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A Short Introduction to Frames, Gabor Systems, and Wavelet Systems

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Abstract. In this article we present a short survey of frame theory in Hilbert spaces. We discuss Gabor frames and wavelet frames, and a recent transform that allows to move results from one setting into the other and vice versa.

Key Words and Phrases: frames, dual pair of frames, wavelet system, Gabor system

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1. Introduction

Frames provide us with a convenient tool to obtain expansions in Hilbert spaces of a similar type as the one that arise via orthonormal bases. However, the frame conditions are significantly weaker, which makes frames much more flexible. For this reason frame theory has attracted much attention in recent years, especially in connection with its concrete manifestations within Gabor analysis and wavelet analysis.

In this article we give a short overview of the general theory for frames in Hilbert spaces, as well as its applications in Gabor analysis and wavelet analysis. Finally, we present a method that allows to construct wavelet frames based on certain Gabor frames, and vice versa. Applying this to Gabor frames generated by exponential B-splines produces a class of attractive dual wavelet frame pairs generated by functions whose Fourier transforms are compactly supported splines with geometrically distributed knots.

2. A survey on frame theory

General frames were introduced already in the paper [17] by Duffin and Schaeffer in 1952. Apparently it did not find much use at that time, until it got re-introduced by Young in his book [30] from 1982. After that, Daubechies, Grossmann and Morlet took the key step of connecting frames with wavelets and Gabor systems in the paper [15].

2.1. General frame theory

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. A countable family of elements $\{f_k\}_{k \in I}$ in $\mathcal{H}$ is a

(i) *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \ \forall f \in \mathcal{H};$$

(ii) *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \ \forall f \in \mathcal{H}; \tag{2.1}$$

The numbers A, B in (2.1) are called *frame bounds*.

(iii) *Riesz basis* for $\mathcal{H}$ if $\overline{\operatorname{span}}\{f_k\}_{k \in I} = \mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A \sum |c_k|^2 \leq \left\| \sum c_k f_k \right\|^2 \leq B \sum |c_k|^2. \tag{2.2}$$

for all finite sequences $\{c_k\}$.

Every orthonormal basis is a Riesz basis, and every Riesz basis is a frame (the bounds A, B in (2.2) are frame bounds). That is, Riesz bases and frames are natural tools to gain more flexibility than possible with an orthonormal basis. For an overview of the general theory for frames and Riesz bases we refer to [1] and [6]; a deeper treatment is given in the books [2], [4]. Here, we just mention that the difference between a Riesz basis and a frame is that the elements in a frame might be dependent. More precisely, a frame $\{f_k\}_{k \in I}$ is a Riesz basis if and only if

$$\sum_{k \in I} c_k f_k = 0, \ \{c_k\} \in \ell^2(I) \Rightarrow c_k = 0, \ \forall k \in I.$$

Given a frame $\{f_k\}_{k \in I}$, the associated *frame operator* is a bounded, self-adjoint, and invertible operator on $\mathcal{H}$, defined by

$$Sf = \sum_{k \in I} \langle f, f_k \rangle f_k.$$

The series defining the frame operator converges unconditionally for all $f \in \mathcal{H}$. Via the frame operator we obtain the *frame decomposition*, representing each $f \in \mathcal{H}$ as an infinite linear combination of the frame elements:

$$f = SS^{-1}f = \sum_{k \in I} \langle f, S^{-1}f_k \rangle f_k. \tag{2.3}$$

The family $\{S^{-1}f_k\}_{k \in I}$ is itself a frame, called the *canonical dual frame*. In case $\{f_k\}_{k \in I}$ is a frame but not a Riesz basis, there exist other frames $\{g_k\}_{k \in I}$ which satisfy

$$f = \sum_{k \in I} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}; \tag{2.4}$$

each family $\{g_k\}_{k \in I}$ with this property is called a *dual frame*.

