Downloaded from orbit.dtu.dk on: Jan 23, 2019

DTU Library

The horizontally homogeneous model equations of incompressible atmospheric flow in general orthogonal coordinates

Jørgensen, Bo Hoffmann

Publication date:
2003

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Jørgensen, B. H. (2003). The horizontally homogeneous model equations of incompressible atmospheric flow in general orthogonal coordinates. Denmark. Forskningscenter Risoe. Risoe-R, No. 1446(EN)

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Risø–R–1446(EN)

The horizontally homogeneous model equations of incompressible atmospheric flow in general orthogonal coordinates

Bo Hoffmann Jørgensen

Risø National Laboratory, Roskilde, Denmark
December 2003

Abstract The goal of this brief report is to express the model equations for an incompressible flow which is horizontally homogeneous. It is intended as a computationally inexpensive starting point of a more complete solution for neutral atmospheric flow over complex terrain. This idea was set forth by Ayotte and Taylor (1995) and in the work of Beljaars, Walmsley and Taylor (1987). Unlike the previous models, the present work uses general orthogonal coordinates. Strong conservation form of the model equations is employed to allow a robust and consistent numerical procedure. An invariant tensor form of the model equations is utilized expressing the flow variables in a transformed coordinate system in which they are horizontally homogeneous. The model utilizes the $k - \epsilon$ model with limited mixing length by Apsley and Castro (1997). This turbulence closure reflects the fact that the atmosphere is only neutral up to a certain height. The horizontally homogeneous flow model is a part of a pertubation solver under development which is hoped to be more accurate than the current standard program WAsP by Troen and Petersen (1989) while achieving a high speed of execution.

ISBN 87–550–3295–8 (Internet)
ISSN 0106–2840

Print: Pitney Bowes Management Services Denmark A/S · 2003

Contents

1 Introduction

What is the purpose of solving for a horizontally homogeneous flow? As a starting point for a more complete incompressible flow solution it is useful to consider a solution for a horizontally homogenous flow. It is computationally inexpensive to obtain because it is a one-dimensional solution instead of three-dimensional. It is of course an advantage that the one-dimensional solution becomes as close as possible to the three-dimensional solution. Hence, it is to be found in a transformed coordinate system. As a consequence, horizontal is to be understood as *horizontal in the transformed coordinate system*.

The present work is intended for a model of the incompressible flow over complex terrain. It was inspired by the Mixed Spectral Finite-Difference model of Ayotte and Taylor (1995) and the work of Beljaars et al. (1987). The idea is the same of finding a simple solution which may be used as a starting point for a pertubation solver. The solution may also be used as an initial condition for a more general flow solver. It is the hope that the pertubation solver, which is under development, may result in a flow model which is more accurate than the current flow model of the industry standard program WAsP described in Troen and Petersen (1989) while it should achieve a reasonably high speed of execution. Unlike the previous models, the present work uses general orthogonal coordinates. Strong conservation form of the model equations is employed in order to allow the development of a robust and consistent numerical procedure based on Finite Differences (FD). The tensor notation applied and the general equations are explained in Jørgensen (2003). It would be out of scope to explain the details related to tensor calculus here. The use of general coordinates and strong conservation form was inspired by the CFD code EllipSys which is described in Michelsen (1989), Michelsen (1992), Michelsen (1994) and Sørensen (1995). Vivand (1974) developed strong conservation form in 2D. However, EllipSys utilizes the flow variables in the physical coordinate system. In the present work, an invariant tensor form of the model equations is utilized expressing the flow variables in the transformed coordinate system in which they are horizontally homogeneous. This form is included in Jørgensen (2003).

Although the flow is actually solved in orthogonal general coordinates, it can still be corrected into an initial solution in general coordinates via a pressure correction of a Fractional Step method utilizing Rhie-Chow interpolation. An appropriate Fractional Step method is described in Jørgensen (in preparation). However, it requires the general coordinate system to be nearly orthogonal. The collocation points describing such a coordinate system can be generated efficiently by employing hyperbolic grid generation. A set of coefficients for the coordinate transformation needed for the horizontally homogeneous flow solver can be generated from the general coordinates by optimization utilizing Lagrange Multipliers. This procedure mimicks an orthogonal coordinate system allowing the horizontally homogeneous flow solver to find a solution. In the case of a pertubation solver, small deviations from orthogonality can be treated as pertubations. The solution can be advanced by using an appropriate Fractional Step method such as the method of Jørgensen (in preparation). The implementation of the generation procedure, the use of the resulting coefficients in the horizontally homogeneous flow solver and the subsequent use of the solution are subjects of future work. Of course, the horizontally homogeneous flow solver can also be applied for a general orthogonal grid, provided that such a grid can be generated.

Because the present model is intended for atmospheric flow over complex terrain, it utilizes a turbulence model which is the $k-\epsilon$ model with limited mixing length by Apsley and Castro (1997). In this model, the limitation of the mixing length is obtained by a modification of the scale determining equation, i.e. the dissipation transport equation. This reflects the fact that the real atmosphere is only neutral up to a certain height. Apart from this, the model assumes a neutral flow. Hence, no transport equation is included for temperature.

2 Horizontal homogeneity

In the following, a few basic concepts are described. Then, in the next sections, the model equations are derived.

2.1 The tangent in a transformed coordinate system

By horizontally homogeneous, it is meant that any horizontal tangent of the solution gradient in the transformed coordinate system is zero. The gradient of a tensor,

$$(g^{ij}u^k)_{|j}\ \mathbf{e}_i \tag{1}$$

becomes, in the transformed system,

$$(\acute{g}^{ij}\acute{u}^k)_{|j}\ \acute{\mathbf{e}}_i \tag{2}$$

The tangential components of the gradient are,

$$(\acute{g}^{ij}\acute{u}^k)_{|j}\ \acute{\mathbf{e}}_i \cdot \acute{\mathbf{e}}_a = (\acute{g}^{ij}\acute{u}^k)_{|j}\ \acute{g}_{ia} = \acute{u}^k_{|j}\ \delta^j_a = \acute{u}^k_{|a}; \quad a = 1, 2 \tag{3}$$

However, as $\acute{u}^k_{|a}$ must be understood as the k'th component of the covariant derivative, the concept of horizontal homogeneity would be too restrictive if including $k = 3$. This will become clear later. Thus, horizontal homogeneity is formulated by

$$\acute{u}^k_{|a} = 0; \quad a = 1, 2; \quad k = 1, 2 \tag{4}$$

3 The continuity equation

Stating the continuity equation for incompressible flow as

$$\acute{U}^j_{|j} = 0 \tag{5}$$

Because of horizontal homogeneity,

$$\acute{U}^1_{|1} = 0, \quad \acute{U}^2_{|2} = 0 \tag{6}$$

Thus,

$$\acute{U}^3_{|3} = 0 \tag{7}$$

i.e.

