diagram depicted in Fig. 5(a) it is seen that at the frequency range of 70 Hz (4200 rpm) to 91 Hz (5460 rpm), at point (A) to point (B), two stable attractors exist. The upper branch (C) depicts the stable 1-period limit cycle, whereas the lower branches, from (A) to (B), depict a quasiperiodic attractor. Fig. 6(a) to Fig. 6(c) depict the dynamics of the period-1 motion. Fig. 6(d) to Fig. 6(f) depict the dynamics of the quasiperiodic attractor. Considering the quasiperiodic attractor it is seen that the phase plane has a non-smooth and discontinuous behavior which is a result of the drastic changes in the stiffness as the journal undergoes different stiffness zones as illustrated in Fig. 3(c). Regarding Fig. 6(e) it is clearly seen that the Poincaré return map fills up a close curve which is typical for quasiperiodic attractors. However, if the journal is attracted to the period-1 stable limit cycle, point (C), it will operate connected to the pedestal as the location of the limit cycle exceeds the bearing clearance, whereas the quasiperiodic attractor will force the rotor to operate within the bearing clearance but with super synchronous responses. Fig. 7(a) shows phase plane portrait together with local trajectories near the two attractors at the same plane, and Fig. 7(b) shows the post transient time series. A disturbance in the system can cause the trajectory to start intersecting one of the so-called sewing surfaces, i.e., surfaces that divide the phase space into domains of different dynamics. This is referred to as border-collision bifurcations. Within each such domain the system is smooth, but the equation of motion changes abruptly from one domain to the next.

At approximately $\Omega = 92.7 \, \text{Hz}$ (5562 rpm), point (B) in Fig. 5(a), a bifurcation takes place. The quasiperiodic attractor loses its stability in what seems to be in a saddle bifurcation of cycles (blue sky bifurcation). This is depicted in Fig. 8. Shortly after the quasi periodic attractor looses its stability there is a saddle-node remnant or ghost leading to a slow passage at the location where the quasiperiodic attractor existed and the trajectory is attracted through a collision of the so called sewing surface to the stable limit cycle. During the dangerous bifurcation the current attractor suddenly disappears and the state jumps to a remote disconnected attractor, from point (A) to point (B) in Fig. 8. Just before the bifurcation the journal undergoes super synchronous oscillations, after the bifurcation the journal undergoes synchronous motion, see the power spectrum depicted in Fig. 6. The bifurcation is always discontinuous. Such non-smooth changes in behavior may represent dangers to the life of the system. Reversing
At the angular velocity of $\Omega = 96\, \text{Hz}$ (5760 rpm), point (D) in Fig. 5(a), the quasiperiodic attractor appears “out of the clear blue sky”, where upon it undergoes a quasiperiodic route to chaos. This route involves a transition into chaos through an interval of quasiperiodic motion, Hopf bifurcations, and into the transition via different forms of torus destruction (Ruelle-Takens-Newhouse scenario). Fig. 10 depicts the dynamics of the chaotic motion at the angular velocity of $\Omega = 115\, \text{Hz}$ (6900 rpm), point (E) in Fig. 5(a), with large amplitude orbits filling the phase plane and a broad-banded frequency spectrum. The corresponding Poincaré section and the correlation dimension given in Fig. 11 and Fig. 12 evaluated by means of the two methods, confirm that the response is chaotic. The correlation dimension is evaluated by using 15,000 points for the two approaches. The correlation dimension given in Fig. 11 is found by employing different embedded dimensions. As the embedding dimension is increased, the linear part of the slope approaches a constant value. The correlation dimension given in Fig. 12 is found by evaluating the correlation integral by employing the point on the strange attractor. The approximated correlation dimension is found to $d_G \approx 0.9890$ and $d_G \approx 0.9710$, respectively. A non-integer fractal dimension of this order indicates that the dynamics of the journal exists on a finite low dimensional attractor.

As the angular velocity is increased the journal undergoes a quasiperiodic motion that leads to a stable period-1 state of motion, point (F) in Fig. 5(a).

