Gabor Frames in 2(\(\mathbb{Z}\)) and Linear Dependence

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Gabor frames in $\ell^2(\mathbb{Z})$ and linear dependence

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Abstract

We prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finitely supported sequence is always linearly dependent. This is a particular case of a general result about linear dependence versus independence for Gabor systems in $\ell^2(\mathbb{Z})$ with modulation parameter $1/M$ and translation parameter $N$ for some $M, N \in \mathbb{N}$, and generated by a finite sequence $g$ in $\ell^2(\mathbb{Z})$ with $K$ nonzero entries.

Keywords: Frames, Gabor system in $\ell^2(\mathbb{Z})$, linear dependency of Gabor systems

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1 Introduction

Linear dependence versus linear independence is a well-studied topic in Gabor analysis. In particular Linnell [11] proved that any Gabor system in $L^2(\mathbb{R})$ generated by a nonzero function and a time-frequency lattice $a\mathbb{Z} \times b\mathbb{Z}$ is linearly independent, hereby confirming a conjecture by Heil, Ramanathan and Topiwala [4]. The analogous problem based on time-frequency shifts on a general locally compact abelian group was studied by Kutyniok in [9] and Gabor systems on finite groups were analyzed in the paper [10] by Lawrence, Pfander, and Walnut. Results by Jitomirskaya [8] imply that the conjecture would fail on $\ell^2(\mathbb{Z})$, as explained by Demeter and Gautam in [3].

The purpose of this short note is to give a more detailed discussion of frame properties and linear independence versus linear dependence for Gabor systems in $\ell^2(\mathbb{Z})$. In particular we prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finite sequence is always linearly dependent. Furthermore we collect and apply various methods for analysis of such frames, e.g., the duality
principle, sampling of Gabor frames for $L^2(\mathbb{R})$, and perturbation methods. For $g \in \ell^2(\mathbb{Z})$ we denote the $j$th coordinate by $g(j)$. For $M \in \mathbb{N}$, define the modulation operators $E_{m/M}, m = 0, \ldots, M-1$, acting on $\ell^2(\mathbb{Z})$ by $E_{m/M}g(j) := e^{2\pi i jm/M}g(j)$; also, define the translation operators $T_n, n \in \mathbb{Z}$, by $T_ng(j) = g(j+n)$. The Gabor system generated by a fixed $g \in \ell^2(\mathbb{Z})$ and some $M, N \in \mathbb{N}$ is $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m=0,\ldots,M-1}$; specifically, $E_{m/M}T_{nN}g$ is the sequence in $\ell^2(\mathbb{Z})$ whose $j$th coordinate is $E_{m/M}T_{nN}g(j) = e^{2\pi i jm/M}g(j-nN)$.

In the rest of this note we will write $\{E_{m/M}T_{nN}g\}$ instead of $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m=0,\ldots,M-1}$. It is well-known [2] that $\{E_{m/M}T_{nN}g\}$ can only be a frame for $\ell^2(\mathbb{Z})$ if $N/M \leq 1$. We prove that if $N/M < 1$, such frames can be constructed with windows $g$ having any number $K \geq N$ of nonzero entries; in contrast to the case of Gabor frames in $L^2(\mathbb{R})$ these frames are always linearly dependent. Similarly, for $M = N$ we can construct Riesz bases for $\ell^2(\mathbb{Z})$ with windows $g$ having any number $K \geq N$ of nonzero entries; however, for exactly the same parameter choices there also exist linearly dependent Gabor systems. More generally, we characterize the parameters $M, N, K$ for which the Gabor system is automatically linearly independent, linear dependent, resp. that both cases can occur depending on the choice of $g \in \ell^2(\mathbb{Z})$.