$$\frac{\partial \acute{U}^3}{\partial \acute{x}^3} + \left\{ {}^{\ 3}_{i\ 3} \right\}' \acute{U}^i = 0 \tag{8}$$

Assuming orthogonality,

$$\acute{g}_{ij} = 0 \quad \text{for } i \neq j, \tag{9}$$

it is obtained that

$$\frac{\partial \acute{U}^3}{\partial \acute{x}^3} + \frac{\partial}{\partial \acute{x}^i}(\log \sqrt{\acute{g}_{33}})\ \acute{U}^i = 0 \tag{10}$$

which can be rewritten as

$$\frac{\partial \acute{U}^3}{\partial \acute{x}^3} + \frac{1}{\sqrt{\acute{g}_{33}}} \frac{\partial}{\partial \acute{x}^i}(\sqrt{\acute{g}_{33}})\ \acute{U}^i = 0 \tag{11}$$

Multiplying by $\sqrt{\acute{g}_{33}}$ and collecting terms,

$$\frac{\partial}{\partial \acute{x}^3}(\sqrt{\acute{g}_{33}}\,\acute{U}^3) + \frac{\partial}{\partial \acute{x}^1}(\sqrt{\acute{g}_{33}})\acute{U}^1 + \frac{\partial}{\partial \acute{x}^2}(\sqrt{\acute{g}_{33}})\acute{U}^2 = 0 \tag{12}$$

Assuming also that

$$\frac{\partial}{\partial \acute{x}^1}(\acute{g}_{33}) = 0, \quad \frac{\partial}{\partial \acute{x}^2}(\acute{g}_{33}) = 0 \tag{13}$$

it is obtained that

$$\frac{\partial}{\partial \acute{x}^3}(\sqrt{\acute{g}_{33}}\,\acute{U}^3) = 0 \tag{14}$$

Since $\acute{U}^3 = 0$ at the lower and upper boundaries, the horizontally homogeneous flow has

$$\acute{U}^3 = 0 \quad \text{everywhere.} \tag{15}$$

4 The equation of motion

4.1 The equation of motion in the transformed system

Like in Jørgensen (2003) the equation of motion is stated on the form

$$\frac{\partial}{\partial t}(\rho \acute{U}^j) + (\rho \acute{U}^i \acute{U}^j)_{|i} = -(\acute{g}^{ij}P)_{|i} + \acute{\Sigma}^{ij}_{|i} + \acute{F}^j \tag{16}$$

For an isotropic fluid with constant density, ρ, the viscous tensor in the transformed coordinate system $(\acute{x}^i)$ is restated

$$\acute{\Sigma}^{ij} = \mu(\acute{g}^{jr}\acute{U}^i + \acute{g}^{ir}\acute{U}^j)_{|r} = \mu(\acute{g}^{jr}\acute{U}^i_{|r} + \acute{g}^{ir}\acute{U}^j_{|r}) \tag{17}$$

It is quickly obtained that the convection terms are zero for the horizontal equations. Because of continuity and horizontal homogeneity, for $j = 1, 2$,

$$\begin{aligned} (\acute{U}^i\acute{U}^j)_{|i} &= \acute{U}^i_{|i}\acute{U}^j + \acute{U}^i\acute{U}^j_{|i} \\ &= \acute{U}^1\acute{U}^j_{|1} + \acute{U}^2\acute{U}^j_{|2} + \acute{U}^3\acute{U}^j_{|3} \\ &= 0 \end{aligned} \tag{18}$$

4.2 Diffusion terms

In order to find the horizontal equations of motion, the viscous tensor ă(17) and its covariant derivatives are evaluated. Assume orthogonality ($\acute{g}_{ij} = 0$ for $i \neq j$).

For $j = 1$,

$$\frac{1}{\rho}\acute{\Sigma}^{11} = \nu(\acute{g}^{1n}\acute{U}^1 + \acute{g}^{1n}\acute{U}^1)_{|n} = 2\nu\acute{g}^{11}\acute{U}^1_{|1} = 0 \tag{19}$$

$$\frac{1}{\rho}\acute{\Sigma}^{21} = 0 \tag{20}$$

$$\begin{aligned} \frac{1}{\rho}\acute{\Sigma}^{31} &= \nu(\acute{g}^{33}\acute{U}^1_{|3} + \acute{g}^{11}\acute{U}^3_{|1}) \\ &= \nu\frac{1}{\acute{g}_{33}}\left(\frac{\partial}{\partial \acute{x}^3}(\acute{U}^1) + \left\{ {}^{\;1}_{3\;r} \right\}'\acute{U}^r\right) + \nu\frac{1}{\acute{g}_{11}}\left(\frac{\partial}{\partial \acute{x}^1}(\acute{U}^3) + \left\{ {}^{\;3}_{1\;r} \right\}'\acute{U}^r\right) \\ &= \nu\frac{1}{\acute{g}_{33}}\left(\frac{\partial}{\partial \acute{x}^3}(\acute{U}^1) + \left\{ {}^{\;1}_{3\;1} \right\}'\acute{U}^1\right) + \nu\frac{1}{\acute{g}_{11}}\left\{ {}^{\;3}_{1\;1} \right\}'\acute{U}^1 \end{aligned} \tag{21}$$

Because of orthogonality, for $j = 1, 2$ (without summation over j),

$$\left\{ {j \atop 3\ j} \right\}' = \frac{1}{2\acute{g}_{jj}} \frac{\partial}{\partial \acute{x}^3} (\acute{g}_{jj}) \tag{22}$$

$$\left\{ {3 \atop j\ j} \right\}' = -\frac{1}{2\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3} (\acute{g}_{jj}) \tag{23}$$

Hence,

$$\begin{aligned} \frac{1}{\rho}\acute{\Sigma}^{31} &= \nu \frac{1}{\acute{g}_{33}} \left(\frac{\partial}{\partial \acute{x}^3}(\acute{U}^1) + \frac{1}{2\acute{g}_{11}} \frac{\partial}{\partial \acute{x}^3} (\acute{g}_{11})\, \acute{U}^1 \right) - \nu \frac{1}{\acute{g}_{11}} \frac{1}{2\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3} (\acute{g}_{11})\, \acute{U}^1 \\ &= \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^1) \end{aligned} \tag{24}$$