The formulas (2.3) and (2.4) are the main reason for considering frames, but they also immediately reveal one of the fundamental problems with frames. In fact, in order for (2.3) to be practically useful, one has to invert the frame operator, which is difficult when $\mathcal{H}$ is infinite-dimensional. One way to avoid this difficulty is to consider *tight* frames, i.e., frames $\{f_k\}_{k \in I}$ for which

$$\sum_{k \in I} |\langle f, f_k \rangle|^2 = A||f||^2, \ \forall f \in \mathcal{H} \tag{2.5}$$

for some $A > 0$. For a tight frame, $\langle Sf, f \rangle = A||f||^2$, which implies that $S = AI$, and therefore

$$f = \frac{1}{A} \sum_{k \in I} \langle f, f_k \rangle f_k, \ \forall f \in \mathcal{H}. \tag{2.6}$$

2.2. Operators on $L^2(\mathbb{R})$

In order to construct concrete frames in the Hilbert space $L^2(\mathbb{R})$, we need to consider some important classes of operators.

Definition 2.1. (Translation, modulation, dilation) *Consider the following classes of linear operators on $L^2(\mathbb{R})$:*

(i) For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$(T_a f)(x) := f(x - a), \ x \in \mathbb{R}. \tag{2.7}$$

(ii) For $b \in \mathbb{R}$, the operator E_b, called modulation by b, is defined by

$$(E_b f)(x) := e^{2\pi i b x} f(x), \ x \in \mathbb{R}. \tag{2.8}$$

(iii) For $c > 0$, the operator D_c, called dilation by c, is defined by

$$(D_c f)(x) := \frac{1}{\sqrt{c}} f(\frac{x}{c}), \ x \in \mathbb{R}. \tag{2.9}$$

All the above operators are linear, bounded, and unitary. We will also need the Fourier transform, for $f \in L^1(\mathbb{R})$ defined by

$$\widehat{f}(\gamma) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx.$$

The Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$ in the usual way.

2.3. Gabor systems in $L^2(\mathbb{R})$

A *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given function $g \in L^2(\mathbb{R})$. Using the translation operators and the modulation operators we can denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

It is easy to show that the Gabor system $\{e^{2\pi imx}\chi_{[0,1]}(x-n)\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. However, the function $\chi_{[0,1]}$ is discontinuous and has very slow decay in the Fourier domain. Thus, the function is not suitable for time-frequency analysis. For the sake of time-frequency analysis, we want the Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be generated by a continuous function g with compact support. This forces us to consider Gabor frames rather than bases:

Lemma 1. *If g is be a continuous function with compact support, then*

- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can not be an ONB.*

- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can not be a Riesz basis.*

- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can be a frame if $0 < ab < 1$;*

- *For each $a, b > 0$ with $ab < 1$, there exists function $g \in C_c(\mathbb{R})$ such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame.*

In order for a frame to be useful, we need a dual frame. The duality conditions for a pair of Gabor systems were obtained by Ron & Shen [25], [26]. We state the formulation due to Janssen [21]:

Theorem 2.2. *Given $b, \alpha > 0$, two Bessel sequences $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$, where $g, \tilde{g} \in L^2(\mathbb{R})$, form dual Gabor frames for $L^2(\mathbb{R})$ if and only if for all $n \in \mathbb{Z}$,*

$$\sum_{j\in\mathbb{Z}} \overline{g(x+j\alpha)}\tilde{g}(x+j\alpha+n/b) = b\delta_{n,0}, \ a.e. \ x \in \mathbb{R}.$$

Theorem 2.2 characterizes pairs of dual Gabor frames, but it does not show how to construct convenient pairs of Gabor frames. A class of convenient dual pairs of frames are constructed in [5] and [9]:

Theorem 2.3. *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function for which $\operatorname{supp} g \subseteq [0, N]$ and*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1. \tag{2.10}$$

Let $b \in]0, \frac{1}{2N-1}]$. Define $\tilde{g} \in L^2(\mathbb{R})$ by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

Figure 1: The generators B_2 and B_3 and some dual generators.

where

$$a_0 = b, \quad a_n + a_{-n} = 2b, \ n = 1, 2, \cdots, N - 1.$$

Then g and h generate dual frames $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n \tilde{g}\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$.