2 Gabor systems in $\ell^2(\mathbb{Z})$

For a finitely supported sequence $g \in \ell^2(\mathbb{Z})$, let $\|\text{supp } g\|$ denote the number of nonzero entries of $g$. For illustrations and concrete examples we will often use the sequences $\delta_k \in \ell^2(\mathbb{Z}), k \in \mathbb{Z}$, given by

$$
\delta_k(j) = \begin{cases} 
1 & \text{if } j = k, \\
0 & \text{if } j \neq k.
\end{cases}
$$

It was observed already by Lopez & Han [12] that for any $M, N \in \mathbb{N}$ with $N \leq M$ there exist frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by windows with $N$ nonzero elements. We will need the following extension, characterizing the existence of Gabor frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ with a given support size $K$.

**Theorem 2.1** Let $M, N, K \in \mathbb{N}$. Then the following hold:

(i) There exists a Gabor frame $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by a window $g$ with $\|\text{supp } g\| = K$ if and only if $N \leq M$ and $K \geq N$. 

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(ii) There exists a Riesz sequence \( \{E_{m/M}T_{nNg}\} \) in \( \ell^2(\mathbb{Z}) \) generated by a window \( g \) with \( |\text{supp } g| = K \) if and only if \( N \geq M \) and \( K \geq M \).

**Proof.** For the proof of (i), the necessity of the condition \( N \leq M \) is obvious. We will now show that if \( K < N \) then \( \{E_{m/M}T_{nNg}\} \) can not be complete in \( \ell^2(\mathbb{Z}) \). We do this by identifying some \( k \in \mathbb{Z} \) such that \( E_{m/M}T_{nNg}(k) = 0 \) for all \( n \in \mathbb{Z} \) and \( m \in \{0, \ldots, M-1\} \). Consider \( I := \{1, \ldots, N\} \); then, for any \( j \in \mathbb{Z} \), there exists exactly one value of \( n \in \mathbb{Z} \) such that \( j + nN \in I \). Since \( g(j) \neq 0 \) only occur for \( K < N \) values of \( j \), there exists some \( k \in I \) such that \( j + nN \neq k \) for all \( n \in \mathbb{Z} \) and all \( j \in \mathbb{Z} \) such that \( g(j) \neq 0 \). That is, \( k - nN \neq j \) for all \( n \in \mathbb{Z} \) and all \( j \in \mathbb{Z} \) such that \( g(j) \neq 0 \). Thus for all \( n \in \mathbb{Z} \), we have that \( g(k - nN) = 0 \). This proves that \( E_{m/M}T_{nNg}(k) = 0 \) for all \( n \in \mathbb{Z} \) and \( m \in \{0, \ldots, M-1\} \) and thus \( \{E_{m/M}T_{nNg}\} \) can not be complete if \( K < N \); in other words, \( K \geq N \) is necessary for \( \{E_{m/M}T_{nNg}\} \) to be a frame for \( \ell^2(\mathbb{Z}) \).

Now assume that \( N \leq M \) and consider any \( g \in \ell^2(\mathbb{Z}) \) for which

\[
\begin{align}
\text{All the vectors in } \{ E_{m/M}g \}_{m=0,\ldots,M-1} \text{ have support in } \{1, \ldots, N\}. \text{ Writing the coordinates for these vectors for } j \in \{1, \ldots, N\} \text{ as rows in an } M \times N \text{ matrix, we get}\n\end{align}
\]

\[
A = \left( \begin{array}{cccc}
g(1) & g(2) & \ldots & g(N) \\
\frac{e^{2\pi i}{1}}{M}g(1) & \frac{e^{2\pi i}{2}}{M}g(2) & \ldots & \frac{e^{2\pi i}{N}}{M}g(N) \\
\frac{e^{2\pi i}{1}}{M}g(1) & \frac{e^{2\pi i}{2}}{M}g(2) & \ldots & \frac{e^{2\pi i}{N}}{M}g(N) \\
\vdots & \ddots & \ddots & \vdots \\
\frac{e^{2\pi i}{M-1}}{M}g(1) & \frac{e^{2\pi i}{(M-1)}}{M}g(2) & \ldots & \frac{e^{2\pi i}{(M-1)N}}{M}g(N) \\
\end{array} \right). \]

Thus, letting \( \omega := e^{2\pi i} \),

\[
A = \left[ w_{(k-1)j} \right]_{k=1,\ldots,M,j=1,\ldots,N} \text{ Diag}(g(1), \ldots, g(N)).
\]