Similarly,

$$\frac{1}{\rho}\acute{\Sigma}^{12} = 0 \tag{25}$$

$$\frac{1}{\rho}\acute{\Sigma}^{22} = 0 \tag{26}$$

$$\frac{1}{\rho}\acute{\Sigma}^{32} = \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^2) \tag{27}$$

and

$$\frac{1}{\rho}\acute{\Sigma}^{13} = \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^1) \tag{28}$$

$$\frac{1}{\rho}\acute{\Sigma}^{23} = \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^2) \tag{29}$$

$$\frac{1}{\rho}\acute{\Sigma}^{33} = 0 \tag{30}$$

The last equation is a consequence of continuity, i.e. because $\acute{U}^3_{|3} = 0$. Thus, since

$$\sqrt{\acute{g}}\ \frac{1}{\rho}\acute{\Sigma}^{ij}_{|i} = \frac{\partial}{\partial \acute{x}^i} \left(\sqrt{\acute{g}}\ \frac{1}{\rho}\acute{\Sigma}^{ij} \right) + \sqrt{\acute{g}}\ \left\{ {j \atop i\ k} \right\}' \frac{1}{\rho}\acute{\Sigma}^{ik} \tag{31}$$

it is obtained (without summation over j) for $j = 1, 2$ that

$$\begin{aligned} \sqrt{\acute{g}}\ \frac{1}{\rho}\acute{\Sigma}^{ij}_{|i} &= \frac{\partial}{\partial \acute{x}^3} \left(\sqrt{\acute{g}}\ \frac{1}{\rho}\acute{\Sigma}^{3j} \right) + \sqrt{\acute{g}}\ \left(\left\{ {j \atop j\ 3} \right\}' \frac{1}{\rho}\acute{\Sigma}^{j3} + \left\{ {j \atop 3\ j} \right\}' \frac{1}{\rho}\acute{\Sigma}^{3j} \right) \\ &= \frac{\partial}{\partial \acute{x}^3} \left(\sqrt{\acute{g}}\ \frac{1}{\rho}\acute{\Sigma}^{3j} \right) + \sqrt{\acute{g}}\ \frac{1}{\acute{g}_{jj}} \frac{\partial}{\partial \acute{x}^3}(\acute{g}_{jj}) \frac{1}{\rho}\acute{\Sigma}^{3j} \end{aligned} \tag{32}$$

Multiplying by $\acute{g}_{jj}$ (without summation over j) and collecting, for $j = 1, 2$ the diffusion terms are obtained on the form,

$$\sqrt{\acute{g}}\ \acute{g}_{jj} \frac{1}{\rho}\acute{\Sigma}^{ij}_{|i} = \frac{\partial}{\partial \acute{x}^3} \left(\sqrt{\acute{g}}\ \acute{g}_{jj} \frac{1}{\rho}\acute{\Sigma}^{3j} \right) \tag{33}$$

Inserting

$$\frac{1}{\rho}\acute{\Sigma}^{3j} = \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^j) \tag{34}$$

the diffusion terms become

$$\sqrt{\acute{g}}\ \acute{g}_{jj} \frac{1}{\rho}\acute{\Sigma}^{ij}_{|i} = \frac{\partial}{\partial \acute{x}^3} \left(\sqrt{\acute{g}}\ \acute{g}_{jj} \nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^j) \right) \tag{35}$$

4.3 Pressure gradient

Using horizontal homogeneity, the pressure gradient reduces,

$$-(\acute{g}^{ij}P)_{|i} = -(\acute{g}^{3j}P)_{|3} \tag{36}$$

As before, assuming orthogonality,

$$\acute{g}_{ij} = 0 \quad \text{for } i \neq j, \tag{37}$$

it is obtained that

$$-(\acute{g}^{i1}P)_{|i} \;=\; -(\acute{g}^{31}P)_{|3} = 0 \tag{38}$$

$$-(\acute{g}^{i2}P)_{|i} \;=\; -(\acute{g}^{32}P)_{|3} = 0 \tag{39}$$

Thus, the horizontal pressure gradient is zero.

4.4 Coriolis force

The Coriolis force in the physical coordinate system is expressed as

$$F^m = -\rho f_c \frac{1}{\sqrt{g}} \varepsilon^{imn} g_{nr} g_{iq} s^r (U^q - U_g^q) \tag{40}$$

where

$$s^r = \delta^{r3} \tag{41}$$

Thus, in the transformed coordinate system,

$$\frac{1}{\rho}\acute{F}^j \;=\; -f_c\ \varepsilon^{im3}\delta_{iq}(\acute{U}^k - \acute{U}_g^k)\beta_k^q\ \alpha_m^j \tag{42}$$

$$\;=\; f_c\ (-\beta_k^1\alpha_2^j + \beta_k^2\alpha_1^j)(\acute{U}^k - \acute{U}_g^k) \tag{43}$$

In order that the Coriolis force term is horizontally homogeneous, special treatment is required, for instance by horizontal averageing. Defining,

$$B_k^j = \frac{1}{\acute{A}} \int_{\acute{A}} (-\beta_k^1\alpha_2^j + \beta_k^2\alpha_1^j)\ \mathrm{d}\acute{A} \tag{44}$$

the Coriolis force term is modified as,

$$\frac{1}{\rho}\acute{F}^j = f_c B_k^j (\acute{U}^k - \acute{U}_g^k) \tag{45}$$

4.5 Resulting equation of motion

Now, the evaluated terms are inserted into Equation (16) multiplied by $\sqrt{\acute{g}}\ \acute{g}_{jj}$ and divided by ρ and the equation of motion for horizontally homogeneous flow is obtained for $j = 1, 2$ (no summation over j),

$$\sqrt{\acute{g}}\ \acute{g}_{jj} \frac{\partial}{\partial t}(\acute{U}^j)$$

$$= \frac{\partial}{\partial \acute{x}^3}\left(\sqrt{\acute{g}}\ \acute{g}_{jj}\nu \frac{1}{\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^3}(\acute{U}^j)\right) + \sqrt{\acute{g}}\ \acute{g}_{jj} f_c B_k^j (\acute{U}^k - \acute{U}_g^k) \tag{46}$$

where B_k is given by

$$B_k^j = \frac{1}{\acute{A}} \int_{\acute{A}} (-\beta_k^1\alpha_2^j + \beta_k^2\alpha_1^j)\ \mathrm{d}\acute{A} \tag{47}$$