Example 2.4. *The conditions in Theorem 2.3 are satisfied for any B-spline B_N, $N \in \mathbb{N}$. Some choices of the coefficients a_n are the following:*

1) *Take*

$$a_0 = b, a_n = 0 \text{ for } n = -N + 1, \ldots, -1, a_n = 2b, n = 1, \ldots N - 1.$$

This choice gives the dual frame generated by the function with shortest support.

2) *Take*

$$a_{-N+1} = a_{-N+2} = \cdots = a_{N-1} = b :$$

if g is symmetric, this leads to a symmetric dual generator

$$\tilde{g}(x) = b \sum_{n=-N+1}^{N-1} g(x + n).$$

$\square$

2.4. Wavelet systems in $L^2(\mathbb{R})$

A *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^j x - kb)\}_{j,k \in \mathbb{Z}}$ for some parameters $a > 1, b > 0$ and a given function $\psi \in L^2(\mathbb{R})$. Introducing the scaling operators and the translation operators, the wavelet system can be written as $\{D_{a^j}T_{kb}\psi\}_{j,k \in \mathbb{Z}}$.

There are also characterizing equations for dual wavelet frames; see [11]. They are formulated in terms of the Fourier transform:

Theorem 2.5. *Given $a > 1$, $b > 0$, two Bessel sequences $\{D_{a^j} T_{kb} \psi\}_{j,k \in \mathbb{Z}}$ and $\{D_{a^j} T_{kb} \widetilde{\psi}\}_{j,k \in \mathbb{Z}}$, where $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$, form dual wavelet frames for $L^2(\mathbb{R})$ if and only if the following two conditions hold:*

(i) $\sum_{j \in \mathbb{Z}} \overline{\widehat{\psi}(a^j \gamma)} \widehat{\widetilde{\psi}}(a^j \gamma) = b$ *for a.e.* $\gamma \in \mathbb{R}$.

(ii) For any number $\alpha \neq 0$ of the form $\alpha = m/a^j$, $m, j \in \mathbb{Z}$,

$$\sum_{(j,m) \in I_\alpha} \overline{\widehat{\psi}(a^j \gamma)} \widehat{\widetilde{\psi}}(a^j \gamma + m/b) = 0, \ \ a.e. \ \gamma \in \mathbb{R},$$

where $I_\alpha := \{(j, m) \in \mathbb{Z}^2 \mid \alpha = m/a^j\}$.

We will present a few aspects of wavelet theory, beginning with the classical multiresolution analysis

2.5. Classical multiresolution analysis

Multiresolution analysis is a tool to construct orthonormal bases for $L^2(\mathbb{R})$ of the form $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ for a suitably chosen function $\psi \in L^2(\mathbb{R})$. Such a function ψ is called a *wavelet*. Its original definition of a multiresolution analysis was given by Mallat and Meyer [22, 23], is as follows:

Definition 2.6. *A multiresolution analysis for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$ such that*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$

(ii) $\cap_j V_j = \{0\}$ *and* $\overline{\cup_j V_j} = L^2(\mathbb{R})$

(iii) $f \in V_j \Leftrightarrow Df \in V_{j+1}$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}$.

(v) $\{T_k \phi\}_{k \in \mathbb{Z}}$ *is an orthonormal basis for V_0.*

A multiresolution analysis is in fact generated just by a suitable choice of the function ϕ : if the conditions in Definition 2.6 are satisfied, then necessarily

$$V_j = \overline{\mathrm{span}} \{D^j T_k \phi\}_{k \in \mathbb{Z}}, \ \forall j \in \mathbb{Z}.$$

The following result, due to Mallat and Meyer [22, 23], shows how to construct a wavelet based on a multiresolution analysis. Other proofs can be found in [14], [31], and [3].

