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# Prewhitening for Rank-Deficient Noise in Subspace Methods for Noise Reduction

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**Abstract**—A fundamental issue in connection with subspace methods for noise reduction is that the covariance matrix for the noise is required to have full rank in order for the prewhitening step to be defined. However, there are important cases where this requirement is not fulfilled, e.g., when the noise has narrowband characteristics or in the case of tonal noise. We extend the concept of prewhitening to include the case when the noise covariance matrix is rank deficient, using a weighted pseudoinverse and the quotient singular value decomposition, and we show how to formulate a general rank-reduction algorithm that works also for rank-deficient noise. We also demonstrate how to formulate this algorithm by means of a quotient ULV decomposition, which allows for faster computation and updating. Finally, we apply our algorithm to a problem involving a speech signal contaminated by narrowband noise.

**Index Terms**—Noise reduction, rank deficient noise, singular value decomposition, subspace methods, ULV decomposition.

## I. INTRODUCTION

SUBSPACE methods and rank reduction have emerged as important techniques for noise reduction in many applications, including speech enhancement; see [5], [6], [8], [13], and [19]. In all these applications, there is a fundamental restriction, namely, that the covariance matrix for the noise must have full rank; this is necessary because the prewhitening step essentially consists of post multiplying the signal matrix with the inverse of the Cholesky factor of the noise covariance matrix.

However, there also exist important applications where the requirement of full rank is not satisfied, for example, in the case of narrowband noise or tonal noise. It is therefore preferable to have a general method that is guaranteed to work in all cases, independently of the rank of the noise correlation matrix. Hence, it is of interest to extend the concept of prewhitening for subspace methods, such that it can also handle rank-deficient noise covariance matrices—in such a way that the new technique is identical to standard prewhitening in the full-rank case.

The underlying mathematics of a general prewhitener for a rank-deficient noise covariance matrix was developed in [11] under a rather technical assumption. In that paper, the quotient (or generalized) singular value decomposition was used to demonstrate that the prewhitening matrix, to be post-multiplied

to the signal matrix, should be a weighted pseudoinverse. An algorithm suited for efficient updating, based on the rank-revealing quotient ULV decomposition, was also outlined in [11].

In this paper, we formulate the abstract algorithm from [11] in signal processing terms (in order to make it easier accessible to this community), and we demonstrate the usefulness of the new algorithm in connection with realistic signals. In addition, we complete the theory of the rank-deficient prewhitening algorithm by extending the results from [11] to the completely general case with no technical assumptions. Our work makes use of a weighted pseudoinverse that originates in work by Mitra and Rao [16] and Eldén [7] and is related to the oblique projection, which is a tool that is currently receiving attention in the signal processing literature [1], [2].

Our paper is organized as follows. In Section II, we discuss full-rank and low-rank prewhitening in terms of the quotient singular value decomposition (SVD). In Section III, we introduce the rank-revealing quotient ULV decomposition, which is a computationally attractive alternative to the quotient SVD, and we demonstrate how to formulate rank-deficient prewhitening in terms of this decomposition. Section IV discusses the removal of the above-mentioned technical assumption, and we demonstrate that in practice, this does not lead to difficulties with the algorithms. Finally, in Section V, we illustrate the performance of our algorithms by an example involving a speech signal with low-rank noise.

Throughout the paper,  $I_q$  denotes the identity matrix of order  $q$ , and  $\mathcal{R}(X)$  denotes the range, or column space, of the matrix  $X$ .

## II. PREWHITENING FOR SUBSPACE METHODS

Standard subspace methods for noise reduction assume that the noise covariance matrix has full rank, and that the pure signal lies in a lower dimensional subspace, such that we can separate the noisy signal into two components lying in orthogonal subspaces—the so-called signal and noise subspaces. The component in the latter subspace consists of pure noise, whereas the other component consists of the pure signal plus noise. The practical implementation of these methods typically involves the formation of a Toeplitz or Hankel signal matrix  $X$  in such a way that the cross product  $X^T X$  is a scaled estimate of the signal's covariance matrix and such that the desired signal subspace is a proper subspace of  $\mathcal{R}(X)$ .

### A. Full-Rank Prewhitener

If the noise in the signal is additive and white and is uncorrelated to the desired signal, then the signal subspace simply

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consists of the principal left singular vectors of the signal matrix  $X$ ; see, e.g., [4] and [18]. If the noise is not white, then it is still possible to use the same basic approach, provided that the covariance matrix  $C$  for the noise has full rank. The key idea is to use prewhitening; if  $C$  has the Cholesky factorization  $C = R^T R$ , then the matrix  $X R^{-1}$  represents a new signal whose noise component is white, and the principal left singular vectors of this matrix form the desired signal subspace.

The matrix  $\hat{X}_{\text{filt}}$  that represents the filtered signal is then obtained by modifying the singular values of  $X R^{-1}$  followed by right multiplication with  $R$  (“dewhitening”). Different optimality criteria lead to different rules for modifying the singular values; for example, applying the least squares (LS) criterion to the prewhitened signal corresponds to solving the problem

$$\min \|(X - \hat{X}_{\text{filt}})R^{-1}\|_F \quad \text{s.t.} \quad \text{rank}(\hat{X}_{\text{filt}}) = k.$$

Similarly, the minimum variance (MV) criterion leads to  $\hat{X}_{\text{filt}} = XT$ , where the matrix  $T$  solves the problem

$$\min \|(XT - X_{\text{pure}})R^{-1}\|_F$$

in which  $X_{\text{pure}}$  is the signal matrix for the pure signal.

We emphasize that this approach requires that we are able to estimate  $C$ , typically by forming a “noise matrix”  $E$  from samples of pure noise (similar to  $X$ ) such that  $E^T E$  is a scaled estimate of  $C$ . This is often possible; e.g., in speech processing applications, the noise can be recorded in speechless frames.

