Bioinspired computation in combinatorial optimization - Algorithms and their computational complexity

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Evolutionary Algorithms and Other Search Heuristics

Most famous search heuristic: Evolutionary Algorithms (EAs)

- a bio-inspired heuristic
- paradigm: evolution in nature, “survival of the fittest”
- actually it’s only an algorithm, a randomized search heuristic (RSH)

Goal: optimization
Here: discrete search spaces, combinatorial optimization, in particular pseudo-boolean functions

Optimize $f : \{0,1\}^n \to \mathbb{R}$

Why Do We Consider Randomized Search Heuristics?

- Not enough resources (time, money, knowledge) for a tailored algorithm
- Black Box Scenario rules out problem-specific algorithms
- We like the simplicity, robustness, … of Randomized Search Heuristics
- They are surprisingly successful.

Point of view
Want a solid theory to understand how (and when) they work.

What RSHs Do We Consider?

Theoretically considered RSHs

- $(1+1)$ EA
- $(1+\lambda)$ EA (offspring population)
- $(\mu+1)$ EA (parent population)
- $(\mu+1)$ GA (parent population and crossover)
- SEMO, DEMO, FEMO, … (multi-objective)
- Randomized Local Search (RLS)
- Metropolis Algorithm/Simulated Annealing (MA/SA)
- Ant Colony Optimization (ACO)
- Particle Swarm Optimization (PSO)
- …

First of all: define the simple ones
The Most Basic RSHs

(1+1) EA and RLS for maximization problems

(1+1) EA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   - Create $y$ by flipping each bit of $x_t$ indep. with probab. $1/n$.
   - If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

RLS

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   - Create $y$ by flipping one bit of $x_t$ uniformly.
   - If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

What Kind of Theory Are We Interested in?

- Not studied here: convergence, local progress, models of EAs (e.g., infinite populations).
- Treat RSHs as randomized algorithm!
- Analyze their “runtime” (computational complexity) on selected problems

**Definition**

Let RSH $A$ optimize $f$. Each $f$-evaluation is counted as a time step. The runtime $T_{A,f}$ of $A$ is the random first point of time such that $A$ has sampled an optimal search point.

- Often considered: expected runtime, distribution of $T_{A,f}$
- Asymptotical results w. r. t. $n$

How Do We Obtain Results?

We use (rarely in their pure form):

- Coupon Collector’s Theorem
- Concentration inequalities: Markov, Chebyshev, Chernoff, Hoeffding, … bounds
- Markov chain theory: waiting times, first hitting times
- Rapidly Mixing Markov Chains
- Random Walks: Gambler’s Ruin, drift analysis, martingale theory, electrical networks
- Random graphs (esp. random trees)
- Identifying typical events and failure events
- Potential functions and amortized analysis
- …

Adapt tools from the analysis of randomized algorithms; understanding the stochastic process is often the hardest task.

Early Results

Analysis of RSHs already in the 1980s:

- Sasaki/Hajek (1988): SA and Maximum Matchings
- Sorkin (1991): SA vs. MA
- Jerrum (1992): SA and Cliques
- …

High-quality results, but limited to SA/MA (nothing about EAs) and hard to generalize.

Since the early 1990s

Systematic approach for the analysis of RSHs, building up a completely new research area
How the Systematic Research Began — Toy Problems

Simple example functions (test functions)
- OneMax \(x_1, \ldots, x_n\) = \(x_1 + \cdots + x_n\)
- LeadingOnes \(x_1, \ldots, x_n\) = \(\sum_{i=1}^{n} \prod_{j=1}^{i} x_j\)
- BinVal \(x_1, \ldots, x_n\) = \(\sum_{i=1}^{n} 2^{n-i} x_j\)
- polynomials of fixed degree

Goal: derive first runtime bounds and methods

Artificially designed functions
- with sometimes really horrible definitions
- but for the first time these allow rigorous statements

Goal: prove benefits and harm of RSH components, e.g., crossover, mutation strength, population size . . .

Example: OneMax

Theorem (e.g., Droste/Jansen/Wegener, 1998)
The expected runtime of the RLS, \((1+1)\) EA, \((\mu+1)\) EA, \((1+\lambda)\) EA on OneMax is \(\Omega(n \log n)\).

Proof by modifications of Coupon Collector’s Theorem.

Theorem (e.g., Mühlenbein, 1992)
The expected runtime of RLS and the \((1+1)\) EA on OneMax is \(O(n \log n)\).