### 5.2 Studying the behavior of the system by reversing $\Omega$

The bifurcation diagram depicted in Fig. 9 is computed by gradually decreasing the parameter $\Omega$ from 180 Hz (10800 rpm) to 70 Hz (4200 rpm). This diagram demonstrates the direct transition from a stable cycle to chaotic behavior, at point (A), where upon the trajectory is attracted to the upper stable period-1 cycle, from point (B).
Figure 9: One dimensional Bifurcation Diagram

Figure 10: (a) Phase plane, (b) Poincaré section, (c) Displacement power spectrum, (c) Trajectory of rotor within the bearing clearance, (b) Time series

to point (C). This chaotic behavior arises via a border-collision bifurcation. Feigin [13] illustrated how chaotic oscillations can arise via border-collision bifurcation. Nusse and Yorke [14], [15] also observed direct transition from period-2 dynamics to chaotic behavior. Particular attention was paid to the analysis of border-collision bifurcation in which cycles of chaotic intervals softly arise from a stable periodic cycle. Therefore, a run down of the compressor can cause the journal to undergo a sudden chaotic state of motion where upon it jumps to the upper attractor, point (C). The bifurcation diagram is computed by employing the final state of motion, position and velocity, as the new set of initial conditions at each simulation. In this way it is clearly seen that the journal remains at the upper attractor, point (C), during the run down.

5.3 Studying the effects of the direct damping $D_1$ in the $x_1$ direction

It is shown in Fig. 5(a) that a critical bifurcation occurs at the angular velocity of $\Omega = 92.7 \text{Hz}$ (5562 rpm). This bifurcation should by all means be avoided since it is crucial to the system. One approach towards avoiding
Figure 11: (a) Variation of $C(r)$ with embedded dimension $m$, (b) Zoom of (a) to illustrate the variation in $m$, (c) Variation of correlation dimension with embedding dimension $m$

Figure 12: Correlation integral evaluated by sampling on the strange attractor

the critical bifurcation is to increase the damping $D_1$. Fig. 5(b) depicts the bifurcation diagram for the case where $D_1$ is increased by a factor of 10 i.e., from $10000 \text{ Ns/m}$ to $100000 \text{ Ns/m}$. It is clearly seen in the bifurcation diagram that the journal stays within the bearing clearance, from point (A) to Point (C). The bifurcation diagram is computed by using the final state of motion of the journal as the new set of initial condition at each simulation. The transition into chaos is also prevented in this case. From point (A) to point (C) the journal undergoes quasiperiodic motion where upon the motion becomes period-1, synchronous oscillation. The additional damping is generated by the squeeze effect of the SFD. Changes in geometry of the SFD can lead to an increase in the damping. Such changes involve increasing the active axial damper length and oil viscosity and reducing the radial clearance.

5.4 Comparison between experimental and numerical result

The dynamic force utilized to shake the test rig is delivered by means of a system of adjustable unbalances. These unbalances are attached to a rotor which rests in a ball bearing inside the bearing cage. Besides the proximity probes, two accelerometers are used to measure the absolute acceleration of the inner cage in vertical and horizontal direction, and a key phasor giving a common reference for the phase measurements. The damper bearing cage is centered before the test is conducted. The experimental results presented in his paper is obtained by setting the driving frequency to the constant value of $\Omega = 90 \text{ Hz}$ (5400 rpm). Fig. 13 depicts a comparison between the experimental and numerical results. The numerical results are in good agreement with the experimental result. It is clearly seen that the damper bearing cage undergoes super synchronous oscillations. This is caused by the fact that the journal operates close to the bend of the stiffness curve. Furthermore, by comparing Fig. 13(b) and Fig. 13(c) it is seen that the effect of mitigating the responses is poor by increasing the damping for this load case by a factor of 10.

6 Conclusion

The nonlinear analyses in this study revealed that the system undergoes an unfavorable bifurcation which is crucial to the system. An approach towards mitigating this effect is to increase the damping. Consequently, this is obtained by centering the journal within the SFD. A further improvement is to load the arc spring and assure that
it operates within a linear range of the stiffness curve, hence avoiding the bend in the stiffness curve. Moreover, good agreement between experimental and numerical result was obtained.

Acknowledgment
The authors would sincerely thank Henning Hartmann and Claus Myllerup, Lloyd’s Register ODS, for their effort of making the test rig and conducting the experiments.

References