The basis for our analysis is the *quotient singular value decomposition* (QSVD)<sup>1</sup> of the two matrices  $X$  and  $E$ . Let  $X \in \mathbb{R}^{m_X \times n}$  and  $E \in \mathbb{R}^{m_E \times n}$  with  $m_X \geq n$  and  $m_E \geq n$  and assume that  $\text{rank}(E) = p \leq n$ . Moreover, assume that the matrix  $(X^T, E^T)$  as well as  $E$  have full rank, i.e.,  $p = n$  (this case was studied in [13]). Then, the QSVD takes the form

$$X = Q_X \Sigma \Theta^T, \quad E = Q_E M \Theta^T$$

where  $Q_X \in \mathbb{R}^{m_X \times n}$  and  $Q_E \in \mathbb{R}^{m_E \times n}$  have orthonormal columns;  $\Theta \in \mathbb{R}^{n \times n}$  is a nonsingular matrix; and  $\Sigma$  and  $M$  are diagonal  $n \times n$  matrices satisfying  $\Sigma^2 + M^2 = I_n$ .

In this case,  $E$  has the QR factorization  $E = QR$ , where  $R$  is the above-mentioned Cholesky factor, and the Moore–Penrose pseudoinverse of  $E$  is given by

$$E^\dagger = R^{-1}Q^T = (\Theta^T)^{-1}M^{-1}Q_E^T.$$

Hence, the matrix quotient

$$X E^\dagger = X R^{-1} Q^T = Q_X (\Sigma M^{-1}) Q_E^T$$

has the same singular values and left singular vectors as the prewhitened matrix  $X R^{-1}$  introduced above. Moreover, we see that when using the prewhitener  $E^\dagger$ , the QSVD immediately provides the SVD of the matrix quotient  $X E^\dagger$ , and thus, the desired signal subspace is immediately available from the QSVD

in the form of the vectors of  $Q_X$ . The prewhitener is only used in the formulation of the problem; neither  $E^\dagger$  nor  $R^{-1}$  is explicitly needed. To see that the prewhitened noise is indeed white, we note that  $(E E^\dagger)^T E E^\dagger = Q_E Q_E^T$  is the covariance matrix for white noise in the subspace  $\mathcal{R}(Q_E)$ .

As shown in [12] and [13], different optimality criteria lead to different formulas for the reconstructed signal, and a common feature is that the filtering is achieved by multiplying the singular values with appropriate factors. Hence, to compute the filtered matrix  $\hat{X}_{\text{filt}}$  via the QSVD, we first modify the singular values of the matrix quotient  $X E^\dagger$  and then right-multiply with  $E$ . Inserting the QSVD, it is easy to see that the complete process can be written as

$$\hat{X}_{\text{filt}} = Q_X \hat{\Psi} \Sigma \Theta^T$$

where  $\hat{\Psi}$  denotes a diagonal filter matrix [12], [13]. It follows immediately that the covariance matrix for the filtered signal is given by

$$\hat{X}_{\text{filt}}^T \hat{X}_{\text{filt}} = \Theta \hat{\Psi}^2 \Sigma^2 \Theta^T.$$

The LS estimate of rank  $k$  is obtained by choosing  $\hat{\Psi} = \text{diag}(I_k, 0)$ , such that the  $k$  largest elements of  $\Sigma$  are retained while the rest are discarded. The MV estimate leads to the choice  $\hat{\Psi} = I_n - (M \Sigma^{-1})^2$ .

### B. Low-Rank Prewhitener

The prewhitening described above breaks down when the covariance matrix  $C$  is rank deficient and the matrix  $R^{-1}$ , therefore, no longer exists. One might be tempted to still use the pseudoinverse  $E^\dagger$ , but the numerical results in [11] demonstrate that there is a better solution.

We now analyze the case with a rank-deficient noise matrix  $E$ , i.e.,  $\text{rank}(E) = p < n$ , still assuming that the matrix  $(X^T, E^T)$  has full rank. Then, the QSVD takes the form

$$X = Q_X \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix} \Theta^T \quad (1)$$

$$E = Q_E \begin{pmatrix} M & 0 \end{pmatrix} \Theta^T \quad (2)$$

where again,  $Q_X \in \mathbb{R}^{m_X \times n}$  and  $Q_E \in \mathbb{R}^{m_E \times p}$  have orthonormal columns;  $\Theta \in \mathbb{R}^{n \times n}$  is a nonsingular matrix; and  $\Sigma$  and  $M$  are diagonal  $p \times p$  matrices satisfying  $\Sigma^2 + M^2 = I_p$ ; see [3, Sec. 4.2.2] for more details. It is convenient to partition the two matrices  $Q_X$  and  $\Theta$  into submatrices

$$Q_X = (Q_{X1}, Q_{X2}), \quad \Theta = (\Theta_1, \Theta_2) \quad (3)$$

with  $p$  and  $n - p$  columns, respectively.

By means of the QSVD (1) and (2), we can express the scaled covariance matrix of the observed signal as

$$\begin{aligned} X^T X &= \Theta \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix}^2 \Theta^T \\ &= \Theta_1 \Sigma^2 \Theta_1^T + \Theta_2 \Theta_2^T. \end{aligned} \quad (4)$$

<sup>1</sup>The QSVD is also known as the generalized SVD (GSVD).

This expression shows that we can consider the observed signal as a sum of two signal components with covariance matrices  $\Theta_1 \Sigma^2 \Theta_1^T$  and  $\Theta_2 \Theta_2^T$ , respectively. Moreover, since the scaled covariance matrix for the noise is

$$E^T E = \Theta_1 M^2 \Theta_1^T,$$

we see that the first signal component of (4) is associated with the same  $p$ -dimensional subspace  $\mathcal{R}(\Theta_1)$  as the rank-deficient noise, whereas the second component is associated with the “noise-free” subspace  $\mathcal{R}(\Theta_2)$ . These two subspaces are disjoint but not orthogonal.

The key observation is that the second signal component, lying in  $\mathcal{R}(\Theta_2)$ , is not influenced by the noise. Only the first component, lying in  $\mathcal{R}(\Theta_1)$ , is affected by the noise, and only this component needs to be filtered.

This analysis also sheds light on the existence and form of a matrix that can take the place of the full-rank prewhitening matrices  $R^{-1}$  and  $E^\dagger$ . Inserting the QSVD, it follows immediately that if we multiply  $X$  from the right with the matrix

$$E_X^\dagger = (\Theta^T)^{-1} \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} Q_E^T \quad (5)$$

then we obtain

$$X E_X^\dagger = Q_X \begin{pmatrix} \Sigma M^{-1} \\ 0 \end{pmatrix} Q_E^T = Q_{X1} (\Sigma M^{-1}) Q_E^T \quad (6)$$

which is the desired SVD of  $X E_X^\dagger$  and is expressed entirely in terms of QSVD quantities. The matrix  $E_X^\dagger$  in (5) is known as the  $X$ -weighted pseudoinverse of  $E$ ; see [7] for details about the weighted pseudoinverse and its relation to the QSVD.