Theorem 2.7. *Assume that the function $\phi \in L^2(\mathbb{R})$ generates a multiresolution analysis. Then the following holds:*

(i) *There exists a 1-periodic function $H_0 \in L^2(0,1)$ such that*

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma), \ \gamma \in \mathbb{R}. \tag{2.11}$$

(ii) *Define the 1-periodic function H_1 by*

$$H_1(\gamma) := \overline{H_0(\gamma + \frac{1}{2})}\, e^{-2\pi i\gamma}. \tag{2.12}$$

Also, define the function ψ via

$$\hat{\psi}(2\gamma) := H_1(\gamma)\hat{\phi}(\gamma). \tag{2.13}$$

Then ψ is a wavelet:

The definition in (2.13) is quite indirect: it defines the function ψ in terms of its Fourier transform, so we have to apply the inverse Fourier transform in order to obtain an expression for ψ. This actually leads to an explicit expression of the function ψ in terms of the given function ϕ:

Proposition 2.8. *Assume that (2.13) holds for a 1-periodic function $H_1 \in L^2(0,1)$,*

$$H_1(\gamma) = \sum_{k \in \mathbb{Z}} d_k e^{2\pi i k\gamma}. \tag{2.14}$$

Then

$$\psi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} d_k DT_{-k}\phi(x) = 2 \sum_{k \in \mathbb{Z}} d_k \phi(2x+k), \ x \in \mathbb{R}. \tag{2.15}$$

The classical example of a wavelet generated by a multiresolution analysis is the *Haar wavelet*,

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}[\\ -1 & \text{if } x \in [\frac{1}{2}, 1[\\ 0 & \text{otherwise} \end{cases}$$

It is generated by the function $\phi = \chi_{[0,1]}$. In 1989 Daubechies managed to construct an important class of compactly supported wavelets with very good approximation properties. We will not go into a detailed discussion of these, but just refer to, e.g., [14] and [31].

2.6. The unitary extension principle

In this section we present results by Ron and Shen, which enables us to construct tight wavelet frames generated by a collection of functions $\psi_1, \ldots, \psi_n$. Our presentation is based on the papers [27], [28], and [29]. Note also that a more flexible tool, the oblique extension principle, has later been introduced by two groups of researchers, see [12] and [16].

The generators $\psi_1, \ldots, \psi_n$ will be constructed on the basis of a function which satisfy a refinement equation, and since we will work with all those functions simultaneously it is convenient to change our previous notation slightly and denote the refinable function by ψ_0.

General setup: Let $\psi_0 \in L^2(\mathbb{R})$. Assume that $\lim_{\gamma \to 0} \hat{\psi}_0(\gamma) = 1$ and that there exists a function $H_0 \in L^\infty(\mathbb{T})$ such that

$$\hat{\psi}_0(2\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma). \tag{2.16}$$

Let $H_1, \ldots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \ldots, \psi_n \in L^2(\mathbb{R})$ by

$$\hat{\psi}_\ell(2\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, \ldots, n. \tag{2.17}$$

Finally, let H denote the $(n+1) \times 2$ matrix-valued function defined by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix}. \tag{2.18}$$

We will frequently suppress the dependence on γ and simply speak about the matrix H. The purpose is to find $H_1, \ldots, H_n$ such that

$$\{D^j T_k \psi_1\}_{j,k \in \mathbb{Z}} \cup \{D^j T_k \psi_2\}_{j,k \in \mathbb{Z}} \cup \cdots \cup \{D^j T_k \psi_n\}_{j,k \in \mathbb{Z}} \tag{2.19}$$

constitute a tight frame. The *unitary extension principle* by Ron and Shen shows that a condition on the matrix H will imply this:

Theorem 2.9. *Let $\{\psi_\ell, H_\ell\}_{\ell=0,\ldots,n}$ be as in the general setup, and assume that the 2×2 matrix $H(\gamma)^*H(\gamma)$ is the identity for a.e. γ. Then the multi-wavelet system $\{D^j T_k \psi_\ell\}_{j,k \in \mathbb{Z}, \ell=1,\ldots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to one.*

As an application, we show how one can construct compactly supported tight spline frames.