Proposition 1.4.3 in [1] shows that the rows in the matrix \( A \) form a frame for \( \text{span}\{\delta_k\}_{k=1}^N \) if and only if the columns in \( A \) are linearly independent; since \( g(j) \neq 0 \) for \( j = 1, \ldots, N \) the linear independence of the columns follows from (2.2). Applying the translation operators \( T_{nN} \) it now follows that \( \{E_{m/M}T_{nNg}\}_{n \in \mathbb{Z}, m=0,\ldots,M-1} \) is a frame for \( \ell^2(\mathbb{Z}) \), with \( K = N \).

Now, consider any \( K > N \) and any \( \epsilon > 0 \) and let \( \tilde{g} := g + \epsilon \sum_{k=N+1}^K \delta_k \). It is easy to see that \( \{E_{m/M}T_{nNg}\} \) is a Bessel sequence with bound \( M \); it follows
that for any finite sequence \( \{c_{m,n}\} \in \ell^2(\{1, \ldots, M-1\} \times \mathbb{Z}) \),

\[
\left\| \sum c_{m,n} E_{m/M} T_{nN} (\tilde{g} - g) \right\| = \left\| \epsilon \sum_{k=N+1}^{K} \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\|
\leq \epsilon \sum_{k=N+1}^{K} \left\| \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\|
\leq \epsilon(K - N) \sqrt{M} \left( \sum |c_{m,n}|^2 \right)^{1/2}.
\]

Let \( A \) denote a lower frame bound for \( \{E_{m/M} T_{nN} g\}_{n \in \mathbb{Z}, m=0,...,M-1} \). If we choose \( \epsilon > 0 \) such that \( \epsilon(K - N) \sqrt{M} < A \), it follows from Theorem 22.1.1 in \([1]\) that \( \{E_{m/M} T_{nN} \tilde{g}\}_{m=0,...,M-1, n \in \mathbb{Z}} \) is a frame for \( \ell^2(\mathbb{Z}) \). By construction, \( K = |\text{supp} \, g| \).

The result in (ii) is a consequence of the duality principle \([7]\), stating that a Bessel sequence \( \{E_{m/M} T_{nN} g\} \) is a frame for \( \ell^2(\mathbb{Z}) \) if and only if the Gabor system \( \{E_{m/N} T_{nM} g\} \) is a Riesz sequence; in particular the finitely supported windows \( g \) generating frames in (i) are precisely the ones that generate Riesz sequences in (ii). A direct proof of the existence can be given along the lines of the proof of (i), as follows. Assume that \( M \leq N \) and consider any \( g \in \ell^2(\mathbb{Z}) \) for which \( g(j) \neq 0 \) for \( j \in \{1, \ldots, M\} \) and \( g(j) = 0 \) for \( j \notin \{1, \ldots, M\} \). Then \( \{E_{m/M} g\}_{m=0,...,M-1} \) is a basis for \( \text{span} \{\delta_k\}_{k=1}^{M} \); since \( N \geq M \) this implies that \( \{E_{m/M} T_{nN} g\} \) is a Riesz sequence in \( \ell^2(\mathbb{Z}) \). A similar perturbation argument as in (i) now yields the conclusion. \( \square \)

Let us mention yet another way of proving the existence of Gabor frames \( \{E_{m/M} T_{nN} g\} \) for \( N/M < 1 \), using sampling of B-spline generated Gabor frames for \( L^2(\mathbb{R}) \). Recall that the B-splines \( B_k, K \in \mathbb{N} \), are defined recursively by convolutions, \( B_1 := \chi_{[0,1]}, B_{K+1}(x) := (B_K * B_1)(x) = \int_0^1 B_K(x - t) \, dt, \, x \in \mathbb{R} \).