We conclude that the covariance matrix for the filtered signal takes the form

$$\hat{X}_{\text{filt}}^T \hat{X}_{\text{filt}} = \Theta_1 \hat{\Psi}^2 \Sigma^2 \Theta_1^T + \Theta_2 \Theta_2^T$$

where  $\hat{\Psi}$  is a diagonal  $p \times p$  filter matrix constructed from the SVD of  $X E_X^\dagger$  (depending on the chosen optimality criterion). This, in turn, corresponds to writing the reconstructed matrix as

$$\hat{X}_{\text{filt}} = Q_X \begin{pmatrix} \hat{\Psi} \Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix} \Theta^T. \quad (7)$$

The important observation here is that we have partitioned the subspace  $\mathcal{R}(X) = \mathcal{R}(Q_X)$  in two *orthogonal* subspaces in such a way that the subspace  $\mathcal{R}(Q_{X2})$ , of dimension  $n - p$ , contains a pure signal component, and the  $p$ -dimensional subspace  $\mathcal{R}(Q_{X1})$  contains the remaining pure signal component plus all the noise. This particular splitting lets us restrict the filtering to the latter subspace, thus preserving the pure signal component in  $\mathcal{R}(Q_{X2})$ .

Again, we must verify that the prewhitened noise is white, and therefore, we consider the matrix  $E E_X^\dagger$ . Inserting the QSVD, we obtain

$$\begin{aligned} E E_X^\dagger &= Q_E (M, 0) \Theta^T (\Theta^T)^{-1} \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} Q_E^T \\ &= Q_E Q_E^T \end{aligned}$$

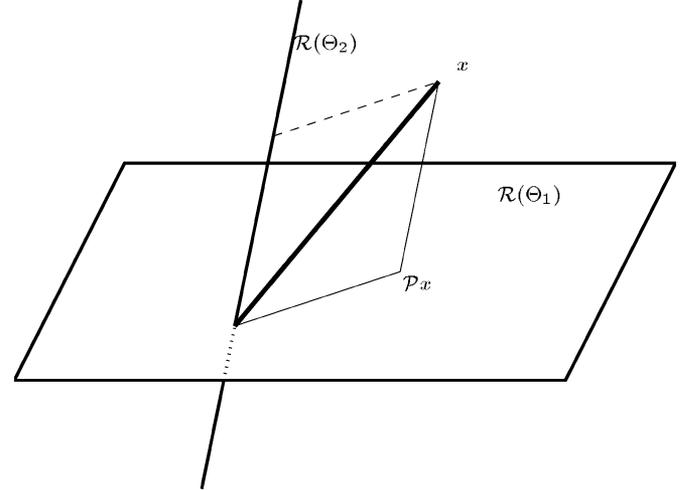


Fig. 1. Oblique projection. The subspaces  $\mathcal{R}(\Theta_1)$  and  $\mathcal{R}(\Theta_2)$  are represented by a plane and a line, intersecting at an angle less than  $90^\circ$ . Any vector  $x$  can be written as a sum  $x = \mathcal{P}x + (I - \mathcal{P})x$ , where  $\mathcal{P}x \in \mathcal{R}(\Theta_1)$  is the oblique projection of  $x$  on  $\mathcal{R}(\Theta_1)$  along  $\mathcal{R}(\Theta_2)$ .

and hence,  $(E E_X^\dagger)^T E E_X^\dagger$  is the covariance matrix for a white-noise signal in the subspace  $\mathcal{R}(Q_E)$ . This theory generalizes the existing theory from [13], and in the full-rank case, the two methods are identical because  $E_X^\dagger = E^\dagger$  when  $E$  has full column rank.

We emphasize that the above-mentioned orthogonal splitting of the subspace  $\mathcal{R}(X)$  is not achieved if we use the matrix  $E^\dagger$  as prewhitener. From the relation  $E^\dagger = \Pi E_X^\dagger$ , where  $\Pi$  is the orthogonal projector on  $\mathcal{R}(E^T)$ , it follows that the SVD of  $X E^\dagger$  is not obtained from the QSVD, and hence, we do not achieve a splitting of  $\mathcal{R}(X)$  with a noise-free component in a subspace of dimension  $n - p$ .

We note that the matrix  $Q_{X1} \Sigma \Theta_1^T$ , which represents the signal component in  $\mathcal{R}(\Theta_1)$ , can be written as

$$Q_{X1} \Sigma \Theta_1^T = X E_X^\dagger E$$

and that the matrix

$$\mathcal{P} = (E_X^\dagger E)^T = \Theta \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \Theta^{-1}$$

is the *oblique projector* onto  $\mathcal{R}(\Theta_1)$  along  $\mathcal{R}(\Theta_2)$ . Fig. 1 illustrates an oblique projection; see, e.g., [2], concerning the use of oblique projections in signal processing.

It is precisely the use of this oblique projection that allows us to prewhiten for rank-deficient noise via a splitting of  $\mathcal{R}(X)$  into two orthogonal subspaces, precisely in such a way that the signal component in the noise-free subspace  $\mathcal{R}(Q_{X2})$  is left unfiltered, whereas only the component in the noisy subspace  $\mathcal{R}(Q_{X1})$  is filtered. The Moore–Penrose pseudoinverse  $E^\dagger$  (for which the symmetric matrix  $E^\dagger E$  is an orthogonal projector) does not provide this favorable subspace splitting.

### C. Examples of Low-Rank Prewhitening

Before turning to computationally efficient methods for working with full-rank and low-rank prewhitening, it is worthwhile to illustrate the ideas of the low-rank prewhitening

TABLE I  
AMPLITUDES  $a_j$  OF THE PURE SIGNAL AND THE PURE NOISE AND  
AMPLITUDES  $\hat{a}_j^{(k)}$  OF THE RECONSTRUCTION

$j$	1	2	3	4	5	6
$f_j$	25	50	21	30	45	57
$a_j$	3.00	5.00	0.80	0.80	0.80	0.80
$\hat{a}_j^{(0)}$	1.58	2.51	0.04	0.11	0.12	0.04
$\hat{a}_j^{(1)}$	1.82	4.36	0.01	0.09	0.23	0.18
$\hat{a}_j^{(2)}$	3.02	5.03	0.09	0.24	0.41	0.25
$\hat{a}_j^{(3)}$	3.01	4.97	0.23	0.30	0.44	0.31
$\hat{a}_j^{(4)}$	2.99	4.99	0.48	0.58	0.52	0.40

approach described above. We do this by two simple examples. The QSVD is computed by Matlab's `gsvd` function.

