



Coupling and reduction of the HAWC equations

Nim, Erik

Publication date:
2001

Document Version
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):
Nim, E. (2001). Coupling and reduction of the HAWC equations. Denmark. Forskningscenter Risoe. Risoe-R, No. 1294(EN)

DTU Library

Technical Information Center of Denmark

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Coupling and Reduction of the HAWC Equations

Erik Nim

Abstract This report contains a description of a general method for coupling and reduction of the so-called HAWC equations, which constitute the basic equations of motion of the aeroelastic model HAWC used widely by research institutes and industrial companies for more than ten years. The principal aim of the work has been to enable the modelling wind turbines with large displacements of the blades in order to predict phenomena caused by geometric non-linear effects. However, the method can also be applied to model the nacelle/shaft structure of a turbine more detailed than the present HAWC model. In addition, the method enables the reduction of the number of degrees of freedom of the structure in order to increase the calculation efficiency and improve the condition of the system.

This work was founded by The Danish Ministry of Energy through the project *Program for forskning i aeroelasticitet*, EFP-2000, contract no ENS-1363/00-0007.

ISBN 87-550-2944-2; 87-550-2945-0 (Internet)
ISSN 0106-2840

Print: Pitney Bowes Management Services Denmark A/S, 2001

Contents

1	Introduction	<i>5</i>
2	The substructure	<i>7</i>
2.1	The substructuring concept	<i>7</i>
2.2	The substructure coordinate system	<i>8</i>
2.3	The finite element method for a substructure	<i>10</i>
3	Transformations of substructure e.o.m.	<i>13</i>
3.1	Assembling the mass matrix	<i>13</i>
3.2	The rigid-body transformation	<i>13</i>
3.3	The Craig-Bampton transformation	<i>15</i>
4	Coupling of substructure e.o.m.	<i>21</i>
4.1	Collecting the d.o.f. of the complete structure	<i>21</i>
4.2	The basic coupling conditions	<i>22</i>
4.3	Selection of dependent constraint d.o.f.	<i>24</i>
4.4	Preparing the coupling conditions for transformation	<i>24</i>
4.5	The coupling transformation	<i>26</i>
4.6	The coupled e.o.m.	<i>28</i>
5	The fixed coupling	<i>33</i>
5.1	Conditions for translation	<i>33</i>
5.2	Conditions for rotation	<i>35</i>
5.3	Coupling conditions in common form	<i>37</i>
6	Conclusion	<i>41</i>
A	Kinematic analysis	<i>45</i>
B	Rearranging the expression for the nodal inertia forces	<i>47</i>
C	The rigid-body displacement modal matrix	<i>49</i>
D	The skew-symmetric matrix associated with a vector	<i>51</i>
E	The Craig-Bampton transformation with a full mass matrix	<i>53</i>
F	Euler rotation parameters	<i>55</i>

1 Introduction

Traditionally aeroelastic models of wind turbines are applied mainly for the calculation of dynamic loads on the structure and these models play an increasingly important role in the process of designing large modern wind turbines. In addition, aeroelastic models are fundamental for the study of the dynamic behaviour of wind turbines in some critical situations such as overspeeds, shutdowns, and startups. For wind turbines with active power regulation, aeroelastic models have also proved to be very efficient in designing and tuning the regulator, which may drastically change the dynamic response of the turbine.

The aeroelastic model HAWC was developed by Petersen [9]¹, and it has served as Risø's in-house model since the first version emerged in 1990. In addition, it has been applied by some wind turbine manufacturers and type approval institutions for calculation of dynamic loads. HAWC are based on the finite element method with a Timoshenko beam element having two end-point nodes with six degrees of freedom. Since the standard finite element method in principle assumes infinitesimally small rotations, the wind turbine was divided into three *substructures*, namely one tower, one shaft/nacelle and one rotor including two or three blades. This concept of substructuring enables the presence of finite rotations between the substructures, whereas the relative displacements within a substructure are assumed to be small. The substructures were then coupled together by means of force equilibrium in the two coupling nodes situated in the tower top and in the rotor centre.

An important property of the HAWC model is that relative displacements within a substructure are described with respect to a local moving coordinate system, which is attached to the substructure. The absolute displacement of a material point is therefore composed of one contribution from the local displacement due to elastic deformations and another contribution from the displacement of the substructure.

In the development of the HAWC model, special attention was paid to the inertia loads, which were modelled by transforming the distributed inertia loads acting on an element to the two nodes consistently with the principle of virtual work. By this method the Coriolis forces and the inertia stiffening forces accounting for the local description of the relative displacements appear directly in the expression for the inertia forces. Generally, the inertia loads are non-linear and depend on the angular velocity and acceleration of the substructure in question.

For wind turbines with very flexible blades the assumption of small displacements within the rotor substructure cannot be fulfilled and the dynamic response therefore becomes subject to geometric non-linearities, which is characterized by large rotations but moderate strains. The most important effects of geometric non-linearities for the blades are that the out-of-plane displacements of the tip are overestimated and that the pitching moments in the root flange are the subject of severe errors due to second order effects of the external loading.

This report describes a general method for coupling of substructures modelled according to the HAWC method. In this context the term "general" covers that the substructures may be selected freely and that various types of couplings between the substructures can be described. The method was developed mainly in order to enable the modelling of geometric non-linearities of the blades, since the effect of this phenomenon increases with increasing size of the wind turbines. However, it is obvious that the generalization of the substructuring concept also enables a more detailed modelling of the nacelle/shaft structure as well as modelling of

¹An overview of the model are given in reference [11]

wind turbines, which structurally differ significantly from the "Danish concept" including most modern horizontal axis wind turbines with three blades.

Further to the general method for coupling of substructures, the standard Craig-Bampton method [3] is applied for the reduction of the number of degrees of freedom within a substructure in order to increase the calculation efficiency and improve the condition of the dynamic system by removing high-frequency modes.

The selection of a method of representation of finite rotations of the substructures is a crucial step in the development of the method. The well known Euler angle representation describes the rotation by three successive rotation about the coordinate axis in a specified order. However, this representation is singular in some cases, which means that an unique relation between the time derivatives of the Euler angles and the angular velocity does not exist. In addition, it is well known that the algebra associated with Euler angles has no symmetric properties. These problems is disregarded completely by resorting to the so-called Euler parameters, which represents the rotation by a normalized vector indicating the direction of the rotation axis and a rotation angle.

The report is structured as follows: Section 2 describes the degrees of freedom associated with a free substructure including the local coordinate system, which is attached to the substructure. At the end of this section, the final form of the applied equations of motion for a substructure is given. These equations were originally developed by Pedersen [9], but rewritten in Appendix B in order to fit into the present context. Section 3 deals with linear transformations of the substructure equations of motion in order to prepare the equations for coupling and to reduce the number of degrees of freedom. A detailed description of the proposed method for coupling of substructure equations of motion is given in Section 4, which ends with the explicit expressions for the resulting mass and stiffness matrices. Section 5 described the so-called fixed coupling, which interconnects two substructures in a common node.

2 The substructure

2.1 The substructuring concept

Fundamental for the proposed method is that the complete structure is divided into a number of structural components or *substructures*. A substructure is assumed to interact with other substructures only in certain *coupling nodes*, which have to be specified when defining the complete structure. For a wind turbine with a simple submodel of the nacelle, the obvious choice of substructures is one tower, one shaft/nacelle and two or three blades. The coupling nodes are then situated in the tower top and in the root flange of the blades.

The described concept of substructuring may be regarded as a generalization of the method used for the aeroelastic model HAWC developed by Petersen [9]. In this model, the wind turbine is divided into three substructures, which is chosen as one tower, one shaft/nacelle and one rotor including two or three blades. The choice of substructures is fixed and they are an integrated part of the model.

Basically it is assumed that the substructures consist of flexible bodies, which deform linearly elastic with respect to a local coordinate system being rigidly attached to a specific node. Within a substructure, the elastic deformations are assumed to be infinitesimally small, which allows the application of the finite element method for modelling of the dynamic response. In addition to the dynamics within a substructure, the local coordinate system translates and rotates with respect to an inertial coordinate system. The absolute displacement of a node is therefore composed of a two parts; one part originating from the translation and rotation of the local coordinate system and another part originating from the elastic deformations. However, the concept of substructuring is more general and additional structural components such as rigid bodies could be the subject for future extensions.

The substructuring of large complicated structures is widely used for both static and dynamic finite element applications [2] mainly as a tool for reducing the number of degrees of freedom (d.o.f.) of the complete structure in order to minimize the calculation time as well as the necessary computer storage. Particularly, the concept has proved to be highly efficient for dynamic applications, as much of the numerical analysis for the generation of a reduced set of generalized d.o.f. may be performed before time integration. For dynamic applications, the method is often referred to as *mode component synthesis* of which the Craig-Bampton method [3] is one of the most effective. This method is used in Section 3.3 in order to reduce the number of d.o.f., which describe the displacement of a substructure.

The main reason for adopting the substructuring concept in the present work is, however, the need for modelling of finite rotations of structural components, as the standard finite element method basically assumes that the rotation of the nodes due to angular deformations are infinitesimally small. For an operating wind turbine, the need for description of finite rotations is essential because the dynamic response to a considerable degree is determined by the rotational motion of the blades.

In addition, the substructuring concept may be applied to model geometric non-linearities, i.e. non-linear effects due to large rotations and displacements but moderate strains, simply by subdividing the components into more substructures, which subsequently are coupled together by a fixed connection ensuring geometric compatibility in the coupling nodes. However, the displacements within the substructures are still assumed to be infinitesimally small, and the method may therefore require a large number of substructures in extreme situations. In that case the proposed method approaches the method developed by Fabian [4]. For

wind turbines, the effects of large blade displacements is of particular interest due to the increasing size of the wind turbines and with that increasing flexibility. However, the displacements of the blade of the largest state-of-the-art wind turbines are still only moderately large, and may possibly be modelled effectively by two or three substructures only.

2.2 The substructure coordinate system

The substructure position d.o.f.

As appearing from the above description, the local substructure coordinate system attached to a particular substructure is dynamic, which means that it translates and rotates with respect to the inertial coordinate system. For this reason, the parameters describing the position and the angular orientation of a substructure coordinate system are generally regarded as independent d.o.f. It is obvious that these d.o.f. are closely related to the rigid-body displacements of the substructure.

The position of the substructure coordinate system are described by the 3×1 vector $\{\mathbf{r}_S^I\}$ of inertial coordinates to the position of origo. The orientation of the substructure is formally described by the 3×1 vector $\{\boldsymbol{\rho}_S\}$ of *independent* rotation parameters, which do not have to be specified further at this stage. However, it may already now be unveiled that finite rotations throughout this report are described by Euler parameters² given by a 4×1 vector $\{\rho_{S,0}^{S,0}\}$, where the scalar component $\rho_{S,0}$ and the vector component $\{\boldsymbol{\rho}_S\}$ are related by $(\rho_{S,0})^2 + |\boldsymbol{\rho}_S|^2 = 1$, leaving three independent parameters left. Euler parameters are described in brief in Appendix F, where the most important formulas are listed.

The substructure position d.o.f. are collected in the 6×1 vector

$$\{\mathbf{R}_S^I\} = \begin{Bmatrix} \mathbf{r}_S^I \\ \boldsymbol{\rho}_S \end{Bmatrix} \quad (2.1)$$

The corresponding vectors of inertial coordinates to the velocity of the substructure coordinate system are collected in the 6×1 vector $\{\mathbf{V}_S^I\}$ defined by

$$\{\mathbf{V}_S^I\} = \begin{Bmatrix} \mathbf{v}_S^I \\ \boldsymbol{\omega}_S^I \end{Bmatrix} \quad (2.2)$$

where $\{\mathbf{v}_S^I\} = \{\dot{\mathbf{r}}_S^I\}$ is the 3×1 vector of inertial coordinates to the velocity of origo of the substructure coordinate system, while $\{\boldsymbol{\omega}_S^I\}$ is the 3×1 vector of inertial coordinates to the angular velocity. Similarly, the vectors of inertial coordinates to the acceleration are collected in the 6×1 vector $\{\mathbf{A}_S^I\}$ defined by

$$\{\mathbf{A}_S^I\} = \begin{Bmatrix} \mathbf{a}_S^I \\ \boldsymbol{\alpha}_S^I \end{Bmatrix} \quad (2.3)$$

where $\{\mathbf{a}_S^I\} = \{\ddot{\mathbf{r}}_S^I\}$ is the 3×1 vector of inertial coordinates to the acceleration of origo of the substructure coordinate system, while $\{\boldsymbol{\alpha}_S^I\} = \{\dot{\boldsymbol{\omega}}_S^I\}$ is the 3×1 vector of inertial coordinates to the angular acceleration.

By comparison of the definitions (2.2) and (2.3) it appears that

$$\{\mathbf{A}_S^I\} = \{\dot{\mathbf{V}}_S^I\} \quad (2.4)$$

The time derivative of rotation parameters

A severe complication of finite rotation in three-dimensional applications is that the time derivatives of the rotation parameters generally differ from the time

²In some references called *unit quaternions*.

derivative of the rotation as noted by Krenk [6]³. Consequently, the time derivative of $\{\mathbf{R}_S^I\}$ do not equal $\{\mathbf{V}_S^I\}$, as is the case when restricting to two-dimensional applications, where the rotation parameter simply consists of a single angle. However, the vectors $\{\dot{\rho}_S\}$ and $\{\omega_S\}$ are closely related by an unique⁴ relation (see [6]), and in order to write the equations of motion (e.o.m.) in a compact form, a special interpretation of the usual time differentiation operator, denoted by a superposed dot ($\dot{}$), is introduced when it is applied to the vector $\{\rho_S\}$ and with that the vector $\{\mathbf{R}_S^I\}$. Then by definition

$$\{\dot{\mathbf{R}}_S^I\} = \{\mathbf{V}_S^I\} \quad (2.5)$$

and

$$\{\ddot{\mathbf{R}}_S^I\} = \{\mathbf{A}_S^I\} \quad (2.6)$$

It is emphasized that the definition of the superposed dot operator introduced by equations (2.5) and (2.6) differs from the usual definition, which have to be taken into account when the system is integrated.

The coordinate transformation matrix

The foundation for the description of the elastic deformations of the substructures in a local substructure coordinate system is the coordinate transformation matrix, which relates a 3×1 vector of substructure coordinates to the corresponding 3×1 vector of inertial coordinates. In the following we only explicitly refer to the 3×3 matrix $[\mathbf{T}_S]$, which transforms from substructure coordinates to inertial coordinates. The inverse of $[\mathbf{T}_S]$, which transforms from inertial coordinates to substructure coordinates, is then given by

$$[\mathbf{T}_S]^{-1} = [\mathbf{T}_S]^T \quad (2.7)$$

as coordinate transformation matrices in general are orthogonal.

It is important to note that the coordinate transformation matrix is described completely by the rotation parameters. In case of Euler rotation parameters given by the 4×1 vector $\{\rho_S^{s,0}\}$, Nikravesh [8] shows that the corresponding transformation matrix becomes

$$[\mathbf{T}_S] = \left(2(\rho_{S,0})^2 - 1 \right) [\mathbf{1}] + 2 \left(\{\rho_S\} \{\rho_S\}^T + \rho_{S,0} [\tilde{\rho}_S] \right) \quad (2.8)$$

where the tilde (\sim) superposed a vector denotes the associated skew-symmetric matrix defined in Appendix D, and the symbol $[\mathbf{1}]$ denotes the 3×3 unity matrix⁵. A more explicit formula for $[\mathbf{T}_S]$ is given in Appendix F.

It appears to be convenient to define the 6×6 expanded coordinate transformation matrix $[\mathbf{T}_S^*]$, which transforms a pair of 3×1 vectors collected in a common 6×1 vector. This matrix is obviously defined by

$$[\mathbf{T}_S^*] = \begin{bmatrix} \mathbf{T}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_S \end{bmatrix} \quad (2.9)$$

Since $[\mathbf{T}_S^*]$ is a diagonal matrix, the inverse is found directly from equation (2.7), which yields

$$[\mathbf{T}_S^*]^{-1} = [\mathbf{T}_S^*]^T \quad (2.10)$$

³Actually Krenk states that the increment of the rotation parameters is different from the increment of the rotation, but this statement is analogous.

⁴This is certainly true for Euler parameters and rotation pseudo vectors. For the popular Euler *angles* the relation is not unique in certain cases, which may cause numerical problems.

⁵This notation for the unity matrix is maintained throughout this report. The actual dimension of $[\mathbf{1}]$ will appear from the context.

2.3 The finite element method for a substructure

The dynamics of a substructure defined in a local translating and rotating coordinate system were modelled by Petersen [9] using the displacement based finite element method. By this method, a substructure is divided into a number of *elements*, which are coupled together by the *nodes*. The e.o.m. are then derived by the equilibrium condition of the nodal forces and moments, which is determined by transforming all forces and moments⁶ acting on the elements to the nodes consistently with the principle of virtual work.

The resulting model, which represents the fundamental structural model of HAWC, has been adopted in the present work almost completely. An exception is the submodel of the geometric stiffness, which automatically enters into the equations when using an adequate large number of substructures. However, the inclusion of the submodel of geometric stiffness would model the centrifugal stiffening of the blades even with one substructure only, which in some cases would be preferable in order to increase the calculation efficiency. Therefore, the inclusion of the submodel for geometric stiffness could be the subject for a future extension of the proposed method.

Definition of the nodes of a substructure

We now consider a single *free* substructure, i.e. the interaction with the other substructure is disregarded. As described in Section 2.2, a local coordinate system defined with respect to the inertial coordinate system is attached to the substructure at a selected node.

It is obvious that the nodal positions in the undeformed state expressed in local substructure coordinates are constant and given by the geometry. The common vector $\{\mathbf{r}^S\}$ of substructure coordinates to the positions of all nodes in the substructure is defined by

$$\{\mathbf{r}^S\} = \begin{Bmatrix} \mathbf{r}_1^S \\ \mathbf{r}_2^S \\ \vdots \\ \mathbf{r}_N^S \end{Bmatrix} \quad (2.11)$$

where $\{\mathbf{r}_n^S\}$ is the 3×1 vector of substructure coordinates to the position of the n 'th node and N is the number of nodes.

Definition of relative nodal displacements

The elastic deformations of the substructure are described by the displacements of the nodes measured relatively to the substructure coordinate system. In the following $\{\mathbf{u}_n^S\}$ denotes the 3×1 vector of substructure coordinates to the translation of the n 'th node, while $\{\boldsymbol{\theta}_n^S\}$ denotes the similar vector for rotation, which is assumed to be infinitesimally small in order to allow the application of the finite element method. A common vector of substructure coordinates to the nodal displacements is then given by the 6×1 vector

$$\{\mathbf{U}_n^S\} = \begin{Bmatrix} \mathbf{u}_n^S \\ \boldsymbol{\theta}_n^S \end{Bmatrix} \quad (2.12)$$

⁶Including inertia forces expressed by D'Alembert's principle.

The deformation pattern of a substructure is then completely described by the $6N \times 1$ nodal displacement vector $\{\mathbf{U}^S\}$ defined by

$$\{\mathbf{U}^S\} = \begin{Bmatrix} \mathbf{U}_1^S \\ \mathbf{U}_2^S \\ \vdots \\ \mathbf{U}_N^S \end{Bmatrix} \quad (2.13)$$

Attaching the local coordinate system to the substructure

For convenience it is assumed that the local substructure coordinate system is attached to the first node, which means that this *attaching node* is chosen by the indexing. By definition, the displacement of the attaching node vanish, which implies that $\{\mathbf{U}_1^S\} = \{\mathbf{0}\}$. It appears to be convenient to formulate this condition by partitioning of the displacement vector $\{\mathbf{U}^S\}$ according to

$$\{\mathbf{U}^S\} = \begin{Bmatrix} \mathbf{U}_o^S \\ \mathbf{U}_f^S \end{Bmatrix} \quad (2.14)$$

The attaching condition may then be written as

$$\{\mathbf{U}_o^S\} = \{\mathbf{0}\} \quad (2.15)$$

It is noted that the condition (2.15) reduces the number of d.o.f. of the substructure by six. The total number of d.o.f. for the substructure then becomes $6N$, i.e. 6 substructure position d.o.f. and $(6N - 6)$ nodal displacement d.o.f.

An important consequence of the described method for attaching the local coordinate system to the substructure is that rigid-body displacements by nodal displacements are prevented due to the condition (2.15). This is essential since rigid-body displacements already are described by displacement of the local coordinate system.

The e.o.m. of a free substructure

The e.o.m. of a free beam element were derived by Pedersen [9], who based the analysis on a Timoshenko theory, which includes coupling between bending and torsion. A detailed description of the chosen element is given by Pedersen and Jensen [12].

Special attention was paid to the modelling of the non-linear inertia forces, which arises from dynamics of the substructure due to nodal displacements as well as dynamics of the substructure coordinate system accounting for the rigid-body displacements of the substructure. The expression for the inertia force acting on a beam element is rearranged in Appendix B in order prepare it for application in the present context.