**Example 2.2** Assume that \( N < M \) and consider the B-spline \( B_{N+1} \). Since \( 1/M \leq 1/(N + 1) \), the system \( \{e^{2\pi imx/M} B_{N+1}(x - nN)\}_{n,m \in \mathbb{Z}} \) is a Gabor frame for \( L^2(\mathbb{R}) \) by Corollary 11.7.1 in \([1]\). Define the discrete sequence \( B_{N+1}^D = \{B_{N+1}(j)\}_{j \in \mathbb{Z}} \). Since \( B_{N+1} \) is a continuous function with compact support, the sampling results in \([6]\) imply that the discrete Gabor system \( \{E_{m/M} T_{nN} B_{N+1}^D\}_{n \in \mathbb{Z}, m=0,...,M-1} \) is a frame for \( \ell^2(\mathbb{Z}) \). Note that \( \text{supp} \, B_{N+1}^D = \{1,2, \ldots, N\} \), i.e., \( |\text{supp} \, B_{N+1}^D| = N \). \( \square \)

The main body of Gabor analysis in \( L^2(\mathbb{R}) \) has a completely parallel version in \( \ell^2(\mathbb{Z}) \), but with regard to linear dependence the two cases are very different. In fact, certain choices of the parameters \( M, N, K \in \mathbb{N} \) imply that the Gabor
system \( \{E_{m/M}T_nNg\} \) is linearly dependent for all windows \( g \in \ell^2(\mathbb{Z}) \) with \( |\text{supp } g| = K \); for other choices of the parameters there exist linearly dependent as well as linearly independent Gabor systems. The precise statement is as follows.

**Theorem 2.3** Let \( M, N \in \mathbb{N} \). Then the following hold:

(i) If \( M = 1 \), the system \( \{E_{m/M}T_nNg\} \) is linearly independent for all \( g \in \ell^2(\mathbb{Z}) \setminus \{0\} \).

(ii) If \( M > |\text{supp } g| \) the Gabor system \( \{E_{m/M}T_nNg\} \) is linearly dependent.

(iii) If \( N < M \), the Gabor system \( \{E_{m/M}T_nNg\} \) is linearly dependent for any finitely supported \( g \in \ell^2(\mathbb{Z}) \).

(iv) For all \( M, N, K \in \mathbb{N} \) there exists a linearly dependent Gabor system \( \{E_{m/M}T_nNg\} \) with \( K = |\text{supp } g| \).

(v) If \( N \geq M \), then there exists for any \( K \geq M \) a linearly independent Gabor system \( \{E_{m/M}T_nNg\} \) with \( K = |\text{supp } g| \).

**Proof.** For \( M = 1 \) the system \( \{E_{m/M}T_nNg\} \) equals the shift-invariant system \( \{T_nNg\}_{n \in \mathbb{Z}} \) and is thus linearly independent whenever \( g \in \ell^2(\mathbb{Z}) \setminus \{0\} \); this proves (i). For the proof of (ii), the vectors \( \{E_{m/M}g\}_{m=1}^{M-1} \) can be considered as \( M \) vectors in a space of dimension \( |\text{supp } g| \); thus they are linearly dependent if \( M > |\text{supp } g| \), and hence \( \{E_{m/M}T_nNg\} \) is linearly dependent.

For the proof of (iii), consider any finitely supported \( g \in \ell^2(\mathbb{Z}) \). Without loss of generality, assume that \( g(j) = 0 \) for \( j \notin \{1, 2, \ldots, L\} \). Now, if \( L < M \), then the finite collection of vectors \( \{E_{m/M}g\}_{m=0}^{M-1} \) is clearly linear dependent. Thus, we now consider the case \( M \leq L \). Considering a finite number of translates of \( g \), i.e., \( \{T_nNg\}_{n=0}^{\ell} \) for some \( \ell \in \mathbb{N} \), there are at most \( L + \ell N \) coordinates where one or more of the vectors are nonzero; thus the system \( \{T_nNg\}_{n=0}^{\ell} \) belongs to an \((L + \ell N)\)-dimensional space. Therefore the collection \( \{E_{m/M}T_nNg\}_{m=0}^{M-1,n=0}^{\ell} \) consists of \((L + 1)M\) vectors in an \((L + \ell N)\)-dimensional space. Clearly they are linearly dependent if we choose \( \ell \in \mathbb{N} \) such that \((L + 1)M > L + \ell N \), i.e., \( \ell > \frac{L - M}{M - N} \). Thus the Gabor system \( \{E_{m/M}T_nNg\} \) is linearly dependent, as claimed.