Holds also for population-based \((\mu+1)\) EA and for \((1+\lambda)\) EA with small populations.
Proof of the $O(n \log n)$ bound

- **Fitness levels:** $L_i := \{x \in \{0, 1\}^n \mid \text{OneMax}(x) = i\}$
- (1+1) EA never decreases its current fitness level.
- From $i$ to some higher-level set with prob. at least
  
  \[
  \left(\frac{n-i}{1}\right) \cdot \left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{n-i}{en}
  \]
  
  - Expected time to reach a higher-level set is at most $\frac{en}{n-i}$.
  - Expected runtime is at most
    \[
    \sum_{i=0}^{n-1} \frac{en}{n-i} = O(n \log n).
    \]

Later Results Using Toy Problems

- Find the theoretically optimal mutation strength ($1/n$ for OneMax!).
- Bound the optimization time for linear functions ($O(n \log n)$).
- Optimal population size (often 1!)
- Crossover vs. no crossover $\rightarrow$ Real Royal Road Functions
- Multistarts vs. populations
- Frequent restarts vs. long runs
- Dynamic schedules
- $\ldots$

RSHs for Combinatorial Optimization

- Analysis of runtime and approximation quality on well-known combinatorial optimization problems, e.g.,
  - Sorting problems (is this an optimization problem?),
  - Covering problems,
  - Cutting problems,
  - Subsequence problems,
  - Traveling salesman problem,
  - Eulerian cycles,
  - Minimum spanning trees,
  - Matching problems,
  - Shortest paths,
  - $\ldots$
- We do not hope: to be better than the best problem-specific algorithms
- Instead: maybe reasonable polynomial running times
- In the following no fine-tuning of the results

Agenda

- The origins: example functions and toy problems
  - A simple toy problem: OneMax for (1+1) EA
- Combinatorial optimization problems
  - Minimum spanning trees
  - Maximum matchings
  - Shortest paths
  - Makespan scheduling
  - Covering problems
  - Traveling salesman problem
- End
- References
Minimum Spanning Trees:

- **Given**: Undirected connected graph $G = (V, E)$ with $n$ vertices and $m$ edges with positive integer weights.
- **Find**: Edge set $E' \subseteq E$ with minimal weight connecting all vertices.
- Search space $\{0,1\}^m$
- Edge $e_i$ is chosen iff $x_i = 1$
- Consider $(1+1)$ EA

Fitness function:

- Decrease number of connected components, find minimum spanning tree.
- $f(s) := (c(s),w(s))$.
  Minimization of $f$ with respect to the lexicographic order.

First goal: Obtain a connected subgraph of $G$.

How long does it take?

Connected graph in expected time $O(m \log n)$ (fitness-based partitions)

Bijection for minimum spanning trees:

\[
\begin{align*}
        e_1 & \quad \alpha(e_1) \\
        e_2 & \quad \alpha(e_2) \\
        e_3 & \quad \alpha(e_3)
\end{align*}
\]

$k := |E(T^*) \setminus E(T) |$

Bijection $\alpha$: $E(T^*) \setminus E(T) \to E(T) \setminus E(T^*)$

$\alpha(e_i)$ on the cycle of $E(T) \cup \{e_i\}$

$w(e_i) \leq w(\alpha(e_i))$

$\Rightarrow k$ accepted 2-bit flips that turn $T$ into $T^*$
Upper Bound

Theorem:
The expected time until (1+1) EA constructs a minimum spanning tree is bounded by $O(m^2(\log n + \log w_{max}))$.

Sketch of proof:
• $w(s)$ weight current solution $s$.
• $w_{opt}$ weight minimum spanning tree $T^*$
• set of $m + 1$ operations to reach $T^*$
• $m' = m - (n - 1) 1$-bit flips concerning non-$T^*$ edges
  ⇒ spanning tree $T$
• k $2$-bit flips defined by bijection
• $n - k$ non accepted $2$-bit flips
• $\Rightarrow$ average distance decrease $(w(s) - w_{opt})/(m + 1)$

Proof

1-step (larger total weight decrease of $1$-bit flips)
2-step (larger total weight decrease of $2$-bit flips)

Consider 2-steps:
• Expected weight decrease by a factor $1 - (1/(2n))$
• Probability $(n/m^2)$ for a good $2$-bit flip
• Expected time until $q$ 2-steps $O(qm^2/n)$

Consider 1-steps:
• Expected weight decrease by a factor $1 - (1/(2m'))$
• Probability $(m'/m)$ for a good $1$-bit flip
• Expected time until $q$ 1-steps $O(qm/m')$

1-steps faster $\Rightarrow$ show bound for 2-steps.

