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Counting Combinatorial Classes of Complex Polynomial Vector Fields

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Summary

A vector field ξ_P in the space Ξ_d of degree d monic, centered complex polynomial vector fields has a combinatorial structure which can be fully described by an equivalence relation on the $2d-2$ separatrices. This equivalence relation can be equivalently represented in a combinatorial disk model, by labelling the points $\exp(\frac{2\pi i\ell}{2d-2})$, $\ell = 0, ..., 2d-3$ on $\mathbb{D}$ by s_ℓ and joining the points in the same equivalence class by geodesics in $\mathbb{D}$ with respect to the Poincaré metric. Here, the combinatorics can be fully described by non-crossing invoutions between even and odd separatrices s_ℓ corresponding to homoclinic separatrices and between the even and odd ends e_ℓ, the principal points defined by the $2d-2$ accesses to infinity between the separatrices $s_{\ell-1}$ and s_ℓ, which correspond to $\alpha\omega$-zones. These disk models can be completely described by bracketing problems: combinatorial problems involving pairings of parentheses placed in a string of elements ($e_0e_1...e_{2d-3}$ and $s_0s_1...s_{2d-3}$ in our case). This specific problem is similar to the generalized bracketing problem of Schröder [C], and a similar method is used to calculate a recursion equation for the number of combinatorial classes $c_{2(d-1)}$ for vector fields in Ξ_d. Furthermore, an implicit expression for the generating function for $c_{2(d-1)}$ is computed, and asymptotic growth questions are considered.

Structurally Stable Vector Fields

For structurally stable vector fields, this disk model is dual to a non-crossing involution on the $2d-2$ ends e_ℓ. An involution on the ends can be seen as one way that $2d-2$ people can shake hands without any handshake crossing another. You might recognize then that the number of combinatorial classes (with the labelling of the separatrices) for the structurally stable vector fields is just the Catalan number $\mathcal{C}(d-1)$ [DES]. Note that this is equivalent to the number of ways to make "valid" pairings of $d-1$ pairs of parentheses [D]. An example is given below to demonstrate.

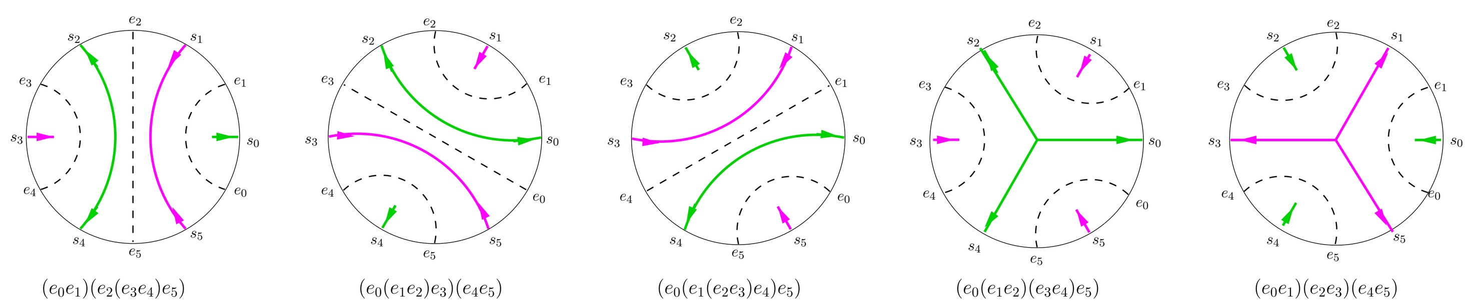

Figure 1: Disk models for the structurally stable vector fields of degree $d=4$. The involution on the ends is marked by the dashed curves. The representation of the involution as a bracketing problem is displayed below each figure.

Valid Bracketings

The bracketing must satisfy certain rules so that they are in accordance with what can happen for a vector field. Pairs of parentheses placed in a string of elements is called a valid bracketing if:

1. *there are an equal number of right and left parentheses*
2. *the number of left parentheses must be greater than or equal to the number of right, reading from left to right* **Example: ())()(not valid**
3. *there must be at least one element between successive left (resp. right) parentheses* **Example:** $((e_0e_1)e_2e_3)$ **not valid**
4. *There must be an even number of elements in each pair of parentheses* **Example:** $(e_0e_1e_2)e_3$ **not valid**

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The Complete Solution

If we allow for non-structurally stable vector fields, then an involution on the ends is not enough to describe the combinatorial structure. We also need to include pairings of separatrices, corresponding to homoclinic separtrices for the vector field. This is represented by a bracketing problem where we distinguish between pairings of separatrices and pairings of ends. This will be represented by placing round brackets () and square brackets [] in the string of the $2(d-1)$ elements $s_0s_1s_2...s_{2s-1}$ in the valid way. An example is given below to demonstrate.