For the proof of (iv), given \( M \in \mathbb{N} \), let \( g := \sum_{k=1}^{K} \delta_{kM} \); then for any \( m' \in \mathbb{N} \),

\[
E_{m'/M}g(j) = e^{2\pi im'j/M} \sum_{k=1}^{K} \delta_{kM}(j) = \sum_{k=1}^{K} \delta_{kM}(j) = g(j), \forall j \in \mathbb{Z},
\]
i.e., \( E_{m'/M}g = g \); thus the Gabor system \( \{ E_{m/M}T_{nN}g \} \) is linearly dependent. The result in (v) is a consequence of Theorem 2.1 (ii).

Let us single out the particular result that indeed motivated us to write this short note. Recall that a frame that is not a basis is said to be overcomplete; for a frame \( \{ E_{m/M}T_{nN}g \} \) in \( \ell^2(\mathbb{Z}) \) this is the case if and only if \( N < M \) [2].

**Corollary 2.4** Any overcomplete Gabor frame \( \{ E_{m/M}T_{nN}g \} \) with a finitely supported window \( g \in \ell^2(\mathbb{Z}) \) is linearly dependent.

**Proof.** The result follows immediately from Theorem 2.3 (iii).

The picture changes if we allow windows with infinite support: linearly independent and overcomplete Gabor frames with infinitely supported windows exist, as we show now. Our construction is inspired by a calculation for Hermite functions in \( L^2(\mathbb{R}) \) given in [4].

**Proposition 2.5** Define \( g \in \ell^2(\mathbb{Z}) \) by \( g(j) = e^{-j^2} \). Then \( \{ E_{m/M}T_{nN}g \} \) is linearly independent for all \( M, N \in \mathbb{N} \) and a frame for \( \ell^2(\mathbb{Z}) \) if \( N < M \).

**Proof.** It is well-known that a Gabor system \( \{ e^{2\pi ibx} \varphi(x-na) \}_{m,n \in \mathbb{Z}} \) in \( L^2(\mathbb{R}) \) is a Gabor frame for \( L^2(\mathbb{R}) \) whenever \( \varphi(x) = e^{-x^2} \) and \( 0 < ab < 1 \). Applying the sampling results by Janssen (see Proposition 2 in [6]) it follows that the sequence \( g \) generates a Gabor frame \( \{ E_{m/M}T_{nN}g \} \) for \( \ell^2(\mathbb{Z}) \) whenever \( N/M < 1 \). Note that this argument uses that the Gaussian satisfies the so-called condition R; we refer to [6] for details.

Now consider any \( M, N \in \mathbb{N} \). In order to show that \( \{ E_{m/M}T_{nN}g \} \) is linearly independent, assume that there is a finite scalar sequence \( \{ c_{n,m} \}_{n=-L,...,L,m=0,...,M-1} \) such that \( \sum_{n=-L}^{L} \sum_{m=0}^{M-1} c_{n,m} E_{m/M}T_{nN}g = 0 \). Thus, for all \( j \in \mathbb{Z} \),

\[
0 = \sum_{n=-L}^{L} \sum_{m=0}^{M-1} c_{n,m} e^{2\pi i jm/M} e^{-(j-nN)^2} = e^{-j^2} \sum_{n=-L}^{L} \sum_{m=0}^{M-1} c_{n,m} e^{2\pi i jm/M} e^{2nNj-(nN)^2}
\]

For \( n = -L, \ldots, L \), defining the functions \( \mathcal{E}_n \) on \( \mathbb{Z} \) by \( \mathcal{E}_n(j) = \sum_{m=0}^{M-1} c_{n,m} e^{2\pi i jm/M} \), \( j \in \mathbb{Z} \), we thus have

\[
\sum_{n=-L}^{L} \mathcal{E}_n(j) e^{2nNj-(nN)^2} = 0, \quad \forall j \in \mathbb{Z}.
\]  

