Bioinspired computation in combinatorial optimization - Algorithms and their computational complexity

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Evolutionary Algorithms and Other Search Heuristics

Most famous search heuristic: Evolutionary Algorithms (EAs)

- a bio-inspired heuristic
- paradigm: evolution in nature, “survival of the fittest”
- actually it’s only an algorithm, a randomized search heuristic (RSH)

Goal: optimization

Here: discrete search spaces, combinatorial optimization, in particular pseudo-boolean functions

Optimize \( f : \{0,1\}^n \rightarrow \mathbb{R} \)

What RSHs Do We Consider?

- Theoretically considered RSHs
  - \((1+1)\) EA
  - \((1+\lambda)\) EA (offspring population)
  - \((\mu+1)\) EA (parent population)
  - \((\mu+1)\) GA (parent population and crossover)
  - SEMO, DEMO, FEMO, \ldots (multi-objective)
  - Randomized Local Search (RLS)
  - Metropolis Algorithm/Simulated Annealing (MA/SA)
  - Ant Colony Optimization (ACO)
  - Particle Swarm Optimization (PSO)
  - \ldots

First of all: define the simple ones

Why Do We Consider Randomized Search Heuristics?

- Not enough resources (time, money, knowledge) for a tailored algorithm
- Black Box Scenario rules out problem-specific algorithms
- We like the simplicity, robustness, \ldots of Randomized Search Heuristics
- They are surprisingly successful.

Point of view

Want a solid theory to understand how (and when) they work.
The Most Basic RSHs

(1+1) EA and RLS for maximization problems

(1+1) EA
1. Choose $x_0 \in \{0, 1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   1. Create $y$ by flipping each bit of $x_t$ indep. with probab. $1/n$.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

RLS
1. Choose $x_0 \in \{0, 1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   1. Create $y$ by flipping one bit of $x_t$ uniformly.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

What Kind of Theory Are We Interested in?

- Not studied here: convergence, local progress, models of EAs (e.g., infinite populations), . . .
- Treat RSHs as randomized algorithm!
- Analyze their “runtime” (computational complexity) on selected problems

**Definition**
Let RSH $A$ optimize $f$. Each $f$-evaluation is counted as a time step. The *runtime* $T_{A,f}$ of $A$ is the random first point of time such that $A$ has sampled an optimal search point.

- Often considered: expected runtime, distribution of $T_{A,f}$
- Asymptotical results w. r. t. $n$

How Do We Obtain Results?

We use (rarely in their pure form):
- Coupon Collector’s Theorem
- Concentration inequalities:
  Markov, Chebyshev, Chernoff, Hoeffding, . . . bounds
- Markov chain theory: waiting times, first hitting times
- Rapidly Mixing Markov Chains
- Random Walks: Gambler’s Ruin, drift analysis, martingale theory, electrical networks
- Random graphs (esp. random trees)
- Identifying typical events and failure events
- Potential functions and amortized analysis
- . . .

Adapt tools from the analysis of randomized algorithms; understanding the stochastic process is often the hardest task.

Early Results

Analysis of RSHs already in the 1980s:
- Sasaki/Hajek (1988): SA and Maximum Matchings
- Sorkin (1991): SA vs. MA
- Jerrum (1992): SA and Cliques
- . . .

High-quality results, but limited to SA/MA (nothing about EAs) and hard to generalize.

Since the early 1990s

Systematic approach for the analysis of RSHs, building up a completely new research area
This Tutorial

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA

2. Combinatorial optimization problems
   - Minimum spanning trees
   - Maximum matchings
   - Shortest paths
   - Makespan scheduling
   - Covering problems
   - Traveling salesman problem

3. End

4. References

How the Systematic Research Began — Toy Problems

Example: OneMax

Simple example functions (test functions)
- OneMax\( (x_1, \ldots, x_n) = x_1 + \cdots + x_n \)
- LeadingOnes\( (x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j \)
- BinVal\( (x_1, \ldots, x_n) = \sum_{i=1}^{n} 2^{n-i} x_i \)
- polynomials of fixed degree

Goal: derive first runtime bounds and methods

Artificially designed functions
- with sometimes really horrible definitions
- but for the first time these allow rigorous statements

Goal: prove benefits and harm of RSH components, e.g., crossover, mutation strength, population size . . .