It must be required that

$$\frac{\partial}{\partial \acute{x}^j}(\acute{g}_{33}) = 0 \quad \text{for } j = 1, 2 \tag{48}$$

in addition to the requirement of orthogonality

$$\acute{g}_{ij} = 0 \quad \text{for } i \neq j. \tag{49}$$

4.6 The vertical equation of motion

For $j = 3$, the convection terms do not in general reduce to zero. Instead, since $\acute{U}^3_{|3} = 0$ (because of continuity),

$$\begin{aligned}(\acute{U}^i\acute{U}^j)_{|i} &= \acute{U}^1\acute{U}^3_{|1} + \acute{U}^2\acute{U}^3_{|2} \\ &= \acute{U}^1 \left\{ {}^{\ 3}_{1\ 1} \right\}' \acute{U}^1 + \acute{U}^2 \left\{ {}^{\ 3}_{2\ 2} \right\}' \acute{U}^2 \\ &= -\frac{1}{2\acute{g}_{33}} \left(\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\acute{U}^1\acute{U}^1 + \frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\acute{U}^2\acute{U}^2 \right) \end{aligned} \tag{50}$$

where it has been used that,

$$\frac{\partial}{\partial \acute{x}^1}(\acute{g}_{33}) = 0, \quad \frac{\partial}{\partial \acute{x}^2}(\acute{g}_{33}) = 0, \tag{51}$$

in addition to the requirement of orthogonality ($\acute{g}_{ij} = 0$ for $i \neq j$).

The divergence of the viscous tensor reduces to zero because of horizontal homogeneity, since

$$\begin{aligned}\sqrt{\acute{g}}\, \frac{1}{\rho}\acute{\Sigma}^{i3}_{|i} &= \frac{\partial}{\partial \acute{x}^1}\left(\sqrt{\acute{g}}\, \frac{1}{\rho}\acute{\Sigma}^{13}\right) + \frac{\partial}{\partial \acute{x}^1}\left(\sqrt{\acute{g}}\, \frac{1}{\rho}\acute{\Sigma}^{13}\right) \\ &\quad + \sqrt{\acute{g}} \left(\left\{ {}^{\ 3}_{1\ 3} \right\}' \frac{1}{\rho}\acute{\Sigma}^{13} + \left\{ {}^{\ 3}_{3\ 1} \right\}' \frac{1}{\rho}\acute{\Sigma}^{31} + \left\{ {}^{\ 3}_{2\ 3} \right\}' \frac{1}{\rho}\acute{\Sigma}^{23} + \left\{ {}^{\ 3}_{3\ 2} \right\}' \frac{1}{\rho}\acute{\Sigma}^{32} \right) \\ &= 0 \end{aligned} \tag{52}$$

where for $j = 1, 2$

$$\left\{ {}^{\ 3}_{j\ 3} \right\}' = \frac{1}{2\acute{g}_{33}} \frac{\partial}{\partial \acute{x}^j}(\acute{g}_{33}) = 0 \tag{53}$$

Horizontal homogeneity also causes the pressure terms to reduce,

$$-(\acute{g}^{13}P)_{|1} = 0 \tag{54}$$

$$-(\acute{g}^{23}P)_{|2} = 0 \tag{55}$$

Hence,

$$\frac{\partial P}{\partial \acute{x}^1} = 0, \quad \frac{\partial P}{\partial \acute{x}^2} = 0 \tag{56}$$

Inserting in the equation of motion (16) for $j = 3$ and rewriting,

$$\frac{\partial P}{\partial \acute{x}^3} = \acute{g}^{33}\acute{F}^3 + \frac{\rho}{2}\left(\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\acute{U}^1\acute{U}^1 + \frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\acute{U}^2\acute{U}^2 \right) \tag{57}$$

where the Coriolis force term is given by (45). Equation (57) can be utilized to find the pressure once $\acute{U}^1$ and $\acute{U}^2$ have been determined. This is most conveniently carried out by solving the vertical derivative of the equation for the pressure,

$$\frac{\partial^2 P}{\partial \acute{x}^3 \partial \acute{x}^3} = \frac{\partial}{\partial \acute{x}^3}\left(\acute{g}^{33}\acute{F}^3\right) + \frac{\rho}{2}\frac{\partial}{\partial \acute{x}^3}\left(\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\acute{U}^1\acute{U}^1 + \frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\acute{U}^2\acute{U}^2 \right) \tag{58}$$

5 The turbulence closure

As a turbulence closure model, consider the k-ϵ model with limited mixing length by Apsley and Castro (1997). Denoting the turbulent kinetic energy by E, the model is formulated in the transformed coordinate system,

$$\frac{\partial}{\partial t}(E) + (\acute{U}^i E)_{|i} = \left(\frac{\nu}{\sigma_E}(\acute{g}^{ij}E)_{|j} \right)_{|i} + \Pi - \epsilon \tag{59}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}(\epsilon) + (\acute{U}^i\epsilon)_{|i} &= \left(\frac{\nu}{\sigma_\epsilon}(\acute{g}^{ij}\epsilon)_{|j}\right)_{|i} + \left(c_{\epsilon 1} + (c_{\epsilon 2} - c_{\epsilon 1})\frac{l}{l_{max}}\right)\frac{\epsilon}{E}\Pi \\
&\quad - c_{\epsilon 2}\frac{\epsilon^2}{E} \qquad (60)\\
\Pi &= \nu(\acute{g}^{kn}\acute{U}^m + \acute{g}^{mn}\acute{U}^k)_{|n}\acute{U}^q_{|m}\acute{g}_{kq} \qquad (61)
\end{aligned}
$$

Turbulent viscosity: $\nu = c_\mu \frac{E^2}{\epsilon}$

Mixing length: $l = c_\mu^{\frac{3}{4}}\frac{E^{\frac{3}{2}}}{\epsilon}$

Maximum mixing length: $l_{max} = \frac{h}{3}$ (neutral conditions)

Height of boundary layer: h

The treatment of the E-equation and the ϵ-equation is similar. Because of continuity and horizontal homogeneity, the convection terms disappear,

$$
(\acute{U}^iE)_{|i} = \acute{U}^i_{|i}E + \acute{U}^1E_{|1} + \acute{U}^2E_{|2} + \acute{U}^3E_{|3} = 0 \qquad (62)
$$