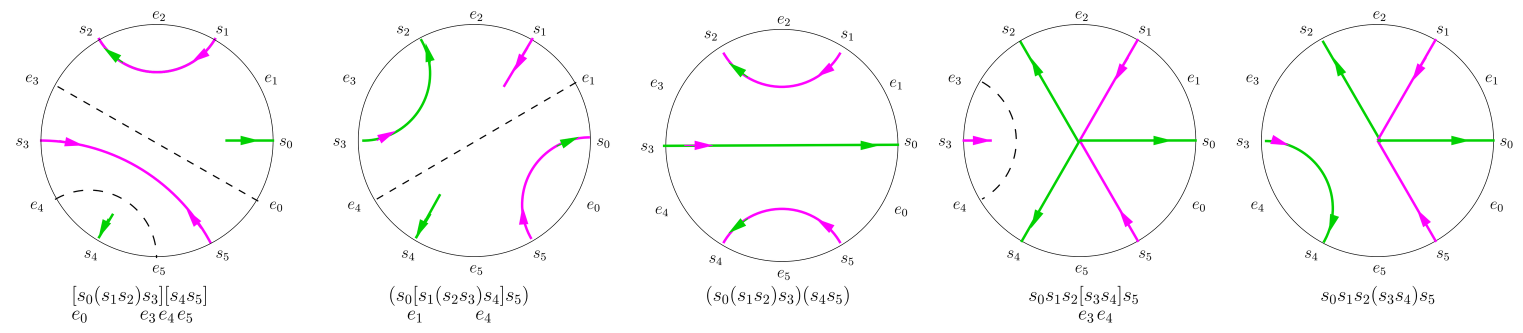


Figure 2: Disk models for some non-structurally stable vector fields of degree $d=4$. The involution on the ends is marked by the dashed curves. The curves representing the separatrices, as well as the dashed curves joining the ends may not cross. The representation of the involution as a bracketing problem is displayed below each figure. A round paring $(s_{\ell_1}...s_{\ell_2})$ represents a homoclinic separatrix $s_{\ell_1} = s_{\ell_2}$, and a square pairing $[s_{\ell_1}...s_{\ell_2}]$ represents a pairing of the ends e_{ℓ_1} and e_{ℓ_2}.

The problem reduces to counting the number of ways c_{2s} one can place pairs of parentheses (some round, some square) in the valid way in a string of length $2s$.
We consider first the situation when no distinction is made between square and round brackets. There are l terms defined by the outermost brackets: l_1 terms consist of one element, and the other terms must consist of an even number of elements of which there are l_i each, $i = 2, 4, ..., 2s$. **Example:** $\underbrace{(s_0(s_1s_2)s_3)}\ \underbrace{s_4}\ \underbrace{s_5}$ **has 3 terms: one of length 4 and 2 of length 1.** *Note that l_1 is always even and the l_i must furthermore satisfy*

$$\begin{aligned} l_1 + l_2 + l_4 + ... + l_{2s} &= l \\ l_1 + 2l_2 + 4l_4 + ... + 2sl_{2s} &= 2s. \end{aligned} \tag{1}$$

There are $l!/(l_1!l_2!l_4!...l_{2s}!)$ ways to arrange these l terms. Within each pair of parentheses having $i \geq 4$ elements, there might be more pairs of parentheses nested inside these. Since the 2 outermost elements contained in this pair of pars is already paired off, there are $i-2$ elements that might contain parentheses. Hence, each term of length i contributes c_{i-2}.
Now if we distinguish between the square and round brackets, and there are $k_{2j} = 0, ..., l_{2j}$ square parentheses in the outermost bracketing of length $2j$, $j = 1, ..., s$, then the number of ways to arrange the terms becomes

$$\frac{l!}{(l_1!l_2!l_4!...l_{2s}!)} \prod_{j=1}^{s} \binom{l_{2j}}{k_{2j}}. \tag{2}$$

Putting these elements together, one can deduce that the number of combinatorial configurations for a sequence having $2s$ elements is

$$c_{2s} = \sum_{l=1}^{2s} \sum_{(l_1,l_2,l_4,...,l_{2s})} \sum_{k_2=0}^{l_2} \sum_{k_4=0}^{l_4} \cdots \sum_{k_{2s}=0}^{l_{2s}} \prod_{j=1}^{s} \binom{l_{2j}}{k_{2j}} \frac{l!}{(l_1!l_2!l_4!...l_{2s}!)} c_0^{l_2} c_2^{l_4} ... c_{2s-2}^{l_{2s}}, \quad s \geq 1, c_0 = 1 \tag{3}$$

where the $(1_1, l_2, l_4, ..., l_{2s})$ satisfy (1). Since $\sum_{k_{2j}=0}^{l_{2j}} \binom{l_{2j}}{k_{2j}} = 2^{l_{2j}}$, equation (4) reduces to

$$c_{2s} = \sum_{l=1}^{2s} \sum_{(l_1,l_2,l_4,...,l_{2s})} \frac{l!}{(l_1!l_2!l_4!...l_{2s}!)} (2c_0)^{l_2} (2c_2)^{l_4} ... (2c_{2s-2})^{l_{2s}}, \quad s \geq 1, c_0 = 1. \tag{4}$$