(2.3)

Note that \( \mathcal{E}_n \) is a bounded and \( M \)-periodic function on \( \ell^2(\mathbb{Z}) \). We will first prove that \( \mathcal{E}_n = 0 \) for all \( n = -L, \ldots, L \). Assume that there is some \( n > 0 \)
such that $E_n(j) \neq 0$ for some $j \in \mathbb{Z}$. Then take the largest such $n$ and a corresponding $j_0 \in \{1, \ldots, M - 1\}$ such that $E_n(j_0) \neq 0$. Then

$$\sum_{n=-L}^{L} E_n(j_0 + \ell M)e^{-(nN)^2}e^{2nN(j_0 + \ell M)} \to \infty \quad \text{as } \ell \to \infty$$

which is contradicting (2.3). Therefore for all $0 < n \leq L$, $E_n = 0$. A similar argument shows that for all $-L \leq n < 0$, we have $E_n = 0$. Now (2.3) implies that also $E_0 = 0$, as claimed.

Considering now any $n = -L \ldots, L$, we thus have $\sum_{m=0}^{M-1} c_{n,m}e^{2\pi im/M} = 0$ for all $j = 0, \ldots, M - 1$. Writing this set of equations in matrix form, the matrix describing the system is a Vandermonde matrix and thus invertible; it follows that $c_{n,m} = 0$ for $m = 0, \ldots, M - 1$. Since $n \in \{-L, \ldots, L\}$ was arbitrary, this proves that the Gabor system is linearly independent. \qed

Let us also give a construction of a linearly dependent Gabor frame for $\ell^2(\mathbb{Z})$ with an infinitely supported window.

**Example 2.6** Assume that $N < M$ and consider the sequence $g \in \ell^2(\mathbb{Z})$ given by $g(j) = 1$ for $j \in \{1, \ldots, N\}$ and $g(j) = 0$ for $j \notin \{1, \ldots, N\}$. As we have seen in the proof of Theorem 2.4(i), the system $\{E_{m/M}T_{n,N}g\}$ is a frame for $\ell^2(\mathbb{Z})$. For $\epsilon > 0$, let $\tilde{g} = g + \sum_{\ell=1}^{\infty} \epsilon \frac{1}{\ell} \delta_{\ell M+1}$. Then $\tilde{g}$ has infinite support and a similar calculation as in the proof of Theorem 2.4(i) shows that for any finite sequence $\{c_{m,n}\}$, $\| \sum c_{m,n}E_{m/M}T_{n,N}(g - \tilde{g}) \| \leq \epsilon M^{1/2}$. Applying again the perturbation results for frames (Theorem 22.1.1 in [H]), it follows that for sufficiently small $\epsilon$, the system $\{E_{m/M}T_{nN}\}$ is a frame for $\ell^2(\mathbb{Z})$. Now, since $N < M$ and the support of $g$ has length $N$, the system $\{E_{m/M}g\}_{m=0,\ldots,M-1}$ is linearly dependent; thus, we can choose a nonzero scalar sequence $\{c_m\}_{m=0}^{M-1}$ such that $\sum_{m=0}^{M-1} c_mE_{m/M}g = 0$, i.e., $\sum_{m=0}^{M-1} c_mE^{2\pi im/M} = 0$ for $j = 1, \ldots, N$. It follows that for any $\ell \in \mathbb{N}$,

$$\sum_{m=0}^{M-1} c_mE_{m/M}\delta_{\ell M+1}(\ell M + 1) = \sum_{m=0}^{M-1} c_mE^{2\pi i(\ell M+1)m/M} = \sum_{m=0}^{M-1} c_mE^{2\pi im/M} = 0,$$

and thus $\sum_{m=0}^{M-1} c_mE_{m/M}\delta_{\ell M+1} = 0$. The construction of the sequence $\tilde{g}$ now shows that $\sum_{m=0}^{M-1} c_mE_{m/M}\tilde{g} = 0$; it follows that the Gabor system $\{E_{m/M}T_{nN}\}$ is linearly dependent, as claimed. \qed

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