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A note on circulatory systems: Old and new results

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It is astonishing that after more than half a century intensive research in the area of non-conservative systems of second order differential equations still new interesting results appear, see [4]. In that paper an old stability criterion by Metelitsyn [8] and Frik [9] was reinvented. We shortly repeat this result in order to emphasize that the criterion is sufficient but not necessary for stability. Afterwards we concentrate on circulatory systems with purely imaginary eigenvalues and investigate the influence of indefinite damping. Finally the possibility of stabilizing circulatory systems by gyroscopic forces will be commented. Examples will demonstrate the developed theory.

1 Introduction

This paper deals mainly with second order systems of linear differential equations of the form

$$M\ddot{x} + (K + N)x = 0, \quad M = M^T > 0,\ K = K^T > 0,\ N = -N^T. \tag{1}$$

Such MKN-systems are sometimes called pseudo-conservative [1], since they show some similarity to conservative systems with $N = 0$. We prefer the name circulatory systems [2,3]. Recently a paper appeared about the influence of damping terms $D\dot{x}$ with $D \geq 0$ on circulatory systems, and on more general non-conservative systems as well, [4]. In our paper we will firstly recapitulate the most important of these results including some additional comments and secondly we will show the influence of indefinite damping on circulatory systems (1) with only purely imaginary eigenvalues. A few words concerning modeling in Mechanics with a circulatory matrix N may be adequate. Stiffness matrices with a non-symmetrical part N are well-known in rotor dynamics due to internal damping or to non-symmetrical steam flow in turbines, due to follower forces in elasto-mechanics, in pipe lines etc. They can lead to self-excited vibrations and therefore to instability of the system. A recent example is the investigation of disc brakes [5]. Moreover, circulatory forces can stabilize an unstable conservative system and destabilize a stable conservative system, see e.g. [2, 3]. Indefinite damping in mechanical systems should be addressed shortly as well. E. g. sliding bearings [6] and cutting of of metals [7] can be modelled with indefinite damping which can lead to self-excited vibrations. If the eigenvalues of a system are symmetrical with respect to both axes we say that the system possesses *Hamiltonian symmetry*. Circulatory systems have Hamiltonian symmetry. We will show how an unstable circulatory system under certain circumstances can be stabilized by adding a gyroscopic term $G\dot{x}$ with $G = -G^T$. To this end we use a theory developed in [15]. With help of this theory we can transform a $IDGKN$-system with Hamiltonian symmetry into an isospectral circulatory IK_0N_0-system to facilitate the investigation of stability.

2 Some known results

In [4] stability of more general linear non-conservative MDGKN-systems of the form

$$M\ddot{x} + (D + G)\dot{x} + (K + N)x = 0,\ M = M^T, D = D^T, K = K^T, G = -G^T, N = -N^T, M, K > 0 \tag{2}$$

was under consideration as well. To this end a theorem by Metelitsyn [8] and Frik [9], also mentioned in [3], was reinvented. Without loss of generality we assume M = I. Then stability of system (2) can be completely understood with help of the eigenvalue problem

$$\left(\lambda^2 I + \lambda(D + G) + K + N\right) u = 0, \quad u \neq 0. \tag{3}$$

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The eigenvalues λ are the roots of the characteristic polynomial $det[\lambda^2 I + \lambda(D+G) + K + N]$, and if all eigenvalues have negative real parts, system (2) is asymptotically stable. Marginal stability is characterized by semi-simple eigenvalues on the imaginary axis (at least one) together with all the other eigenvalues having negative real parts. Pre-multiplying (3) by the conjugate transposed u^* of the eigenvector u and normalizing by $u^* u = 1$, we receive (one equation for each eigenvector)

$$\lambda^2 + (d + ig)\lambda + (k + in) = 0, \tag{4}$$

where the coefficients $d = u^* Du$, $ig = u^* Gu$, $k = u^* Ku$, $in = u^* Nu$ are the Rayleigh quotients of the matrices D, G, K, and N, respectively. Their numerical values are limited by smallest and largest eigenvalues of the respective matrices. (For example $\lambda(D)_{min} \le d \le \lambda(D)_{max}$, $-|\lambda(G)|_{max} \le g \le |\lambda(G)|_{max}$, etc.). If we now claim that *both* roots of (4) have negative real parts we receive the following theorem:

Metelitsyn-Frik
System (2) *is asymptotically stable if*

$$d > 0 \quad \wedge \quad d^2 k + d\, g\, n - n^2 > 0. \tag{5}$$

By taking advantage of the properties of the Rayleigh quotients these inequalities lead to

$$d_{min} > 0 \quad \wedge \quad d_{min}\,(d_{min}\, k_{min} - g_{max}\, n_{max}) - n_{max}^2 > 0. \tag{6}$$

Although these two conditions (5) were already derived in [4], we repeated shortly the procedure in order to point out that the criterion is only *sufficient* for asymptotic stability, but *not nescessary*, since the eigenvalue λ is one of the roots of Eq. (4), but the other root need not to be an eigenvalue. Actually, it is more an exception than a rule that both roots of (4) are eigenvalues. Discussion and examples can be found in [10]. Any $IDGKN$-system with $K > 0$ and $D > 0$ will be asymptotically stable if dissipative and/or potential forces are sufficiently large. It is a crucial point that the inequalities of the criterion are not necessary for asymptotic stability. To claim sufficient large d_{min} may lead to an undesirable large damping. Therefore Theorem 2 in [4] is of importance, where asymptotic stabilization of a IKN-system with purely imaginary eigenvalues can be achieved by a damping $D > 0$ with arbitrarily small norm.

3 Indefinite damping in circulatory systems with imaginary eigenvalues

In order to investigate the influence of *indefinite* damping matrices on circulatory systems with *purely imaginary* eigenvalues we recall the concept of *symmetrizability*. A real square matrix C is called (real) symmetrizable if any of the four equivalent conditions is satisfied:

1. There exists a non-singular matrix W such that $W^{-1}CW$ is symmetric.
2. C possesses only real eigenvalues and a full set of eigenvectors.
3. C is the product of two symmetric matrices, one of which is is positive definite. (If both symmetric matrices are positive definite, then $W^{-1}CW > 0$ and all eigenvalues of C are positive.)
4. C becomes symmetric when multiplied by a suitable positive definite matrix.

The circulatory systems IKN (1) under consideration are connected to the eigenvalue problem $(\lambda^2 I + K + N)u = 0$, where we assume $\lambda \in i\mathbb{R}$. This is a necessary (but not sufficient) condition for the IKN-system to be stable. Moreover, if we add damping, necessary conditions for asymptotic stability of the $IDKN$-system are $trace(D) > 0$, $trace(K^{-1}D) > 0$, and $det(K + N) > 0$, [3]. The following investigation is divided into 2 cases:

i) The matrix $C = K + N$ is real symmetrizable and both symmetric matrices in condition 3. are positive definite. Then also $W^{-1}CW > 0$ and the IKN-system is completely marginally stable.
ii) The matrix $C = K + N$ is not symmetrizable. Then the IKN-system is not stable.

Case i)
In [11] the influence of an indefinite damping matrix on a conservative and stable IK-system was investigated by a perturbation approach. The perturbation analysis assumes small damping terms compared to the conservative forces described by K. We will show that a similar procedure works for a circulatory IKN-system.

Let the damping matrix $D = D^T$ depend smoothly on the vector $p = (p_1, p_2, \ldots, p_n)$ of real parameters (for example $D = \mathrm{diag}(p_1, p_2, \ldots, p_n)$). Consider p as a point in the parameter space. At $p = p_0$ we have $D(p_0) = 0$ and system IKN posesses a purely imaginary eigenvalue $\lambda_0 = i\omega_0$ with a full set of real eigenvectors. We want to study the behavior of the eigenvalue in the vicinity of the initial point p_0. To this end we consider a parameter variation $p = p_0 + \epsilon e$, where

$e = (e_1, .., e_n)$ is a direction vector of unit norm, $|e| = 1$, and $\epsilon > 0$ is a small perturbation parameter. Hence we obtain

$$D = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots$$

where

$$D_0 = D(p_0), \quad D_1 = \sum_{i=1}^{n} \frac{\partial D}{\partial p_i} e_i, \quad D_2 = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 D}{\partial p_i \partial p_j} e_i e_j, \quad \ldots \tag{7}$$

Then the eigenvalue, the right eigenvector u and the left eigenvector v of the $IDKN$-system can be expanded in power series of ϵ

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \ldots, \quad u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, \quad v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots. \tag{8}$$