Expected Multiplicative Distance Decrease (aka Drift Analysis)

Maximum distance: $w(s) - w_{opt} \leq D := m \cdot w_{max}$

1 step: Expected distance at most $(1 - 1/(2n))(w(s) - w_{opt})$

t steps: Expected distance at most $(1 - 1/(2n))^t(w(s) - w_{opt})$

$t := \lceil 2 \cdot (\ln 2)n(\log D + 1) \rceil : (1 - 1/(2n))^t(w(s) - w_{opt}) \leq 1/2$

Expected number of 2-steps $2t = O(n(\log n + \log w_{max}))$ (Markov)

Expected optimization time $O(tm^2/n) = O(m^2(\log n + \log w_{max}))$. 
### Agenda

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA

2. Combinatorial optimization problems
   - Minimum spanning trees
   - Maximum matchings
   - Shortest paths
   - Makespan scheduling
   - Covering problems
   - Traveling salesman problem

3. End

4. References

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### Maximum Matchings

A **matching** in an undirected graph is a subset of pairwise disjoint edges; aim: find a maximum matching (solvable in poly-time)

Simple example: path of odd length

![Maximum Matching Example](image)

Maximum matching with more than half of edges

#### Suboptimal matching

**Concept**: augmenting path
- Alternating between edges being inside and outside the matching
- Starting and ending at “free” nodes not incident on matching
- Flipping all choices along the path improves matching

**Example**: whole graph is augmenting path

Interesting: how simple EAs find augmenting paths

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### Maximum Matchings: Upper Bound

Fitness function $f : \{0, 1\}^\# \text{edges} \to \mathbb{R}$:
- one bit for each edge, value 1 iff edge chosen
- value for legal matchings: size of matching
- otherwise penalty leading to empty matching

**Example**: path with $n + 1$ nodes, $n$ edges: bit string selects edges

![Fitness Function Example](image)

**Theorem**

*The expected time until (1+1) EA finds a maximum matching on a path of $n$ edges is $O(n^4)$.*
Maximum Matchings: Upper Bound (Ctnd.)

Proof idea for $O(n^4)$ bound
- Consider the level of second-best matchings.
- Fitness value does not change (walk on plateau).
- If “free” edge: chance to flip one bit! $\rightarrow$ probability $\Theta(1/n)$.
- Else steps flipping two bits $\rightarrow$ probability $\Theta(1/n^2)$.
- Shorten or lengthen augmenting path
- At length 1, chance to flip the free edge!

Length changes according to a fair random walk
$\rightarrow$ equal probability for lengthenings and shortenings

Fair Random Walk

Scenario: fair random walk
- Initially, player A and B both have $\frac{2}{3}$ USD
- Repeat: flip a coin
- If heads: A pays 1 USD to B, tails: other way round
- Until one of the players is ruined.

How long does the game take in expectation?

Theorem:
Fair random walk on $\{0, \ldots, n\}$ takes in expectation $O(n^2)$ steps.

Maximum Matchings: Lower Bound

Worst-case graph $G_{h, \ell}$

Augmenting path can get shorter but is more likely to get longer. (unfair random walk)

Theorem
For $h \geq 3$, $(1+1) EA$ has exponential expected optimization time $2^{\Omega(\ell)}$ on $G_{h, \ell}$.

Proof requires analysis of negative drift (simplified drift theorem).
Maximum Matching: Approximations

**Insight:** do not hope for exact solutions but for approximations

For maximization problems: solution with value $a$ is called $(1 + \varepsilon)$-approximation if $\frac{\text{OPT}}{a} \leq 1 + \varepsilon$, where OPT optimal value.

**Theorem**

For $\varepsilon > 0$, $(1+1) \text{ EA}$ finds a $(1 + \varepsilon)$-approximation of a maximum matching in expected time $O(m^{2/\varepsilon + 2})$ ($m$ number of edges).

**Proof idea:** If current solution worse than $(1 + \varepsilon)$-approximate, there is a “short” augmenting path (length $\leq 2/\varepsilon + 1$); flip it in one go.

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**All-pairs-shortest-path (APSP) problem**

Given: Connected directed graph $G = (V, E)$, $|V| = n$ and $|E| = m$, and a function $w: E \to \mathbb{N}$ which assigns positive integer weights to the edges.

Compute from each vertex $v_i \in V$ a shortest path (path of minimal weight) to every other vertex $v_j \in V \setminus \{v_i\}$

**Representation:**

Individuals are paths between two particular vertices $v_i$ and $v_j$

Initial Population: $P := \{I_{u, v} = (u, v) \mid (u, v) \in E\}$
Mutation:

Pick individual \( I_{u,v} \) uniformly at random

\[ E^-(u) : \text{incoming edges of } u \quad E^+(v) : \text{outgoing edges of } v \]

Pick uniformly at random an edge \( E \). Mutation:

Pick an edge \( E \) uniformly at random.

