Riemann zeta function from wave-packet dynamics

We show that the time evolution of a thermal phase state of an anharmonic oscillator with logarithmic energy spectrum is intimately connected to the generalized Riemann zeta function \( \zeta(s, a) \). Indeed, the autocorrelation function at a time \( t \) is determined by \( \zeta(\sigma + i \tau, a) \), where \( \sigma \) is governed by the temperature of the thermal phase state and \( \tau \) is proportional to \( t \). We use the JWKB method to solve the inverse spectral problem for a general logarithmic energy spectrum; that is, we determine a family of potentials giving rise to such a spectrum. For large distances, all potentials display a universal behavior; they take the shape of a logarithm. However, their form close to the origin depends on the value of the Hurwitz parameter \( a \) in \( \zeta(s, a) \). In particular, we establish a connection between the value of the potential energy at its minimum, the Hurwitz parameter and the Maslov index of JWKB. We compare and contrast exact and approximate eigenvalues of purely logarithmic potentials. Moreover, we use a numerical method to find a potential which leads to exact logarithmic eigenvalues. We discuss possible realizations of Riemann zeta wave-packet dynamics using cold atoms in appropriately tailored light fields.