We introduce the matrix valued functions

$$L(\lambda) = \lambda^2 I + \lambda D + K + N, \quad L_1(\lambda) = \sum_{i=1}^{n} \frac{\partial L}{\partial p_i} e_i = \lambda_0 D_1 . \tag{9}$$

Inserting (7) and (8) into $L(\lambda)u = 0$ and comparing expressions with the same power of ϵ leads to

$$\lambda_1 = -\frac{v_0^T L_1 u_0}{2\lambda_0 v_0^T u_0} = -\frac{v_0^T D_1 u_0}{2 v_0^T u_0}. \tag{10}$$

λ_1 is real since the nominator and denominator both are real. Therefore the eigenvalue λ has a negative real part in the vicinity of λ_0, if $\lambda_1 < 0$ which means *stability*.

Example 3.1. Consider the circulatory system IKN with

$$K = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then matrix $C = K + N$ is real symmetrizable with positive eigenvalues and eigenvectors and system IKN is completely marginally stable. We choose $\lambda_0 = 2.3028i$ with right eigenvector

$$u_0 = \begin{bmatrix} 0.9172 \\ 0.3983 \end{bmatrix}$$

and the left eigenvector

$$v_0 = \begin{bmatrix} 0.6089 \\ 0.7933 \end{bmatrix}$$

and a direction vector $e = (e_1, e_2) = \frac{1}{\sqrt{5}}(2, -1)$. Then

$$D(p) = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, \; D_1 = \frac{\partial D}{\partial p_1} e_1 + \frac{\partial D}{\partial p_2} e_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\lambda_1 = -\frac{v_0^T D_1 u_0}{2 v_0^T u_0} = -0.2048$$

implies stability. If

$$D = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.01 \end{bmatrix},$$

we get $\epsilon = \frac{\sqrt{5}}{2} \cdot 0.02 = 0.02236$ and $\lambda = \lambda_0 + \epsilon\lambda_1 + o(\epsilon) = 2.3028i - 0.00458 + o(\epsilon)$.

Respectively, for $\lambda_0 = 1.3028i$ we receive

$$u_0 = \begin{bmatrix} -0.7933 \\ 0.6089 \end{bmatrix}, \quad v_0 = \begin{bmatrix} -0.3983 \\ 0.9172 \end{bmatrix}$$

and

$$\lambda_1 = -\frac{v_0^T D_1 u_0}{2 v_0^T u_0} = -0.0188$$

implies stability. Finally $\lambda = \lambda_0 + \epsilon\lambda_1 + o(\epsilon) = 1.3028i - 0.00042 + o(\epsilon)$.

Example 1 showed stability of a $IDKN$-system for an indefinite damping matrix D. But a positive definite damping can also cause instability if $K + N$ is symmetrizable as the next example from [4] shows.

Example 3.2. Consider the IKN-system with

$$K = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0.499 \\ -0.499 & 0 \end{bmatrix}.$$

Assume

$$D_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then we receive for eigenvalue $\lambda_0 = 1.5991i$ and respective right and left eigenvectors according to (10) $\lambda_1 = -1.4436$ which means stability. But for eigenvalue $\lambda_0 = 1.5711i$ we receive $\lambda_1 = 0.7502$ which implies instability, as also shown graphically in [4].

Actually in [4] theorem 1 it was shown that for any circulatory IKN-system with only purely imaginary eigenvalues there exists a damping matrix $D \geq 0$ such the the resulting $IDKN$-system is unstable.

Case ii)

We assume that at $p = p_0$ the IKN-system posesses a double eigenvalue with only one right eigenvector u_0 and one left eigenvector v_0. Then there excists a Jordan chain of lenght 2, [12]

$$L_0 u_0 = 0, \quad L_0 w_0 = -\frac{\partial L}{\partial \lambda} u_0, \quad L_0 = \lambda_0^2 I + K + N, \quad v_0^T L_0 = 0, \quad \frac{\partial L}{\partial \lambda} = 2\lambda_0 I$$

where w_0 is a generalized right eigenvector. The IKN-system is not stable and a solution includes a secular term. I order to study the behavior of eigenvalues in the vicinity of $p = p_0$ we assume like in case i) a variation $p = p_0 + \epsilon e$. According to the perturbation theory developed in [13, 14] for case ii), expansions of eigenvalues and eigenvectors contain fractional powers of ϵ

$$\lambda = \lambda_0 + \epsilon^{1/2} \lambda_1 + \epsilon \lambda_2 + \ldots, \qquad u = u_0 + \epsilon^{1/2} u_1 + \epsilon u_2 + \ldots,$$
$$v = v_0 + \epsilon^{1/2} v_1 + \epsilon v_2 + \ldots.$$