Because of horizontal homogeneity and orthogonality, $\acute{g}^{ij} = 0$ for $i \neq j$, the diffusion terms become,

$$
\begin{aligned}
\left(\frac{\nu}{\sigma_E}(\acute{g}^{ij}E)_{|j}\right)_{|i} &= \left(\frac{\nu}{\sigma_E}\acute{g}^{ij}\frac{\partial}{\partial \acute{x}^j}(E)\right)_{|i} \\
&= \frac{1}{\sqrt{\acute{g}}}\frac{\partial}{\partial \acute{x}^i}\left(\sqrt{\acute{g}}\frac{\nu}{\sigma_E}\acute{g}^{ij}\frac{\partial}{\partial \acute{x}^j}(E)\right) \\
&= \frac{1}{\sqrt{\acute{g}}}\frac{\partial}{\partial \acute{x}^3}\left(\sqrt{\acute{g}}\frac{\nu}{\sigma_E}\acute{g}^{33}\frac{\partial}{\partial \acute{x}^3}(E)\right) \qquad (63)
\end{aligned}
$$

Thus, the transport equations for the turbulence closure become,

$$
\begin{aligned}
\sqrt{\acute{g}}\ \frac{\partial}{\partial t}(E) &= \frac{\partial}{\partial \acute{x}^3}\left(\sqrt{\acute{g}}\ \frac{1}{\acute{g}_{33}}\frac{\nu}{\sigma_E}\frac{\partial}{\partial \acute{x}^3}(E)\right) + \sqrt{\acute{g}}\ \Pi - \sqrt{\acute{g}}\ \epsilon \qquad (64)\\
\sqrt{\acute{g}}\ \frac{\partial}{\partial t}(\epsilon) &= \frac{\partial}{\partial \acute{x}^3}\left(\sqrt{\acute{g}}\ \frac{1}{\acute{g}_{33}}\frac{\nu}{\sigma_\epsilon}\frac{\partial}{\partial \acute{x}^3}(\epsilon)\right) \\
&\quad + \sqrt{\acute{g}}\left(c_{\epsilon 1} + (c_{\epsilon 2} - c_{\epsilon 1})\frac{l}{l_{max}}\right)\frac{\epsilon}{E}\Pi - \sqrt{\acute{g}}\ c_{\epsilon 2}\frac{\epsilon^2}{E} \qquad (65)
\end{aligned}
$$

The production evaluates to,

$$
\begin{aligned}
\Pi &= \nu(\acute{g}^{kn}U^m)_{|n}\acute{U}^q_{|m}\acute{g}_{kq} + \nu(\acute{g}^{mn}U^k)_{|n}\acute{U}^q_{|m}\acute{g}_{kq} \\
&= \Big[\nu(\acute{g}^{11}U^3)_{|1}\acute{U}^1_{|3}\acute{g}_{11} + \nu(\acute{g}^{22}U^3)_{|2}\acute{U}^2_{|3}\acute{g}_{22} \\
&\quad + \nu(\acute{g}^{33}U^1)_{|3}\acute{U}^3_{|1}\acute{g}_{33} + \nu(\acute{g}^{33}U^2)_{|3}\acute{U}^3_{|2}\acute{g}_{33}\Big] \\
&+ \Big[\nu(\acute{g}^{11}U^3)_{|1}\acute{U}^3_{|1}\acute{g}_{33} + \nu(\acute{g}^{22}U^3)_{|2}\acute{U}^3_{|2}\acute{g}_{33} \\
&\quad + \nu(\acute{g}^{33}U^1)_{|3}\acute{U}^1_{|3}\acute{g}_{11} + \nu(\acute{g}^{33}U^2)_{|3}\acute{U}^2_{|3}\acute{g}_{22}\Big] \\
&= \Bigg[-\frac{\nu}{\acute{g}_{33}}\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\acute{U}^1\frac{1}{\sqrt{\acute{g}_{11}}}\frac{\partial}{\partial \acute{x}^3}(\sqrt{\acute{g}_{11}}\acute{U}^1) \\
&\quad - \frac{\nu}{\acute{g}_{33}}\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\acute{U}^2\frac{1}{\sqrt{\acute{g}_{22}}}\frac{\partial}{\partial \acute{x}^3}(\sqrt{\acute{g}_{22}}\acute{U}^2)\Bigg] \\
&+ \Bigg[\frac{\nu}{4}\frac{1}{\acute{g}_{33}\ \acute{g}_{11}}\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{11})\acute{U}^1\acute{U}^1 + \frac{\nu}{4}\frac{1}{\acute{g}_{33}\ \acute{g}_{22}}\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\frac{\partial}{\partial \acute{x}^3}(\acute{g}_{22})\acute{U}^2\acute{U}^2
\end{aligned}
$$

$$+\nu\frac{1}{\acute{g}_{33}}\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g_{11}}}\acute{U}^1)\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g_{11}}}\acute{U}^1)+\nu\frac{1}{\acute{g}_{33}}\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g_{22}}}\acute{U}^2)\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g_{22}}}\acute{U}^2)\Bigg] \tag{66}$$

Reorganizing the terms, the production reduces to

$$\begin{aligned}\Pi &= \frac{\nu}{\acute{g}_{33}}\left(-\frac{1}{2\sqrt{\acute{g}_{11}}}\frac{\partial}{\partial\acute{x}^3}(\acute{g}_{11})\acute{U}^1+\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g}_{11}}\acute{U}^1)\right)^2 \\ &\quad +\frac{\nu}{\acute{g}_{33}}\left(-\frac{1}{2\sqrt{\acute{g}_{22}}}\frac{\partial}{\partial\acute{x}^3}(\acute{g}_{22})\acute{U}^2+\frac{\partial}{\partial\acute{x}^3}(\sqrt{\acute{g}_{22}}\acute{U}^2)\right)^2 \\ &= \nu\frac{\acute{g}_{11}}{\acute{g}_{33}}\frac{\partial}{\partial\acute{x}^3}(\acute{U}^1)\frac{\partial}{\partial\acute{x}^3}(\acute{U}^1)+\nu\frac{\acute{g}_{22}}{\acute{g}_{33}}\frac{\partial}{\partial\acute{x}^3}(\acute{U}^2)\frac{\partial}{\partial\acute{x}^3}(\acute{U}^2)\end{aligned} \tag{67}$$

6 Boundary conditions for a rough surface using the wall law

6.1 Surface distance

The wall law for a rough surface depends on the distance to the surface in the physical coordinate system. Define the position on the surface by $\mathbf{h}$. Choosing the transformed coordinate system $(\acute{x})$ so that $\acute{x}^3 = 0$ coincides with the lower surface, the surface distance is given by

$$\begin{aligned}\mathbf{n}\cdot\mathbf{r} &= \mathbf{n}\cdot(\mathbf{x}-\mathbf{h}) \\ &= n_k\mathbf{e}^k\cdot\mathbf{e}_m(x^m-h^m) \\ &= \acute{n}_i\acute{\mathbf{e}}^i\cdot\acute{\mathbf{e}}_j\acute{x}^j \\ &= \acute{n}_i\acute{g}_{qj}\acute{g}^{iq}\acute{x}^j\end{aligned} \tag{68}$$