*Sinusoids in Low-Rank Noise:* The pure signal  $\bar{x}$  has length  $N = 128$  and is a sum of two sampled sinusoids

$$\bar{x}_i = a_1 \sin\left(i \frac{2\pi}{N} f_1\right) + a_2 \sin\left(i \frac{2\pi}{N} f_2\right), \quad i = 1, \dots, N$$

whose amplitudes and frequencies are given in Table I. The noise is an interfering signal consisting of a sum of four sinusoids

$$e_i = \sum_{j=3}^6 a_j \sin\left(i \frac{2\pi}{N} f_j + \phi_j\right), \quad i = 1, \dots, N$$

whose amplitudes and frequencies are also given in Table I, whereas their phases are chosen randomly. The observed signal is  $x = \bar{x} + e$ . The signal matrix  $X$  is the  $119 \times 10$  Hankel matrix defined by the vector  $x$ , and this matrix has full rank, i.e.,  $\text{rank}(X) = n = 10$ . The matrix corresponding to the pure signal  $\bar{x}$  has rank 4. Finally, the matrix  $E$  for the pure noise has  $\text{rank}(E) = p = 8$ , i.e.,  $E$  is rank deficient.

Since the pure signal matrix has rank 4, we expect that a signal subspace of dimension 4 will lead to the best reconstruction. In our experiments, we choose the LS filter matrix  $\hat{\Psi}$  that selects the  $k$  largest values of  $\Sigma$ , thus ensuring that  $\text{rank}(\hat{X}_{\text{filt}}) = n - p + k = 2 + k$ . Finally the reconstructed signal is obtained by averaging along the antidiagonals of  $\hat{X}_{\text{filt}}$  [13].

Table I lists the amplitudes  $\hat{a}_j^{(k)}$  of the reconstructed signal at the six relevant frequencies, computed via the FFT. For  $k = 0$  and  $k = 1$ , the dimension of the signal subspace is not large enough to capture the desired signal, whereas it is well reconstructed for  $k = 2, 3$ , and 4. As  $k$  increases, the signal subspace captures an increasing amount of the low-rank noise. For  $k = 2$ , the errors in  $a_1^{(2)}$  and  $a_2^{(2)}$  are less than 1%, and all four noise amplitudes  $\hat{a}_j^{(2)}$  for  $j = 3, 4, 5$ , and 6 are damped.

For  $k = 2$ , the dimensions of the estimated and the pure signal subspaces are both equal to 4, and we can compute the angle between these two subspaces. The angle is 0.24 rad (about  $14^\circ$ ), which is quite small compared with the large SNR in the noisy signal.

*Voiced Speech in Additive Low-Rank Noise:* The pure signal is a voiced speech signal of length  $N = 160$  and sampled at 8 kHz, whereas the low-rank noise is an interfering signal con-

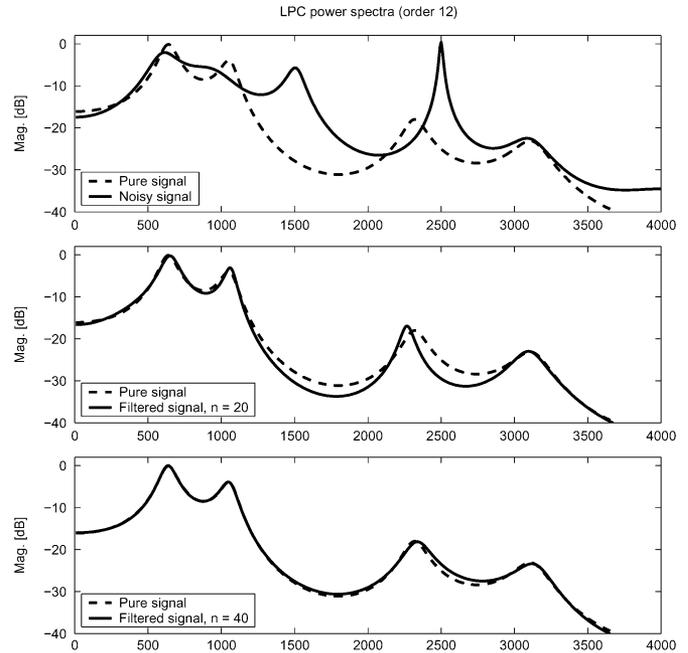


Fig. 2. LPC power spectra of order 12. (Top) Pure and noisy speech signals, the noise consisting of two sinusoids at 1.5 kHz and 2.5 kHz. (Middle and bottom) Pure and filtered signals using  $k = 0$ , for  $n = 20$  and  $n = 40$ .

sisting of a sum of two sinusoids with unit amplitude, random phase, and frequencies  $f_1 = 1.5$  kHz and  $f_2 = 2.5$  kHz. These two frequencies are selected such that  $f_1$  is between the second and the third formant, whereas  $f_2$  is close to the fourth formant. The signal-to-noise ratio is 5 dB.

The data matrix  $X$  and the noise matrix  $E$  are again Hankel matrices with  $n$  columns. The noise matrix has rank  $p = 4$ , whereas the data matrix has full rank. In order to suppress the interference as much as possible, we choose  $k = 0$ , i.e., our reconstructed signal lies solely in the noise-free subspace  $\mathcal{R}(\Theta_2)$  of dimension  $n - 4$ . Moreover, we use  $n = 20$  and  $n = 40$  to illustrate the relation between matrix dimensions and noise-reduction performance.

The 12th-order LPC spectra of the pure signal, the observed noisy signal, and the two reconstructed signals are shown in Fig. 2. Clearly, we are able to suppress the noise by the QSVD approach.

### III. IMPLEMENTATION BY THE RANK-REVEALING QUOTIENT ULV DECOMPOSITION

Although the QSVD is ideal for defining the weighted pseudoinverse and the low-rank prewhitening algorithm, the QSVD algorithm may be too computationally demanding for realtime applications. Hence, we need an alternative decomposition, which is easier to compute and update, and yields good approximations to the quantities in the QSVD. The rank-revealing quotient ULV (QULV) decomposition, which is also referred to as the ULLV decomposition, is such a tool.