With the same definitions for L, L_1, and D_1 as in case i) we receive

$$\lambda_1^2 = -\frac{\lambda_0 \, v_0^T D_1 u_0}{v_0^T \frac{\partial L}{\partial \lambda} w_0 + v_0^T u_0}.$$

Since L_0 is singular, a generalized eigenvector w_0 as a solution of $L_0 w_0 = -2\lambda_0 \, u_0$ only exists if $v_0^T u_0 = 0$. Then w_0 is purely imaginary and therefore also λ_1^2 is purely imaginary and

$$\lambda = \lambda_0 + \epsilon^{1/2} \lambda_1 + o(\epsilon^{1/2}) = \lambda_0 \pm \sqrt{\epsilon \lambda_1^2} + o(\epsilon^{1/2}).$$

This shows that the double eigenvalue $\lambda_0 = i\omega_0$ bifurcates into two eigenvalues, one of which has a positive real part, which means *instability by flutter* for all kind of damping matrices $D(p)$. We should mention that if in case ii) the matrix $K + N$ depends on a load parameter q, a small variation δq of this parameter causes flutter instability as well.

Example 3.3. Consider the IKN-system

$$K = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}, \quad N = \frac{\sqrt{17}}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The system IKN has a double eigenvalue $\lambda_0 = 1.87083i$ with only one right eigenvector u_0 and one left eigenvector v_0 and a generalized eigenvector w_0 given by

$$u_0 = \begin{bmatrix} 0.99251 \\ -0.12218 \end{bmatrix}, \quad v_0 = \begin{bmatrix} 0.12218 \\ 0.99251 \end{bmatrix}, \quad w_0 = \begin{bmatrix} 4659.74822i \\ -574.55556i \end{bmatrix}.$$

If we choose the direction $e = \frac{1}{\sqrt{5}} \cdot (2, -1)$ in the parameter space and the indefinite damping matrix

$$D = \begin{bmatrix} 0.02 & 0.00 \\ 0.00 & -0.01 \end{bmatrix},$$

then this leads to $\epsilon = \frac{\sqrt{5}}{2} \cdot 0.02 = 0.02236$ and $\lambda_1^2 = 0.08964i$. Finally we receive

$$\lambda \simeq \lambda_0 + \epsilon^{1/2}\lambda_1 = \begin{cases} 0.03166 + 1.90249i \\ -0.03166 + 1.83917i \end{cases},$$

which shows a bifurcation of the double eigenvalue $\lambda_0 = 1.87083i$ and therefore *instability* of the $IDKN$-system by flutter.

4 Damping in circulatory systems

In this section we shall study the influence of damping on a general IKN-system. As earlier mentioned, the circulatory system is marginally stable if $K + N$ is real symmetrizable with positive eigenvalues and otherwise unstable. An unstable circulatory system can under certain circumstances be marginally stabilized by adding a gyroscopic damping matrix G where $G = -G^T$. To show this we use that an $IDGKN$-system wich possesses Hamiltonian symmetry is isospectral with a circulatory IK_0N_0-system through the following relations found in [15]

$$K + N = H - HZH^{-1}Z, \tag{11}$$

$$D + G = HZH^{-1} - Z, \tag{12}$$

where

$$H = Z^2 + K_0 + N_0, \quad Z,\ H,\ K_0,\ N_0 \in \mathbb{R}^{n\times n}, \tag{13}$$

and Z and H are non-singular. If $K_0 + N_0$ is symmetrizable and has positive eigenvalues, then all the eigenvalues of the $IDGKN$-system lie on the imaginary axis and the system is *marginally* stable. We now formulate

Theorem
An $IDGKN$-system with Hamiltonian symmetry and a full set of eigenvectors can be generated by Eqs. (11) *and* (12) *where $K_0,\ N_0,\ Z \in \mathbb{R}^{n\times n}$ and $K_0 + N_0$ and Z are chosen such that Z and $H = Z^2 + K_0 + N_0$ are non-singular. If $K_0 + N_0$ has positive eigenvalues then the $IDGKN$-system is marginally stable.*

Conversely, if an $IDGKN$-system with Hamiltonian symmetry is given then one can find an implicit equation for $K_0 + N_0$ which defines the isospectral IK_0N_0-system expressed by $K + N$ and $D + G$, see [15].