Consider two vertices \( u \) from \( \gamma \) and \( v \) from \( \gamma \):

- \( u \) to \( v \) consisting of at most \( \ell \) edges in \( G \).
- The sub-path \( \gamma' = (v^1 = u, v^2, \ldots, v^j) \) is a shortest path from \( u \) to \( v^j \).

Lemma:

Let \( \ell \geq \log n \). The expected time until has found all shortest paths with at most \( \ell \) edges is \( O(n^3 \ell) \).

Proof idea:

Consider two vertices \( u \) and \( v \), \( u \neq v \).

Let \( \gamma := (v^1 = u, v^2, \ldots, v^\ell = v) \) be a shortest path from \( u \) to \( v \) consisting of \( \ell' \leq \ell \), edges in \( G \).

Probability to pick \( I_{u,v} \) is at least \( 1/n^2 \).

Pick shortest path from \( u \) to \( v_j \) and append edge \((v_j, v_{j+1})\).

Shortest path from \( u \) to \( v_{j+1} \).

Success with probability at least \( 1/(2n^3) \).

Population size is upper bounded \( n^2 \).

(For each pair of vertices at most one path)

Steady State EA

1. Set \( P = \{I_{u,v} = (u,v) \mid (u,v) \in E\} \).
2. Choose an individual \( I_{x,y} \in P \) uniformly at random.
3. Mutate \( I_{x,y} \) to obtain an individual \( I'_{s,t} \).
4. If there is no individual \( I_{s,t} \in P \), \( P = P \cup \{I'_{s,t}\} \).
   else if \( f(I'_{s,t}) \leq f(I_{s,t}) \), \( P = (P \cup \{I'_{s,t}\}) \setminus \{I_{s,t}\} \).
5. Repeat Steps 2–4 forever.

Mutation-based EA
Consider typical run consisting of $T = cn^3 l$ steps.

What is the probability that the shortest path from $u$ to $v$ has been obtained?

We need at most $l$ successes, where a success happens in each step with probability at least $p = 1/(2n^3)$

Define for each step $i$ a random variable $X_i$.

- $X_i = 1$ if step $i$ is a success
- $X_i = 0$ if step $i$ is not a success

**Theorem**

The expected optimization time of Steady State EA for the APSP problem is $O(n^4)$.

**Remark:**

There are instances where the expected optimization of $(\mu + 1)$-EA is $\Omega(n^4)$

**Question:**

Can crossover help to achieve a better expected optimization time?

\[ \text{Prob}(X_i = 1) \geq p = 1/(2n^3) \quad X = \sum_{i=1}^{T} X_i \quad X \geq \ell \]

Expected number of successes $E(X) \geq T/(2n^3) = \frac{cn^3 l}{2n^3} = \frac{c}{2}$

Chernoff: $\text{Prob}(X < (1 - \delta)E(x)) \leq e^{-E(X)\delta^2/2}$

\[ \delta = \frac{1}{2} \]

$\text{Prob}(X < (1 - \frac{1}{2})E(x)) \leq e^{-E(X)/8} \leq e^{-T/(16n^3)} = e^{-cn^2/(16n^2)} = e^{-c\ell/(16)}$

Probability for failure of at least one pair of vertices at most: $n^2 \cdot e^{-c\ell/16}$

$e$ large enough and $\ell \geq \log n$:

No failure in any path with probability at least $\alpha = 1 - n^2 \cdot e^{-c\ell/16} = 1 - o(1)$

Holds for any phase of $T$ steps

**Expected time upper bound by** $T/\alpha = O(n^3 \ell)$

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**Analysis**

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**Crossover**

Pick two individuals $I_{u,v}$ and $I_{s,t}$ from population uniformly at random.

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**Shortest paths have length at most $n-1$.**

Set $l = n-1$
Steady State GA
1. Set \( P = \{ I_{u,v} = (u,v) \mid (u,v) \in E \} \).
2. Choose \( r \in [0,1] \) uniformly at random.
3. If \( r \leq p_c \), choose two individuals \( I_{x,y} \in P \) and \( I_{x',y'} \in P \) uniformly at random and perform crossover to obtain an individual \( I_{x,t} \),
   else choose an individual \( I_{x,y} \in P \) uniformly at random and mutate \( I_{x,y} \) to obtain an individual \( I_{x,t} \).
4. If \( I'_{s,t} \) is a path from \( s \) to \( t \) then
   \( \star \) if there is no individual \( I_{s,t} \in P \), \( P = P \cup \{ I'_{s,t} \} \),
   \( \star \) else if \( f(I'_{s,t}) \leq f(I_{s,t}) \), \( P = (P \cup \{ I'_{s,t} \}) \setminus \{ I_{s,t} \} \).
5. Repeat Steps 2–4 forever.