#### A. ULV and QULV Decompositions

Before introducing the QULV, we first briefly describe the ULV decomposition [17], which was introduced as a computa-

tionally attractive alternative to the SVD. The ULV decomposition of  $X$  takes the form

$$X = ULV^T = (U_r, U_o) \begin{pmatrix} L_r & 0 \\ F & G \end{pmatrix} (V_r, V_o)^T$$

where  $U$  and  $V$  have orthonormal columns,  $L$  is lower triangular, and the numerical rank of  $X$  is revealed in  $L$  in the sense that both norms  $\|F\|_2$  and  $\|G\|_2$  are small. Hence, the matrix  $U_r L_r V_r^T$  is a low-rank approximation to  $X$ , and the range of  $U_r$  is an approximation to the desired signal subspace. The ULV decomposition can therefore replace the SVD in subspace algorithms for the white-noise case.

The QULV decomposition<sup>2</sup> [11], [15] factors the two matrices  $X$  and  $E$  as products of a left orthogonal matrix, one or two lower triangular matrices, and a common right orthogonal matrix. Specifically, the QULV decomposition takes the form

$$X = U_X L \begin{pmatrix} \bar{L} & 0 \\ 0 & I_{n-p} \end{pmatrix} V^T \quad (8)$$

$$E = U_E (\bar{L}, 0) V^T \quad (9)$$

in which  $U_X \in \mathbb{R}^{m_X \times n}$ ,  $U_E \in \mathbb{R}^{m_E \times p}$ , and  $V \in \mathbb{R}^{n \times n}$  have orthonormal columns, whereas  $L \in \mathbb{R}^{n \times n}$  and  $\bar{L} \in \mathbb{R}^{p \times p}$  are lower triangular.<sup>3</sup> Similar to the QSVD, it is convenient to work with the partitionings

$$U_X = (U_{X1}, U_{X2}), \quad V = (V_1, V_2)$$

and

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

where  $U_{X1}$  and  $V_1$  have  $p$  columns, and  $L_{11}$  is  $p \times p$ .

The similarity between the QULV and the QSVD is perhaps better revealed by rewriting the QULV decomposition in the form

$$X = U_X \begin{pmatrix} L_{11} & 0 \\ 0 & I_{n-p} \end{pmatrix} (\bar{\Theta}_1, \bar{\Theta}_2)^T$$

$$E = U_E (I_p, 0) (\bar{\Theta}_1, \bar{\Theta}_2)^T$$

where we have defined the matrix  $\bar{\Theta} = (\bar{\Theta}_1, \bar{\Theta}_2)$  with

$$\bar{\Theta}_1 = V_1 \bar{L}^T$$

and

$$\bar{\Theta}_2 = V_1 \bar{L}^T L_{21}^T + V_2 L_{22}^T = \bar{\Theta}_1 L_{21}^T + V_2 L_{22}^T.$$

The column spaces of the QULV matrices  $U_{X1}$ ,  $U_{X2}$ ,  $U_E$ ,  $\bar{\Theta}_1$  and  $\bar{\Theta}_2$  are approximations to the column spaces of the corresponding QSVD matrices  $Q_{X1}$ ,  $Q_{X2}$ ,  $Q_E$ ,  $\Theta_1$  and  $\Theta_2$ , respectively.

When  $E$  has full rank, the matrices  $U_{X2}$ ,  $I_{n-p}$ , and  $V_2$  vanish, and the QULV takes the simpler form  $X = U_X \bar{L} \bar{L}^T$  and  $E = U_E \bar{L} V^T$ . This is the original version of the QULV from [14].

<sup>2</sup>Matlab software for computing the rank-revealing QULV decomposition in the full-rank and rank-deficient cases is available in [10] and [9], respectively.

<sup>3</sup>We emphasize that  $L$  and  $V$  in the ULV and QULV decompositions are not the same matrices.

The QULV decomposition is rank-revealing in the following sense. As shown in [11], the  $X$ -weighted pseudoinverse of  $E$  can be written in terms of the QULV decomposition as

$$E_X^\dagger = (\bar{\Theta}^T)^{-1} \begin{pmatrix} I_p \\ 0 \end{pmatrix} U_E^T = V \begin{pmatrix} \bar{L}^{-1} \\ -L_{22}^{-1} L_{21} \end{pmatrix} U_E^T.$$

Consequently, the matrix quotient  $X E_X^\dagger$  can be expressed in terms of the QULV factors simply as

$$X E_X^\dagger = U_{X1} L_{11} U_E^T$$

which is a rank-revealing ULV decomposition of  $X E_X^\dagger$ . Hence, the numerical rank of  $X E_X^\dagger$  is immediately revealed in the  $p \times p$  triangular submatrix  $L_{11}$  in the QULV. To incorporate filtering, we must filter or truncate this submatrix. Hence, the QULV-based reconstruction takes the form

$$\hat{X}_{\text{filt}} = U_X \begin{pmatrix} \hat{\Psi} L_{11} & 0 \\ 0 & I_{n-p} \end{pmatrix} (\bar{\Theta}_1, \bar{\Theta}_2)^T$$

$$= U_{X1} \hat{\Psi} L_{11} \bar{\Theta}_1^T + U_{X2} \bar{\Theta}_2^T$$

where  $\hat{\Psi}$  is the filter matrix [12].

The covariance matrix for  $\hat{X}_{\text{filt}}$  thus takes the form

$$\hat{X}_{\text{filt}}^T \hat{X}_{\text{filt}} = \bar{\Theta}_1 L_{11}^T \hat{\Psi}^T \hat{\Psi} L_{11} \bar{\Theta}_1^T + \bar{\Theta}_2 \bar{\Theta}_2^T.$$

This expression shows that, again, the reconstructed signal has a filtered component lying in the  $p$ -dimensional subspace  $\mathcal{R}(\bar{\Theta}_1)$  and an unfiltered component in the subspace  $\mathcal{R}(\bar{\Theta}_2)$  of dimension  $n - p$ . The two subspaces are disjoint but not orthogonal, and in the Appendix, we prove that if  $\angle(\bar{\Theta}_1, \bar{\Theta}_2)$  denotes the subspace angle between the column spaces of  $\bar{\Theta}_1$  and  $\bar{\Theta}_2$ , then

$$\cos \angle(\bar{\Theta}_1, \bar{\Theta}_2) \leq \|L_{22}^{-1} L_{21} \bar{L}\|_2 \quad (10)$$

showing that the smaller the norm of  $L_{21}$ , the larger the angle between the two subspaces.