To illustrate the above Theorem, we will show that it is possible to marginally stabilize a circulatory system only by adding a *gyroscopic* damping $G = -G^T$. From (12) we get the condition, $D = 0$, for the damping matrix to be skew-symmetric

$$H^T(Z + Z^T)H = H^T HZ + Z^T H^T H. \tag{14}$$

The above equation is fullfilled if we write $Z = A,\ A = -A^T$ and $H^T H = I$. Then Eqs. (11) and (12) leads to

$$K + N = H - HAH^T A, \quad H^T H = I, \quad A = -A^T, \tag{15}$$

$$G = HAH^T - A = -G^T. \tag{16}$$

The system is *marginally* stable if $K_0 + N_0 = H + A^T A$ is symmetrizable with positive eigenvalues. An example is given below

Example 4.1. If we use the orthogonal matrix

$$H = \begin{bmatrix} 0.656 & 0.231 & 0.552 & 0.459 \\ 0.335 & -0.151 & -0.770 & 0.522 \\ -0.512 & -0.507 & 0.301 & 0.625 \\ -0.442 & 0.816 & -0.111 & 0.355 \end{bmatrix}, \quad HH^T = I$$

and the skew-symmetric matrix

$$Z = A = \begin{bmatrix} 0.00 & -6.00 & -2.20 & -8.90 \\ 6.00 & 0.00 & -6.30 & 0.40 \\ 2.20 & 6.30 & 0.00 & 2.40 \\ 8.90 & -0.40 & -2.40 & 0.00 \end{bmatrix},$$

we find

$$K_0 + N_0 = H - Z^2 = \begin{bmatrix} 121.0 & 10.5 & -58.6 & 8.14 \\ 10.6 & 75.7 & 13.4 & 69.0 \\ -59.7 & 13.7 & 50.6 & 17.7 \\ 7.24 & 69.3 & 16.9 & 85.5 \end{bmatrix}$$

with the eigenvalues $\{154.0,\ 155.0,\ 12.3,\ 11.1\}$ which shows that the IK_0N_0-system is *marginally* stable. Further we have according to (15)

$$K + N = \begin{bmatrix} 38.5 & 42.9 & 9.06 & 16.2 \\ 76.0 & -16.9 & -27.5 & -25.1 \\ -40.5 & 43.4 & 30.5 & 63.1 \\ -41.6 & 53.2 & 62.3 & 47.4 \end{bmatrix}$$

which has the eigenvalues $\{-23.21,\ -25.52,\ 74.17 \pm 0.5573\,i\}$. Thus the IKN-system is unstable. For G we obtain

$$G = -G^T = \begin{bmatrix} 0 & 9.105 & -4.862 & 4.297 \\ -9.105 & 0 & 5.470 & -8.695 \\ 4.862 & -5.470 & 0 & 1.532 \\ -4.297 & 8.695 & -1.532 & 0 \end{bmatrix}.$$

The eigenvalues of the system are $\{\pm 12.44\,i,\ \pm 12.42\,i,\ \pm 3.51\,i,\ \pm 3.33\,i\}$. This shows that the $IGKN$-system has Hamiltonian symmetry and is marginally stable. □

The above example shows that there exists unstable IKN-systems which can be stabilized by adding a gyroscopic damping matrix. But it seems that the procedure does not work for all kind of circulatory systems. Further investigations are necessary to clarify this point.

5 Conclusions

Firstly we payed attention to an old stability result of Metelitsyn and Frik for non-conservative systems, since it has been often misunderstood as a necessary condition for stability. Secondly we presented the main goal of this paper, namely the study of some new aspects of circulatory systems. We investigated the influence of indefinite damping on IKN-systems with purely imaginary eigenvalues. Simple examples demonstrated the theory. Systems with for example triple eigenvalues where $K + N$ is not symmetrizable are not taken under consideration. In such a case fractional powers of $\epsilon^{1/3}$ in the respective series have to be used. Finally the connection between circulatory systems and more general systems with Hamiltonian symmetry has been addressed with the aim to show how an unstable circulatory system under certain circumstances may be stabilized by gyroscopic damping.

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