\( p_c \) is a constant

Theorem:
The expected optimization time of Steady State GA is \( O(n^{3.5} \sqrt{\log n}) \).

Mutation and \( \ell^* := \sqrt{n \log n} \)

All shortest path of length at most \( \ell^* \) edges are obtained

Show: Longer paths are obtained by crossover within the stated time bound.

Analysis Crossover
Long paths by crossover:
Assumption: All shortest paths with at most \( \ell^* \) edges have already been obtained.

Assume that all shortest paths of length \( k \leq \ell^* \) have been obtained.

What is the expected time to obtain all shortest paths of length at most \( 3k/2 \)?

Analysis Crossover
Consider pair of vertices \( x \) and \( y \) for which a shortest path of \( r, k < r \leq 3k/2 \), edges exists.

There are \( 2k \)-r pairs of shortest paths of length at most \( k \) that can be joined to obtain shortest path from \( x \) to \( y \).

Probability for one specific pair: at least \( 1/n^4 \)
At least \( 2k+1-r \) possible pairs: probability at least \( (2k+1-r)/n^4 \) \( \geq k/(2n^4) \)
At most \( n^2 \) shortest paths of length \( r, k < r \leq 3k/2 \)
Time to collect all paths \( O(n^4 \log n/ k) \)
(similar to Coupon Collectors Theorem)
Analysis Crossover

Sum up over the different values of \( k \), namely
\[
\sqrt{n \log n}, c \cdot \sqrt{n \log n}, c^2 \cdot \sqrt{n \log n}, \ldots, c^{\log_2(n/\sqrt{n \log n})} \cdot \sqrt{n \log n},
\]
where \( c = 3/2 \).

Expected Optimization
\[
\sum_{s=0}^{\infty} \left( O\left( \frac{n^3}{\sqrt{n \log n}} \right) c^{-s} \right) = O(n^3) \sum_{s=0}^{\infty} c^{-s} = O(n^3 \sqrt{\log n})
\]

Makespan Scheduling

What about NP-hard problems? \( \rightarrow \) Study approximation quality

Makespan scheduling on 2 machines:
- \( n \) objects with weights/processing times \( w_1, \ldots, w_n \)
- 2 machines (bins)
- Minimize the total weight of fuller bin = makespan.

Formally, find \( I \subseteq \{1, \ldots, n\} \) minimizing
\[
\max \left\{ \sum_{i \in I} w_i, \sum_{i \notin I} w_i \right\}.
\]

Sometimes also called the Partition problem.
This is an “easy” NP-hard problem, good approximations possible.

Fitness Function

- Problem encoding: bit string \( x_1, \ldots, x_n \) reserves a bit for each object, put object \( i \) in bin \( x_i + 1 \).
- Fitness function
\[
f(x_1, \ldots, x_n) := \max \left\{ \sum_{i=1}^{n} w_i x_i, \sum_{i=1}^{n} w_i (1 - x_i) \right\}
\]
to be minimized.
- Consider (1+1) EA and RLS.
Types of Results

- Worst-case results
- Success probabilities and approximations
- An average-case analysis
- A parameterized analysis

Sufficient Conditions for Progress

Abbreviate $S := w_1 + \cdots + w_n \Rightarrow$ perfect partition has cost $\frac{S}{2}$.
Suppose we know
- $s^* =$ size of smallest object in the fuller bin,
- $f(x) > \frac{S}{2} + \frac{s^*}{2}$ for the current search point $x$
then the solution is improvable by a single-bit flip.

If $f(x) < \frac{S}{2} + \frac{s^*}{2}$, no improvements can be guaranteed.

Lemma
If smallest object in fuller bin is always bounded by $s^*$ then $(1+1)$ EA and RLS reach $f$-value $\leq \frac{S}{2} + \frac{s^*}{2}$ in expected $O(n^2)$ steps.
**Sufficient Conditions for Progress**
Abbreviate $S := w_1 + \cdots + w_n \Rightarrow$ perfect partition has cost $\frac{S}{2}$. Suppose we know
- $s^* =$ size of smallest object in the fuller bin,
- $f(x) > \frac{S}{2} + \frac{s^*}{2}$ for the current search point $x$ then the solution is improvable by a single-bit flip.