The resulting e.o.m. for a substructure are found by assemblage of the e.o.m. for the elements using the standard linear method (see e.g. reference [2]). The final form of the e.o.m. used in the following sections is

$$\begin{aligned} [\mathbf{M}] \{\ddot{\mathbf{U}}^S\} + \left([\mathbf{C}] + [\mathbf{C}_I]\right) \{\dot{\mathbf{U}}^S\} + \left([\mathbf{K}] + [\mathbf{K}_I]\right) \{\mathbf{U}^S\} \\ + [\mathbf{M}_R] \{\mathbf{A}_S^S\} + \{\mathbf{F}_R\} = \{\mathbf{F}_{\text{ext}}^S\} \end{aligned} \quad (2.16)$$

where

$[\mathbf{M}]$ is the constant $6N \times 6N$ mass matrix, which is symmetric and positive definit.

$[\mathbf{C}]$ is the constant $6N \times 6N$ structural damping matrix.

$[\mathbf{C}_I]$ is the time-dependent $6N \times 6N$ inertia damping matrix (the Coriolis matrix), which is a function of $\{\boldsymbol{\omega}_S^S\}$. This matrix is skew-symmetric.

$[\mathbf{K}]$ is the $6N \times 6N$ structural stiffness matrix, which is symmetric.

$[\mathbf{K}_I]$ is the time-dependent $6N \times 6N$ inertia stiffening matrix, which is a function of both $\{\boldsymbol{\omega}_S^S\}$ and $\{\boldsymbol{\alpha}_S^S\}$. Generally, this matrix is neither symmetric nor skew-symmetric.

$[\mathbf{M}_R]$ is the constant $6N \times 6$ mass matrix accounting for rigid-body dynamics imposed by the substructure coordinate system. The matrix is defined in Appendix B.

$\{\mathbf{F}_R\}$ is the time-dependent $6N \times 1$ inertia force vector, which is a function of $\{\boldsymbol{\omega}_S^S\}$. This force vector accounts for rigid-body dynamics imposed by the substructure coordinate system.

$\{\mathbf{F}_{\text{ext}}^S\}$ is a $6N \times 1$ vector of substructure coordinates to the external forces arising from e.g. aerodynamic interaction and gravity.

It is noted that the inclusion of an explicit submodel of geometric stiffness would result in an additional stiffness matrix in equation (2.16).

Structural damping

Structural damping is modelled by the standard Rayleigh method expressing the structural damping matrix $[\mathbf{C}]$ as a linear combination of the structural stiffness matrix $[\mathbf{K}]$ and the mass matrix $[\mathbf{M}]$.

As the structural damping is the key in many dynamic problems the standard Rayleigh method is probably not suitable for describing the damping for a complete wind turbine. Thus more elaborated methods such as the general methods described by Clough and Penzien [1] or the more specific method described by Hansen [5] should be considered for future improvements.

3 Transformations of substructure e.o.m.

In this section the e.o.m. for a free substructure is transformed into a form, which is suitable for coupling as described in Section 4. In addition, a set of generalized d.o.f. for the substructure is introduced in order to enable the reduction of the total number of the d.o.f. of the substructure.

3.1 Assembling the mass matrix

For convenience, the following analysis only explicitly includes the two mass matrices $[\mathbf{M}]$ and $[\mathbf{M}_R]$ in the e.o.m. for a substructure as given by equation (2.16). However, the analysis could easily be carried through with the e.o.m. in a form, which additionally includes the structural stiffness matrix $[\mathbf{K}]$ as well as the structural damping matrix $[\mathbf{C}]$. However, the inclusion of the damping matrix requires that structural damping is not affected by rigid-body motion as noted in Section 3.2). On the other hand it seems impossible to include the two inertia matrices $[\mathbf{C}_i]$ and $[\mathbf{K}_i]$ explicitly, due to the non-linear appearance of the angular velocity in the definition of these matrices. Thus, the e.o.m. (2.16) are employed in the contracted form

$$[\mathbf{M}]\{\ddot{\mathbf{U}}^S\} + [\mathbf{M}_R]\{\mathbf{A}_S^S\} = \{\mathbf{F}_{rhs}^S\} \quad (3.1)$$

where the right-hand side force vector $\{\mathbf{F}_{rhs}^S\}$ simply equals $\{\mathbf{F}_{ext}^S\}$ minus the remaining terms on the left-hand side of equation (2.16).

Equation (3.1) is now partitioned according to the partitioning of the nodal displacement vector as given by equation (2.13). Thus

$$\begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{of} \\ \mathbf{M}_{fo} & \mathbf{M}_{ff} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}}_o^S \\ \ddot{\mathbf{U}}_f^S \end{Bmatrix} + \begin{bmatrix} \mathbf{M}_{Ro} \\ \mathbf{M}_{Rf} \end{bmatrix} \{\mathbf{A}_S^S\} = \{\mathbf{F}_{rhs}^S\} \quad (3.2)$$

where $[\mathbf{M}_{oo}]$ and $[\mathbf{M}_{Ro}]$ both are 6×6 submatrices. It is obvious that the attachment condition (2.15) implies that $\{\ddot{\mathbf{U}}_o^S\} = \{\mathbf{0}\}$. Inserting this relation into equation (3.2) and rearranging slightly yields

$$\begin{bmatrix} \mathbf{M}_{Ro} & \mathbf{M}_{of} \\ \mathbf{M}_{Rf} & \mathbf{M}_{ff} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_S^S \\ \ddot{\mathbf{U}}_f^S \end{Bmatrix} = \{\mathbf{F}_{rhs}^S\} \quad (3.3)$$

3.2 The rigid-body transformation

The relation between the mass matrices

Apparently, the assembled mass matrix in equation (3.3) is neither symmetric nor positive definite, which is the properties generally required for the mass matrix of a dynamic system. In the following we therefore transform the e.o.m. into a form, where the mass matrix may be proved to be symmetric and positive definite.

The key for this transformation is a relation between the two mass matrices $[\mathbf{M}]$ and $[\mathbf{M}_R]$. Initially it is realized that an acceleration $\{\mathbf{A}_S^S\}$ imposed by the substructure coordinate system is equivalent to the nodal accelerations $[\boldsymbol{\psi}]\{\mathbf{A}_S^S\}$, where $[\boldsymbol{\psi}]$ is the modal matrix of rigid-body displacements, which is defined in Appendix C. Consequently, the inertia forces arising from these accelerations are equal in every node, which implies that $[\mathbf{M}][\boldsymbol{\psi}]\{\mathbf{A}_S^S\} = [\mathbf{M}_R]\{\mathbf{A}_S^S\}$. As the vector $\{\mathbf{A}_S^S\}$ is arbitrarily chosen, this relation holds for any $\{\mathbf{A}_S^S\}$, which consequently may be cancelled on both sides of the equation. Thus

$$[\mathbf{M}][\boldsymbol{\psi}] = [\mathbf{M}_R] \quad (3.4)$$

Equation (3.4) constitutes the desired relation between the two mass matrices $[\mathbf{M}]$ and $[\mathbf{M}_R]$, which enter into the e.o.m. (3.1). The relation has been verified numerically in a number of different cases using the expressions derived by Pedersen [9] for the mass matrix $[\mathbf{M}]$ and the quantities, which enter into the definition of $[\mathbf{M}_R]$ given by equation (B.7) for an element. However, an analytical proof would be preferable and should be carried through in the future.

Relations of other system matrices

For theoretical reasons it might be interesting, whether relations similar to that of equation (3.4) exist for other system matrices, which enter into the e.o.m. in the full form given by equation (2.16).

As elastic forces arising from rigid-body displacements generally vanish⁷, it is obvious that such a relation for the stiffness matrix is

$$[\mathbf{K}][\psi] = [\mathbf{0}] \quad (3.5)$$

Equation (3.5) is just the relation which is needed if the stiffness matrix were included explicitly in the contracted form of the e.o.m. (3.1) and with that included in the analysis throughout this section.

Regarding the damping matrix, it is obvious that a relation similar to (3.5) is required if the damping matrix were included explicitly in equation (3.1). Physically, this relation is equivalent to specify that damping should be independent of rigid-body motion, which seems to be a natural property of the structural damping phenomenon. Therefore, it is natural to require

$$[\mathbf{C}][\psi] = [\mathbf{0}] \quad (3.6)$$

which have to be taken into account when modelling the structural damping. Obviously the requirement (3.6) is fulfilled for stiffness proportional damping but not for Rayleigh damping in general.

The symmetric and positive definite form of the e.o.m.

The relation (3.4) between the two mass matrices is now employed to transform the e.o.m. into a form having a symmetric and positive definite mass matrix. Initially equation (3.4) is partitioned according to

$$\begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{of} \\ \mathbf{M}_{fo} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \psi_o \\ \psi_f \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{Ro} \\ \mathbf{M}_{Rf} \end{bmatrix} \quad (3.7)$$

Inserting equation (3.7) into the e.o.m. (3.3) and rearranging then yields

$$\begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{of} \\ \mathbf{M}_{fo} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \psi_o & \mathbf{0} \\ \psi_f & \mathbf{1} \end{bmatrix} \left\{ \begin{matrix} \mathbf{A}_S^S \\ \dot{\mathbf{U}}_f^S \end{matrix} \right\} = \{\mathbf{F}_{rhs}^S\} \quad (3.8)$$

In order to write equation (3.8) in a compact form we define the $6N \times 6N$ matrix

$$[\mathbf{\Psi}] = \begin{bmatrix} \psi_o & \mathbf{0} \\ \psi_f & \mathbf{1} \end{bmatrix} \quad (3.9)$$

and the $6N \times 1$ vector

$$\{\mathbf{D}^S\} = \left\{ \begin{matrix} \mathbf{R}_S^S \\ \mathbf{U}_f^S \end{matrix} \right\} \quad (3.10)$$

where the 6×1 vector $\{\mathbf{R}_S^S\}$, defined by equation (2.1), contains the six substructure position d.o.f., i.e. three substructure coordinates to the position of origo and three independent rotation parameters. With the special definition of the symbol

⁷This is a general property of the finite element formulation [2].

$\{\ddot{\mathbf{R}}_S^S\}$ introduced by equation (2.6), the e.o.m. (3.8) may be written in a compact form as

$$[\mathbf{M}][\Psi]\{\ddot{\mathbf{D}}^S\} = \{\mathbf{F}_{\text{rhs}}^S\} \quad (3.11)$$

It is obvious that the e.o.m. given by equation (3.11) are transformed into a symmetric and positive definite form simply by pre-multiplying both sides of the equation by the matrix $[\Psi]^T$. However, for reasons that will become clear in Section 4, where the substructures are coupled, the transformation will be carried through more rigorously using the principle of virtual work. Initially, we define the transformation

$$\{\mathbf{U}^S\} = [\Psi]\{\dot{\mathbf{D}}^S\} \quad (3.12)$$

where

$$\{\dot{\mathbf{D}}^S\} = \begin{Bmatrix} \dot{\mathbf{R}}_S^S \\ \mathbf{U}_f^S \end{Bmatrix} \quad (3.13)$$

contains a new 6×1 vector $\{\dot{\mathbf{R}}_S^S\}$, which imposes rigid-body displacement of the substructure by nodal displacements, whereas rigid-body displacement by moving the local substructure coordinate system is imposed by the vector $\{\mathbf{R}_S^S\}$ defined by equation (2.1).

The nodal forces acting on the substructure are now considered being in equilibrium, which implies that the e.o.m. are fulfilled. It is noted that $\{\dot{\mathbf{R}}_S^S\} = \{\mathbf{0}\}$ in this state, which is readily shown by inserting the attachment condition (2.15) into equation (3.12). Let $\{\delta\dot{\mathbf{D}}^S\}$ denote a small virtual displacement from equilibrium, while keeping the substructure coordinate system fixed in the equilibrium position. The corresponding virtual displacement $\{\delta\mathbf{U}^S\}$ of the nodes is then found by equation (3.12), implying that

$$\{\delta\mathbf{U}^S\} = [\Psi]\{\delta\dot{\mathbf{D}}^S\} \quad (3.14)$$

Equating the work done by the nodal forces on the left-hand side and the right-hand side of the e.o.m. (3.11) during the virtual displacement yields

$$\{\delta\dot{\mathbf{D}}^S\}^T [\Psi]^T [\mathbf{M}][\Psi]\{\dot{\mathbf{D}}^S\} = \{\delta\dot{\mathbf{D}}^S\}^T [\Psi]^T \{\mathbf{F}_{\text{rhs}}^S\} \quad (3.15)$$

As the virtual displacement $\{\delta\dot{\mathbf{D}}^S\}$ is arbitrarily chosen, the factors $\{\delta\dot{\mathbf{D}}^S\}^T$ may be cancelled on both sides of equation (3.15). This yields

$$[\Psi]^T [\mathbf{M}][\Psi]\{\dot{\mathbf{D}}^S\} = [\Psi]^T \{\mathbf{F}_{\text{rhs}}^S\} \quad (3.16)$$

It is readily shown⁸ that the transformed mass matrix $[\Psi]^T [\mathbf{M}][\Psi]$ is symmetric and positive definite for any transformation matrix $[\Psi]$ different from zero.

3.3 The Craig-Bampton transformation

The Craig-Bampton method [3] is one of the most popular methods for mode component synthesis [2] applied in order to reduce the number of d.o.f. of a structure, which is divided into substructures. It is important to note that the method originally was developed for substructures defined in an inertial local coordinate system, which implies that the resulting e.o.m. for the complete structure becomes linear. In the present context, the local coordinate systems are generally dynamic

⁸If $[\mathbf{M}]$ is symmetric then by definition $[\mathbf{M}]^T = [\mathbf{M}]$. Ordinary linear algebra then yields: $([\Psi]^T [\mathbf{M}][\Psi])^T = [\Psi]^T [\mathbf{M}]^T ([\Psi]^T)^T = [\Psi]^T [\mathbf{M}][\Psi]$, which shows that $[\Psi]^T [\mathbf{M}][\Psi]$ is symmetric.

If $[\mathbf{M}]$ is positive definite then by definition $\{\mathbf{V}\}^T [\mathbf{M}]\{\mathbf{V}\} > 0$ for any $\{\mathbf{V}\} \neq \{\mathbf{0}\}$. Particularly, if $\{\mathbf{V}\} = [\Psi]\{\mathbf{V}'\}$ then $([\Psi]\{\mathbf{V}'\})^T [\mathbf{M}](\Psi)\{\mathbf{V}'\} = \{\mathbf{V}'\}^T ([\Psi]^T [\mathbf{M}][\Psi])\{\mathbf{V}'\} > 0$ for any $\{\mathbf{V}'\} \neq \{\mathbf{0}\}$, which shows that $[\Psi]^T [\mathbf{M}][\Psi]$ is positive definite for any $[\Psi] \neq [\mathbf{0}]$.

and consequently the coupling of substructures cannot be accomplished as originally described in [3]. For this reason, the Craig-Bampton method is applied only for the reduction of the number of d.o.f. within a substructure in order to increase the calculation efficiency and to improve the condition of system by removing inactive high frequency modes. The coupling of the substructures, which is described in Section 4, is then carried through using a modified method, which could be regarded as an extension of the original Craig-Bampton method.

Coupling nodes and interior nodes

The Craig-Bampton method takes advantages of the fact that only a subset of the nodes of a particular substructure couples to nodes of other substructures. A requirement for efficiency of the method is that the number of these *coupling nodes* is small compared to the number of the remaining *interior nodes*.

For a wind turbine with conventionally defined substructures, i.e. one tower, one shaft and two or three blades, the coupling nodes may easily be identified as the tower top node and nodes in the root flanges of the blades. The number of beam elements necessary for describing the dynamics of these substructure with adequately accuracy varies but typically it amounts to 5–10 elements for the tower and the shaft to 10–20 elements for the blades. Consequently, the number of d.o.f. may effectively be reduced by application of the Craig-Bampton method for a typical wind turbine.

The attaching node, i.e. the first node used for attaching the local coordinate system to the substructure (see Section 2.3), constitutes a separate coupling node, which needs special attention in the analysis. The remaining coupling nodes are termed *constraint nodes*. Denoting the number of constraint nodes by N_c and the number of interior nodes by N_i , the total number of nodes then becomes $N = 1 + N_c + N_i$. It is assumed that the indices of the constraint nodes are numbered from 2 to $(N_c + 1)$ which means that the constraint nodes are chosen by the indexing. With this assumption, the nodal displacement vector of the substructure may be partitioned according to

$$\{\mathbf{U}^S\} = \begin{Bmatrix} \mathbf{U}_o^S \\ \mathbf{U}_c^S \\ \mathbf{U}_i^S \end{Bmatrix} \quad (3.17)$$

where $\{\mathbf{U}_c^S\}$ is the $N_c \times 1$ vector of substructure coordinates to the displacements of the coupling nodes and $\{\mathbf{U}_i^S\}$ is the $N_i \times 1$ vector of substructure coordinates to the displacements of the interior nodes.

The mass matrix is partitioned to comply with the partitioning of the nodal displacement vector according to equation (3.17). Thus

$$[\mathbf{M}] = \begin{bmatrix} \mathbf{M}_{oo} & \mathbf{M}_{oc} & \mathbf{M}_{oi} \\ \mathbf{M}_{co} & \mathbf{M}_{cc} & \mathbf{M}_{ci} \\ \mathbf{M}_{io} & \mathbf{M}_{ic} & \mathbf{M}_{ii} \end{bmatrix} \quad (3.18)$$

where the dimensions of the submatrices are given by the dimensions of the subvectors on the right-hand side of equation (3.17). Similarly, the structural stiffness matrix is partitioned according to

$$[\mathbf{K}] = \begin{bmatrix} \mathbf{K}_{oo} & \mathbf{K}_{oc} & \mathbf{K}_{oi} \\ \mathbf{K}_{co} & \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{io} & \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix} \quad (3.19)$$

Constraint modes and interior modes

The *constraint modes* are defined as the mode shapes of the interior nodes due to successive unit displacement of the constraint nodes in every direction keeping all

other constraint nodes (and the attaching node) fixed. The number of constraint modes is denoted by M_c and obviously we have $M_c = 6N_i$. It is readily shown (see reference [3]) that the matrix of the coupling modes arranged as columns becomes

$$[\phi_c] = -[\mathbf{K}_{ii}]^{-1} [\mathbf{K}_{ic}] \quad (3.20)$$

The dimension of $[\phi_c]$ is $6N_i \times M_c$.

The *normal modes* are defined as the normal modes of the substructure with fixed coupling nodes. These modes are then the eigenvectors $\{\mathbf{V}\}$ obtained from the eigenvalue problem

$$[\mathbf{K}_{ii}] \{\mathbf{V}\} = \lambda [\mathbf{M}_{ii}] \{\mathbf{V}\} \quad (3.21)$$

The fundamental assumption of the Craig-Bampton method is that the contribution to the displacement of interior nodes can be approximated by only a subset of the interior d.o.f. This reduced basis is chosen as the first M_n eigenvectors arranged by increasing eigenfrequency and stored as columns in the normal mode matrix $[\phi_n]$. The dimension of $[\phi_n]$ then becomes $6N_i \times M_n$. Eigenvectors corresponding to multiple eigenfrequencies are orthogonalized, and all eigenmodes are finally normalized with respect to the mass matrix. This means that all eigenvectors are mutually orthonormal⁹ with respect to the mass matrix, i.e.

$$[\phi_n]^T [\mathbf{M}_{ii}] [\phi_n] = [\mathbf{1}] \quad (3.22)$$

Denoting the amplitude of the coupling modes by $\{\mathbf{p}_c\}$ and the amplitude of the normal modes by $\{\mathbf{p}_n\}$, the corresponding displacements of the constraint nodes and the interior nodes become

$$\{\mathbf{U}_c\} = \{\mathbf{p}_c\} \quad (3.23)$$

$$\{\mathbf{U}_i\} = [\phi_c] \{\mathbf{p}_c\} + [\phi_n] \{\mathbf{p}_n\} \quad (3.24)$$

It is noted that the two modal matrices $[\phi_c]$ and $[\phi_n]$ of constraint modes and normal modes are constant and may be calculated before time integration.

Definition of the Craig-Bampton transformation

The vectors $\{\mathbf{p}_c\}$ and $\{\mathbf{p}_n\}$ together with substructure position $\{\mathbf{R}_S^S\}$ represent a set of generalized d.o.f. of the substructure. A common vector $\{\mathbf{p}\}$ containing all generalized d.o.f. of the substructure is then given by

$$\{\mathbf{p}\} = \begin{Bmatrix} \mathbf{R}_S^S \\ \mathbf{p}_c \\ \mathbf{p}_n \end{Bmatrix} \quad (3.25)$$

The Craig-Bampton transformation relates the generalized d.o.f. to the original d.o.f. $\{\mathbf{D}^S\}$ defined by equation (3.10). According to equations (3.23) and (3.24) the transformation is then defined by

$$\{\mathbf{D}^S\} = [\Phi] \{\mathbf{p}\} \quad (3.26)$$

where

$$[\Phi] = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \phi_c & \phi_n \end{bmatrix} \quad (3.27)$$

is the *Craig-Bampton transformation matrix*. In the present context this matrix differs slightly compared to the original definition elaborated by Craig and Bampton [3] in order to allow for the presence of the substructure position $\{\mathbf{R}_S^S\}$ in the transformation.