Since the normal vector satisfies

$$\acute{n}_1 = 0,\ \acute{n}_2 = 0, \acute{n}_3 = \frac{1}{\sqrt{\acute{g}^{33}}} \tag{69}$$

the following surface distance is obtained

$$\mathbf{n}\cdot\mathbf{r} = \frac{1}{\sqrt{\acute{g}^{33}}}\acute{g}_{qj}\acute{g}^{3q}\acute{x}^j \tag{70}$$

For the implementation of the wall law, the lower boundary is placed at a surface distance equivalent to the roughness length, z_0. However, as this creates a discontinuous boundary at the roughness changes, the actual surface is depressed so that the surface distance is increased by z_0. This has very little effect on the geometry of the surface and is assumed to have negligible influence on the flow. The surface displacement is achieved internally in the flow model by adding z_0 to the surface distance,

$$\mathbf{n}\cdot\mathbf{r}+z_0 = \frac{1}{\sqrt{\acute{g}^{33}}}\acute{g}_{qj}\acute{g}^{3q}\acute{x}^j+z_0 \tag{71}$$

For the horizontally homogeneous flow, assuming orthogonality, i.e. $\acute{g}^{ij} = 0$ for $i \neq j$, the following is obtained,

$$\mathbf{n}\cdot\mathbf{r}+z_0 = \frac{1}{\sqrt{\acute{g}^{33}}}\acute{g}_{33}\acute{g}^{33}\acute{x}^3+z_0 = \sqrt{\acute{g}_{33}}\ \acute{x}^3+z_0 \tag{72}$$

6.2 Friction velocity

It is assumed that the friction velocity U_τ is constant in the region where the wall law is enforced, so that the absolute velocity U is described by

$$\frac{U}{U_\tau} = \frac{1}{\kappa} \log\left(\frac{\mathbf{n}\cdot\mathbf{r} + z_0}{z_0}\right) \tag{73}$$

i.e.

$$U_\tau = \frac{U\kappa}{\log\left(\dfrac{\mathbf{n}\cdot\mathbf{r} + z_0}{z_0}\right)} \tag{74}$$

For the horizontally homogeneous solution it is necessary to use the average $\overline{z_0}$ of the roughness over the surface. Thus,

$$U_\tau = \frac{U\kappa}{\log\left(\dfrac{\sqrt{\acute{g}_{33}}\;\acute{x}^3 + \overline{z_0}}{\overline{z_0}}\right)} \tag{75}$$

In a finite difference formulation, this expression can be applied in the center of the boundary cell.

6.3 Absolute velocity

The absolute velocity is given by

$$U = \sqrt{U^k U^k} = \sqrt{\beta^k_q \acute{U}^q \beta^k_s \acute{U}^s} = \sqrt{\acute{g}_{qs}\acute{U}^q\acute{U}^s} \tag{76}$$

For the horizontally homogeneous flow, the absolute velocity becomes

$$U = \sqrt{\acute{g}_{11}\acute{U}^1\acute{U}^1 + \acute{g}_{22}\acute{U}^2\acute{U}^2} \tag{77}$$

Note that on the lower boundary, $U = 0$, i.e. $U(z_0) = 0$.

6.4 Surface stress

Now, consider the stress tensor,

$$\Sigma^{ij} \tag{78}$$

The corresponding force, F^j, on the surface with the normal vector $\mathbf{n}$ is then

$$F^j = \Sigma^{ij} g_{ik} n^k = \Sigma^{ij} n_i \tag{79}$$

which is the inner product between the stress tensor and the normal vector. In the transformed coordinate system,

$$\acute{F}^q = \acute{\Sigma}^{sq}\acute{g}_{sm}\acute{n}^m = \acute{\Sigma}^{sq}\acute{n}_s \tag{80}$$

Thus,

$$F^j = \beta^j_q \acute{\Sigma}^{sq}\acute{n}^s \tag{81}$$

Inserting the transformed coordinates of the normal vector, the surface shear stress is obtained,

$$\tau^j_w = \beta^j_q \acute{\Sigma}^{3q} \frac{1}{\sqrt{\acute{g}^{33}}} \tag{82}$$

With the orthogonality assumption for the horizontally homogeneos flow this can be written,

$$\tau^j_w = \beta^j_q \acute{\Sigma}^{3q} \sqrt{\acute{g}_{33}} \tag{83}$$

Applying the definition of friction velocity,

$$\tau_w = \rho U_\tau U_\tau \tag{84}$$

assume that the surface stress has the same direction as the velocity near the surface,

$$\tau_w^j = \rho U_\tau \frac{U_\tau}{U} U^j = \left(\rho U_\tau \frac{U_\tau}{U}\right) \beta_q^j \acute{U}^q \tag{85}$$

Hence, the surface stress is given by

$$\frac{1}{\rho}\acute{\Sigma}^{3q} = \sqrt{\acute{g}^{33}} \left(U_\tau \frac{U_\tau}{U}\right) \acute{U}^q \tag{86}$$

which for the horizontally homogeneous flow can also be formulated

$$\frac{1}{\rho}\acute{\Sigma}^{3q} = \frac{1}{\sqrt{\acute{g}_{33}}} \left(U_\tau \frac{U_\tau}{U}\right) \acute{U}^q \tag{87}$$

This expression for the surface stress must be applied at the boundary for the momentum equations. Implicit treatment of the velocity appearing in the term is possible.

6.5 Turbulent kinetic energy

The surface boundary condition of the turbulent kinetic energy is,

$$\frac{\partial}{\partial \acute{x}^3}(E) = 0 \tag{88}$$

Let a finite difference representation be considered. When using a wall law for a rough surface, the above condition is not applied directly. Rather, it is substituted in place of the relevant term at the boundary in the discretized equation for E for the boundary cell. Also, the production of turbulent kinetic energy and the dissipation must be evaluated for the boundary cell in order to solve the equation for E for the boundary cell.

Refer to the surface distance as

$$\xi = \mathbf{n} \cdot \mathbf{r} + z_0 \tag{89}$$

For the horizontally homogeneous flow,

$$\xi = \sqrt{\acute{g}_{33}}\ \acute{x}^3 + z_0 \tag{90}$$

Introduce the vertical index k, so that the center of the boundary cell is placed at k and the lower boundary at $k - \frac{1}{2}$. Thus,

$$\xi(k - \tfrac{1}{2}) = z_0 \tag{91}$$

Details regarding the choice of $\xi(k)$ can be found elsewhere. See for instance Sørensen (1995). However, past experience show that a minimum of approximately

$$\xi^+(k) = \frac{\xi(k) U_\tau}{\nu_{\mathrm{mol}}} \geq 100 \tag{92}$$

is needed although formally only a minimum value of 11.6 is required. Here, ν_{mol} denotes the molecular kinematic viscosity.