Example 2.10. *Fix any $m = 1, 2, \ldots$, and consider the function $\psi_0 := B_{2m}$, i.e., the B-spline of order m. It is defined by*

$$\psi_0 = \chi_{[-\frac{1}{2}, \frac{1}{2}]} * \chi_{[-\frac{1}{2}, \frac{1}{2}]} * \cdots * \chi_{[-\frac{1}{2}, \frac{1}{2}]} \quad (2m \text{ factors}).$$

Note that

$$\hat{\psi}_0(\gamma) = \frac{\sin^{2m}(\pi\gamma)}{(\pi\gamma)^{2m}}.$$

ψ_0 *is known as the B-spline of order* $2m$ *It is clear that* $\lim_{\gamma\to 0}\hat{\psi}_0(\gamma) = 1$, *and by direct calculation,*

$$\hat{\psi}_0(2\pi\gamma) = \cos^{2m}(\pi\gamma)\hat{\psi}_0(\gamma).$$

Thus ψ_0 *satisfies the refinement equation with*

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

Let $\begin{pmatrix} 2m \\ \ell \end{pmatrix}$ *denote the binomial coefficients* $\frac{(2m)!}{(2m-\ell)!\ell!}$ *and define the 1-periodic bounded functions* $H_1, H_2, \ldots, H_{2m}$ *by*

$$H_\ell(\gamma) = \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \sin^\ell(\pi\gamma) \cos^{2m-\ell}(\pi\gamma).$$

Then

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} \cos^{2m}(\pi\gamma) & \sin^{2m}(\pi\gamma) \\ \sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}}\sin(\pi\gamma)\cos^{2m-1}(\pi\gamma) & -\sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}}\cos(\pi\gamma)\sin^{2m-1}(\pi\gamma) \\ \sqrt{\begin{pmatrix} 2m \\ 2 \end{pmatrix}}\sin^2(\pi\gamma)\cos^{2m-2}(\pi\gamma) & \sqrt{\begin{pmatrix} 2m \\ 2 \end{pmatrix}}\cos^2(\pi\gamma)\sin^{2m-2}(\pi\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}}\sin^{2m}(\pi\gamma) & \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}}\cos^{2m}(\pi\gamma) \end{pmatrix}.$$

Now consider the 2×2 *matrix* $M := H(\gamma)^*H(\gamma)$. *Using the binomial formula*

$$(x + y)^{2m} = \sum_{\ell=0}^{2m} \begin{pmatrix} 2m \\ \ell \end{pmatrix} x^\ell y^{2m-\ell}$$

we see that the first entry in the first row of M is

$$M_{1,1} = \sum_{\ell=0}^{2m} \binom{2m}{\ell} \sin^{2\ell}(\pi\gamma) \cos^{2(2m-\ell)}(\pi\gamma) = 1.$$

A similar argument gives that $M_{2,2} = 1$. Also,

$$
\begin{aligned}
M_{1,2} &= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)\left(1 - \binom{2m}{1} + \binom{2m}{2} - \cdots + \binom{2m}{2m}\right) \\
&= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)(1-1)^{2m} = 0.
\end{aligned}
$$

Thus M is the identity on $\mathbb{C}^2$ for all γ; by Theorem 2.9 this implies that the $2m$ functions $\psi_1, \ldots, \psi_{2m}$ defined by

$$
\begin{aligned}
\hat{\psi}_\ell(\gamma) &= H_\ell(\gamma/2)\hat{\psi}_0(\gamma/2) \\
&= \sqrt{\binom{2m}{\ell}} \frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}}
\end{aligned}
$$

generate a multiwavelet frame for $L^2(\mathbb{R})$.