⁹Orthogonal and normal

The transformed e.o.m. for a substructure

As was done in Section 3.2, the transformation of the e.o.m. is carried through somewhat rigorously using the principle of virtual work.

Initially we define the vector

$$\{\dot{\mathbf{p}}\} = \begin{Bmatrix} \dot{\mathbf{R}}_S^S \\ \mathbf{p}_c \\ \mathbf{p}_n \end{Bmatrix} \quad (3.28)$$

where $\{\dot{\mathbf{R}}_S^S\}$ was introduced in equation (3.13). With $\{\delta\dot{\mathbf{p}}\}$ denoting a virtual displacement of the generalized d.o.f. from equilibrium where $\{\dot{\mathbf{R}}_S^S\} = \{\mathbf{0}\}$, the corresponding virtual displacement of the original d.o.f. $\{\delta\dot{\mathbf{D}}^S\}$ defined by equation (3.13) becomes

$$\{\delta\dot{\mathbf{D}}\} = [\Phi] \{\delta\dot{\mathbf{p}}\} \quad (3.29)$$

Substituting Eqs. (3.26) and (3.29) into the e.o.m. (3.15) yields

$$\{\delta\dot{\mathbf{p}}\}^T [\Phi]^T [\Psi]^T [\mathbf{M}] [\Psi] [\Phi] \{\dot{\mathbf{p}}\} = \{\delta\dot{\mathbf{p}}\}^T [\Phi]^T [\Psi]^T \{\mathbf{F}_{\text{rhs}}\} \quad (3.30)$$

Finally, the factors containing the arbitrarily chosen $\{\delta\dot{\mathbf{p}}\}$ are cancelled on both sides of equation (3.30), which results in

$$[\Phi]^T [\Psi]^T [\mathbf{M}] [\Psi] [\Phi] \{\dot{\mathbf{p}}\} = [\Phi]^T [\Psi]^T \{\mathbf{F}_{\text{rhs}}\} \quad (3.31)$$

Equation (3.31) represents the final form of the e.o.m. for a substructure.

The combined transformation matrix for a substructure

Obviously, the matrix product $[\Psi] [\Phi]$ appearing in the transformed e.o.m. (3.31) may be regarded as a single transformation matrix. In this connection, the rigid-body transformation matrix $[\Psi]$ defined by equation (3.9) is further partitioned according to

$$[\Psi] = \begin{bmatrix} \psi_o & \mathbf{0} & \mathbf{0} \\ \psi_c & \mathbf{1} & \mathbf{0} \\ \psi_i & \mathbf{0} & \mathbf{1} \end{bmatrix} \quad (3.32)$$

The combined transformation matrix $[\Psi] [\Phi]$ is then found by usual matrix multiplication, which yields

$$[\Psi] [\Phi] = \begin{bmatrix} \psi_o & \mathbf{0} & \mathbf{0} \\ \psi_c & \mathbf{1} & \mathbf{0} \\ \psi_i & \phi_c & \phi_n \end{bmatrix} \quad (3.33)$$

It should be noted that the combined transformation matrix is constant and may be calculated before time integration.

The transformed mass matrix

The transformed mass matrix $[\bar{\mathbf{M}}] = [\Phi]^T [\Psi]^T [\mathbf{M}] [\Psi] [\Phi]$, which appears in equation (3.31), may be expressed explicitly applying the usual rules for matrix products of submatrices. This yields

$$[\bar{\mathbf{M}}] = \begin{bmatrix} \bar{\mathbf{M}}_{oo} & \bar{\mathbf{M}}_{oc} & \bar{\mathbf{M}}_{on} \\ \bar{\mathbf{M}}_{co} & \bar{\mathbf{M}}_{cc} & \bar{\mathbf{M}}_{cn} \\ \bar{\mathbf{M}}_{no} & \bar{\mathbf{M}}_{nc} & \bar{\mathbf{M}}_{nn} \end{bmatrix} \quad (3.34)$$

where

$$[\bar{\mathbf{M}}_{oo}] = [\boldsymbol{\psi}]^T [\mathbf{M}] [\boldsymbol{\psi}] \quad (3.35)$$

$$[\bar{\mathbf{M}}_{oc}] = [\bar{\mathbf{M}}_{co}]^T = [\mathbf{M}_{Rc}]^T + [\mathbf{M}_{Ri}]^T [\boldsymbol{\phi}_c] \quad (3.36)$$

$$[\bar{\mathbf{M}}_{on}] = [\bar{\mathbf{M}}_{no}]^T = [\mathbf{M}_{Ri}]^T [\boldsymbol{\phi}_n] \quad (3.37)$$

$$[\bar{\mathbf{M}}_{cc}] = [\mathbf{M}_{cc}] + [\mathbf{M}_{ci}] [\boldsymbol{\phi}_c] + \left([\mathbf{M}_{ci}] [\boldsymbol{\phi}_c]\right)^T + [\boldsymbol{\phi}_c]^T [\mathbf{M}_{ii}] [\boldsymbol{\phi}_c] \quad (3.38)$$

$$[\bar{\mathbf{M}}_{cn}] = [\bar{\mathbf{M}}_{nc}]^T = \left([\mathbf{M}_{ci}] + [\boldsymbol{\phi}_c]^T [\mathbf{M}_{ii}]\right) [\boldsymbol{\phi}_n] \quad (3.39)$$

$$[\bar{\mathbf{M}}_{nn}] = [\boldsymbol{\phi}_n]^T [\mathbf{M}_{ii}] [\boldsymbol{\phi}_n] = [\mathbf{1}] \quad (3.40)$$

and where

$$[\mathbf{M}_{Rc}] = [\mathbf{M}_{co}] [\boldsymbol{\psi}_o] + [\mathbf{M}_{cc}] [\boldsymbol{\psi}_c] + [\mathbf{M}_{ci}] [\boldsymbol{\psi}_i] \quad (3.41)$$

$$[\mathbf{M}_{Ri}] = [\mathbf{M}_{io}] [\boldsymbol{\psi}_o] + [\mathbf{M}_{ic}] [\boldsymbol{\psi}_c] + [\mathbf{M}_{ii}] [\boldsymbol{\psi}_i] \quad (3.42)$$

The transformed stiffness matrix

As noted in Section 3.2 the stiffness matrix could easily be included explicitly in the analysis, which would result in the appearance of the transformed stiffness matrix $[\bar{\mathbf{K}}] = [\boldsymbol{\Phi}]^T [\boldsymbol{\Psi}]^T [\mathbf{K}] [\boldsymbol{\Psi}] [\boldsymbol{\Phi}]$ on the left-hand side of equation (3.31). It may be shown that the transformed stiffness matrix becomes

$$[\bar{\mathbf{K}}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}}_{cc} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_{nn} \end{bmatrix} \quad (3.43)$$

where the submatrices are defined by

$$[\bar{\mathbf{K}}_{cc}] = [\mathbf{K}_{cc}] + [\mathbf{K}_{ci}] [\boldsymbol{\phi}_c] \quad (3.44)$$

$$[\bar{\mathbf{K}}_{nn}] = [\boldsymbol{\phi}_n]^T [\mathbf{K}_{ii}] [\boldsymbol{\phi}_n] \quad (3.45)$$

It is noted that $[\bar{\mathbf{K}}_{nn}]$ is a diagonal matrix due to the orthogonality of the substructure normal modes, and that the diagonal elements equal the eigenvalues calculated from equation (3.21).

Transformation of a force vector

In the following we consider an arbitrary force vector $\{\mathbf{F}\}$, which is partitioned according to

$$\{\mathbf{F}\} = \begin{Bmatrix} \mathbf{F}_o \\ \mathbf{F}_c \\ \mathbf{F}_i \end{Bmatrix} \quad (3.46)$$

It is noted that this is similar to the partitioning of the displacement vector given by equation (3.17). It is readily shown that the corresponding transformed force vector $\{\bar{\mathbf{F}}\} = [\boldsymbol{\Phi}]^T [\boldsymbol{\Psi}]^T \{\mathbf{F}\}$ becomes

$$\{\bar{\mathbf{F}}\} = \begin{Bmatrix} \bar{\mathbf{F}}_o \\ \bar{\mathbf{F}}_c \\ \bar{\mathbf{F}}_n \end{Bmatrix} \quad (3.47)$$

where

$$\{\bar{\mathbf{F}}_o\} = [\boldsymbol{\psi}] \{\mathbf{F}\} \quad (3.48)$$

$$\{\bar{\mathbf{F}}_c\} = \{\mathbf{F}_c\} + [\boldsymbol{\phi}_c]^T \{\mathbf{F}_i\} \quad (3.49)$$

$$\{\bar{\mathbf{F}}_n\} = [\boldsymbol{\phi}_n]^T \{\mathbf{F}_i\} \quad (3.50)$$

4 Coupling of substructure e.o.m.

In the previous sections, a single substructure was considered, and the interaction with other substructures was therefore neglected. This section considers a system of K substructures, which are coupled together in selected nodes, which means that the interaction of the substructures is taken into account.

In order to differ between the substructures, we introduce an expanded notation compared to what is used previously. By this notation, a quantity of a particular substructure is denoted by an upper index surrounded by parentheses added to the appropriate symbol, which in many cases already is defined in Sections 2 or 3. As an example, the nodal displacement vector defined by equation (2.13) for the i 'th substructure is denoted by the symbol $\{\mathbf{U}^{\text{S}(i)}\}$.

4.1 Collecting the d.o.f. of the complete structure

Every substructure is provided with a local coordinate system rigidly attached to the attaching node, in which the relative displacements consequently vanish. The local substructure coordinate system is described in detail in Section 2.2. The vector containing the position and rotation of all substructure coordinate systems is defined by

$$\{\mathbf{R}_S^{\text{I}}\} = \left\{ \begin{array}{c} \mathbf{R}_S^{\text{I}(1)} \\ \mathbf{R}_S^{\text{I}(2)} \\ \vdots \\ \mathbf{R}_S^{\text{I}(K)} \end{array} \right\} \quad (4.1)$$

where $\{\mathbf{R}_S^{\text{I}(i)}\}$, defined by equation (2.1), denotes the 6×1 vector of inertial coordinates to the position and angular orientation of the i 'th substructure. The definition of $\{\mathbf{R}_S^{\text{I}}\}$ by equation (4.1) implicitly defines the vectors $\{\mathbf{V}_S^{\text{I}}\}$ and $\{\mathbf{A}_S^{\text{I}}\}$ of inertial coordinates to the angular velocities and angular accelerations by application of the special interpretation of the symbols $\{\dot{\mathbf{R}}_S^{\text{I}(i)}\}$ and $\{\ddot{\mathbf{R}}_S^{\text{I}(i)}\}$ introduced by equations (2.5) and (2.6), respectively.

Besides the selection of the attaching node, the nodes of the substructure are divided into interior nodes and constraint nodes as described in Section 3.3. Only attaching nodes and constraint nodes are capable of coupling to nodes of other substructures for which reason these nodes together are termed coupling nodes. In the following, the number of constraint nodes for the i 'th substructure is denoted by $N_c^{(i)}$, and since every node has six d.o.f., the corresponding number of constraint d.o.f. becomes $M_c^{(i)} = 6N_c^{(i)}$. The number of constraint d.o.f. for the complete structure is then $M_c = \sum_{i=1}^K M_c^{(i)} = 6 \sum_{i=1}^K N_c^{(i)}$.

The displacements of the coupling nodes for all substructures are collected in the $(6K + M_c) \times 1$ vector $\{\mathbf{U}_b^{\text{S}}\}$ defined by

$$\{\mathbf{U}_b^{\text{S}}\} = \left\{ \begin{array}{c} \mathbf{U}_b^{\text{S}(1)} \\ \mathbf{U}_b^{\text{S}(2)} \\ \vdots \\ \mathbf{U}_b^{\text{S}(K)} \end{array} \right\} \quad (4.2)$$

where the subvectors appearing on the right-hand side are defined by

$$\{\mathbf{U}_b^{\text{S}(i)}\} = \left\{ \begin{array}{c} \mathbf{U}_o^{\text{S}(i)} \\ \mathbf{U}_c^{\text{S}(i)} \end{array} \right\}, \quad i = 1, 2, \dots, K \quad (4.3)$$

Similarly, the $M_c \times 1$ vector of displacements of the constraint nodes is defined by

$$\{\mathbf{U}_c^S\} = \begin{Bmatrix} \mathbf{U}_c^{S(1)} \\ \mathbf{U}_c^{S(2)} \\ \vdots \\ \mathbf{U}_c^{S(K)} \end{Bmatrix} \quad (4.4)$$

4.2 The basic coupling conditions

Geometric compatibility

The coupling of the substructures is carried through solely from the geometric compatibility between the involved nodes, and the corresponding equilibrium conditions of the forces is derived directly from the compatibility conditions applying the principle of virtual work. This is somewhat different to the method used by Petersen [9], who bases the coupling on force equilibrium in the coupling nodes.

Imposing geometric compatibility between two coupling nodes belonging to two different substructures results in a number of relations between the absolute translation and rotation of the coupling nodes. The total number of such conditions or *constraints* for the complete structure is denoted by M_d . As the absolute displacements of the nodes are composed of displacements of the local substructure coordinate systems as well as relative displacements of the nodes, the geometric compatibility conditions generally result in M_d relations between the vectors $\{\mathbf{R}_S^I\}$ and $\{\mathbf{U}_b^S\}$ defined by equations (4.1) and (4.2).

In addition to the relations between absolute displacements of the coupling nodes, relations between the absolute velocity and acceleration are needed. These additional relations are obtained by absolute differentiation of the basic relations for displacement with respect to time. Differentiation once yields M_d relations between the vectors $\{\mathbf{V}_S^I\}$ and $\{\dot{\mathbf{U}}_b^S\}$. Differentiation once more yields M_d relations between $\{\mathbf{A}_S^I\}$ and $\{\ddot{\mathbf{U}}_b^S\}$. An example of the derivation of geometric compatibility conditions may be found in Section 5, where the fixed coupling between two substructures is considered.

A fundamental assumption for the proposed method for coupling of the substructures is that the geometric compatibility conditions for displacement, velocity and acceleration may be expressed in the general forms

$$[\mathbf{C}_b]\{\mathbf{U}_b\} + \{\mathbf{d}_0\} = \{\mathbf{0}\} \quad (4.5)$$

$$[\mathbf{C}_S]\{\mathbf{V}_S^I\} + [\mathbf{C}_b]\{\dot{\mathbf{U}}_b^S\} + \{\mathbf{d}_1\} = \{\mathbf{0}\} \quad (4.6)$$

$$[\mathbf{C}_S]\{\mathbf{A}_S^I\} + [\mathbf{C}_b]\{\ddot{\mathbf{U}}_b^S\} + \{\mathbf{d}_2\} = \{\mathbf{0}\} \quad (4.7)$$

where

$[\mathbf{C}_S]$ is a time-dependent $M_d \times 6K$ matrix, which is a function of the rotation parameters of the substructures.

$[\mathbf{C}_b]$ is a time-dependent $M_d \times (6K + M_c)$ matrix being a function of the rotation parameters too.

$\{\mathbf{d}_0\}$, $\{\mathbf{d}_1\}$, and $\{\mathbf{d}_2\}$ are time-dependent $M_d \times 1$ vectors, which contain the remaining ‘‘small’’ terms. These vectors are functions of both substructure position d.o.f. and nodal displacements including their time derivatives.

In the following, $[\mathbf{C}_S]$ and $[\mathbf{C}_c]$ are termed *basic coupling matrices* while $\{\mathbf{d}_0\}$, $\{\mathbf{d}_1\}$ and $\{\mathbf{d}_2\}$ are termed *basic coupling vectors*. These quantities completely describe all couplings between the substructures.

In the original Craig-Bampton method [3], the geometric compatibility conditions are assumed to have the general form $[\mathbf{C}_b]\{\mathbf{U}_b\} = \{\mathbf{0}\}$, where $[\mathbf{C}_b]$ is a

constant matrix. Consequently, the compatibility conditions given in the forms of equations (4.5) to (4.7) may be regarded as an extension of the Craig-Bampton method in case of dynamic local substructure coordinate systems.

The common form of the geometric compatibility conditions

It appears that the geometric compatibility condition expressed in the form of equations (4.5) to (4.7) have to be transformed into a common form in order to ensure symmetry and positive definiteness of the final mass matrix of the complete structure.

First, the basic coupling matrices are partitioned according to

$$[\mathbf{C}_S] = \begin{bmatrix} \mathbf{C}_S^{(1)} & \mathbf{C}_S^{(2)} & \dots & \mathbf{C}_S^{(K)} \end{bmatrix} \quad (4.8)$$

$$[\mathbf{C}_b] = \begin{bmatrix} \mathbf{C}_b^{(1)} & \mathbf{C}_b^{(2)} & \dots & \mathbf{C}_b^{(K)} \end{bmatrix} \quad (4.9)$$

where $[\mathbf{C}_S^{(i)}]$, are $M_d \times 6$ submatrices and $[\mathbf{C}_b^{(i)}]$, are $M_d \times (6 + M_c^{(i)})$ submatrices defined by

$$[\mathbf{C}_b^{(i)}] = \begin{bmatrix} \mathbf{C}_o^{(i)} & \mathbf{C}_c^{(i)} \end{bmatrix}, \quad i = 1, 2, \dots, K \quad (4.10)$$

Furthermore, we define the $M_d \times M_c$ matrix

$$[\mathbf{C}_c] = \begin{bmatrix} \mathbf{C}_c^{(1)} & \mathbf{C}_c^{(2)} & \dots & \mathbf{C}_c^{(K)} \end{bmatrix} \quad (4.11)$$

which will be used below.

As noted in the beginning of this section, the geometric compatibility conditions for velocity and acceleration in the forms of equations (4.6) and (4.7) are derived by differentiation of the displacement conditions (4.5). Therefore it is obvious to assume that the basic coupling matrices $[\mathbf{C}_S]$ and $[\mathbf{C}_c]$ are mutual dependent by some relations. It appears that the form of such relations between the basic coupling matrices have to be

$$\begin{bmatrix} \mathbf{C}_o^{(i)} & \mathbf{C}_c^{(i)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_o^{(i)} \\ \boldsymbol{\psi}_c^{(i)} \end{bmatrix} = [\mathbf{C}_S^{(i)}] [\mathbf{T}_S^{*(i)}], \quad i = 1, 2, \dots, K \quad (4.12)$$

where $[\mathbf{T}_S^{*(i)}]$ is the expanded coordinate transformation matrix defined by equation (2.9), while $[\boldsymbol{\psi}_o^{(i)}]$ and $[\boldsymbol{\psi}_c^{(i)}]$ are submatrices of the rigid-body modal matrix defined by equation (C.6) and partitioned as

$$[\boldsymbol{\psi}^{(i)}] = \begin{bmatrix} \boldsymbol{\psi}_o^{(i)} \\ \boldsymbol{\psi}_c^{(i)} \\ \boldsymbol{\psi}_n^{(i)} \end{bmatrix} \quad (4.13)$$

At first sight the relations given by equation (4.12) may seem somewhat surprising. However, as it will appear from the following, they are necessary to ensure symmetry and positive definiteness of the resulting mass matrix of the complete structure. Furthermore, the analysis in Section 5 for the fixed coupling between two substructures reveals that the relations so to speak are naturally derived by the differentiation operation. At least this is true for this particular coupling type, and generally the relations given by equation (4.12) must be verified for every new type of coupling, which may be elaborated in the future.

Now it is noted that the transformation given by equation (3.12) applied solely for the coupling nodes may be written as

$$\begin{Bmatrix} \mathbf{U}_o^S(i) \\ \mathbf{U}_c^S(i) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_o^{(i)} & \mathbf{0} \\ \boldsymbol{\psi}_c^{(i)} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} \mathbf{R}_S^S(i) \\ \mathbf{U}_c^S(i) \end{Bmatrix}, \quad i = 1, 2, \dots, K \quad (4.14)$$

where the 6×1 vector $\{\dot{\mathbf{R}}_S^{S(i)}\}$ imposes rigid-body displacement on the i 'th substructure by nodal displacements. Inserting equation (4.14) into the geometric compatibility conditions (4.5) for displacement, employing the relation (4.12) between the basic coupling matrices, and finally rearranging yield

$$\left[\mathbf{C}_S^{(1)} \quad \mathbf{C}_c^{(1)} \mid \mathbf{C}_S^{(2)} \quad \mathbf{C}_c^{(2)} \mid \dots \mid \mathbf{C}_S^{(K)} \quad \mathbf{C}_c^{(K)} \right] \left\{ \begin{array}{c} \dot{\mathbf{R}}_S^{I(1)} \\ \mathbf{U}_c^{S(1)} \\ \dot{\mathbf{R}}_S^{I(2)} \\ \mathbf{U}_c^{S(2)} \\ \vdots \\ \dot{\mathbf{R}}_S^{I(K)} \\ \mathbf{U}_c^{S(K)} \end{array} \right\} + \{\mathbf{d}_0\} = \{\mathbf{0}\} \quad (4.15)$$

where $\{\dot{\mathbf{R}}_S^{I(i)}\} = [\mathbf{T}_S^{*(i)}] \{\dot{\mathbf{R}}_S^{S(i)}\}$. The geometric compatibility conditions (4.5) to (4.7) may then be written in a common form as

$$[\mathbf{C}_S] \{\dot{\mathbf{R}}_S^I\} + [\mathbf{C}_c] \{\mathbf{U}_c^S\} + \{\mathbf{d}_0\} = \{\mathbf{0}\} \quad (4.16)$$

$$[\mathbf{C}_S] \{\mathbf{V}_S^I\} + [\mathbf{C}_c] \{\dot{\mathbf{U}}_c^S\} + \{\mathbf{d}_1\} = \{\mathbf{0}\} \quad (4.17)$$

$$[\mathbf{C}_S] \{\mathbf{A}_S^I\} + [\mathbf{C}_c] \{\ddot{\mathbf{U}}_c^S\} + \{\mathbf{d}_2\} = \{\mathbf{0}\} \quad (4.18)$$

Again it is noted that the vector $\{\dot{\mathbf{R}}_S^I\}$ appearing in equation (4.16) vanishes due to the attaching condition (2.15). However, the term including this vector is introduced in order to make possible the derivation of the resulting e.o.m. for the complete structure.