6.6 Production of turbulent kinetic energy

Define,

$$\phi = \frac{U(\xi(k+\frac{1}{2}))}{U(\xi(k))} = \frac{\log\left(\dfrac{\xi(k+\frac{1}{2})}{z_0}\right)}{\log\left(\dfrac{\xi(k)}{z_0}\right)} \tag{93}$$

which for the horizontally homogeneous flow becomes,

$$\phi = \frac{\log\left(\dfrac{\acute{x}^3(k+\frac{1}{2}) + \overline{z_0}}{\overline{z_0}}\right)}{\log\left(\dfrac{\acute{x}^3(k) + \overline{z_0}}{\overline{z_0}}\right)} \tag{94}$$

Now, the production is

$$\Pi = \frac{1}{\rho}\tau_w \frac{\partial U}{\partial \xi} \tag{95}$$

where τ_w is considered constant. The finite difference form is analogous to the finite volume form, where the average of Π is calculated for the boundary cell and applied in the center of the cell, i.e.

$$\begin{aligned}\overline{\Pi} &= \frac{1}{\rho}\tau_w \frac{1}{\xi(k+\frac{1}{2}) - \xi(k-\frac{1}{2})} \int_{\xi(k-\frac{1}{2})}^{\xi(k+\frac{1}{2})} \frac{\partial U}{\partial \xi}\, d\xi \\ &= \frac{1}{\rho}\tau_w \frac{1}{\xi(k+\frac{1}{2}) - z_0} \left(U(\xi(k+\tfrac{1}{2})) - U(z_0)\right) \end{aligned} \tag{96}$$

Utilizing ϕ and inserting $\tau_w = \rho U_\tau U_\tau$ the following is obtained,

$$\overline{\Pi} = \frac{\phi U_\tau U_\tau}{\xi(k+\frac{1}{2}) - z_0} U(\xi(k)) \tag{97}$$

For the horizontally homogeneous flow, this can also be written as

$$\overline{\Pi} = \frac{\phi U_\tau U_\tau}{[\sqrt{\acute{g}_{33}}\acute{x}^3](k+\frac{1}{2})} U([\sqrt{\acute{g}_{33}}\acute{x}^3](k) + \overline{z_0}) \tag{98}$$

6.7 Dissipation of turbulent kinetic energy

Using the relationship

$$U_\tau = c_\mu^{\frac{1}{4}} E^{\frac{1}{2}} \tag{99}$$

together with the wall law under the assumption of a balance between the dissipation and production near the surface, the dissipation can be expressed as

$$\epsilon = c_\mu^{\frac{3}{4}} \frac{E^{\frac{3}{2}}}{\kappa \xi} \tag{100}$$

For the boundary cell, the equation for the dissipation is replaced entirely by the average dissipation for the boundary cell which is then applied in the cell center, i.e.

$$\begin{aligned}\overline{\epsilon} &= \frac{1}{\xi(k+\frac{1}{2}) - \xi(k-\frac{1}{2})} \int_{\xi(k-\frac{1}{2})}^{\xi(k+\frac{1}{2})} \frac{c_\mu^{\frac{3}{4}} E^{\frac{3}{2}}}{\kappa \xi}\, d\xi \\ &= \frac{1}{\xi(k+\frac{1}{2}) - z_0} \frac{c_\mu^{\frac{3}{4}} E^{\frac{3}{2}}}{\kappa} \log\left(\frac{\xi(k+\frac{1}{2})}{z_0}\right) \\ &= \frac{\phi}{\xi(k+\frac{1}{2}) - z_0} \frac{c_\mu^{\frac{3}{4}} E^{\frac{3}{2}}}{\kappa} \log\left(\frac{\xi(k)}{z_0}\right) \end{aligned} \tag{101}$$

Utilizing $U_\tau = c_\mu^{\frac{1}{4}} E^{\frac{1}{2}}$ and the wall law,

$$U(\xi(k)) = \frac{U_\tau}{\kappa} \log\left(\frac{\xi(k)}{z_0}\right) \tag{102}$$

it is obtained that

$$\overline{\epsilon} = \frac{\phi}{\xi(k+\frac{1}{2}) - z_0}\, c_\mu^{\frac{1}{2}}\, E\, U(\xi(k)) \tag{103}$$

This is the expression that is applied for the dissipation in the boundary cell. Note that E can be referenced implicitely in the above expression for the average dissipation of the boundary cell when used in the formulation of the surface boundary condition for the turbulent kinetic energy. For the horizontally homogeneous flow, it becomes

$$\overline{\epsilon} = \frac{\phi}{[\sqrt{\acute{g}_{33}}\acute{x}^3](k+\frac{1}{2})} \; c_\mu^{\frac{1}{2}} \; E \, U([\sqrt{\acute{g}_{33}}\acute{x}^3](k) + \overline{z_0}) \qquad (104)$$

6.8 Pressure

Boundary conditions for the pressure are not needed to solve for the horizontally homogeneous flow. However, it is needed if a pressure solution is desired. From the vertical equation of motion the following is obtained for the lower boundary,

$$\frac{\partial P}{\partial \acute{x}^3} = \acute{g}^{33}\acute{F}^3 = -\frac{1}{\acute{g}_{33}}\rho f_c B_j^3 \acute{U}_g^j \qquad (105)$$

where B_j^3 is defined in Equation (44).

7 Upper boundary conditions

7.1 Velocity

At the upper boundary, the Coriolis force is zero. Hence, the velocity must satisfy

$$\acute{U}^j - \acute{U}_g^j = 0 \quad \text{for } j = 1,2 \qquad (106)$$

where $\acute{U}_g^j$ is the transformed geostrophic wind.

7.2 Turbulence properties

At the upper boundary, let

$$\frac{\partial E}{\partial \acute{x}^3} = 0 \qquad (107)$$

$$\frac{\partial \epsilon}{\partial \acute{x}^3} = 0 \qquad (108)$$

which constitutes the boundary conditions for the turbulence properties.

7.3 Pressure

At the upper boundary, let

$$\frac{\partial P}{\partial \acute{x}^3} = 0 \qquad (109)$$

which constitutes the boundary condition for the pressure needed if a pressure solution is desired.