Generating Function and Asymptotic Growth

The generating function for c_{2s} can be computed (after simplification) to be

$$\mathcal{G}(t) = \sum_{s \geq 0} c_{2s} t^{2s} = \frac{1}{2} \left[\frac{1}{1 - (2\mathcal{G}t^2 - t)} + \frac{1}{1 - (2\mathcal{G}t^2 + t)} \right]. \tag{5}$$

Now $\mathcal{G}(t)$ is algebraic since equation (5) gives $4t^4\mathcal{G}^3 - 4t^2\mathcal{G}^2 + (1+t^2)\mathcal{G} - 1 = 0$. A simple calculation and application of the implicit function theorem shows that $\mathcal{G}(t)$ has positive radius of convergence at $t = 0$, where the radius can be determined by solving the system of equations

$$\begin{aligned} & 4t^4\mathcal{G}^3 - 4t^2\mathcal{G}^2 + (1+t^2)\mathcal{G} - 1 = 0 \\ & \frac{\partial}{\partial \mathcal{G}} \left[4t^4\mathcal{G}^3 - 4t^2\mathcal{G}^2 + (1+t^2)\mathcal{G} - 1 \right] = 12t^4\mathcal{G}^2 - 8t^2\mathcal{G} + 1 + t^2 = 0, \end{aligned} \tag{6}$$

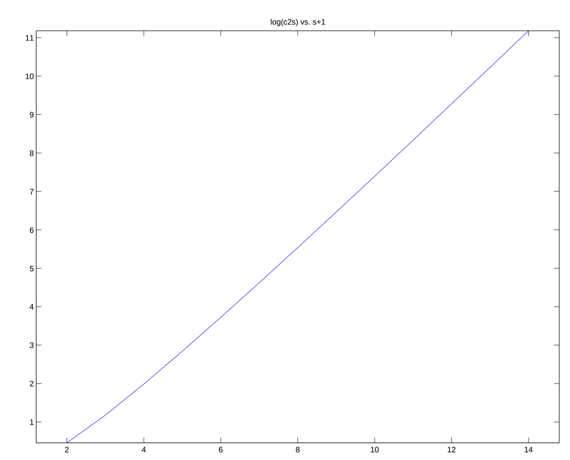
giving $R = \frac{1}{2}\sqrt{-22 + 10\sqrt{5}} \approx .3002830998$. Now

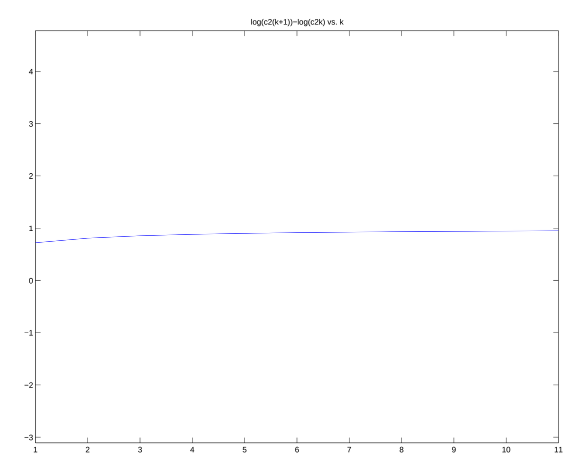
$$\frac{1}{R} = \limsup_{s \to \infty} \sqrt[2s]{|c_{2s}|} = \lim_{s \to \infty} \sqrt[2s]{c_{2s}} \tag{7}$$

since $c2s$ is a strictly positive and increasing sequence. By the definition of limit, one can conclude that for large enough s,

$$\left(\frac{1}{R} - \epsilon \right)^{2s} < c_{2s} < \left(\frac{1}{R} + \epsilon \right)^{2s}, \tag{8}$$

that is, $c_{2s} \approx \left(\frac{1}{R}\right)^{2s}$ for s large enough. Numerical evidence is given below.





Conclusions/Future work

A recursion equation is successfully computed for the number of combinatorial classes for vector fields in Ξ_d when the separatrices are labeled. An estimate for the asymptotic growth of the sequence is given.
I have also computed a recursion equation for the number of combinatorial classes of a given dimension within $\mathbb{C}^{d-1}$, however, a two-varirable generating function has not successfully computed as of date.

References

[C] Louis Comtet. *Advanced Combinatorics.* D. Reidel Publishing Company, Dordrecht-Holland, 1974.

[D] Tom Davis. *Catalan Numbers.*, **http://mathcircle.berkeley.edu/BMC6/ps/catalan.pdf**, 2006.

[DES] A. Douady, F. Estrada, and P. Sentenac. *Champs de vecteurs polynomiaux sur $\mathbb{C}$.* Unpublished manuscript.