4.3 Selection of dependent constraint d.o.f.

As the geometric compatibility conditions constitute M_d relations between the d.o.f., the number of independent or *general* d.o.f. has to be reduced by M_d . Thus we must select a set of M_d *dependent* d.o.f., which subsequently is eliminated (condensed out) and expressed by the general d.o.f.

A fundamental assumption for the proposed method is that the dependent d.o.f. are chosen among the constraint d.o.f. Thus all substructure position d.o.f. are regarded as independent.

The selection of the M_d dependent constraint d.o.f. in $\{\mathbf{U}_c^S\}$ is expressed by the equation

$$[\mathbf{P}_d \quad \mathbf{P}_g] \left\{ \begin{array}{c} \mathbf{U}_{cd}^S \\ \mathbf{U}_{cg}^S \end{array} \right\} = \{\mathbf{U}_c^S\} \quad (4.19)$$

where $[\mathbf{P}_d \quad \mathbf{P}_g]$ is a $M_c \times M_c$ permutation matrix in which $[\mathbf{P}_d]$ is a $M_c \times M_d$ submatrix and $[\mathbf{P}_g]$ is a $M_c \times (M_c - M_d)$ submatrix. The dependent constraint d.o.f. are collected in the $M_d \times 1$ vector $\{\mathbf{U}_{cd}^S\}$, while $\{\mathbf{U}_{cg}^S\}$ is the $(M_c - M_d) \times 1$ vector of the remaining general constraint d.o.f.

In the implementation, the permutation matrix is formed automatically from the parameters, which specify the couplings in terms of indices of the involved coupling nodes and substructures.

4.4 Preparing the coupling conditions for transformation

The coupling conditions for displacement, velocity and acceleration, given by equations (4.16) to (4.18), are now transformed into a form, which prepares the coupling conditions for being expressed by a transformation similar to that used by Craig

and Bampton [3]. As the forms of equations (4.16) to (4.18) are identical, we may restrict the analysis for a condition having the general form

$$[\mathbf{C}_S]\{\mathbf{R}_S^I\} + [\mathbf{C}_c]\{\mathbf{U}_c^S\} + \{\mathbf{d}\} = \{\mathbf{0}\} \quad (4.20)$$

Inserting equation (4.19) into equation (4.20) yields

$$[\mathbf{C}_S]\{\mathbf{R}_S^I\} + [\mathbf{C}_c][\mathbf{P}_d]\{\mathbf{U}_{cd}^S\} + [\mathbf{C}_c][\mathbf{P}_g]\{\mathbf{U}_{cg}^S\} + \{\mathbf{d}\} = \{\mathbf{0}\} \quad (4.21)$$

In the following it is assumed that $M_d > 0$, which always is true if two or more coupled substructures are considered. Furthermore it is assumed that the $M_d \times M_d$ matrix-matrix product $[\mathbf{C}_c][\mathbf{P}_d]$ is non-singular, which is true if the constraints given by equation (4.20) are linearly independent and if the dependent coupling d.o.f. are chosen carefully. With these assumptions, it is apparent that equation (4.21) may be written as

$$\{\mathbf{U}_{cd}^S\} = [\mathbf{\Upsilon}_c] \left([\mathbf{C}_S]\{\mathbf{R}_S^I\} + [\mathbf{C}_c][\mathbf{P}_g]\{\mathbf{U}_{cg}^S\} + \{\mathbf{d}\} \right) \quad (4.22)$$

where

$$[\mathbf{\Upsilon}_c] = -([\mathbf{C}_c][\mathbf{P}_d])^{-1} \quad (4.23)$$

Inserting equation (4.22) into equation (4.19) and rearranging then finally yield

$$\{\mathbf{U}_c^S\} = [\gamma_S \ \gamma_c] \left\{ \begin{matrix} \mathbf{R}_S^I \\ \mathbf{U}_{cg}^S \end{matrix} \right\} + \{\delta\} \quad (4.24)$$

where

$$[\gamma_S] = [\mathbf{P}_d][\mathbf{\Upsilon}_c][\mathbf{C}_S] \quad (4.25)$$

$$[\gamma_c] = [\mathbf{P}_d][\mathbf{\Upsilon}_c][\mathbf{C}_c][\mathbf{P}_g] + [\mathbf{P}_g] \quad (4.26)$$

$$\{\delta\} = [\mathbf{P}_d][\mathbf{\Upsilon}_c]\{\mathbf{d}\} \quad (4.27)$$

In the following $[\gamma_S]$ and $[\gamma_c]$ are termed *transformed coupling matrices*, while $\{\delta\}$ is termed *transformed coupling vector*.

Special case with no constraints

If only one substructure is considered then $M_d = 0$, and equation (4.20) vanishes. The vector $\{\mathbf{U}_{cd}^S\}$ of dependent coupling d.o.f. is then equal to the null-vector and equation (4.19) simplifies to $[\mathbf{P}_g]\{\mathbf{U}_{cg}^S\} = \{\mathbf{U}_c^S\}$. In this special case, the transformed coupling matrices and vectors then become

$$[\gamma_S] = [\mathbf{0}] \quad (4.28)$$

$$[\gamma_c] = [\mathbf{P}_g] \quad (4.29)$$

$$\{\delta\} = \{\mathbf{0}\} \quad (4.30)$$

Partitioned equations

Partitioning of equation (4.24) according to the partitioning of $\{\mathbf{R}_S^I\}$ and $\{\mathbf{U}_c^S\}$ given by equations (4.1) and (4.4) yields

$$\left\{ \begin{matrix} \mathbf{U}_c^{S(1)} \\ \mathbf{U}_c^{S(2)} \\ \vdots \\ \mathbf{U}_c^{S(K)} \end{matrix} \right\} = \left[\begin{matrix} \gamma_S^{(1,1)} & \gamma_S^{(1,2)} & \dots & \gamma_S^{(1,K)} \\ \gamma_S^{(2,1)} & \gamma_S^{(2,2)} & \dots & \gamma_S^{(2,K)} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_S^{(K,1)} & \gamma_S^{(K,2)} & \dots & \gamma_S^{(K,K)} \end{matrix} \right] \left[\begin{matrix} \gamma_c^{(1)} \\ \gamma_c^{(2)} \\ \vdots \\ \gamma_c^{(K)} \end{matrix} \right] \left\{ \begin{matrix} \mathbf{R}_S^{I(1)} \\ \mathbf{R}_S^{I(2)} \\ \vdots \\ \mathbf{R}_S^{I(K)} \\ \mathbf{U}_{cg}^S \end{matrix} \right\} + \left\{ \begin{matrix} \delta^{(1)} \\ \delta^{(2)} \\ \vdots \\ \delta^{(K)} \end{matrix} \right\} \quad (4.31)$$

Furthermore, the two submatrices of the permutation matrix are partitioned as

$$[\mathbf{P}_d] = \begin{bmatrix} \mathbf{P}_d^{(1)} \\ \mathbf{P}_d^{(2)} \\ \vdots \\ \mathbf{P}_d^{(K)} \end{bmatrix}, \quad [\mathbf{P}_g] = \begin{bmatrix} \mathbf{P}_g^{(1)} \\ \mathbf{P}_g^{(2)} \\ \vdots \\ \mathbf{P}_g^{(K)} \end{bmatrix} \quad (4.32)$$

It is then straightforward to show that the submatrices and subvectors, which enter in equation (4.31), are given by

$$[\gamma_S^{(i,j)}] = [\mathbf{P}_d^{(i)}][\boldsymbol{\Upsilon}_c][\mathbf{C}_S^{(j)}] \quad (4.33)$$

$$[\gamma_c^{(i)}] = [\mathbf{P}_d^{(i)}][\boldsymbol{\Upsilon}_c] \sum_{k=1}^K [\mathbf{C}_c^{(k)}][\mathbf{P}_g^{(k)}] + [\mathbf{P}_g^{(i)}] \quad (4.34)$$

$$\{\boldsymbol{\delta}^{(i)}\} = [\mathbf{P}_d^{(i)}][\boldsymbol{\Upsilon}_c]\{\mathbf{d}\} \quad (4.35)$$

where

$$[\boldsymbol{\Upsilon}_c] = - \left(\sum_{k=1}^K [\mathbf{C}_c^{(k)}][\mathbf{P}_d^{(k)}] \right)^{-1} \quad (4.36)$$

Final result of the analysis

Equation (4.24) represents the final result of the analysis in a general compact form, whereas the corresponding expanded form including matrix partitioning is given by equation (4.31). The coupling conditions for displacement, velocity, and acceleration may then be written in the compact forms as

$$\{\mathbf{U}_c\} = [\gamma_S \ \gamma_c] \begin{Bmatrix} \dot{\mathbf{R}}_S^I \\ \mathbf{U}_{cg}^I \end{Bmatrix} + \{\boldsymbol{\delta}_0\} \quad (4.37)$$

$$\{\dot{\mathbf{U}}_c\} = [\gamma_S \ \gamma_c] \begin{Bmatrix} \mathbf{V}_S^I \\ \dot{\mathbf{U}}_{cg}^I \end{Bmatrix} + \{\boldsymbol{\delta}_1\} \quad (4.38)$$

$$\{\ddot{\mathbf{U}}_c\} = [\gamma_S \ \gamma_c] \begin{Bmatrix} \mathbf{A}_S^I \\ \ddot{\mathbf{U}}_{cg}^I \end{Bmatrix} + \{\boldsymbol{\delta}_2\} \quad (4.39)$$

where $\{\boldsymbol{\delta}_m^{(i)}\}$, $m = 0, 1, 2$ are related to the basic coupling vectors by equation (4.35) in which the vector $\{\mathbf{d}\}$ have to be replaced by $\{\mathbf{d}_m\}$.

4.5 The coupling transformation

The geometric compatibility conditions is now expressed by means of a global transformation, which relates the generalized d.o.f. of the coupled structure to the generalized d.o.f. of the free substructures as defined in Section 3.3.

First, the vector of generalized d.o.f. for all substructures is defined by

$$\{\mathbf{p}\} = \begin{Bmatrix} \mathbf{p}^{(1)} \\ \mathbf{p}^{(2)} \\ \vdots \\ \mathbf{p}^{(K)} \end{Bmatrix} \quad (4.40)$$

where the subvectors $\{\mathbf{p}^{(i)}\}$ are defined by equation (3.25). Furthermore, the generalized d.o.f. of the coupled structure are collected in the vector

$$\{\mathbf{q}\} = \begin{Bmatrix} \mathbf{R}_S^I \\ \mathbf{q}_c \\ \mathbf{q}_n \end{Bmatrix} \quad (4.41)$$

where

$$\{\mathbf{q}_c\} = \{\mathbf{U}_{cg}^S\} \quad (4.42)$$

represents the $(M_c - M_d) \times 1$ vector of general constraint d.o.f. and

$$\{\mathbf{q}_n\} = \begin{Bmatrix} \mathbf{p}_n^{(1)} \\ \mathbf{p}_n^{(2)} \\ \vdots \\ \mathbf{p}_n^{(K)} \end{Bmatrix} \quad (4.43)$$

is a collection of the vectors of normal modes d.o.f. described in Section 3.3.

With the definitions of the vectors of generalized d.o.f. given by equation (4.40) for the free substructures and equation (4.41) for the complete structure, the coupling conditions in the form of equations (4.37), (4.38), and (4.39) may be expressed by the three transformations defined by

$$\{\dot{\mathbf{p}}\} = [\mathbf{\Gamma}]\{\dot{\mathbf{q}}\} + \{\Delta_0\} \quad (4.44)$$

$$\{\mathbf{p}\} = [\mathbf{\Gamma}]\{\mathbf{q}\} + \{\Delta_1\} \quad (4.45)$$

$$\{\ddot{\mathbf{p}}\} = [\mathbf{\Gamma}]\{\ddot{\mathbf{q}}\} + \{\Delta_2\} \quad (4.46)$$

where the new symbols are defined below.

The vector $\{\dot{\mathbf{q}}\}$ appearing in equation (4.44) is defined by

$$\{\dot{\mathbf{q}}\} = \begin{Bmatrix} \dot{\mathbf{R}}_S^I \\ \mathbf{q}_c \\ \mathbf{q}_n \end{Bmatrix} \quad (4.47)$$

where $\{\dot{\mathbf{R}}_S^I\}$ denotes the collection of the vectors $\{\dot{\mathbf{R}}_S^{I(i)}\}$ introduced by equation (3.13). It should be noted that $\{\dot{\mathbf{R}}_S^{I(i)}\}$ imposes rigid-body displacement on the i 'th substructure and that $\{\dot{\mathbf{R}}_S^{I(i)}\} = \{\mathbf{0}\}$ at equilibrium.

The *global coupling transformation matrix* $[\mathbf{\Gamma}]$ appearing on the right-hand side of equations (4.44) to (4.46) is defined by

$$[\mathbf{\Gamma}] = \begin{bmatrix} [\mathbf{T}_S^{*(1)}]^T & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \gamma_S^{(1,1)} & \gamma_S^{(1,2)} & \dots & \gamma_S^{(1,K)} & \gamma_c^{(1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathbf{0} & [\mathbf{T}_S^{*(2)}]^T & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \gamma_S^{(2,1)} & \gamma_S^{(2,2)} & \dots & \gamma_S^{(2,K)} & \gamma_c^{(2)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \dots & [\mathbf{T}_S^{*(K)}]^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \gamma_S^{(K,1)} & \gamma_S^{(K,2)} & \dots & \gamma_S^{(K,K)} & \gamma_c^{(K)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{bmatrix} \quad (4.48)$$

This matrix is the key in the proposed method for coupling the substructures. It is obvious that $[\mathbf{\Gamma}]$ generally is time-dependent, as the expanded coordinate transformation matrices $[\mathbf{T}_S^{*(i)}]$ as well as the transformed coupling matrices $[\gamma_S^{(i,j)}]$ and $[\gamma_c^{(i)}]$ defined by equations (4.33) and (4.34) include the time-dependent rotation parameters of the substructure coordinate systems.

The *global coupling vectors* appearing on the right-hand side of equations (4.44) to (4.46) are defined by

$$\{\Delta_m\} = \begin{Bmatrix} \Delta_m^{(1)} \\ \Delta_m^{(2)} \\ \vdots \\ \Delta_m^{(K)} \end{Bmatrix} \quad (4.49)$$

where

$$\{\Delta_m^{(i)}\} = \begin{Bmatrix} \mathbf{0} \\ \delta_m^{(i)} \\ \mathbf{0} \end{Bmatrix}, \quad i = 1, 2, \dots, K, \quad m = 0, 1, 2 \quad (4.50)$$

Like the global coupling matrix, the global coupling vectors $\{\Delta_m^{(i)}\}$ are time-dependent, as $\{\delta_m^{(i)}\}$ defined by equation (4.35) are time-dependent.

4.6 The coupled e.o.m.

We are now ready for coupling the e.o.m. for all substructures into a common e.o.m. for the complete structure.

First, a collection of the transformed mass matrices for all substructures is defined by the diagonal matrix

$$[\bar{\mathbf{M}}] = \begin{bmatrix} \bar{\mathbf{M}}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{M}}^{(2)} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \bar{\mathbf{M}}^{(K)} \end{bmatrix} \quad (4.51)$$

where $[\bar{\mathbf{M}}^{(i)}]$, defined by equations (3.34), denotes the transformed mass matrix of the i 'th substructure. In addition, a collection of the transformed force vectors for all substructures is defined by

$$\{\bar{\mathbf{F}}_{\text{rhs}}\} = \begin{Bmatrix} \bar{\mathbf{F}}_{\text{rhs}}^{(1)} \\ \bar{\mathbf{F}}_{\text{rhs}}^{(2)} \\ \vdots \\ \bar{\mathbf{F}}_{\text{rhs}}^{(K)} \end{Bmatrix} \quad (4.52)$$

where the transformed vector $\{\bar{\mathbf{F}}_{\text{rhs}}^{(i)}\}$ of the i 'th substructure is defined through equation (3.47).

The derivation of the e.o.m. of a free substructure was carried through in Section 3, where the main result of the analysis is given by equation (3.31). It is obvious that the e.o.m. for all substructures regarded as uncoupled may be written by a single matrix equation as

$$[\bar{\mathbf{M}}]\{\ddot{\mathbf{p}}\} = \{\bar{\mathbf{F}}_{\text{rhs}}\} \quad (4.53)$$

Let $\{\delta\dot{\mathbf{q}}\}$ denote a virtual displacements of $\{\dot{\mathbf{q}}\}$ from equilibrium with *fixed* substructure coordinate systems. The corresponding virtual displacement of the vector $\{\dot{\mathbf{p}}\}$, which denotes the collection of the vectors $\{\dot{\mathbf{p}}^{(i)}\}$ defined by equation (3.28), may then be expressed by means of equation (4.44). Since the rotation parameters are constant for fixed substructure coordinate systems this yields

$$\{\delta\dot{\mathbf{p}}\} = [\mathbf{\Gamma}]\{\delta\dot{\mathbf{q}}\} \quad (4.54)$$

Inserting equation (4.46) into equation (4.53), equating the virtual work done by the forces of both sides of the resulting equation during the virtual displacement, and finally cancelling the factors containing the arbitrarily chosen $\{\delta\dot{\mathbf{q}}\}$ on both sides yield

$$[\mathbf{\Gamma}]^T [\bar{\mathbf{M}}] [\mathbf{\Gamma}]\{\ddot{\mathbf{q}}\} = [\mathbf{\Gamma}]^T \left(\{\bar{\mathbf{F}}_{\text{rhs}}\} - [\bar{\mathbf{M}}]\{\Delta_2\} \right) \quad (4.55)$$

Equation (4.55) constitutes the e.o.m. for the complete structure, and it is main result of the analysis in this section.