We note in passing that there is also an equivalent quotient URV decomposition with upper triangular matrices. However, the analysis in [11] shows that this decomposition is impractical in connection with the applications that we have in mind.

## B. Examples of the QULV-Based Algorithm

We illustrate the use of the QULV-based algorithm by means of the two examples from the previous section. The QULV decomposition is computed with the Matlab function `ulliv` from [9].

*Sinusoids in Low-Rank Noise:* We applied the QULV-based algorithm to the first test problem and computed a least squares estimate by keeping the leading  $k \times k$  block of  $L_{11}$  and setting the remaining elements of  $L_{11}$  to zero. The reconstructions are of essentially the same quality as those computed by means of the QSVD; cf. Table II. The subspace angle (for  $k = 2$ ) between the exact and estimated signal subspaces is, again, 0.24 rad. This illustrates that the QULV decomposition is indeed able to yield good approximations to the quantities defined by the QSVD.

*Voiced Speech in Additive Low-Rank Noise:* We also applied the QULV algorithm to the second test problem, using only the component of the solution in  $\mathcal{R}(\bar{\Theta}_2)$ . We obtained reconstructed

TABLE II  
AMPLITUDES  $a_j$  OF THE PURE SIGNAL AND THE PURE NOISE AND AMPLITUDES  $\hat{a}_j^{(k)}$  OF THE QULV-BASED RECONSTRUCTION

$j$	1	2	3	4	5	6
$f_j$	25	50	21	30	45	57
$a_j$	3.00	5.00	0.80	0.80	0.80	0.80
$\hat{a}_j^{(0)}$	1.58	2.51	0.04	0.11	0.13	0.04
$\hat{a}_j^{(1)}$	2.45	3.59	0.07	0.22	0.19	0.10
$\hat{a}_j^{(2)}$	2.94	4.94	0.07	0.24	0.38	0.20
$\hat{a}_j^{(3)}$	3.00	5.01	0.26	0.37	0.50	0.34
$\hat{a}_j^{(4)}$	2.95	4.98	0.36	0.41	0.58	0.42

signals whose LPC power spectra are very similar to those obtained by means of the QSVD; the spectral distance between the QSVD and QULV spectra is of the order 1 dB for  $n = 20$  and less than 1 dB for  $n = 40$ .

#### IV. RANK-DEFICIENT CASE

So far, we have assumed that the matrix  $(X^T, E^T)$  has full rank. A rank-deficient  $(X^T, E^T)$  implies that the order  $n$  of the model used for describing the system ( $n$  is the size of the covariance matrices of  $X$  and  $E$ ) is larger than necessary; the noisy signal lies in a subspace whose dimension is less than  $n$ . Therefore, one cure for rank deficiency is to reduce the order  $n$ .

However, for the generality of our algorithms, it is important to be able to treat the rank-deficient case because this allows an implementation with a fixed  $n$ . We will now demonstrate that the QSVD and QULV decompositions described above can also be used to handle this case.

The QSVD and QULV are not unique when  $(X^T, E^T)$  is rank deficient, and different algorithms lead to different formulations. We use the following approach: 1) Compute a rank-revealing ULV decomposition  $E = UL_E V_E^T$  with  $L_E \in \mathbb{R}^{p \times p}$ , 2) compute the QSVD or QULV of the matrix pair  $(X, L_E V_E^T)$  using either Matlab's `gsvd` function or the QULV function `ulliv` from [9], and 3) left-multiply  $U$  on the left orthogonal factor of  $L_E V_E^T$ . The advantage is that the resulting QSVD and QULV have the forms<sup>4</sup> (1)–(2) and (8)–(9), respectively, which lets us easily extend the previous results.

From the condition  $\Sigma^2 + M^2 = I_p$ , it follows that the middle matrix in the expression

$$\begin{pmatrix} X \\ E \end{pmatrix} = \begin{pmatrix} Q_X & 0 \\ 0 & Q_E \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \\ M & 0 \end{pmatrix} \Theta^T$$

has full rank. The left-most matrix has orthonormal columns and, therefore, full rank as well. Hence,  $\text{rank}((X^T, E^T)) = \text{rank}(\Theta)$ , i.e., any rank deficiency must manifest itself in the matrix  $\Theta$ . Consequently, when  $(X^T, E^T)$  is rank deficient, we cannot infer about the ranks of  $X$  and  $E$  merely from inspection of  $\Sigma$  and  $M$  (we refer again to [3] for details).

<sup>4</sup>Computing the QSVD of  $(X, E)$  directly, e.g., via Matlab's `gsvd` function, does not yield the form (1) and (2)

The situation is the same in the QULV setting, in which

$$\begin{pmatrix} X \\ E \end{pmatrix} = \begin{pmatrix} U_X & 0 \\ 0 & U_E \end{pmatrix} \begin{pmatrix} L_{11} & 0 \\ 0 & I_{n-p} \\ I_p & 0 \end{pmatrix} \bar{\Theta}^T$$

showing that  $\text{rank}((X^T, E^T)) = \text{rank}(\bar{\Theta})$ . A closer look at

$$\bar{\Theta}^T = \begin{pmatrix} I_p & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \bar{L} & 0 \\ 0 & I_{n-p} \end{pmatrix} V^T$$

reveals that any rank deficiency in  $\text{rank}((X^T, E^T))$  manifests itself in  $L_{22}$  being singular because  $\bar{L}$  has full rank.

#### A. QSVD Algorithm

To extend the QSVD algorithm from Section II-B to the case where  $\Theta$  is rank deficient, we seek a prewhitening matrix  $E_X^\oplus$  of the form

$$E_X^\oplus = ZM^{-1}Q_E^T$$

where  $Z \in \mathbb{R}^{n \times p}$  is a matrix to be determined. There are two requirements to  $E_X^\oplus$ , namely, that  $EE_X^\oplus$  must represent white noise and that  $XE_X^\oplus$  must represent a prewhitened signal with no component in the noise-free subspace. From the expressions

$$\begin{aligned} EE_X^\oplus &= Q_E M \Theta_1^T Z M^{-1} Q_E^T \\ XE_X^\oplus &= Q_X \begin{pmatrix} \Sigma \Theta_1^T Z \\ \Theta_2^T Z \end{pmatrix} M^{-1} Q_E^T \end{aligned}$$

we see that the two requirements are achieved if we choose  $Z$  such that  $M \Theta_1^T Z M^{-1}$  is an orthogonal projection matrix and such that  $\Theta_2^T Z = 0$ .