If $f(x) < \frac{S}{2} + \frac{s^*}{2}$, no improvements can be guaranteed.

**Lemma**
If smallest object in fuller bin is always bounded by $s^*$ then $(1+1)$ EA and RLS reach $f$-value $\leq \frac{S}{2} + \frac{s^*}{2}$ in expected $O(n^2)$ steps.

**Worst-Case Results**

**Theorem**
On any instance to the makespan scheduling problem, the $(1+1)$ EA and RLS reach a solution with approximation ratio $\frac{4}{3}$ in expected time $O(n^2)$.

Use study of object sizes and previous lemma.

**Theorem**
There is an instance $W_{\varepsilon}^* = \{w_1, \ldots, w_n\}$ such that the $(1+1)$ EA and RLS need with prob. $\Omega(1)$ at least $n^{\Omega(n)}$ steps to find a solution with a better ratio than $4/3 - \varepsilon$.

**Worst-Case Instance**
Instance $W_{\varepsilon}^* = \{w_1, \ldots, w_n\}$ is defined by $w_1 := w_2 := \frac{1}{3} - \frac{\varepsilon}{2}$ (big objects) and $w_i := \frac{1/3 + 2/3}{n-2}$ for $3 \leq i \leq n$, $\varepsilon$ very small constant; $n$ even
Sum is 1; there is a perfect partition.
But if one bin with big and one bin with small objects: value $\frac{2}{3} + \frac{\varepsilon}{2}$.
Move a big object in the emptier bin $\Rightarrow$ value $(\frac{1}{3} + \frac{\varepsilon}{2}) + (\frac{1}{3} - \frac{\varepsilon}{2}) = \frac{2}{3} + \frac{\varepsilon}{2}!$
Need to move $\geq \varepsilon n$ small objects at once for improvement: very unlikely.

With constant probability in this situation, $n^{\Omega(n)}$ needed to escape.
Worst Case – PRAS by Parallelism

Previous result shows: success dependent on big objects

**Theorem**

On any instance, the \((1+1)\) EA and RLS with prob. \(\geq 2^{-c\frac{1}{\ln(1/\epsilon)}}\) find a \((1 + \epsilon)\)-approximation within \(O(n\ln(1/\epsilon))\) steps.

- \(2^{O(\frac{1}{\ln(1/\epsilon)})}\) parallel runs find a \((1 + \epsilon)\)-approximation with prob. \(\geq 3/4\) in \(O(n\ln(1/\epsilon))\) parallel steps.
- Parallel runs form a polynomial-time randomized approximation scheme (PRAS)!

Worst Case – PRAS by Parallelism (Proof Idea)

Set \(s := \lceil \frac{n}{2} \rceil\)

Assuming \(w_1 \geq \cdots \geq w_n\), we have \(w_i \leq \epsilon \frac{s}{2}\) for \(i \geq s\).

analyze probability of distributing
- large objects in an optimal way,
- small objects greedily \(\Rightarrow\) error \(\leq \epsilon \frac{s}{2}\),

Random search rediscovers algorithmic idea of early algorithms.

Average-Case Analyses

Models: each weight drawn independently at random, namely
- uniformly from the interval \([0, 1]\).
- exponentially distributed with parameter 1 (i.e., \(\text{Prob}(X \geq t) = e^{-t}\) for \(t \geq 0\)).

Approximation ratio no longer meaningful, we investigate:
**discrepancy** = absolute difference between weights of bins.

How close to discrepancy 0 do we come?

Makespan Scheduling – Known Average-Case Results

**Deterministic, problem-specific heuristic LPT**

Sort weights decreasingly, put every object into currently emptier bin.

Known for both random models:
LPT creates a solution with discrepancy \(O((\log n)/n)\).

What discrepancy do the \((1+1)\) EA and RLS reach in poly-time?
Average-Case Analysis of the (1+1) EA

In both models, the (1+1) EA reaches discrepancy $O((\log n)/n)$ after $O(n^{c+4}\log^2 n)$ steps with probability $1 - O(1/n^c)$.

Almost the same result as for LPT!

Proof exploits order statistics:
If $X(i)$ (i-th largest) in fuller bin, $X(i+1)$ in emptier one, and discrepancy $> 2(X(i) - X(i+1)) > 0$, then objects can be swapped; discrepancy falls
Consider such “difference objects”.

W. h. p. $X(i) - X(i+1) = O((\log n)/n)$ (for $i = \Omega(n)$).

Value of Optimal Solution

Recall approximation result: decent chance to distribute $k$ big jobs optimally if $k$ small.

Since $w_1 \geq \cdots \geq w_n$, already $w_k \leq S/k$.