The structural mass matrix

The transformed mass matrix $[\hat{\mathbf{M}}] = [\mathbf{\Gamma}]^T [\bar{\mathbf{M}}] [\mathbf{\Gamma}]$ appearing on the left-hand side of equation (4.55) for the complete structure may be expressed explicitly using the expressions (4.48) and (4.51) for the involved matrices $[\mathbf{\Gamma}]$ and $[\bar{\mathbf{M}}]$. Furthermore, it is assumed that every substructure mass matrix, which enters in $[\bar{\mathbf{M}}]$, is partitioned according to equation (3.34). Usual linear algebra then yields

$$[\hat{\mathbf{M}}] = \left[\begin{array}{c|ccc|ccc} \hat{\mathbf{M}}_{oo}^{(1,1)} & \hat{\mathbf{M}}_{oo}^{(1,2)} & \cdots & \hat{\mathbf{M}}_{oo}^{(1,K)} & \hat{\mathbf{M}}_{oc}^{(1)} & \hat{\mathbf{M}}_{on}^{(1,1)} & \hat{\mathbf{M}}_{on}^{(1,2)} & \cdots & \hat{\mathbf{M}}_{on}^{(1,K)} \\ & \hat{\mathbf{M}}_{oo}^{(2,2)} & \cdots & \hat{\mathbf{M}}_{oo}^{(2,K)} & \hat{\mathbf{M}}_{oc}^{(2)} & \hat{\mathbf{M}}_{on}^{(2,1)} & \hat{\mathbf{M}}_{on}^{(2,2)} & \cdots & \hat{\mathbf{M}}_{on}^{(2,K)} \\ & & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & \hat{\mathbf{M}}_{oo}^{(K,K)} & \hat{\mathbf{M}}_{oc}^{(K)} & \hat{\mathbf{M}}_{on}^{(K,1)} & \hat{\mathbf{M}}_{on}^{(K,2)} & \cdots & \hat{\mathbf{M}}_{on}^{(K,K)} \\ \hline & & & & \hat{\mathbf{M}}_{cc} & \hat{\mathbf{M}}_{cn}^{(1)} & \hat{\mathbf{M}}_{cn}^{(2)} & \cdots & \hat{\mathbf{M}}_{cn}^{(1)} \\ \hline \text{Sym-} & & & & & \hat{\mathbf{M}}_{nn}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ & \text{me-} & & & & & \hat{\mathbf{M}}_{nn}^{(2)} & \cdots & \mathbf{0} \\ & \text{tric} & & & & & & \ddots & \vdots \\ & & & & & & & & \hat{\mathbf{M}}_{nn}^{(2)} \end{array} \right] \quad (4.56)$$

where

$$[\hat{\mathbf{M}}_{oo}^{(i,j)}] = \left\{ \begin{array}{l} [\mathbf{T}_S^{*(i)}] [\bar{\mathbf{M}}_{oo}^{(i)}] [\mathbf{T}_S^{*(i)}]^T \quad \text{for } i = j \\ [\mathbf{0}] \quad \text{for } i \neq j \\ + [\mathbf{T}_S^{*(i)}] [\bar{\mathbf{M}}_{oc}^{(i)}] [\boldsymbol{\gamma}_S^{(i,j)}] + \left([\mathbf{T}_S^{*(j)}] [\bar{\mathbf{M}}_{oc}^{(j)}] [\boldsymbol{\gamma}_S^{(j,i)}] \right)^T \\ + \sum_{k=1}^K [\boldsymbol{\gamma}_S^{(k,i)}]^T [\bar{\mathbf{M}}_{cc}^{(k)}] [\boldsymbol{\gamma}_S^{(k,j)}] \end{array} \right\} \quad (4.57)$$

$$[\hat{\mathbf{M}}_{oc}^{(i)}] = [\mathbf{T}_S^{*(i)}] [\bar{\mathbf{M}}_{oc}^{(i)}] [\boldsymbol{\gamma}_c^{(i)}] + \sum_{k=1}^K [\boldsymbol{\gamma}_S^{(k,i)}]^T [\bar{\mathbf{M}}_{cc}^{(k)}] [\boldsymbol{\gamma}_c^{(k)}] \quad (4.58)$$

$$[\hat{\mathbf{M}}_{on}^{(i,j)}] = \left\{ \begin{array}{l} [\mathbf{T}_S^{*(i)}] [\bar{\mathbf{M}}_{on}^{(i)}] \quad \text{for } i = j \\ [\mathbf{0}] \quad \text{for } i \neq j \\ + [\boldsymbol{\gamma}_S^{(j,i)}]^T [\bar{\mathbf{M}}_{cn}^{(j)}] \end{array} \right\} \quad (4.59)$$

$$[\hat{\mathbf{M}}_{cc}] = \sum_{k=1}^K [\boldsymbol{\gamma}_c^{(k)}]^T [\bar{\mathbf{M}}_{cc}^{(k)}] [\boldsymbol{\gamma}_c^{(k)}] \quad (4.60)$$

$$[\hat{\mathbf{M}}_{cn}^{(i)}] = [\boldsymbol{\gamma}_c^{(i)}]^T [\bar{\mathbf{M}}_{cn}^{(i)}] \quad (4.61)$$

$$[\hat{\mathbf{M}}_{nn}^{(i)}] = [\bar{\mathbf{M}}_{nn}^{(i)}] = [\mathbf{1}] \quad (4.62)$$

It should be noted that the submatrices $[\hat{\mathbf{M}}_{cc}]$, $[\hat{\mathbf{M}}_{cn}^{(i)}]$, and $[\hat{\mathbf{M}}_{nn}^{(i)}]$ are similar to those originally derived by Craig and Bampton [3]. This emphasises that the proposed method may be regarded as an extension to the classical Craig-Bampton method.

The structural stiffness matrix

The transformed stiffness matrix $[\hat{\mathbf{K}}] = [\mathbf{\Gamma}]^T [\bar{\mathbf{K}}] [\mathbf{\Gamma}]$ which would appear on the left-hand side of equation (4.55) if the stiffness matrix were explicitly included in the analysis, becomes

$$[\hat{\mathbf{K}}] = \begin{array}{c|ccc} \begin{array}{cccc} \hat{\mathbf{K}}_{oo}^{(1,1)} & \hat{\mathbf{K}}_{oo}^{(1,2)} & \dots & \hat{\mathbf{K}}_{oo}^{(1,K)} \\ & \hat{\mathbf{K}}_{oo}^{(2,2)} & \dots & \hat{\mathbf{K}}_{oo}^{(2,K)} \\ & & \ddots & \vdots \\ & & & \hat{\mathbf{K}}_{oo}^{(K,K)} \end{array} & \begin{array}{c} \hat{\mathbf{K}}_{oc}^{(1)} \\ \hat{\mathbf{K}}_{oc}^{(2)} \\ \vdots \\ \hat{\mathbf{K}}_{oc}^{(K)} \end{array} & \begin{array}{ccc} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{array} \\ \hline & \hat{\mathbf{K}}_{cc} & \begin{array}{ccc} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \hat{\mathbf{K}}_{nn}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ & \hat{\mathbf{K}}_{nn}^{(2)} & \dots & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \hat{\mathbf{K}}_{nn}^{(2)} \end{array} \\ \hline \begin{array}{ccc} \text{Sym-} & & \\ & \text{me-} & \\ & & \text{tric} \end{array} & & \end{array} \quad (4.63)$$

where

$$[\hat{\mathbf{K}}_{oo}^{(i,j)}] = \sum_{k=1}^K [\gamma_S^{(k,i)}]^T [\bar{\mathbf{K}}_{cc}^{(k)}] [\gamma_S^{(k,j)}] \quad (4.64)$$

$$[\hat{\mathbf{K}}_{oc}^{(i)}] = \sum_{k=1}^K [\gamma_S^{(k,i)}]^T [\bar{\mathbf{K}}_{cc}^{(k)}] [\gamma_c^{(k)}] \quad (4.65)$$

$$[\hat{\mathbf{K}}_{cc}] = \sum_{k=1}^K [\gamma_c^{(k)}]^T [\bar{\mathbf{K}}_{cc}^{(k)}] [\gamma_c^{(k)}] \quad (4.66)$$

$$[\hat{\mathbf{K}}_{nn}^{(i)}] = [\bar{\mathbf{K}}_{nn}^{(i)}] \quad (4.67)$$

Transformation of a force vector

The left-hand side of equation (4.55) has the form of a transformed force vector $\{\hat{\mathbf{F}}\} = [\mathbf{\Gamma}]^T \{\bar{\mathbf{F}}\}$, where $\{\bar{\mathbf{F}}\}$ denotes the collection of substructure force vectors partitioned according to equation (3.47). Usual linear algebra yields

$$\{\hat{\mathbf{F}}\} = \left\{ \begin{array}{c} \hat{\mathbf{F}}_o^{(1)} \\ \hat{\mathbf{F}}_o^{(2)} \\ \vdots \\ \hat{\mathbf{F}}_o^{(K)} \\ \hline \hat{\mathbf{F}}_c \\ \hline \hat{\mathbf{F}}_n^{(1)} \\ \hat{\mathbf{F}}_n^{(2)} \\ \vdots \\ \hat{\mathbf{F}}_n^{(K)} \end{array} \right\} \quad (4.68)$$

where

$$[\hat{\mathbf{F}}_o^{(i)}] = [\mathbf{T}_S^{*(i)}] [\bar{\mathbf{F}}_o^{(i)}] + \sum_{k=1}^K [\gamma_S^{(k,i)}]^T [\bar{\mathbf{F}}_c^{(i)}] \quad (4.69)$$

$$[\hat{\mathbf{F}}_c] = \sum_{k=1}^K [\gamma_c^{(i)}]^T [\bar{\mathbf{F}}_c^{(i)}] \quad (4.70)$$

$$[\hat{\mathbf{F}}_n^{(i)}] = [\bar{\mathbf{F}}_n^{(i)}] \quad (4.71)$$

Global boundary conditions

For an usual wind turbine, the global boundary condition dictates that the absolute displacement of the node at the tower bottom vanish. Assuming that the lower part of the tower generally are selected as the first substructure this condition are fulfilled simply by selecting the attaching node as the tower bottom node. In the implementation the global boundary condition is then introduced by removing the six first equations in the global e.o.m. (4.55), which means that the six first rows and columns in the global mass and stiffness matrices given by equations (4.56) and (4.63) are removed.

It should be noted that a consequence of an inertial local coordinate system for the first substructure is that a linear system is achieved when the complete structure is defined within one substructure only.

5 The fixed coupling

In Section 4 the substructure e.o.m. was coupled by means of a method, which was based on the geometric compatibility conditions between the substructures. These conditions for displacement, velocity and acceleration were assumed to have the general forms given by equations (4.5) to (4.7). No assumptions were made regarding the type of couplings, and generally it is possible to incorporate more different types. Here we only consider the so-called fixed coupling, which rigidly connects two substructures with a specified intermediate rotation, which is assumed to be constant.

Figure 5.1 shows the two substructures denoted by 1 and 2, which are coupled in a common point $P'_\alpha = P'_\beta$. The position of this point equals the position of the deformed state of the coupling nodes α and β belonging to substructure 1 and substructure 2, respectively. In addition the figure shows the inertial coordinate system (O, x, y, z) and the two dynamic coordinate systems (O_1, x_1, y_1, z_1) and (O_2, x_2, y_2, z_2) , which are attached to either substructure.

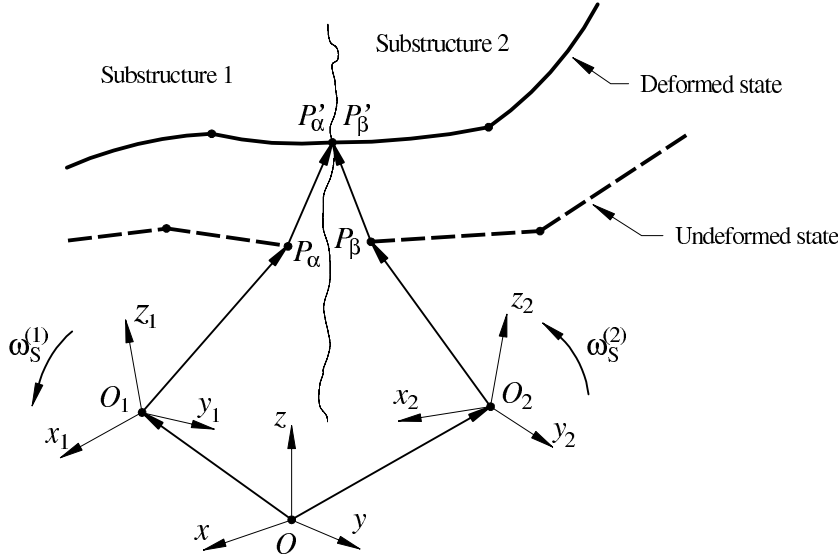


Figure 5.1. The fixed coupling between two substructures. The substructures are coupled together in the nodes denoted by α and β with an specified intermediate rotation. The undeformed state of the substructures is indicated by dashed lines, while the deformed state is indicated by full lines.

5.1 Conditions for translation

In the following the coupling conditions for translation are derived directly from the geometry, while the corresponding conditions for velocity and acceleration are derived by time differentiation of the fundamental condition for translation.

Initially we define the position vector to the origo of local coordinate system attached to substructure 1 by $\vec{r}_S^{(1)} = \overrightarrow{OO_1}$, the position vector to the coupling node α of substructure 1 by $\vec{r}_\alpha^{(1)} = \overrightarrow{O_1P'_\alpha}$, and the vector of linear deformation of coupling node α by $\vec{u}_\alpha^{(1)} = \overrightarrow{P_\alpha P'_\alpha}$. Similarly we define the three vectors $\vec{r}_S^{(2)} = \overrightarrow{OO_2}$, $\vec{r}_\beta^{(2)} = \overrightarrow{O_2P'_\beta}$ and $\vec{u}_\beta^{(2)} = \overrightarrow{P_\beta P'_\beta}$ associated with substructure 2. From Figure 5.1 it

then appears that the fundamental relation for translation may be written as

$$\vec{r}_S^{(1)} + \vec{r}_\alpha^{(1)} + \vec{u}_\alpha^{(1)} = \vec{r}_S^{(2)} + \vec{r}_\beta^{(2)} + \vec{u}_\beta^{(2)} \quad (5.1)$$

Displacement

The coupling conditions for translation are given by equation (5.1) in coordinate free form. Expressing the position vectors to origo in inertial coordinates and the node position vectors and the deformation vectors in local substructure coordinates, the coordinate form of equation (5.1) may be written as

$$\begin{aligned} & \{\mathbf{r}_S^{I(1)}\} + [\mathbf{T}_S^{(1)}] \left(\{\mathbf{r}_\alpha^{S(1)}\} + \{\mathbf{u}_\alpha^{S(1)}\} \right) \\ &= \{\mathbf{r}_S^{I(2)}\} + [\mathbf{T}_S^{(2)}] \left(\{\mathbf{r}_\beta^{S(2)}\} + \{\mathbf{u}_\beta^{S(2)}\} \right) \end{aligned} \quad (5.2)$$

where $[\mathbf{T}_S^{(i)}]$, $i = 1, 2$ is the 3×3 transformation matrices, which transform from local substructure coordinates to inertial coordinates. These matrices are formed solely from the Euler parameters describing the rotation of the local substructure coordinate systems as described in Appendix F.

Velocity

The coupling conditions for velocity are derived by absolute differentiation of equation (5.2) with respect to time. Noting that $\{\mathbf{r}_\alpha^{S(1)}\}$ and $\{\mathbf{r}_\beta^{S(1)}\}$ are constant this yields after a slight rearrangement

$$\begin{aligned} & \{\mathbf{v}_S^{I(1)}\} - [\mathbf{T}_S^{(1)}] [\tilde{\mathbf{r}}_\alpha^{S(1)}] [\mathbf{T}_S^{(1)}]^T \{\boldsymbol{\omega}_S^{I(1)}\} + [\mathbf{T}_S^{(1)}] \{\dot{\mathbf{u}}_\alpha^{S(1)}\} \\ & \quad + \{\boldsymbol{\omega}_S^{I(1)}\} \times \{\mathbf{u}_\alpha^{I(1)}\} \\ &= \{\mathbf{v}_S^{I(2)}\} - [\mathbf{T}_S^{(2)}] [\tilde{\mathbf{r}}_\beta^{S(2)}] [\mathbf{T}_S^{(2)}]^T \{\boldsymbol{\omega}_S^{I(2)}\} + [\mathbf{T}_S^{(2)}] \{\dot{\mathbf{u}}_\beta^{S(2)}\} \\ & \quad + \{\boldsymbol{\omega}_S^{I(2)}\} \times \{\mathbf{u}_\beta^{I(2)}\} \end{aligned} \quad (5.3)$$

where $\{\mathbf{v}_S^{I(i)}\} = \{\dot{\mathbf{r}}_S^{I(i)}\}$ and the cross (\times) denotes the usual vector product. The derivation of equation (5.3) involves the time differentiation of the coordinate transformation matrix. In that connection the following relation is employed:

$$[\dot{\mathbf{T}}_S^{(i)}] = [\tilde{\boldsymbol{\omega}}_S^{I(i)}] [\mathbf{T}_S^{(i)}] \quad (5.4)$$

where $[\tilde{\boldsymbol{\omega}}_S^{I(i)}]$ are the 3×3 skew-symmetric matrix associated with the 3×1 vector $\{\boldsymbol{\omega}_S^{I(i)}\}$ of substructure coordinates to the angular velocity of the i 'th substructure.

It should be noted that the expressions for the velocity on both sides of equation (5.3) correspond to the usual formulas for relative motion described in standard textbooks (see e.g. reference [7]).

Acceleration

The coupling conditions for acceleration are derived by time differentiation of equation (5.3). After a slight rearrangement this yields

$$\begin{aligned}
& \{\mathbf{a}_S^{I(1)}\} - [\mathbf{T}_S^{(1)}] [\tilde{\mathbf{r}}_\alpha^{S(1)}] [\mathbf{T}_S^{(1)}]^T \{\boldsymbol{\alpha}_S^{I(1)}\} + [\mathbf{T}_S^{(1)}] \{\ddot{\mathbf{u}}_\alpha^{S(1)}\} \\
& \quad + \{\boldsymbol{\alpha}_S^{I(1)}\} \times \{\mathbf{u}_\alpha^{I(1)}\} + 2\{\boldsymbol{\omega}_S^{I(1)}\} \times \{\dot{\mathbf{u}}_\alpha^{I(1)}\} \\
& \quad + \{\boldsymbol{\omega}_S^{I(1)}\} \times \left(\{\boldsymbol{\omega}_S^{I(1)}\} \times \left(\{\mathbf{r}_\alpha^{I(1)}\} + \{\mathbf{u}_\alpha^{I(1)}\} \right) \right) \\
& = \{\mathbf{a}_S^{I(2)}\} - [\mathbf{T}_S^{(2)}] [\tilde{\mathbf{r}}_\beta^{S(2)}] [\mathbf{T}_S^{(2)}]^T \{\boldsymbol{\alpha}_S^{I(2)}\} + [\mathbf{T}_S^{(2)}] \{\ddot{\mathbf{u}}_\beta^{S(2)}\} \\
& \quad + \{\boldsymbol{\alpha}_S^{I(2)}\} \times \{\mathbf{u}_\beta^{I(2)}\} + 2\{\boldsymbol{\omega}_S^{I(2)}\} \times \{\dot{\mathbf{u}}_\beta^{I(2)}\} \\
& \quad + \{\boldsymbol{\omega}_S^{I(2)}\} \times \left(\{\boldsymbol{\omega}_S^{I(2)}\} \times \left(\{\mathbf{r}_\beta^{I(2)}\} + \{\mathbf{u}_\beta^{I(2)}\} \right) \right)
\end{aligned} \tag{5.5}$$

5.2 Conditions for rotation

The fundamental coupling conditions for rotation cannot be derived directly from the geometry as was the case for translation. As it appears from Appendix F, finite rotations are generally described by means of Euler parameters or alternatively by means of a transformation (rotation) matrix. It is convenient to employ the transformation matrix description for deriving the fundamental relations between the rotation and then rewrite these relations in terms of Euler parameters.

In the following we consider the transformation matrix $[\mathbf{T}_{\alpha'}^{(1)}]$, which transforms from a coordinate system attached to the deformed state of the coupling node α within substructure 1, to inertial coordinates. Similarly, we consider the transformation matrix $[\mathbf{T}_{\beta'}^{(2)}]$ associated with coupling node β within substructure 2. The fundamental condition for rotation may then be written in terms of transformation matrices as

$$[\mathbf{T}_{\alpha'}^{(1)}] [\mathbf{T}_{\beta'\alpha'}] = [\mathbf{T}_{\beta'}^{(2)}], \tag{5.6}$$

where $[\mathbf{T}_{\beta'\alpha'}]$ represents the relative rotation of substructure 2 with respect to substructure 1. As noted in the introduction to this section this intermediate rotation is assumed to be constant and the corresponding Euler parameters, denoted by $\{\boldsymbol{\rho}_\Delta^*\}$, are geometric parameters for the structure.

The transformation matrices $[\mathbf{T}_{\alpha'}^{(1)}]$ and $[\mathbf{T}_{\beta'}^{(2)}]$ introduced in equation (5.6) represent the rotation of the coupling nodes in the deformed positions. These rotations are the compound rotation of the undeformed coupling nodes and an additional relative rotation due to the elastic deformation. This may be expressed by the equation

$$[\mathbf{T}_{\gamma'}^{(i)}] = [\mathbf{T}_\gamma^{(i)}] [\mathbf{T}_{\gamma'\gamma}^{(i)}], \quad (\gamma, i) = (\alpha, 1), (\beta, 2) \tag{5.7}$$

As the undeformed state of the nodes are rigidly connected to the local substructure coordinate system, the rotations of the two are equal. This means that $[\mathbf{T}_{\gamma'}^{(i)}] = [\mathbf{T}_\gamma^{(i)}]$, where $[\mathbf{T}_S^{(i)}]$ denote the transformation matrices, which transforms from local coordinates of the i 'th substructure to inertial coordinates.