8 Conclusion

The model equations have been derived for an incompressible flow which is horizontally homogeneous. The model solution can be used as a computationally inexpensive starting point of a more complete solution for neutral atmospheric flow over complex terrain. Unlike the previous models of Ayotte and Taylor (1995) and Beljaars et al. (1987), the present work uses general orthogonal coordinates. Strong conservation form of the model equations allows a robust and consistent numerical procedure to be developed. An invariant tensor form of the model equations is utilized expressing the flow variables in the transformed coordinate system in which they are horizontally homogeneous.

Although the flow is actually solved in orthogonal general coordinates, it can be corrected into an initial solution in general coordinates via an appropriate Fractional Step method such as the method described in Jørgensen (in preparation). This is provided that the general coordinate system is nearly orthogonal as when using a hyperbolic grid generation method. A pertubation solver yielding a more complete solution may treat small deviations from orthogonality as pertubations. Another possibility is to apply the horizontal homogeneous flow model for a general orthogonal grid, provided that such a grid can be generated.

The horizontal equations of motion can be solved independently of the vertical equation of motion, which then can be utilized to determine the pressure. The model employs the $k-\epsilon$ model with limited mixing length by Apsley and Castro (1997). This turbulence closure reflects the fact that the atmosphere is only neutral up to a certain height. The limitation of the mixing length is obtained by a modification of the transport equation for dissipation. Boundary conditions for a rough surface using the wall law and upper boundary conditions for the atmosphere are expressed.

The horizontally homogeneous flow model is a part of a pertubation solver under development which is hoped to be more accurate than the current flow model of the industry standard program WAsP by Troen and Petersen (1989) while achieving a high speed of execution.

Acknowledgements

This work has been supported by UVE under the contract 5117/00-0038. The author gratefully acknowledges this opportunity.
Thanks to Niels N. Sørensen at Risø National Laboratory for many helpful discussions about the implementation of EllipSys3D - most notably about general coordinates, turbulence modelling, discretization and hyperbolic grid generation.
Thanks to Lisbeth Myllerup at Risø National Laboratory for pointing out the existence of the turbulence closure model by Apsley and Castro.
Thanks to Ole Rathmann at Risø National Laboratory for pointing for pointing out the fact, that a flow can be horizontally homogeneous in a transformed coordinate system instead of in the physical system.
Thanks to Keith W. Ayotte from CSIRO, Australia, for meeting with me for breakfast very early in the morning in Paris to discuss his work on flow models for complex terrain. Also thanks for the other discussions we have had.
Thanks to Jens N. Sørensen at the Technical University of Denmark for friendly advice about numerical fluid mechanics.
Thanks to my family and in particular my wife, Anita, for taking care of our children and other help during my preparation of this document.

References

Apsley, D. D. and Castro, I. P.: 1997, A limited-length-scale $k-\epsilon$ model for the neutral and stably-stratified atmospheric boundary layer, *Boundary-Layer*

Meteorol. **83**, 75–98.

Ayotte, K. W. and Taylor, P. A.: 1995, A Mixed Spectral Finite-Difference Model of Neutral Planetary Boundary-Layer Flow over Topography, *J. Atmos. Sci.* **52**(20), 3523–3537.

Beljaars, A. C. M., Walmsley, J. L. and Taylor, P. A.: 1987, A mixed spectral finite-difference model for neutrally stratified boundary-layer flow over roughness changes and topography, *Boundary-Layer Meteorol.* **38**, 273–303.

Jørgensen, B. H.: 2003, Tensor formulation of the model equations on strong conservation form for an incompressible flow in general coordinates, *Technical Report Risø-R-1445(EN)*, Risø National Laboratory, Denmark.

Jørgensen, B. H.: in preparation, A consistent formulation of the fractional step method using Rhie-Chow interpolation.

Michelsen, J. A.: 1989, Solution of the Navier-Stokes equations on general non-orthogonal meshes, *Technical Report AFM 89-08*, Technical University of Denmark.

Michelsen, J. A.: 1992, Basis3D - a platform for development of multiblock PDE solvers, *Technical Report AFM 92-05*, Technical University of Denmark.

Michelsen, J. A.: 1994, Block structured multigrid solution of 2D and 3D elliptic PDE's, *Technical Report AFM 94-06*, Technical University of Denmark.

Sørensen, N. N.: 1995, *General Purpose Flow Solver Applied to Flow over Hills*, PhD thesis, Risø National Laboratory, Denmark.

Troen, I. and Petersen, E. L.: 1989, *European Wind Atlas*, Risø National Laboratory, Denmark.

Vivand, H.: 1974, Formes conservatives des équations de la dynamique des gaz, *Rech. Aérosp.* **1**, 65–66.

Title and author(s)

The horizontally homogeneous model equations of incompressible atmospheric flow in general orthogonal coordinates

Bo Hoffmann Jørgensen

ISBN	ISSN
87–550–3295–8 (Internet)	0106–2840

Dept. or group	Date
Wind Energy	December 22, 2003

Groups own reg. number(s)	Project/contract No.
1105058–1	UVE 5117/00-0038

Pages	Tables	Illustrations	References
18	0	0	11

Abstract (Max. 2000 char.)

The goal of this brief report is to express the model equations for an incompressible flow which is horizontally homogeneous. It is intended as a computationally inexpensive starting point of a more complete solution for neutral atmospheric flow over complex terrain. This idea was set forth by Ayotte and Taylor (1995) and in the work of Beljaars et al. (1987). Unlike the previous models, the present work uses general orthogonal coordinates. Strong conservation form of the model equations is employed to allow a robust and consistent numerical procedure. An invariant tensor form of the model equations is utilized expressing the flow variables in a transformed coordinate system in which they are horizontally homogeneous. The model utilizes the $k-\epsilon$ model with limited mixing length by Apsley and Castro (1997). This turbulence closure reflects the fact that the atmosphere is only neutral up to a certain height. The horizontally homogeneous flow model is a part of a pertubation solver under development which is hoped to be more accurate than the current standard program WAsP by Troen and Petersen (1989) while achieving a high speed of execution.

Descriptors

BOUNDARY LAYERS; COMPLEX TERRAIN; COORDINATES; FLOW MODELS; FLUID MECHANICS; INCOMPRESSIBLE FLOW; NAVIER-STOKES EQUATIONS; NUMERICAL SOLUTION; TRANSFORMATIONS; WIND

Available on request from:
Information Service Department, Risø National Laboratory
(Afdelingen for Informationsservice, Forskningscenter Risø)
P.O. Box 49, DK–4000 Roskilde, Denmark
Phone (+45) 46 77 46 77, ext. 4004/4005 · Fax (+45) 46 77 40 13
E-mail: risoe@risoe.dk