   Frequently one takes a slightly different choice of H_ℓ, namely,

$$H_\ell(\gamma) = i^\ell \sqrt{\binom{2m}{\ell}} \sin^\ell(\pi\gamma) \cos^{2m-\ell}(\pi\gamma).$$

Inserting this expression in $\hat{\psi}_\ell(\gamma) = H_\ell(\gamma/2)\hat{\psi}_\ell(\gamma/2)$ and using the commutator relations for the operators $\mathcal{F}, D, T_k$ shows that ψ_ℓ is a finite linear combination with real coefficients of the functions

$$DT_k\psi_0, \ \ k = -m, \ldots, m.$$

It follows that ψ_ℓ is a real-valued spline with support in $[-m, m]$, degree $2m-1$, smoothness class C^{2m-2}, and knots at $\mathbb{Z}/2$. Note in particular that we obtain smoother generators by starting with higher order splines, but that the price to pay is that the number of generators increases as well. $\square$

3. From Gabor frames to wavelet frames

   The goal of this section is to show how we can construct dual wavelet frame pairs based on certain dual Gabor frame pairs. The presented results form a survey of results by Christensen and Goh, and are taken from [7] and [8]. The key is the following transform that allows us to move the Gabor structure into the wavelet structure. We note that the first idea of such a transform appears in the paper [15] by Daubechies, Grossmann, and

Morlet; recently it has also been used by Feichtinger, Holighaus and Wiesmayr in their preprent [18].

Let $\theta > 1$ be given. Associated with a function $g \in L^2(\mathbb{R})$ for which $g(\log_\theta |\cdot|) \in L^2(\mathbb{R})$, we define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma) = \begin{cases} g(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \tag{3.1}$$

Note that by (3.1), for any $a > 0, j \in \mathbb{Z}$ and $\gamma \in \mathbb{R} \setminus \{0\}$,

$$\widehat{\psi}(a^j \gamma) = g(j \log_\theta(a) + \log_\theta(|\gamma|)). \tag{3.2}$$

For fixed parameters $b, \alpha > 0$ we will consider two bounded compactly supported functions $g, \tilde{g} \in L^2(\mathbb{R})$ and the associated Gabor systems $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$. For a fixed $\theta > 1$, define the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ by (3.1) for $g, \tilde{g}$ respectively. The following result shows how certain support conditions allow us to construct a pair of dual wavelet frames based on two dual Gabor frames. For the proof we refer to [7].

Theorem 3.1. *Let $b > 0$, $\alpha > 0$, and $\theta > 1$ be given. Assume that $g, \tilde{g} \in L^2(\mathbb{R})$ are bounded functions with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$ and that $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. With $a := \theta^\alpha$, if $b \leq \frac{1}{2\theta^N}$, then $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.*

We can now use Theorem 2.2 to obtain explicit constructions of dual wavelet frame pairs. We again refer to [7] for the proof.

Proposition 3.2. *Let $g \in L^2(\mathbb{R})$ be a bounded real-valued function with support in the interval $[M, N]$ for some $M, N \in \mathbb{Z}$. Suppose that g satisfies the partition of unity condition (2.10). Let $a > 1$ and $b \in (0, \min(\frac{1}{2(N-M)-1}, 2^{-1}a^{-N})]$ be given, and take any real sequence $\{c_n\}_{n=-N+M+1}^{N-M-1}$ such that*

$$c_0 = b, c_n + c_{-n} = 2b, \ n = 1, \ldots, N - M - 1.$$

Then the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ defined by (3.1) and