The transformation matrices $[\mathbf{T}_{\gamma'\gamma}^{(i)}]$ appearing in equation (5.7) represent the relative rotation of the deformed coupling nodes with respect to the undeformed state. These rotations are assumed to be infinitesimally small and they are determined completely by the angular displacement $\{\boldsymbol{\theta}_\gamma^{S(i)}\} = \{\theta_{\gamma,x}^{S(i)} \ \theta_{\gamma,y}^{S(i)} \ \theta_{\gamma,z}^{S(i)}\}^T$ of the coupling nodes. The corresponding transformation matrices are then given

by the well-known first order approximation

$$[\mathbf{T}_{\gamma'\gamma}^{(i)}] = \begin{bmatrix} 1 & -\theta_{\gamma,z}^{S(i)} & \theta_{\gamma,y}^{S(i)} \\ \theta_{\gamma,z}^{S(i)} & 1 & -\theta_{\gamma,x}^{S(i)} \\ -\theta_{\gamma,y}^{S(i)} & \theta_{\gamma,x}^{S(i)} & 1 \end{bmatrix}, \quad (\gamma, i) = (\alpha, 1), (\beta, 2) \quad (5.8)$$

By comparison of equations (5.8) and (F.7) it is apparent that the Euler parameters corresponding to the transformation matrices $[\mathbf{T}_{\gamma'\gamma}^{(i)}]$ are given by¹⁰

$$\{\rho_{\gamma'\gamma}^{*(i)}\} = \left\{ \frac{1}{2} \theta_{\gamma}^{S(i)} \right\}, \quad (i, \gamma) = (1, \alpha), (2, \beta) \quad (5.9)$$

The Euler parameters corresponding to the transformation matrices $[\mathbf{T}_{\gamma'}^{(i)}]$ on the left-hand side of equation (5.7) may then be expressed directly in terms of Euler parameters by means of equation (F.36). Employing equation (5.9) for the relative rotation due to the deformation then yields

$$\begin{aligned} \{\rho_{\gamma'}^{*(i)}\} &= [\mathbf{L}^*(\rho_S^{*(i)})]^T \{\rho_{\gamma'\gamma}^{*(i)}\} \\ &= \{\rho_S^{*(i)}\} + \frac{1}{2} [\mathbf{L}(\rho_S^{*(i)})]^T \{\theta_{\gamma}^{S(i)}\} \end{aligned} \quad (5.10)$$

Displacement

The fundamental condition for rotation is given in terms of rotation matrices by equation (5.6), which expresses that the rotation of the coupling node α of substructure 2 equals the compound rotation of the coupling node β of substructure 2 and the intermediate rotation. Expressing the fundamental condition (5.6) in terms of Euler parameters by application of equation (F.37) and employing equation (5.10) for the compound rotations of the deformed coupling nodes yield

$$[\mathbf{G}_{\Delta}^*]^T \left(\{\rho_S^{*(1)}\} + \frac{1}{2} [\mathbf{L}_S^{(1)}]^T \{\theta_{\alpha}^{S(1)}\} \right) = \{\rho_S^{*(2)}\} + \frac{1}{2} [\mathbf{L}_S^{(2)}]^T \{\theta_{\beta}^{S(2)}\} \quad (5.11)$$

where $[\mathbf{L}_S^{(i)}] = [\mathbf{L}(\rho_S^{*(i)})]$, $i = 1, 2$ are defined by equation (F.10) and $[\mathbf{G}_{\Delta}^*] = [\mathbf{G}^*(\rho_{\Delta}^*)]$ is defined by equation (F.18). As noted previously, $\{\rho_{\Delta}^*\}$ denotes the Euler parameters of the constant intermediate rotation corresponding to the rotation matrix $[\mathbf{T}_{\beta'\alpha'}]$ in equation (5.6).

It is clear that equation (5.11) constitutes four relations between the Euler parameters describing the orientation of the two substructures. Thus we have one superfluous condition, which basically expresses the constraint equation (F.4) of the Euler parameters. In order to reduce the number of coupling condition to three it is necessary to rewrite equation (5.11) in a relative form by means of equation (F.35) after which the condition for the scalar component of the Euler parameters may be removed. From the definition (F.19) of the $[\mathbf{L}^*]$ matrix it appears that this may be achieved simply by pre-multiplying both sides of equation (5.11) by the 3×4 matrix $[\mathbf{L}_S^{(2)}]$. This yields after multiplying both sides by two

$$[\mathbf{L}_S^{(2)}] [\mathbf{G}_{\Delta}^*]^T \left(2\{\rho_S^{*(1)}\} + [\mathbf{L}_S^{(1)}]^T \{\theta_{\alpha}^{S(1)}\} \right) = \{\theta_{\beta}^{S(2)}\} \quad (5.12)$$

where the relations (F.12) and (F.14) has been applied.

¹⁰Obviously equation (5.9) expresses the first terms of an Taylor expansion of the set of Euler parameters given by

$$\left\{ \begin{array}{l} \cos\left(\frac{1}{2}|\theta_{\gamma}^{S(i)}|\right) \\ \{\theta_{\gamma}^{S(i)}\} \sin\left(\frac{1}{2}|\theta_{\gamma}^{S(i)}|\right) \end{array} \right\}$$

Velocity

Differentiation of equation (5.11) with respect to time, noting that $[\mathbf{G}_\Delta^*]$ is constant, yields

$$\begin{aligned} & \frac{1}{2}[\mathbf{G}_\Delta^*]^T \left([\mathbf{G}_S^{(1)}]^T \{\boldsymbol{\omega}_S^{I(1)}\} + [\mathbf{L}_S^{(1)}]^T \{\dot{\boldsymbol{\theta}}_\alpha^{S(1)}\} + [\dot{\mathbf{L}}_S^{(1)}]^T \{\boldsymbol{\theta}_\alpha^{S(1)}\} \right) \\ &= \frac{1}{2} \left([\mathbf{G}_S^{(2)}]^T \{\boldsymbol{\omega}_S^{I(2)}\} + [\mathbf{L}_S^{(2)}]^T \{\dot{\boldsymbol{\theta}}_\beta^{S(2)}\} + [\dot{\mathbf{L}}_S^{(2)}]^T \{\boldsymbol{\theta}_\beta^{S(2)}\} \right) \end{aligned} \quad (5.13)$$

where $[\mathbf{G}_S^{(i)}] = [\mathbf{G}(\boldsymbol{\rho}_S^{*(i)})]$ is defined by equation (F.9) and equation (F.26) has been employed for expressing the time derivatives of the Euler parameters.

The corresponding relative form of equation (5.13) is derived by pre-multiplying both sides by the matrix $[\mathbf{L}_S^{(2)}]$ ¹¹. This yields after multiplying both sides by two

$$\begin{aligned} & [\mathbf{L}_S^{(2)}][\mathbf{G}_\Delta^*]^T \left([\mathbf{G}_S^{(1)}]^T \{\boldsymbol{\omega}_S^{I(1)}\} + [\mathbf{L}_S^{(1)}]^T \{\dot{\boldsymbol{\theta}}_\alpha^{S(1)}\} + [\dot{\mathbf{L}}_S^{(1)}]^T \{\boldsymbol{\theta}_\alpha^{S(1)}\} \right) \\ &= [\mathbf{T}_S^{(2)}]^T \{\boldsymbol{\omega}_S^{I(2)}\} + \{\dot{\boldsymbol{\theta}}_\beta^{S(2)}\} + [\mathbf{L}_S^{(2)}][\dot{\mathbf{L}}_S^{(2)}]^T \{\boldsymbol{\theta}_\beta^{S(2)}\} \end{aligned} \quad (5.14)$$

where equation (F.17) has been employed.

Acceleration

Differentiation of equation (5.13) with respect to time yields

$$\begin{aligned} & \frac{1}{2}[\mathbf{G}_\Delta^*]^T \left([\mathbf{G}_S^{(1)}]^T \{\boldsymbol{\alpha}_S^{I(1)}\} + [\mathbf{L}_S^{(1)}]^T \{\ddot{\boldsymbol{\theta}}_\alpha^{S(1)}\} \right. \\ & \quad \left. + [\dot{\mathbf{G}}_S^{(1)}]^T \{\boldsymbol{\omega}_S^{I(1)}\} + 2[\dot{\mathbf{L}}_S^{(1)}]^T \{\dot{\boldsymbol{\theta}}_\alpha^{S(1)}\} + [\ddot{\mathbf{L}}_S^{(1)}]^T \{\boldsymbol{\theta}_\alpha^{S(1)}\} \right) \\ &= \frac{1}{2} \left([\mathbf{G}_S^{(2)}]^T \{\boldsymbol{\alpha}_S^{I(2)}\} + [\mathbf{L}_S^{(2)}]^T \{\ddot{\boldsymbol{\theta}}_\beta^{S(2)}\} \right. \\ & \quad \left. + [\dot{\mathbf{G}}_S^{(2)}]^T \{\boldsymbol{\omega}_S^{I(2)}\} + 2[\dot{\mathbf{L}}_S^{(2)}]^T \{\dot{\boldsymbol{\theta}}_\beta^{S(2)}\} + [\ddot{\mathbf{L}}_S^{(2)}]^T \{\boldsymbol{\theta}_\beta^{S(2)}\} \right) \end{aligned} \quad (5.15)$$

The corresponding relative form equation (5.15) becomes

$$\begin{aligned} & [\mathbf{L}_S^{(2)}][\mathbf{G}_\Delta^*]^T \left([\mathbf{G}_S^{(1)}]^T \{\boldsymbol{\alpha}_S^{I(1)}\} + [\mathbf{L}_S^{(1)}]^T \{\ddot{\boldsymbol{\theta}}_\alpha^{S(1)}\} \right. \\ & \quad \left. + [\dot{\mathbf{G}}_S^{(1)}]^T \{\boldsymbol{\omega}_S^{I(1)}\} + 2[\dot{\mathbf{L}}_S^{(1)}]^T \{\dot{\boldsymbol{\theta}}_\alpha^{S(1)}\} + [\ddot{\mathbf{L}}_S^{(1)}]^T \{\boldsymbol{\theta}_\alpha^{S(1)}\} \right) \\ &= [\mathbf{T}_S^{(2)}]^T \{\boldsymbol{\alpha}_S^{I(2)}\} + \{\ddot{\boldsymbol{\theta}}_\beta^{S(2)}\} \\ & \quad + [\mathbf{L}_S^{(2)}] \left([\dot{\mathbf{G}}_S^{(2)}]^T \{\boldsymbol{\omega}_S^{I(2)}\} + 2[\dot{\mathbf{L}}_S^{(2)}]^T \{\dot{\boldsymbol{\theta}}_\beta^{S(2)}\} + [\ddot{\mathbf{L}}_S^{(2)}]^T \{\boldsymbol{\theta}_\beta^{S(2)}\} \right) \end{aligned} \quad (5.16)$$

5.3 Coupling conditions in common form

The coupling conditions (5.2) and (5.12) for displacement, (5.3) and (5.14) for velocity, and (5.5) and (5.16) for acceleration are now rewritten on order to fit into the common form of the coupling conditions (4.5) to (4.7).

Initially we define the six 6×1 vectors

$$\{\mathbf{V}_S^{I(i)}\} = \left\{ \begin{array}{c} \mathbf{v}_S^{I(i)} \\ \boldsymbol{\omega}_S^{I(i)} \end{array} \right\}, \quad i = 1, 2 \quad (5.17)$$

$$\{\mathbf{A}_S^{I(i)}\} = \left\{ \begin{array}{c} \mathbf{a}_S^{I(i)} \\ \boldsymbol{\alpha}_S^{I(i)} \end{array} \right\}, \quad i = 1, 2 \quad (5.18)$$

¹¹When $\{\boldsymbol{\theta}_\beta^{S(2)}\} = \{\mathbf{0}\}$ this operation results in the angular velocity, which appears directly from equation (F.28).

$$\{\mathbf{U}_\gamma^{\text{S}(i)}\} = \begin{Bmatrix} \mathbf{u}_\gamma^{\text{S}(i)} \\ \boldsymbol{\theta}_\gamma^{\text{S}(i)} \end{Bmatrix}, \quad (i, \gamma) = (1, \alpha), (2, \beta) \quad (5.19)$$

and the two 6×6 matrices

$$[\mathbf{C}_\text{S}^{(1)}] = \begin{bmatrix} \mathbf{1} & \mathbf{C}_{\text{S},12}^{(1)} \\ \mathbf{0} & \mathbf{C}_{\text{S},22}^{(1)} \end{bmatrix} \quad (5.20)$$

$$[\mathbf{C}_\text{S}^{(2)}] = \begin{bmatrix} -\mathbf{1} & \mathbf{C}_{\text{S},12}^{(2)} \\ \mathbf{0} & \mathbf{C}_{\text{S},22}^{(2)} \end{bmatrix} \quad (5.21)$$

where

$$[\mathbf{C}_{\text{S},12}^{(1)}] = -[\mathbf{T}_\text{S}^{(1)}] [\tilde{\mathbf{r}}_\alpha^{\text{S}(1)}] [\mathbf{T}_\text{S}^{(1)}]^T \quad (5.22)$$

$$[\mathbf{C}_{\text{S},22}^{(1)}] = [\mathbf{L}_\text{S}^{(2)}] [\mathbf{G}_\Delta^*]^T [\mathbf{G}_\text{S}^{(1)}]^T \quad (5.23)$$

$$[\mathbf{C}_{\text{S},12}^{(2)}] = [\mathbf{T}_\text{S}^{(2)}] [\tilde{\mathbf{r}}_\beta^{\text{S}(2)}] [\mathbf{T}_\text{S}^{(2)}]^T \quad (5.24)$$

$$[\mathbf{C}_{\text{S},22}^{(2)}] = -[\mathbf{T}_\text{S}^{(2)}]^T \quad (5.25)$$

Furthermore, we define the two 6×6 matrices

$$[\mathbf{C}_\text{b}^{(1)}] = \begin{bmatrix} \mathbf{T}_\text{S}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\text{b},22}^{(1)} \end{bmatrix} \quad (5.26)$$

$$[\mathbf{C}_\text{b}^{(2)}] = \begin{bmatrix} -\mathbf{T}_\text{S}^{(2)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \quad (5.27)$$

where

$$[\mathbf{C}_{\text{b},22}^{(1)}] = [\mathbf{L}_\text{S}^{(2)}] [\mathbf{G}_\Delta^*]^T [\mathbf{L}_\text{S}^{(1)}]^T \quad (5.28)$$

Finally, it is assumed that the vectors $\{\mathbf{d}_m\}$, $m = 0, 1, 2$ appearing in equations (4.5) to (4.7) are partitioned according to

$$\{\mathbf{d}_m\} = \begin{Bmatrix} \mathbf{d}_{m,1} \\ \mathbf{d}_{m,2} \end{Bmatrix}, \quad m = 0, 1, 2 \quad (5.29)$$

Displacement

By comparing equations (5.2) and (5.12) with the definitions (5.26) and (5.27) of the matrices $[\mathbf{C}_\text{b}^{(1)}]$ and $[\mathbf{C}_\text{b}^{(2)}]$ it appears that the condition for displacement may be written as

$$\begin{bmatrix} \mathbf{C}_\text{b}^{(1)} & \mathbf{C}_\text{b}^{(2)} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_\alpha^{\text{S}(1)} \\ \mathbf{U}_\beta^{\text{S}(2)} \end{Bmatrix} + \{\mathbf{d}_0\} = \{\mathbf{0}\} \quad (5.30)$$

where

$$\{\mathbf{d}_{0,1}\} = \{\mathbf{r}_\text{S}^{\text{I}(1)}\} + [\mathbf{T}_\text{S}^{(1)}] \{\mathbf{r}_\alpha^{\text{S}(1)}\} - \left(\{\mathbf{r}_\text{S}^{\text{I}(2)}\} + [\mathbf{T}_\text{S}^{(2)}] \{\mathbf{r}_\beta^{\text{S}(2)}\} \right) \quad (5.31)$$

$$\{\mathbf{d}_{0,2}\} = 2[\mathbf{L}_\text{S}^{(2)}] [\mathbf{G}_\Delta^*]^T \{\boldsymbol{\rho}_\text{S}^{*(1)}\} \quad (5.32)$$

Velocity

The coupling conditions for velocity given by the equations (5.3) and (5.14) may be written in the common form as

$$\begin{bmatrix} \mathbf{C}_\text{S}^{(1)} & \mathbf{C}_\text{S}^{(2)} \end{bmatrix} \begin{Bmatrix} \mathbf{V}_\text{S}^{\text{I}(1)} \\ \mathbf{V}_\text{S}^{\text{I}(2)} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_\text{b}^{(1)} & \mathbf{C}_\text{b}^{(2)} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}}_\alpha^{\text{S}(1)} \\ \dot{\mathbf{U}}_\beta^{\text{S}(2)} \end{Bmatrix} + \{\mathbf{d}_1\} = \{\mathbf{0}\} \quad (5.33)$$

where

$$\{\mathbf{d}_{1,1}\} = \{\boldsymbol{\omega}_\text{S}^{\text{I}(1)}\} \times \{\mathbf{u}_\alpha^{\text{I}(1)}\} - \{\boldsymbol{\omega}_\text{S}^{\text{I}(2)}\} \times \{\mathbf{u}_\beta^{\text{I}(2)}\} \quad (5.34)$$

$$\{\mathbf{d}_{1,2}\} = [\mathbf{L}_\text{S}^{(2)}] [\mathbf{G}_\Delta^*]^T [\dot{\mathbf{L}}_\text{S}^{(1)}]^T \{\boldsymbol{\theta}_\alpha^{\text{S}(1)}\} - [\mathbf{L}_\text{S}^{(2)}] [\dot{\mathbf{L}}_\text{S}^{(2)}]^T \{\boldsymbol{\theta}_\beta^{\text{S}(2)}\} \quad (5.35)$$

Acceleration

For acceleration the coupling conditions are given by equations (5.5) and (5.16), which may be written in the common form as

$$\left[\mathbf{C}_S^{(1)} \quad \mathbf{C}_S^{(2)} \right] \begin{Bmatrix} \mathbf{A}_S^{I(1)} \\ \mathbf{A}_S^{I(2)} \end{Bmatrix} + \left[\mathbf{C}_b^{(1)} \quad \mathbf{C}_b^{(2)} \right] \begin{Bmatrix} \ddot{\mathbf{U}}_\alpha^{I(1)} \\ \ddot{\mathbf{U}}_\beta^{I(2)} \end{Bmatrix} + \{\mathbf{d}_2\} = \{\mathbf{0}\} \quad (5.36)$$

where

$$\left. \begin{aligned} \{\mathbf{d}_{2,1}\} = & \left\{ \begin{aligned} & \{\boldsymbol{\alpha}_S^{I(1)}\} \times \{\mathbf{u}_\alpha^{I(1)}\} + 2\{\boldsymbol{\omega}_S^{I(1)}\} \times \{\dot{\mathbf{u}}_\alpha^{I(1)}\} \\ & + \{\boldsymbol{\omega}_S^{I(1)}\} \times \left(\{\boldsymbol{\omega}_S^{I(1)}\} \times \left(\{\mathbf{r}_\alpha^{I(1)}\} + \{\mathbf{u}_\alpha^{I(1)}\} \right) \right) \\ & - \left[\{\boldsymbol{\alpha}_S^{I(2)}\} \times \{\mathbf{u}_\beta^{I(2)}\} + 2\{\boldsymbol{\omega}_S^{I(2)}\} \times \{\dot{\mathbf{u}}_\beta^{I(2)}\} \right. \\ & \left. + \{\boldsymbol{\omega}_S^{I(2)}\} \times \left(\{\boldsymbol{\omega}_S^{I(2)}\} \times \left(\{\mathbf{r}_\beta^{I(2)}\} + \{\mathbf{u}_\beta^{I(2)}\} \right) \right) \right] \end{aligned} \right\} \quad (5.37) \end{aligned}$$

$$\left. \begin{aligned} \{\mathbf{d}_{2,2}\} = & \left[\mathbf{L}_S^{(2)} \right] \left[\mathbf{G}_\Delta^* \right]^T \left(\left[\dot{\mathbf{G}}_S^{(1)} \right]^T \{\boldsymbol{\omega}_S^{I(1)}\} \right. \\ & \left. + 2 \left[\dot{\mathbf{L}}_S^{(1)} \right]^T \{\boldsymbol{\theta}_\alpha^S(1)\} + \left[\ddot{\mathbf{L}}_S^{(1)} \right]^T \{\boldsymbol{\theta}_\alpha^S(1)\} \right) \\ & - \left[\mathbf{L}_S^{(2)} \right] \left(\left[\dot{\mathbf{G}}_S^{(2)} \right]^T \{\boldsymbol{\omega}_S^{I(2)}\} \right. \\ & \left. + 2 \left[\dot{\mathbf{L}}_S^{(2)} \right]^T \{\boldsymbol{\theta}_\beta^S(2)\} + \left[\ddot{\mathbf{L}}_S^{(2)} \right]^T \{\boldsymbol{\theta}_\beta^S(2)\} \right) \end{aligned} \right\} \quad (5.38)$$

Relation between coupling matrices

The final step in the derivation of the coupling conditions is to check the validity of equation (4.12).

By ordinary linear algebra it may readily be shown that

$$\left[\mathbf{C}_b^{(1)} \right] \left[\boldsymbol{\sigma}(\mathbf{r}_\alpha^S(1)) \right] = \left[\mathbf{C}_S^{(1)} \right] \left[\mathbf{T}_S^*(1) \right] \quad (5.39)$$

and

$$\left[\mathbf{C}_b^{(2)} \right] \left[\boldsymbol{\sigma}(\mathbf{r}_\beta^S(2)) \right] = \left[\mathbf{C}_S^{(2)} \right] \left[\mathbf{T}_S^*(2) \right] \quad (5.40)$$

where the 6×6 matrix $[\boldsymbol{\sigma}]$ is defined by equation (C.3). Since the coupling conditions for the complete structure are formed by simple linear assemblage of the coupling conditions for each individual coupling, the relations (5.39) and (5.40) ensure that equation (4.12) is fulfilled for the fixed coupling.

6 Conclusion

A new method for the coupling and reduction of the HAWC equations of motions for a wind turbine and other similar structures is proposed.

The method is based on the concept of substructuring, which was applied for the HAWC model as well, but generalized in the present formulation. By this concept the structure is subdivided into a number of structural components or substructures, where the local displacements may be assumed to be infinitesimally small. The displacements within a substructure is described in a local coordinate system attached to the substructure in a selected node.