It is straightforward to show that if  $W$  is a matrix whose columns span the null space of  $\Theta_2^T$ , then the choice

$$Z = W (M \Theta_1^T W)^\dagger M$$

satisfies both requirements. Specifically, we obtain

$$\begin{aligned} EE_X^\oplus &= Q_E P Q_E^T \\ XE_X^\oplus &= Q_{X1} \Sigma M^{-1} P Q_E^T \end{aligned}$$

with the orthogonal projection matrix  $P$  given by

$$P = M \Theta_1^T W (M \Theta_1^T W)^\dagger.$$

We remark that if  $\Theta$  has full rank, then  $W$  consists of the first  $p$  columns of  $(\Theta^T)^{-1}$ , and consequently,  $Z = W, \Theta_1^T Z = I_p, P = I_p$ , and  $E_X^\oplus = E_X^\dagger$ . Therefore, our choice of the prewhitening matrix  $E_X^\oplus$  is a natural extension of the weighted pseudoinverse  $E_X^\dagger$ .

The QSVD algorithm from Section II-B never forms the matrices  $E_X^\dagger$  and  $XE_X^\dagger$  explicitly; it only needs the diagonal matrix  $\Sigma M^{-1}$  to reveal the rank of  $XE_X^\dagger$  in (6). The desired signal is then reconstructed from  $\hat{X}_{\text{filt}}$  in (7).

When  $\Theta$  is rank deficient, we should ideally work with the prewhitened matrix  $XE_X^\oplus$ . However, it is not practical to compute the matrix  $P$ , and instead, we prefer to use the

original QSVD algorithm and ignore  $P$ . To understand the consequence of this, we need to examine  $XE_X^\oplus$  closer. Assume that  $\text{rank}(\Theta_1) = q < p$  and write  $P = QQ^T$  with  $Q \in \mathbb{R}^{p \times q}$ . Then,  $XE_X^\oplus$  takes the form

$$XE_X^\oplus = Q_{X1}(\Sigma M^{-1}Q)(U_E Q)^T.$$

If we ignore  $P$  (and thus  $Q$ ), then the dimension of the signal subspace and the filter matrix  $\hat{\Psi}$  are computed solely from  $\Sigma M^{-1}$ . Ideally, however, they should be computed from an SVD of the matrix  $\Sigma M^{-1}Q$ .

Hence, if the number of columns of  $\Sigma M^{-1}Q$  with large norm is smaller than the number of large elements in  $\Sigma M^{-1}$ , then the signal subspace based on  $\Sigma M^{-1}$  may be too large, i.e., it may include noise components. The opposite situation, where the dimension is chosen too small such that genuine signal components are ignored, cannot happen. For this reason, we believe that it is safe to use the original QSVD algorithm, independent of the rank of  $(X^T, E^T)$  and, thus, avoiding working with the projection matrix  $P$ .

### B. QULV Algorithm

We now repeat the above analysis for the QULV algorithm from Section III-A. When  $\bar{\Theta}$  is rank deficient, we seek a matrix  $\bar{Z}$  such that  $E_X^\oplus = V\bar{Z}U_E^T$  and such that the two previous requirements on

$$EE_X^\oplus = U_E(\bar{L}, 0)\bar{Z}U_E^T$$

and

$$XE_X^\oplus = U_X \begin{pmatrix} (L_{11}\bar{L}, 0)\bar{Z} \\ (L_{21}\bar{L}, L_{22})\bar{Z} \end{pmatrix} U_E^T$$

are again satisfied, i.e., such that  $(L_{11}\bar{L}, 0)\bar{Z}$  is an orthogonal projection matrix, and  $(L_{21}\bar{L}, L_{22})\bar{Z} = 0$ . If  $\bar{W}$  is a matrix whose columns span the null space of  $(L_{21}\bar{L}, L_{22})$ , then

$$\bar{Z} = \bar{W}((L_{11}\bar{L}, 0)\bar{W})^\dagger$$

and it follows that

$$\begin{aligned} EE_X^\oplus &= U_E \bar{P} U_E^T \\ XE_X^\oplus &= U_{X1} L_{11} \bar{P} U_E^T \end{aligned}$$

with the orthogonal projection matrix  $\bar{P}$  given by

$$\bar{P} = (L_{11}\bar{L}, 0)\bar{W}((L_{11}\bar{L}, 0)\bar{W})^\dagger.$$

We note that when  $\bar{\Theta}$  has full rank, then  $\bar{W} = \begin{pmatrix} I_p \\ -L_{22}^{-1}L_{21} \end{pmatrix}$ , and we obtain the results from Section III-A, showing that the above approach is a natural extension of the original QULV algorithm.

Let us now examine the influence of neglecting the matrix  $\bar{P}$  in the QULV algorithm. We write  $\bar{P} = \bar{Q}\bar{Q}^T$  such that

$$XE_X^\oplus = U_{X1}(L_{11}\bar{Q})(U_E\bar{Q})^T$$

showing that the decision about the signal subspace should ideally be based on the matrix  $L_{11}\bar{Q}$ . Hence, if the number of

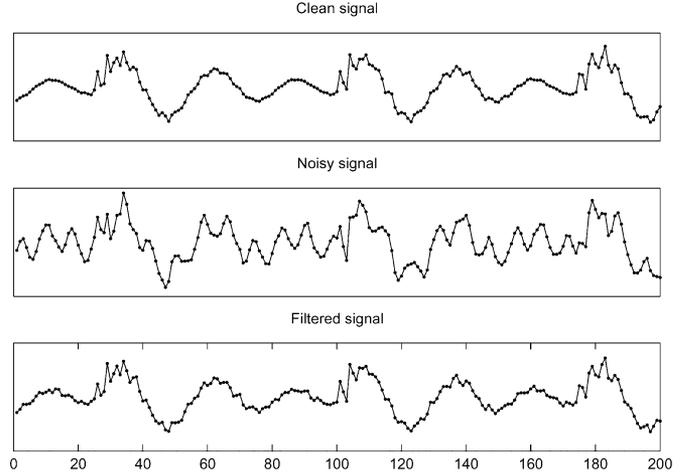


Fig. 3. Example with a speech signal and additive noise from a buzz saw. (Top) Pure voiced signal. (Middle) Noisy signal (SNR = 5 dB). (Bottom) Filtered signal (SNR = 18.6 dB) obtained with  $k = 12$ .

columns in  $L_{11}\bar{Q}$  with large norm is smaller than the number of large-norm columns  $L_{11}$ , then we might include noise components in the signal subspace. As before, the opposite situation cannot happen, i.e., there is no danger that we omit important signal components.