$$\widehat{\widetilde{\psi}}(\gamma) = \sum_{n=-N+M+1}^{N-M-1} c_n g(\log_a(|\gamma|) + n), \ \gamma \neq 0, \tag{3.3}$$

generate dual wavelet frames $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proposition 3.2 can of course be applied to B-splines. However, much more elegant constructions are obtained using exponential B-splines which yield pairs of wavelet frames for which the Fourier transform of the generators are compactly supported splines with geometrically distributed knots and desired smoothness. Exponential splines are of the form

$$\mathcal{E}_N(\cdot) := e^{\beta_1(\cdot)}\chi_{[0,1]}(\cdot) * \cdots * e^{\beta_N(\cdot)}\chi_{[0,1]}(\cdot),$$

where $\beta_k = (k-1)\beta$, $k = 1, \dots, N$, for some $\beta > 0$.

Exponential B-splines are well studied in the literature, see, e.g., [13], [24], [20], [10] (note that there is a typo in the expression for $\mathcal{E}_N(x)$ for $x \in [k-1, k]$ on page 304 of [10]: the expression $e^{a_{j_1}} + \cdots + e^{a_{j_{k-1}}}$ should be $e^{a_{j_1} + \cdots + a_{j_{k-1}}}$). Without going into details (for which we refer to [8]), we state the following explicit construction.

Example 3.3. Consider the exponential B-spline $\mathcal{E}_3$ with $N = 3$ and $\beta = 1$. Then

$$\mathcal{E}_3(x) = \begin{cases} \frac{1 - 2e^x + e^{2x}}{2}, & x \in [0, 1], \\ \frac{-(e + e^2) + 2(e^{-1} + e)e^x - (e^{-2} + e^{-1})e^{2x}}{2}, & x \in [1, 2], \\ \frac{e^3 - 2e^x + e^{-3}e^{2x}}{2}, & x \in [2, 3], \\ 0, & x \notin [0, 3]. \end{cases}$$

Let $a := e^{\beta} = e$, and define the function ψ by

$$\widehat{\psi}(\gamma) = \begin{cases} \frac{1 - 2|\gamma| + \gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [1, e], \\ \frac{-(e + e^2) + 2(e^{-1} + e)|\gamma| - (e^{-2} + e^{-1})\gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [e, e^2], \\ \frac{e^3 - 2|\gamma| + e^{-3}\gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [e^2, e^3], \\ 0, & |\gamma| \notin [1, e^3]. \end{cases}$$

The function $\widehat{\psi}$ is a geometric spline with knots at the points $\pm 1, \pm e, \pm e^2, \pm e^3$.

The construction in Proposition 3.2 works for $b \leq 2^{-1}e^{-3}$. Taking $b = 41^{-1}$ and $c_n = 41^{-1}$ for $n = -2, \dots, 2$, it follows from (3.2) and (3.3) that the resulting dual frame generator $\widetilde{\psi}$ satisfies

$$\widehat{\widetilde{\psi}}(\gamma) = \frac{1}{41} \sum_{n=-2}^{2} \widehat{\psi}(e^n \gamma), \ \gamma \in \mathbb{R}.$$

Figure 2: Plot of the geometric spline $\widehat{\psi}$ in Example 3.3.

Figure 3: Plot of the geometric spline $\widehat{\widetilde{\psi}}$ in Example 3.3.

The function $\widehat{\widetilde{\psi}}$ is a geometric spline with knots at the points $\pm e^{-2}, \pm e^{-1}, \pm 1,$ $\pm e^3, \pm e^4, \pm e^5$. Figures 2–3 show the graphs of the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$. $\square$

It is possible to reverse the process discussed so far, and obtain a way to obtain Gabor frames based on certain wavelet frames. The result can, e.g., be applied to the Meyer wavelet, which yields a construction of a tight Gabor frame generated by a $C^\infty(\mathbb{R})$, compactly supported function. Details of this are provided in [7].

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