The generalization of the substructuring concept enables the modelling of geometric non-linearities of the blades. This phenomenon, which is caused by large rotation but moderate strain, becomes significant at large displacements of the blades and the effect generally increases with increasing size of the wind turbine. In addition the generalization of substructuring concept makes possible a detailed modelling of the nacelle/shaft structure, which may be significant when designing new wind turbines with an modified nacelle arrangement, and even modelling of wind turbines based on new concepts.

In order to enable the reduction of the number of degrees of freedom for the structure, the Craig-Bampton method is applied. However, the standard Craig-Bampton method assumes that the substructures are described in a common coordinate system and the method is therefore modified to deal with the general case, where the substructures are described in local dynamic coordinate systems.

References

- [1] R. W. Clough and J. Penzien: *Dynamics of Structures. Second Edition*. McGraw-Hill Inc., New York, 1993
- [2] R. D. Cook, D. S. Malcus, and M. E. Plesha: *Concepts and Applications of Finite Element Analysis. Third Edition*. John Wiley & Sons, New York, 1989
- [3] R. R. Craig Jr. and M. C. C. Bampton: “Coupling of Substructures for Dynamic Analysis”, *AIAA Journal*, Vol. 6, no 7, pp. 1313–1319. 1968
- [4] O. Fabian: “A New Method for Aeroelastic Analysis of Wind Turbines”. Technical report AFM 81–06, Department of Fluid Mechanics, Technical University of Denmark, Lyngby, 1981
- [5] M. H. Hansen: *Anisotropic Damping of Timoshenko Beam Elements*. Technical report R-1276. Wind Energy Department, Risø National Laboratory, Roskilde, 2001
- [6] S. Krenk: “Finite Rotation Beam Analysis”. Chapter 3 in *Non-linear Analysis with Finite Elements*, lecture notes, Technical University of Denmark, Lyngby
- [7] J. L. Meriam: *Statics and Dynamics*. John Wiley & Sons, Inc. New York, 1980
- [8] P. E. Nikravesh: *Computer-Aided Analysis of Mechanical Systems*. Prentice-Hall International Editions, Englewood Cliffs, 1988
- [9] J. T. Petersen: *Kinematically Nonlinear Finite Element Model of a Horizontal Axis Wind Turbine. Part 1: Mathematical Model and Results*. Wind Energy Department, Risø National Laboratory, Roskilde, 1990
- [10] J. T. Petersen: *Kinematically Nonlinear Finite Element Model of a Horizontal Axis Wind Turbine. Part 2: Supplement. Inertia Matrices and Aerodynamic Model*. Wind Energy Department, Risø National Laboratory, Roskilde, 1990
- [11] J. T. Petersen: “The aeroelastic code HAWC – model and comparisons”, In B.M. Pedersen (editor): *State of the Art of Aeroelastic Codes for Wind Turbine Calculations*, 28th Meeting of Experts, International Energy Agency, Annex XI, pp. 129–135, Lyngby, 1996
- [12] P. T. Pedersen and J. J. Jensen: *Styrkeberegning af maritime konstruktioner. Del 2 Numeriske Metoder*. Institut for Skibs- og Havteknik, DTU, Lyngby, 1982. (In Danish)

A Kinematic analysis

This appendix provides a kinematic analysis of a flexible body defined in a local coordinate system, which translates and rotates with respect to an inertial system. The analysis follows the theory given in standard textbooks [7]. The purpose of the appendix is to derive equations for the absolute velocity and acceleration for a material point on the body.

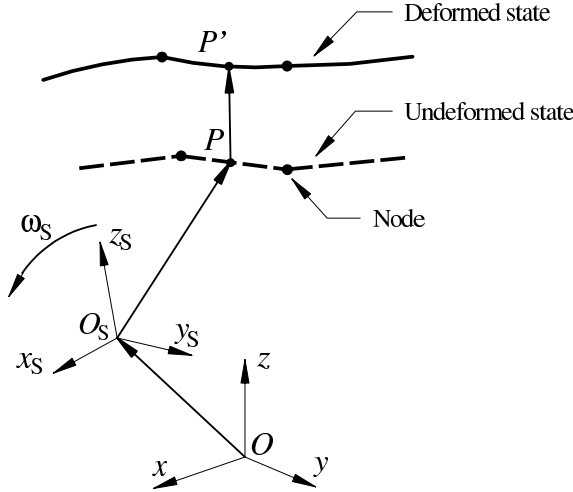


Figure A.1. A flexible body shown in the deformed state (full lines) and in the undeformed state (dashed lines). The body is defined in a local coordinate system (O_S, x_S, y_S, z_S) , which translates and rotates, the angular velocity given by $\vec{\omega}_S$, with respect to an inertial system (O, x, y, z) .

Figure A.1 shows the flexible body in the undeformed and in the deformed state. Here the deformed position of the considered point P is denoted by P' . While the position vector $\vec{O}_S P$ to the undeformed point is constant when measured in the local coordinate system, the relative displacement $\vec{u}_P = \vec{P}P'$ is generally time-dependent. Defining the vectors $\vec{r}_P = \vec{O}P'$, $\vec{r}_S = \vec{O}O_S$ and $\vec{s} = \vec{O}_S P'$, the basic relation for the position vector to the point of the deformed body may be written as

$$\vec{r}_P = \vec{r}_S + \vec{s}_P \quad (\text{A.1})$$

Absolute differentiation of equation (A.1) with respect to time yields

$$\dot{\vec{r}}_P = \dot{\vec{r}}_S + \vec{\omega}_S \times \vec{s}_P + \dot{\vec{u}}_P \quad (\text{A.2})$$

where $\vec{\omega}_S$ is the angular velocity of the local coordinate system with respect to the inertial coordinate system. Differentiation once more with respect to time yields

$$\ddot{\vec{r}}_P = \ddot{\vec{r}}_S + \dot{\vec{\omega}}_S \times \vec{s}_P + \vec{\omega}_S \times (\vec{\omega}_S \times \vec{s}_P) + 2\vec{\omega}_S \times \dot{\vec{u}}_P + \ddot{\vec{u}}_P \quad (\text{A.3})$$

Equations (A.3) constitutes the basic relation for the absolute acceleration of a material point on a flexible body in coordinate free form.

We now introduce a special notation, where the vector of coordinates to a vector is denoted by the corresponding bolded symbol with an upper right index indicating the coordinate system, which is used for reference. In order to keep the notation consistent throughout this report (except Appendix F) the local coordinate system is denoted by the letter S (the substructure coordinate system). By

this convention the vector of local coordinates to the acceleration vector given by equation (A.3) may be written as

$$\{\ddot{\mathbf{r}}_P^S\} = \{\mathbf{a}_S^S\} + [\mathbf{A}'_S]\{\mathbf{s}_P^S\} + [\mathbf{B}'_S]\{\dot{\mathbf{u}}_P^S\} + [\mathbf{C}'_S]\{\ddot{\mathbf{u}}_P^S\} \quad (\text{A.4})$$

where $\{\mathbf{a}_S^S\} = \{\ddot{\mathbf{r}}_S^S\}$ is the vector of local coordinates to the acceleration of origo. The three new 3×3 matrices $[\mathbf{A}'_S]$, $[\mathbf{B}'_S]$, and $[\mathbf{C}'_S]$ appearing in equation (A.4) are defined by

$$[\mathbf{A}'_S] = [\tilde{\boldsymbol{\omega}}_S^S]^2 + [\tilde{\boldsymbol{\alpha}}_S^S] \quad (\text{A.5})$$

$$[\mathbf{B}'_S] = 2[\tilde{\boldsymbol{\omega}}_S^S] \quad (\text{A.6})$$

$$[\mathbf{C}'_S] = [\mathbf{1}] \quad (\text{A.7})$$

where the tilde ($\tilde{}$) superposed a vector denotes the associated skew-symmetric matrix described in Appendix D.

The form of equation (A.4) is identical to the corresponding expression derived by Petersen [9]. Consequently, the matrices $[\mathbf{A}_S]$, $[\mathbf{B}_S]$, and $[\mathbf{C}_S]$ used intensively by Petersen may simply be replaced by the matrices $[\mathbf{A}'_S]$, $[\mathbf{B}'_S]$ and $[\mathbf{C}'_S]$ in the equations for the inertia forces. This is what is done in Appendix B, where the expression for the inertia forces is evaluated.

B Rearranging the expression for the nodal inertia forces

The inertia forces acting on a moving and deforming beam element were modelled by Petersen [9] by transforming the distributed inertia forces, expressed by Newton's second law, consistently with the principle of virtual work to the nodes of the beam element. The element is defined in a local coordinate system (the substructure system), which translates as well as rotates with respect to an inertial system. In the following, $\{\mathbf{a}_S^S\}$ denotes the 3×1 vector of substructure coordinates to the acceleration of origo of the substructure coordinate system, while $\{\boldsymbol{\omega}_S^S\}$ and $\{\boldsymbol{\alpha}_S^S\} = \{\dot{\boldsymbol{\omega}}_S^S\}$ denote the 3×1 vectors of substructure coordinates to the angular velocity and angular acceleration, respectively.

The orientation of the beam element is given by the 3×1 vectors $\{\mathbf{r}_1^S\}$ and $\{\mathbf{r}_2^S\}$ of substructure coordinates to the nodal positions. It is noted that the two vectors are constant and given by the geometry.

Further to the substructure coordinate system, a local coordinate system (the element system) is defined specifically for every beam element of the substructure. The element z -axis is chosen as the elastic axis of the beam, while the element x - and y -axis coincide with the axis of principal bending of the cross-section. The transformation matrix $[\mathbf{T}_{ES}]$, transforming from element coordinates to substructure coordinates, is then constant and given solely by the geometry, i.e. $\{\mathbf{r}_1^S\}$ and $\{\mathbf{r}_2^S\}$ besides the rotation of the beam element about the element z -axis (structural pitch).

As usual for the displacement based finite element method, the deformed state of the beam is expressed by the nodal displacements, which in the present context consist of translations as well as rotations, the later assumed to be infinitesimally small. The nodal displacements of a beam element may then be described by the 12×1 vector $\{\mathbf{U}^E\}$ of element coordinates to the nodal displacements.

The final expression, derived by Petersen [9], for the 12×1 vector of element coordinates to the inertia forces at the two node may be written as

$$\begin{aligned}
 -\{\mathbf{F}_I^E\} &= [\mathbf{M}^{el}] \{\ddot{\mathbf{U}}^E\} + [\mathbf{C}_I^{el}] \{\dot{\mathbf{U}}^E\} + [\mathbf{K}_I^{el}] \{\mathbf{U}^E\} \\
 &+ \{\mathbf{F}_4^{el}\} \\
 &+ [\mathbf{F}_5^{el}] [\mathbf{T}_{ES}]^T [\mathbf{A}_S] \left(\{\mathbf{r}_2^S\} - \{\mathbf{r}_1^S\} \right) \\
 &+ [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T \left([\mathbf{A}_S] \{\mathbf{r}_1^S\} + \{\mathbf{a}_S^S\} \right)
 \end{aligned} \tag{B.1}$$

where

$[\mathbf{M}^{el}]$ is the constant 12×12 mass matrix for the beam element.

$[\mathbf{C}_I^{el}]$ is the time-dependent 12×12 inertia damping (Coriolis) matrix, which is a function of $\{\boldsymbol{\omega}_S^S\}$.

$[\mathbf{K}_I^{el}]$ is the time-dependent 12×12 inertia softening matrix, which is a function of $\{\boldsymbol{\omega}_S^S\}$ and $\{\boldsymbol{\alpha}_S^S\}$.

$\{\mathbf{F}_4^{el}\}$ is a time-dependent 12×1 vector, which is a function of both $\{\boldsymbol{\omega}_S^S\}$ and $\{\boldsymbol{\alpha}_S^S\}$.

$[\mathbf{F}_5^{el}]$ and $[\mathbf{F}_6^{el}]$ are constant 12×3 matrices.

The minus sign was added to the left-hand side of equation (B.1) in order to express the inertia force in accordance with D'Alembert's principle.

The time-dependent matrix $[\mathbf{A}_S]$ appearing in equation (B.1) was evaluated in Appendix A for the present representation of the local substructure coordinate system. The result of this analysis was:

$$[\mathbf{A}_S] = [\tilde{\omega}_S^S]^2 + [\tilde{\alpha}_S^S] \quad (\text{B.2})$$

where the tilde ($\tilde{}$) superposed a vector denotes the associated skew-symmetric matrix defined in Appendix D.

Inserting equation (B.2) into equation (B.1) and employing the general rule (D.5) for the product of a skew-symmetric matrix and a vector yields

$$\begin{aligned} -\{\mathbf{F}_I^E\} &= [\mathbf{M}^{el}] \{\dot{\mathbf{U}}^E\} + [\mathbf{C}_I^E] \{\dot{\mathbf{U}}^E\} + [\mathbf{K}_I^{el}] \{\mathbf{U}^E\} \\ &\quad + \{\mathbf{F}_4^{el}\} \\ &\quad - \left([\mathbf{F}_5^{el}] [\mathbf{T}_{ES}]^T [\tilde{\mathbf{r}}_{12}^S] + [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T [\tilde{\mathbf{r}}_1^S] \right) \{\alpha_S^S\} \\ &\quad + [\mathbf{F}_5^{el}] [\mathbf{T}_{ES}]^T [\tilde{\omega}_S^S]^2 \{\mathbf{r}_{12}^S\} + [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T [\tilde{\omega}_S^S]^2 \{\mathbf{r}_1^S\} \\ &\quad + [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T \{\mathbf{a}_S^S\} \end{aligned} \quad (\text{B.3})$$

where $\{\mathbf{r}_{12}^S\} = \{\mathbf{r}_2^S\} - \{\mathbf{r}_1^S\}$ is the direction vector of the beam element.

As the time-dependent inertia force vector $\{\mathbf{F}_4^{el}\}$ only includes terms with linear expressions of $\{\alpha_S^S\}$ (see reference [9]), the following decomposition is valid:

$$\{\mathbf{F}_4^{el}\} = [\mathbf{I}_4^{el}] \{\alpha_S^S\} + \{\mathbf{R}_4^{el}\} \quad (\text{B.4})$$

where $[\mathbf{I}_4^{el}]$ is a constant 12×3 inertia matrix, and the remaining component $\{\mathbf{R}_4^{el}\}$ is a time-dependent 12×1 vector, which is a function of $\{\omega_S^S\}$ only. Inserting equation (B.4) into equation (B.3) and rearranging then finally yields

$$\begin{aligned} -\{\mathbf{F}_I^E\} &= [\mathbf{M}^{el}] \{\dot{\mathbf{U}}^E\} + [\mathbf{C}_I^{el}] \{\dot{\mathbf{U}}^E\} + [\mathbf{K}_I^{el}] \{\mathbf{U}^E\} \\ &\quad + [\mathbf{M}_R^{el}] \{\mathbf{A}_S^S\} + \{\mathbf{F}_R^{el}\} \end{aligned} \quad (\text{B.5})$$

where the acceleration vectors of the substructure coordinate system has been collected in the common vector

$$\{\mathbf{A}_S^S\} = \begin{Bmatrix} \mathbf{a}_S^S \\ \alpha_S^S \end{Bmatrix} \quad (\text{B.6})$$

and the following two new quantities has been introduced:

$$[\mathbf{M}_R^{el}] = \left[[\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T \mid [\mathbf{I}_4^{el}] - [\mathbf{F}_5^{el}] [\mathbf{T}_{ES}]^T [\tilde{\mathbf{r}}_{12}^S] - [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T [\tilde{\mathbf{r}}_1^S] \right] \quad (\text{B.7})$$

$$\{\mathbf{F}_R^{el}\} = \{\mathbf{R}_4^{el}\} + [\mathbf{F}_5^{el}] [\mathbf{T}_{ES}]^T [\tilde{\omega}_S^S]^2 \{\mathbf{r}_{12}^S\} + [\mathbf{F}_6^{el}] [\mathbf{T}_{ES}]^T [\tilde{\omega}_S^S]^2 \{\mathbf{r}_1^S\} \quad (\text{B.8})$$

Equation (B.5) constitutes the final result of the analysis. It should be noted that $[\mathbf{M}_R^{el}]$ is a constant 12×6 matrix, while $\{\mathbf{F}_R^{el}\}$ is a time-dependent 12×1 vector, which is a function of $\{\omega_S^S\}$ only.

It is apparent that the last two terms on the right-hand side of equation (B.5) are the inertia forces arising from rigid-body dynamics of the substructure, and basically these terms express Newton's second law for translation and Euler's equations for rotation.

C The rigid-body displacement modal matrix

A fundamental property of the finite element method is that rigid-body displacements may be described by certain patterns of deformations. The number of these rigid-body displacement modes is six, namely three modes of translation and three modes of rotation. In this appendix, the modal matrix for rigid-body displacement of a substructure is constructed from the position vectors to the nodes of the substructure.

The nodes of the substructure are defined with respect to a local coordinate system (x_S, y_S, z_S) , with throughout this report is termed *the substructure coordinate system*. The coordinates of the nodes in this system is then constant and given by the geometry.

A single node

Initially, a single node is considered. The displacements of the node are described by the *nodal displacement vector*

$$\{\mathbf{U}^S\} = \begin{Bmatrix} u_x^S \\ u_y^S \\ u_z^S \\ \theta_x^S \\ \theta_y^S \\ \theta_z^S \end{Bmatrix} \quad (\text{C.1})$$

where

u_x^S , u_y^S and u_z^S are the linear displacements (translations) of the node in the x_S -, y_S - and z_S -direction

θ_x^S , θ_y^S and θ_z^S are the angular displacements (rotations) of the node about the x_S -, y_S - and z_S -axis

The substructure coordinates of the node position vector are given by

$$\{\mathbf{r}^S\} = \begin{Bmatrix} r_x^S \\ r_y^S \\ r_z^S \end{Bmatrix} \quad (\text{C.2})$$

The three first modes are defined as translations in the three main directions. Mode no 1 is then given by $u_x = u_{x0}$, while the remaining displacements are zero. Here u_{x0} denotes an arbitrary displacement in the x_S direction. Mode no 2 and 3 are defined similarly.

The remaining three modes, describing rotation about the three axis, are somewhat more complicated. Figure C.1 shows the geometry for mode no 4, where $\theta_x^S = \theta_{0x}^S$, and $\theta_y^S = \theta_z^S = 0$. Here θ_{0x}^S denotes an arbitrary infinitesimal angular displacement about the x_S -axis. The corresponding translations are derived directly from the geometry. From the figure it appears that $u_y^S = -\sin(\alpha) (\theta_{0x}^S L_{yz}) = -r_z^S \theta_{0x}^S$, and $u_z^S = \cos(\alpha) (\theta_{0x}^S L_{yz}) = r_y^S \theta_{0x}^S$, while $u_x^S = 0$. For mode no 5, where $\theta_y^S = \theta_{y0}^S$ and $\theta_x^S = \theta_z^S = 0$, similar arguments yield $u_x^S = +r_z^S \theta_{y0}^S$, $u_y^S = -r_x^S \theta_{y0}^S$ and $u_z^S = 0$. Finally for mode no 6, where $\theta_z^S = \theta_{z0}^S$ and $\theta_x^S = \theta_y^S = 0$, the displacements become $u_x^S = -r_y^S \theta_{z0}^S$, $u_y^S = +r_x^S \theta_{z0}^S$ and $u_z^S = 0$.

We now consider unit displacements in all directions, i.e. $u_{0x}^S = u_{0y}^S = u_{0z}^S = 1$ and $\theta_{0x}^S = \theta_{0y}^S = \theta_{0z}^S = 1$. In that case the six modes may be described by a 6×6

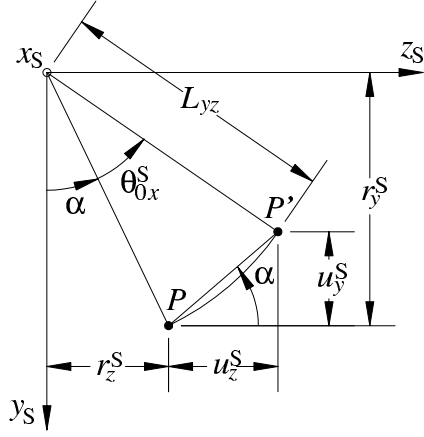


Figure C.1. Rigid-body rotation of a single node. The node is rotated an infinitesimally small angle of θ_{0x}^S from the position P to P' .

matrix $[\sigma(\mathbf{r}^S)]$ defined by

$$[\sigma(\mathbf{r}^S)] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & r_z^S & -r_y^S \\ 0 & 1 & 0 & -r_z^S & 0 & r_x^S \\ 0 & 0 & 1 & r_y^S & -r_x^S & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] = \begin{bmatrix} \mathbf{1} & -\tilde{\mathbf{r}}^S \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \quad (\text{C.3})$$

This matrix has some remarkable properties. For two different nodes with the node position vectors $\{\mathbf{r}^S\}$ and $\{\mathbf{s}^S\}$ it is readily shown that

$$[\sigma(\mathbf{r}^S + \mathbf{s}^S)] = [\sigma(\mathbf{r}^S)][\sigma(\mathbf{s}^S)] \quad (\text{C.4})$$

It then follows that

$$[\sigma(\mathbf{r}^S)][\sigma(\mathbf{s}^S)] = [\sigma(\mathbf{s}^S)][\sigma(\mathbf{r}^S)] \quad (\text{C.5})$$

A complete substructure

For the complete substructure having N nodes, the rigid-body displacements are described by the $6N \times 6$ rigid-body modal matrix $[\psi]$ defined by

$$[\psi] = \begin{bmatrix} \sigma(\mathbf{r}_1^S) \\ \sigma(\mathbf{r}_2^S) \\ \vdots \\ \sigma(\mathbf{r}_N^S) \end{bmatrix} \quad (\text{C.6})$$

where $\{\mathbf{r}_n^S\}$ denotes the 3×1 vector of substructure coordinates to the position of the n 'th node.