In conclusion, we find also in the QULV setting that it is safe to ignore  $\bar{P}$  (and, thus,  $\bar{Q}$ ) and use the original QULV algorithm, independent of the rank of  $(X^T, E^T)$ .

## V. NUMERICAL EXAMPLE

We illustrate the use of our algorithm with samples of a male voice signal contaminated by noise originating from a buzz saw with an overall signal-to-noise ratio of 5 dB. The sampling frequency is 8 kHz. We process the signal by splitting the full-time signal into frames of length 200 samples each and applying the QSVD algorithm in each signal frame using  $n = 40$ .

The noise signal from the buzz saw is dominated by a few harmonics whose frequency vary with time. Hence, the noise matrix  $E$  changes in each time frame; it is always rank deficient, and its rank changes between time frames. The noise reduction is achieved by maintaining the largest  $k$  values of  $\Sigma$  in (7) and discarding the rest. We use a different value of  $k$  each time.

Fig. 3 shows an example of the involved signals: the clean signal, the noisy signal, and the filtered signal obtained with  $k = 12$ . In this frame, the SNR has been improved by about 13 dB.

Fig. 4 shows LPC spectra for the signals in three different time frames. We used an LPC order of 20 in order to capture the spikes in the noise spectra. Above each plot, we give the SNR that was obtained in the corresponding time frame, together with the value of  $k$  that was used. In most cases, the QSVD algorithm is able to adaptively suppress the harmonics of the rank-deficient noise, but in the case of high-energy noise peaks, the algorithm may fail.

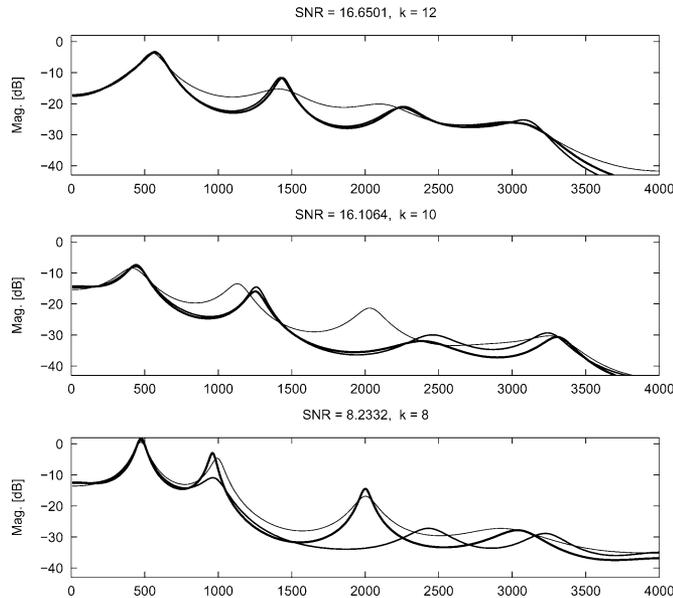


Fig. 4. LPC spectra (order 20) for three signal frames. (Thin line) Noisy signal. (Medium line) Clean signal. (Thick line) Filtered signal. For each time frame, we give the achieved SNR and the number of QSVD components  $k$  that was used.

## VI. CONCLUSION

We have used the results from [11] to extend the QSVD algorithm [13] to the case with narrowband noise, where the covariance matrix of the noise is rank deficient. In particular, we demonstrated that the QSVD of the signal and noise matrices produces all the quantities necessary to perform the rank reduction and construct the signal subspace for the reconstruction. We also demonstrated how the algorithm can be formulated in terms of the rank-revealing QULV algorithm, which has lower computational complexity and is better suited for updating. Finally, we demonstrated the efficiency of the QSVD and QULV algorithms with numerical examples involving speech signals and rank-deficient noise.

## APPENDIX

To prove (10), we first note that a left orthogonal transformation does not change the subspace angle, and therefore, the angle between  $\bar{\Theta}_1$  and  $\bar{\Theta}_2$  is identical to the angle between the ranges of

$$\begin{pmatrix} \bar{L}^T \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{L}^T L_{21}^T \\ L_{22}^T \end{pmatrix}.$$

From the assumption that  $(X^T, E^T)$  has full rank, it follows that both  $\bar{L}$  and  $L_{22}$  have full rank. The range of a matrix is not altered by right-multiplication with a full-rank matrix, and hence, we seek the subspace angle between the ranges of

$$\begin{pmatrix} I_p \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{L}^T L_{21}^T L_{22}^{-T} \\ I_{n-p} \end{pmatrix}. \quad (11)$$

Let  $QR$  denote the ‘‘skinny’’ QR factorization of the latter matrix. Then

$$\begin{aligned} \cos \angle(\bar{\Theta}_1, \bar{\Theta}_2) &= \left\| \begin{pmatrix} I_p \\ 0 \end{pmatrix} \begin{pmatrix} I_p \\ 0 \end{pmatrix}^T Q \right\|_2 \\ &= \left\| \begin{pmatrix} \bar{L}^T L_{21}^T L_{22}^{-T} \\ 0 \end{pmatrix} R^{-1} \right\|_2 \\ &\leq \|\bar{L}^T L_{21}^T L_{22}^{-T}\|_2 \|R^{-1}\|_2. \end{aligned}$$

Since the singular values of the second matrix in (11) are greater than or equal to one, it follows that  $\|R^{-1}\|_2 \leq 1$ , and thus,  $\cos \angle(\bar{\Theta}_1, \bar{\Theta}_2) \leq \|\bar{L}^T L_{21}^T L_{22}^{-T}\|_2$ .

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