Consequence: optimal distribution of first $k$ objects $\rightarrow$ can reach makespan $S/2 + S/k$ by greedily treating the other objects.

(1+1) EA and RLS find solution of makespan $\leq S/2 + S/k$ with probability $\Omega((2k)^{-e_k})$ in time $O(n \log k)$. Multistarts have success probability $\geq 1/2$ after $O(2^{(e+1)}k^{e_k}n \log k)$ evaluations.

$2^{(e+1)}k^{e_k} \log k$ does not depend on $n \rightarrow$ a randomized FPT-algorithm.

A Parameterized Analysis

Have seen: problem is hard for (1+1) EA/RLS in the worst case, but not so hard on average.

What parameters make the problem hard?

Definition

A problem is fixed-parameter tractable (FPT) if there is a problem parameter $k$ such that it can be solved in time $f(k) \cdot \text{poly}(n)$, where $f(k)$ does not depend on $n$.

Intuition: for small $k$, we have an efficient algorithm.

Considered parameters (Sutton and Neumann, 2012):
- Value of optimal solution
- No. jobs on fuller machine in optimal solution
- Unbalance of optimal solution

No. Objects on Fuller Machine

Suppose: optimal solution puts only $k$ objects on fuller machine.

Notion: $k$ is called critical path size.

Intuition:
- Good chance of putting $k$ objects on same machine if $k$ small,
- other objects can be moved greedily.

Theorem

For critical path size $k$, multistart RLS finds optimum in $O(2^k (en)^{e_k} k^{e_k} n \log n)$ evaluations with probability $\geq 1/2$.

Due to term $n^{e_k}$, result is somewhat weaker than FPT (a so-called XP-algorithm). Still, for constant $k$ polynomial.

Remark: with (1+1)-EA, get an additional $\log w_1$-term.
**Unbalance of Optimal Solution**

Consider discrepancy of optimum \( \Delta^* := 2(\text{OPT} - S/2) \).

**Question/decision problem:** Is \( w_k \geq \Delta^* \geq w_{k+1} \)?

**Observation:** If \( \Delta^* \geq w_{k+1} \), optimal solution will put \( w_{k+1}, \ldots, w_n \) on emptier machine. Crucial to distribute first \( k \) objects optimally.

**Theorem**

*Multistart RLS with biased mutation (touches objects \( w_1, \ldots, w_k \) with prob. \( 1/(kn) \) each) solves decision problem in \( O(2^k n^3 \log n) \) evaluations with probability \( \geq 1/2 \).*

Again, a randomized FPT-algorithm.

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**The Problem**

**The Vertex Cover Problem:**

Given an undirected graph \( G=(V,E) \).

Find a minimum subset of vertices such that each edge is covered at least once.

NP-hard, several 2-approximation algorithms.

Simple single-objective evolutionary algorithms fail!!!

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**Decision problem:** Is there a set of vertices of size at most \( k \) covering all edges?

**Our parameter:** Value of an optimal solution (OPT)
**Evolutionary Algorithm**

Representation: Bitstrings of length n

- $x_1 = 1$
- $x_2 = 0$
- $x_3 = 1$
- $x_4 = 0$
- $x_5 = 0$
- $x_6 = 0$

Minimize fitness function:

- $f_1(x) = (|x_1|, U(x))$
- $f_1(x) = (2, 2)$
- $f_2(x) = (|x_1|, LP(x))$
- $f_2(x) = (2, 1)$

$U(x)$: Edges not covered by $x$

$G(x) = G(V, U(x))$

$LP(x)$: value of LP applied to $G(x)$

**Multi-Objective Approach:**

Treat the different objectives in the same way

Keep trade-offs of the two criteria

1. Standard bit mutation with probability $1/n$
2. Mutation probability $1/2$ for vertices adjacent to edges of $U(x)$. Otherwise mutation probability $1/n$. Decide uniformly at random which operator to use in next iteration

Empty set included in the population

$|U(x)|$
Linear Programming

Combination with Linear Programming

- LP-relaxation is half integral, i.e.

\[ x_i \in \{0, 1/2, 1\}, 1 \leq i \leq n \]

Theorem (Nemhauser, Trotter (1975)):

Let \( x^* \) be an optimal solution of the LP. Then there is a minimum vertex cover that contains all vertices \( v_i \) where \( x_i^* = 1 \).

Lemma:

All search points \( x \) with \( LP(x) = LP(0^n) - \|x\|_1 \) are Pareto optimal. They can be extended to minimum vertex cover by selecting additional vertices.

Can we also say something about approximations?