It is apparent that the rigid-body displacements may be expressed by

$$\{\mathbf{U}^S\} = [\psi]\{\mathbf{R}_S^S\} \quad (\text{C.7})$$

where $\{\mathbf{R}_S^S\}$ is a 6×1 vector, which contains the amplitudes of each individual rigid-body mode.

D The skew-symmetric matrix associated with a vector

The usual vector product between two 3×1 vectors $\{\mathbf{u}\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$ and $\{\mathbf{v}\} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$ is defined by

$$\{\mathbf{u}\} \times \{\mathbf{v}\} = \begin{Bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{Bmatrix} \quad (\text{D.1})$$

From this definition it follows directly that the vector product may be expressed by the matrix-vector product as

$$\{\mathbf{u}\} \times \{\mathbf{v}\} = [\tilde{\mathbf{u}}]\{\mathbf{v}\} \quad (\text{D.2})$$

where the new 3×3 matrix $[\tilde{\mathbf{u}}]$ is defined by

$$[\tilde{\mathbf{u}}] = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \quad (\text{D.3})$$

From the above definition it follows that

$$[\tilde{\mathbf{u}}]^T = -[\tilde{\mathbf{u}}] \quad (\text{D.4})$$

which means that $[\tilde{\mathbf{u}}]$ is *skew-symmetric*.

It is clear that equation (D.3) defines an operator, which associates a 3×1 vector $\{\mathbf{u}\}$ with the 3×3 skew-symmetric matrix $[\tilde{\mathbf{u}}]$. This operator, denoted by a superposed tilde ($\tilde{}$), has a number remarkable properties, which may be derived from the usual equations from the discipline of vector analysis. Here we only explicitly quotes the equation

$$[\tilde{\mathbf{u}}]\{\mathbf{v}\} = -[\tilde{\mathbf{v}}]\{\mathbf{u}\} \quad (\text{D.5})$$

which follows directly from the fundamental relation $\{\mathbf{u}\} \times \{\mathbf{v}\} = -\{\mathbf{v}\} \times \{\mathbf{u}\}$.

E The Craig-Bampton transformation with a full mass matrix

In the original paper [3] written by Craig and Bampton, a diagonal mass matrix, obtained from a lumped-mass formulation, was used. In this this Appendix the Craig-Bampton transformation will be extended to cope with a full mass matrix, obtained from a consistent formulation.

Using a notation approximately similar to what is used by Craig and Bampton, the full mass matrix is written

$$[\mathbf{m}] = \begin{bmatrix} \mathbf{m}_{BB} & \mathbf{m}_{BI} \\ \mathbf{m}_{IB} & \mathbf{m}_{II} \end{bmatrix} \quad (\text{E.1})$$

where $[\mathbf{m}_{BI}] = [\mathbf{m}_{IB}]^T$ due to symmetry. The Craig-Bampton transformation matrix $[\boldsymbol{\alpha}]$ is defined by

$$[\boldsymbol{\alpha}] = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \bar{\boldsymbol{\phi}}_C & \bar{\boldsymbol{\phi}}_N \end{bmatrix} \quad (\text{E.2})$$

where $[\bar{\boldsymbol{\phi}}_C]$ is the constraint (coupling) mode matrix and $[\bar{\boldsymbol{\phi}}_N]$ is the normal mode matrix.

The transformed mass matrix is expressed by

$$[\boldsymbol{\alpha}]^T [\mathbf{m}] [\boldsymbol{\alpha}] = \begin{bmatrix} \bar{\mathbf{m}}_{BB} & \bar{\mathbf{m}}_{BN} \\ \bar{\mathbf{m}}_{NB} & \bar{\mathbf{m}}_{NN} \end{bmatrix} \quad (\text{E.3})$$

Simple matrix manipulations then yield

$$[\bar{\mathbf{m}}_{BB}] = [\mathbf{m}_{BB}] + [\mathbf{m}_{BI}] [\bar{\boldsymbol{\phi}}_C] + ([\mathbf{m}_{BI}] [\bar{\boldsymbol{\phi}}_C])^T + [\bar{\boldsymbol{\phi}}_C]^T [\mathbf{m}_{II}] [\bar{\boldsymbol{\phi}}_C] \quad (\text{E.4})$$

$$[\bar{\mathbf{m}}_{BN}] = [\bar{\mathbf{m}}_{NB}]^T = ([\mathbf{m}_{BI}] + [\bar{\boldsymbol{\phi}}_C]^T [\mathbf{m}_{II}]) [\bar{\boldsymbol{\phi}}_N] \quad (\text{E.5})$$

$$[\bar{\mathbf{m}}_{NN}] = [\bar{\boldsymbol{\phi}}_N]^T [\mathbf{m}_{II}] [\bar{\boldsymbol{\phi}}_N] \quad (\text{E.6})$$

Obviously the terms including $[\mathbf{m}_{BI}]$ in equations (E.4) to (E.5) are new compared to the original Craig-Bampton formulation.

F Euler rotation parameters

The problem of describing finite rotations of a *local* coordinate system with respect to a *global* system plays a central role in the modelling of the dynamics of rotating structures and other structures, with are exposed to large non-linear deformations. Traditionally, finite rotations have been represented by means of three Euler angles, which describe the complete rotation as three successive rotations about the coordinate axis. As finite rotations is non-commutative, the order of the rotations have to be specified in advance. However, it is well known that the Euler angle representation is singular for some combinations of angles and that the rotated position of the body (or the coordinate system) depends on the current position. Furthermore, the algebra associated with Euler angles is rather complicated, and do not exhibit any kind of symmetric properties. These problems may be disregarded completely by resort to the so-called Euler parameters described in detail by Nikravesh [8].

This appendix describes the Euler parameters in brief and lists the most important formulars used in the main sections of this report. The text is closely related to Nikravesh's formulation [8], which is particular convenient in connection with the implementation.

For convenience the notation in this appendix are modified slightly compared to what is used in the rest of this report. Thus the upper index I added to a symbol, indicating that the inertial (global) coordinate system are used for reference, is omitted. Quantities which reference to the local co-rotating coordinate system are denoted by primed (') symbols.

Definition

With the representation of a finite rotation by Euler parameters, the rotation are fundamentally described by means of a normalized rotation axis \vec{u} and a rotation angle ϕ . The corresponding Euler parameters are the elements of the 4×1 vector

$$\{\rho^*\} = \begin{Bmatrix} \rho_0 \\ \rho \end{Bmatrix}, \quad \{\rho\} = \begin{Bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{Bmatrix} \quad (\text{F.1})$$

where the scalar ρ_0 is defined by

$$\rho_0 = \cos \frac{\phi}{2} \quad (\text{F.2})$$

and $\{\rho\}$ is the 3×1 vector of global coordinates¹² to the vector $\vec{\rho}$ defined by

$$\vec{\rho} = \vec{u} \sin \frac{\phi}{2} \quad (\text{F.3})$$

The four Euler parameters are not independent and from the above equations it is apparent that

$$\rho_0^2 + |\rho|^2 = 1 \quad (\text{F.4})$$

or in the explicit form

$$\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 1 \quad (\text{F.5})$$

¹²Actually global and local coordinates for this particular vector are equal, which appears from the following

Transformation matrix

The 3×3 transformation matrix $[\mathbf{T}]$ relates a 3×1 vector $\{\mathbf{a}\}$ of global coordinates to a corresponding vector $\{\mathbf{a}'\}$ of local coordinates. More specific

$$\{\mathbf{a}\} = [\mathbf{T}]\{\mathbf{a}'\} \quad (\text{F.6})$$

It is straightforward to show, that the coordinate transformation matrix $[\mathbf{T}]$ corresponding to a set of Euler parameters $\{\boldsymbol{\rho}^*\}$ may be expressed by

$$\begin{aligned} [\mathbf{T}(\boldsymbol{\rho}^*)] &= (2\rho_0^2 - 1) [\mathbf{1}] + 2 (\{\boldsymbol{\rho}\}\{\boldsymbol{\rho}\}^T + \rho_0[\tilde{\boldsymbol{\rho}}]) \\ &= \begin{bmatrix} \rho_0^2 + \rho_1^2 - \rho_2^2 - \rho_3^2 & 2(\rho_1\rho_2 - \rho_0\rho_3) & 2(\rho_1\rho_3 + \rho_0\rho_2) \\ 2(\rho_1\rho_2 + \rho_0\rho_3) & \rho_0^2 - \rho_1^2 + \rho_2^2 - \rho_3^2 & 2(\rho_2\rho_3 - \rho_0\rho_1) \\ 2(\rho_3\rho_1 - \rho_0\rho_2) & 2(\rho_3\rho_2 + \rho_0\rho_1) & \rho_0^2 - \rho_1^2 - \rho_2^2 + \rho_3^2 \end{bmatrix} \end{aligned} \quad (\text{F.7})$$

where the 3×3 skew-symmetric matrix $[\tilde{\boldsymbol{\rho}}]$ is defined in Appendix D.

Fundamentally we have

$$[\mathbf{T}]\{\boldsymbol{\rho}\} = \{\boldsymbol{\rho}\} \quad (\text{F.8})$$

which may be proved directly by means of equation (F.7). However, the relation immediately follows from the fact that the rotation axis given by the vector \vec{u} is not affected by the rotation.

The $[\mathbf{G}]$ and $[\mathbf{L}]$ matrices

It appears to be convenient to define the two 3×4 matrices $[\mathbf{G}]$ and $[\mathbf{L}]$ by

$$\begin{aligned} [\mathbf{G}(\boldsymbol{\rho}^*)] &= [-\{\boldsymbol{\rho}\} \mid [\tilde{\boldsymbol{\rho}} + \rho_0[\mathbf{1}]] \\ &= \begin{bmatrix} -\rho_1 & \rho_0 & -\rho_3 & \rho_2 \\ -\rho_2 & \rho_3 & \rho_0 & -\rho_1 \\ -\rho_3 & -\rho_2 & \rho_1 & \rho_0 \end{bmatrix} \end{aligned} \quad (\text{F.9})$$

$$\begin{aligned} [\mathbf{L}(\boldsymbol{\rho}^*)] &= [-\{\boldsymbol{\rho}\} \mid -[\tilde{\boldsymbol{\rho}} + \rho_0[\mathbf{1}]] \\ &= \begin{bmatrix} -\rho_1 & \rho_0 & \rho_3 & -\rho_2 \\ -\rho_2 & -\rho_3 & \rho_0 & \rho_1 \\ -\rho_3 & \rho_2 & -\rho_1 & \rho_0 \end{bmatrix} \end{aligned} \quad (\text{F.10})$$

It is straightforward to show the following relations:

$$[\mathbf{G}]\{\boldsymbol{\rho}^*\} = \{\mathbf{0}\} \quad (\text{F.11})$$

$$[\mathbf{L}]\{\boldsymbol{\rho}^*\} = \{\mathbf{0}\} \quad (\text{F.12})$$

$$[\mathbf{G}][\mathbf{G}]^T = [\mathbf{1}] \quad (\text{F.13})$$

$$[\mathbf{L}][\mathbf{L}]^T = [\mathbf{1}] \quad (\text{F.14})$$

$$[\mathbf{G}]^T[\mathbf{G}] = [\mathbf{1}] - \{\boldsymbol{\rho}^*\}\{\boldsymbol{\rho}^*\}^T \quad (\text{F.15})$$

$$[\mathbf{L}]^T[\mathbf{L}] = [\mathbf{1}] - \{\boldsymbol{\rho}^*\}\{\boldsymbol{\rho}^*\}^T \quad (\text{F.16})$$

$$[\mathbf{G}][\mathbf{L}]^T = [\mathbf{T}] \quad (\text{F.17})$$

where $[\mathbf{G}] = [\mathbf{G}(\boldsymbol{\rho}^*)]$, $[\mathbf{L}] = [\mathbf{L}(\boldsymbol{\rho}^*)]$, and $[\mathbf{T}] = [\mathbf{T}(\boldsymbol{\rho}^*)]$. It should be noted that $[\mathbf{G}][\mathbf{G}]^T$ and $[\mathbf{L}][\mathbf{L}]^T$ are 3×3 matrices, while $[\mathbf{G}]^T[\mathbf{G}]$ and $[\mathbf{L}]^T[\mathbf{L}]$ are 4×4 matrices.

The $[\mathbf{G}^*]$ and $[\mathbf{L}^*]$ matrices

Further to the $[\mathbf{G}]$ and $[\mathbf{L}]$ matrices described above we define the two 4×4 matrices $[\mathbf{G}^*]$ and $[\mathbf{L}^*]$ by

$$[\mathbf{G}^*(\boldsymbol{\rho}^*)] = \begin{bmatrix} \boldsymbol{\rho}^{*T} \\ \mathbf{G}(\boldsymbol{\rho}^*) \end{bmatrix} \quad (\text{F.18})$$

$$[\mathbf{L}^*(\boldsymbol{\rho}^*)] = \begin{bmatrix} \boldsymbol{\rho}^{*T} \\ \mathbf{L}(\boldsymbol{\rho}^*) \end{bmatrix} \quad (\text{F.19})$$

These matrices have some remarkable properties. Employing equations (F.13) to (F.16) it is readily shown that $[\mathbf{G}^*]$ and $[\mathbf{L}^*]$ both are orthogonal, i.e.

$$[\mathbf{G}^*]^{-1} = [\mathbf{G}^*]^T \quad (\text{F.20})$$

$$[\mathbf{L}^*]^{-1} = [\mathbf{L}^*]^T \quad (\text{F.21})$$

For two sets of Euler parameters $\{\boldsymbol{\rho}^*\}$ and $\{\boldsymbol{\sigma}^*\}$ it appears that

$$[\mathbf{G}^*(\boldsymbol{\rho}^*)]^T \{\boldsymbol{\sigma}^*\} = [\mathbf{L}^*(\boldsymbol{\sigma}^*)]^T \{\boldsymbol{\rho}^*\} \quad (\text{F.22})$$

This relation was not quoted by Nikravesh but it can be proved directly from the definitions (F.18) and (F.19) of the $[\mathbf{G}^*]$ and $[\mathbf{L}^*]$ matrices.

Time derivatives of the transformation matrix

The time derivative of the transformation matrix is closely related to the angular velocity $\vec{\omega}$ of local coordinate system. The fundamental relations are

$$[\dot{\mathbf{T}}] = [\tilde{\omega}][\mathbf{T}] \quad (\text{F.23})$$

$$[\dot{\mathbf{T}}] = [\mathbf{T}][\tilde{\omega}'] \quad (\text{F.24})$$

where $\{\boldsymbol{\omega}\}$ and $\{\boldsymbol{\omega}'\}$ are vectors of global and local coordinates of $\vec{\omega}$, respectively.

Time derivatives of Euler rotation parameters

In two-dimensional cases, finite rotations are generally described by a single angle and the corresponding angular velocity certainly equals the angular velocity. Unfortunately, this is not the case in the three-dimensional world regardless of the choice of rotation parameters. However, the time derivative are closely related to the angular velocity of the rotation parameters. For Euler parameters these relations are particular simple and may be written as

$$\{\boldsymbol{\omega}\} = 2[\mathbf{G}]\{\dot{\boldsymbol{\rho}}^*\} \quad (\text{F.25})$$

$$\{\dot{\boldsymbol{\rho}}^*\} = \frac{1}{2}[\mathbf{G}]^T \{\boldsymbol{\omega}\} \quad (\text{F.26})$$

These relations may alternatively be expressed in local coordinates as

$$\{\dot{\boldsymbol{\rho}}^*\} = \frac{1}{2}[\mathbf{L}]^T \{\boldsymbol{\omega}'\} \quad (\text{F.27})$$

$$\{\boldsymbol{\omega}'\} = 2[\mathbf{L}]\{\dot{\boldsymbol{\rho}}^*\} \quad (\text{F.28})$$

The sizes of the two involved vectors are related by

$$|\dot{\boldsymbol{\rho}}^*| = \frac{1}{2}|\boldsymbol{\omega}| \quad (\text{F.29})$$

Moreover, it appears that the time derivative of the angular velocity (the angular acceleration) is related to the second time derivative of the Euler parameters by

$$\{\dot{\boldsymbol{\omega}}\} = 2[\mathbf{G}]\{\ddot{\boldsymbol{\rho}}^*\} \quad (\text{F.30})$$

$$\{\ddot{\boldsymbol{\rho}}^*\} = \frac{1}{2}[\mathbf{G}]^T \{\dot{\boldsymbol{\omega}}\} - \frac{1}{4}|\boldsymbol{\omega}|^2 \{\boldsymbol{\rho}^*\} \quad (\text{F.31})$$

or expressed in local coordinates by

$$\{\dot{\boldsymbol{\omega}}'\} = 2[\mathbf{L}]\{\ddot{\boldsymbol{\rho}}^*\} \quad (\text{F.32})$$

$$\{\ddot{\boldsymbol{\rho}}^*\} = \frac{1}{2}[\mathbf{L}]^T \{\dot{\boldsymbol{\omega}}'\} - \frac{1}{4}|\boldsymbol{\omega}'|^2 \{\boldsymbol{\rho}^*\} \quad (\text{F.33})$$

Relative and compound rotations

We now consider two local coordinate systems α and β , which are described by the Euler parameters $\{\rho_\alpha^*\}$ and $\{\rho_\beta^*\}$ with the corresponding transformation matrices $[\mathbf{T}_\alpha]$ and $[\mathbf{T}_\beta]$. The transformation matrix

$$[\mathbf{T}_{\alpha\beta}] = [\mathbf{T}_\beta]^T [\mathbf{T}_\alpha] \quad (\text{F.34})$$

obviously describes the relative rotation of the α system with respect to the β system, and it is applied for the transformation of a vector from α coordinates to β coordinates.

For theoretical reasons it is essential to establish a relation, which describes the relative rotation directly in terms of Euler parameters. It is straightforward but rather lengthy to show that the Euler parameters $\{\rho_{\alpha\beta}^*\}$ corresponding to the transformation matrix $[\mathbf{T}_{\alpha\beta}]$ in equation (F.34) may be written as

$$\{\rho_{\alpha\beta}^*\} = [\mathbf{L}^*(\rho_\beta^*)]\{\rho_\alpha^*\} \quad (\text{F.35})$$

where the 4×4 matrix $[\mathbf{L}^*]$ is defined by equation (F.19). Employing equation (F.20) it becomes apparent that equation (F.35) may be written as

$$\{\rho_\alpha^*\} = [\mathbf{L}^*(\rho_\beta^*)]^T \{\rho_{\alpha\beta}^*\} \quad (\text{F.36})$$

This equation may be rewritten by employing equation (F.22), which yields

$$\{\rho_\alpha^*\} = [\mathbf{G}^*(\rho_{\alpha\beta}^*)]^T \{\rho_\beta^*\} \quad (\text{F.37})$$

Obviously equations (F.36) and (F.37) correspond to equation (F.34) written as $[\mathbf{T}_\alpha] = [\mathbf{T}_\beta][\mathbf{T}_{\alpha\beta}]$, which describes the rotation of the α system as the compound rotation of the β system and an additional relative rotation. These equations may easily be generalized to describe more relative rotations.

Title and authors

Coupling and Reduction of the HAWC Equations

Erik Nim

ISBN	ISSN
87-550-2944-2; 87-550-2945-0 (Internet)	0106-2840

Department or group	Date
Wind Energy Department	September 2001

Groups own reg. number(s)	Project/contract No(s)
1110028-0	ENS-1363/00-0007

Pages	Tables	Illustrations	References
60	0	3	12

Abstract (max. 2000 characters)

This report contains a description of a general method for coupling and reduction of the so-called HAWC equations, which constitute the basis equations of motion of the aeroelastic model HAWC used widely by research institutes and industrial companies for more than the ten years. The principal aim of the work has been to enable the modelling wind turbines with large displacements of the blades in order to predict phenomena caused by geometric non-linear effects. However, the method can also be applied to model the nacelle/shaft structure of a turbine more detailed than the present HAWC model. In addition, the method enables the reduction of the number of degrees of freedom of the structure in order to increase the calculation efficiency and improve the condition of the system.

Descriptors INIS/EDB

AERODYNAMICS; ELASTICITY; FINITE ELEMENT METHOD;
HORIZONTAL AXIS TURBINES; MATHEMATICAL MODELS;
MECHANICAL STRUCTURES; NONLINEAR PROBLEMS; ROTATION

Available on request from Information Service Department, Risø National Laboratory,
(Afdelingen for Informationsservice, Forskningscenter Risø), P.O.Box 49, DK-4000 Roskilde, Denmark.
Telephone +45 4677 4004, Telefax +45 4677 4013