Topology of streamlines and vorticity contours for two-dimensional flows

Considering a coordinate-free formulation of helical symmetry rather than more traditional definitions based on coordinates, we discuss basic properties of helical vector fields and compare results from the literature. For inviscid flow where a velocity field is generated by a sum of helical vortex filaments with same pitch we use the established results to prove briefly that the velocity field is helical. We discuss the role of the stream function for the topology of the streamlines in incompressible, helical flows. On this basis, we perform a comprehensive study of the topology of the flow field generated by a helical vortex filament in an ideal fluid. The classical expression for the stream function obtained by Hardin (Phys. Fluids 25, 1982) contains an infinite sum of modified Bessel functions. Using the approach by Okulov (Russ. J. Eng. Thermophys. 5, 1995) we obtain a closed-form approximation which is considerably easier to analyse. Critical points of the stream function can be found from the zeroes of a single real function of one variable, and we show that three different flow topologies can occur, depending on a single dimensionless parameter. Including the self-induced velocity on the vortex filament by the localised induction approximation the stream function is slightly modified and an extra parameter is introduced. In this setting two new flow topologies arise, but not more than two critical points occur for any combination of the parameters. The analysis of the closed form show promise for analysing more complex flow with helical symmetry e.g. multiple helical vortex filaments inside a cylinder which has industrial relevance.

We then change focus and study creation, destruction and interaction of vortices in two dimensional flow. A vortex is advected above a wall causing a viscous response near the wall which generates a new vortex structure. The problem is studied numerically relying on the code developed by Prof. M. Thompson and his group at Monash University, Australia. We also investigate the problem analytically using normal form theory. It is not a simple task to define a vortex in a proper way that allow the study of creation and destruction of vortices. We investigate three sound choices: the vorticity extrema, the streamline centers in a coordinate system with zero wall speed and the streamline centers in a frame moving with constant velocity as predicted by a point vortex above a wall in inviscid fluid. There is no reason to a priori expect equivalent results of the three vortex definitions. However, the study is mainly motivated by the findings of Kudela & Malecha (Fluid Dyn. Res. 41, 2009) who find good agreement between the vorticity and streamlines in the fixed wall system. For small Re no new vortices are observed. Creation of a vortex occurs for sufficiently large Re for all the applied vortex definitions. The new vortex alters the generating vortex motion by slowing its horizontal motion and lifting it further from the wall. In the fixed wall system vortex eruption happens through a characteristic ‘figure 8’ bifurcation. Considering the other coordinate system there is no topological change indicating when a vortex has left the boundary layer. However, here there is remarkable good agreement between streamlines and the vorticity contours even for short-lived vortices close to the wall.

The normal form approach does not reveal simple connections between the streamline topology and the vorticity contour topology. Only for a non simple degenerate on wall critical point may a bifurcation occur in both the streamlines and the vorticity contours. The streamline bifurcations in this normal form contain the lower part of the ‘figure 8’ bifurcation observed in numerics. The similarities and differences of the streamlines in the two different coordinate systems are well described by normal form theory.

We derive the criterion, \( u \cdot \nabla \omega = 0 \), for exactly matching contours of the vorticity contours and streamlines. This is fulfilled when the Navier-Stokes equations and the heat equation have identical solutions.

Finally we focus on the superposition of two rotational invariant vortices in \( \mathbb{R}^2 \). The topology of the streamlines and the topology of the vorticity contours are determined by the zeroes of a single real function. For the canonical example of two Gaussian vortices three parameters exist. Three structurally stable topologies are observed. For the streamlines two of the topologies are well known for the corresponding situation of two point vortices when the singularities are treated as centers. The last topology is a single center which is consistent with the powerful result on the long time behaviour proved by Gallay & Wayne (Comm. in Math. Phys. 255, 2005). The case of three critical points of the streamlines is a subset of three critical points of the vorticity. This explains an observation in the simulations of vortex generation near a wall. Here, a long living erupted vortex disappears due to viscosity. This happens first considering the streamlines while being more robust when considering the vorticity formulation.

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