Approximations

\[ |x|_1 \leq (1 + \epsilon)OPT \]

Kernelization in expected polynomial time

- Subset of a minimum vertex cover
- \( G(x) \) has maximum degree at most \( OPT \)
- \( G(x) \) has at most \( OPT + OPT^2 \) non-isolated vertices

Expected time \( O(2^{OPT} \cdot \text{poly}(n)) \)

\[ |LP(x)| \]
### Agenda

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA

2. Combinatorial optimization problems
   - Minimum spanning trees
   - Maximum matchings
   - Shortest paths
   - Makespan scheduling
   - Covering problems
   - Traveling salesman problem

### References

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**Representation and Mutation**

**Representation:** Permutation of the n cities

For example: (3, 4, 1, 2, 5)

**Inversion (inv) as mutation operator:**
- Select i,j from \{1, ...n\} uniformly at random and invert the part from position i to position j.
- Inv(2,5) applied to (3, 4, 1, 2, 5) yields (3, 5, 2, 1, 4)

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**Euclidean TSP**

Given n points in the plane and Euclidean distances between the cities.

Find a shortest tour that visits each city exactly once and return to the origin.

NP-hard, PTAS, FPT when number of inner points is the parameter.

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**(1+1) EA**

\[ x \leftarrow \text{random permutation of } [n]. \]

**repeat** forever

\[ y \leftarrow \text{MUTATE}(x) \]

if \( f(y) < f(x) \) then \( x \leftarrow y \)

**Mutation:**

(1+1) EA: k random inversion,

k chosen according to 1+Pois(1)
Convex hull containing n-k points

Angle bounded set of points

There may be an exponential number of inversions to end up in a local optimum if points are in arbitrary positions (Englert et al., 2007).

We assume that the set $V$ is angle bounded $V$ is angle-bounded by $\epsilon > 0$ if for any three points $u, v, w \in V$, $0 < \theta < \pi - \epsilon$ where $\theta$ denotes the angle formed by the line from $u$ to $v$ and the line from $v$ to $w$.

If $V$ is angle-bounded then we get a lower bound on an improvement depending on $\epsilon$.

Progress

Assumptions:

- $d_{\text{max}}$: Maximum distance between any two points
- $d_{\text{min}}$: Minimum distance between any two points
- $V$ is angle-bounded by $\epsilon$

Whenever the current tour is not intersection-free, we can guarantee a certain progress.

Lemma:

Let $x$ be a permutation such that it is not intersection-free. Let $y$ be the permutation constructed from an inversion on $x$ that replaces two intersecting edges with two non-intersecting edges. Then, $f(x) - f(y) > 2d_{\text{min}} \left( \frac{1 - \cos(\epsilon)}{\cos(\epsilon)} \right)$.
**Tours**

A tour $x$ is either
- Intersection free
- Non intersection free

Intersection free tour are good. The points on the convex hull are already in the right order (Quintas and Supnick, 1965).

**Claim:** We do not spend too much time on non intersection free tours.

**Time spend on intersecting tours**

**Lemma:**
Let $(x^{(1)}, x^{(2)}, \ldots, x^{(l)}, \ldots)$ denote the sequence of permutations generated by the (1+1)-EA. Let $\alpha$ be an indicator variable defined on permutations of $[n]$ as

$$\alpha(x) = \begin{cases} 1 & x \text{ contains intersections;} \\ 0 & \text{otherwise.} \end{cases}$$

Then $E\left(\sum_{l=1}^{\infty} \alpha(x^{(l)})\right) = O\left(n^3 \left(\frac{d_{\text{max}}}{d_{\text{min}}} - 1\right) \left(\frac{\cos(\theta)}{\cos(\theta)}\right)\right)$.

For an $m \times m$ grid:
For points on an $m \times m$ grid this bound becomes $O(n^3 m^5)$.

**Parameterized Result**

**Lemma:**
Suppose $V$ has $k$ inner points and $x$ is an intersection-free tour on $V$. Then there is a sequence of at most $2k$ inversions that transforms $x$ into an optimal permutation.

**Theorem:**
Let $V$ be a set of points quantized on an $m \times m$ and $k$ be the number of inner points. Then the expected optimisation time of the (1+1)-EA on $V$ is $O(n^3 m^5) + O(n^4 (2k - 1)!)$. 

**Summary and Conclusions**

- Runtime analysis of RSHs in combinatorial optimization
- Starting from toy problems to real problems
- Insight into working principles using runtime analysis
- General-purpose algorithms successful for wide range of problems
- Interesting, general techniques
- Runtime analysis of new approaches possible
- An exciting research direction.

Thank you!
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