Agenda

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Proof of the $O(n \log n)$ bound

- **Fitness levels**: $L_i := \{x \in \{0,1\}^n \mid \text{OneMax}(x) = i\}$
- $(1+1)$ EA never decreases its current fitness level.
- From $i$ to some higher-level set with prob. at least
  \[
  \left(\frac{n-i}{1}\right) \cdot \left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{n-i}{en}
  \]
    - choose a 0-bit
    - flip this bit
    - keep the other bits
- Expected time to reach a higher-level set is at most $\frac{en}{n-i}$.
- Expected runtime is at most
  \[
  \sum_{i=0}^{n-1} \frac{en}{n-i} = O(n \log n).
  \]

Later Results Using Toy Problems

- Find the theoretically optimal mutation strength $(1/n$ for OneMax$)$.
- Bound the optimization time for linear functions ($O(n \log n)$).
- Optimal population size (often 1!)
- Crossover vs. no crossover $\rightarrow$ Real Royal Road Functions
- Multistarts vs. populations
- Frequent restarts vs. long runs
- Dynamic schedules
  - …

RSHs for Combinatorial Optimization

- Analysis of runtime and approximation quality on well-known combinatorial optimization problems, e.g.,
  - Sorting problems (is this an optimization problem?)
  - Covering problems
  - Cutting problems
  - Subsequence problems
  - Traveling salesman problem
  - Eulerian cycles
  - Minimum spanning trees
  - Maximum matchings
  - Scheduling problems
  - Shortest paths
  - …
- We do not hope: to be better than the best problem-specific algorithms
- Instead: maybe reasonable polynomial running times
- In the following no fine-tuning of the results

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- End
- References
Minimum Spanning Trees:

- **Given:** Undirected connected graph $G = (V, E)$ with $n$ vertices and $m$ edges with positive integer weights.
- **Find:** Edge set $E' \subseteq E$ with minimal weight connecting all vertices.

- Search space $\{0,1\}^m$
- Edge $e_i$ is chosen iff $x_i=1$
- Consider (1+1) EA

Fitness function:

- Decrease number of connected components, find minimum spanning tree.

- $f(s) := (c(s),w(s))$.
  Minimization of $f$ with respect to the lexicographic order.

First goal: Obtain a connected subgraph of $G$.

How long does it take?

Connected graph in expected time $O(m\log n)$ (fitness-based partitions)

Bijection for minimum spanning trees:

- $k := |E(T^*) \setminus E(T)|$
- Bijection $\alpha: E(T^*) \setminus E(T) \rightarrow E(T) \setminus E(T^*)$
- $\alpha(e_i)$ on the cycle of $E(T) \cup \{e_i\}$
- $w(e_i) \leq w(\alpha(e_i))$

$\Rightarrow k$ accepted 2-bit flips that turn $T$ into $T^*$
Upper Bound

**Theorem:**
The expected time until (1+1) EA constructs a minimum spanning tree is bounded by $O(m^2(\log n + \log w_{max}))$.

**Sketch of proof:**
- $w(s)$ weight current solution $s$.
- $w_{opt}$ weight minimum spanning tree $T^*$
- set of $m + 1$ operations to reach $T^*$
- $m' = m - (n - 1)$ 1-bit flips concerning non-$T^*$ edges
  $\Rightarrow$ spanning tree $T$
- $k$ 2-bit flips defined by bijection
- $n - k$ non accepted 2-bit flips
- $\Rightarrow$ average distance decrease $(w(s) - w_{opt})/(m + 1)$

Expected Multiplicative Distance Decrease (aka Drift Analysis)

Maximum distance: $w(s) - w_{opt} \leq D := m \cdot w_{max}$

1 step: Expected distance at most $(1 - 1/(2n))(w(s) - w_{opt})$

$t$ steps: Expected distance at most $(1 - 1/(2n))^t(w(s) - w_{opt})$

$t := \lceil 2 \cdot (\ln 2)n(\log D + 1) \rceil$: $(1 - 1/(2n))^t(w(s) - w_{opt}) \leq 1/2$

Expected number of 2-steps $2t = O(n(\log n + \log w_{max}))$(Markov)

Expected optimization time $O(tm^2/n) = O(m^2(\log n + \log w_{max}))$. 

Proof

1-step (larger total weight decrease of 1-bit flips)
2-step (larger total weight decrease of 2-bit flips)

Consider 2-steps:
- Expected weight decrease by a factor $1 - (1/(2n))$
- Probability $(n/m^2)$ for a good 2-bit flip
- Expected time until $q$ 2-steps $O(qm^2/n)$

Consider 1-steps:
- Expected weight decrease by a factor $1 - (1/(2m'))$
- Probability $(m'/m)$ for a good 1-bit flip
- Expected time until $q$ 1-steps $O(qm/m')$

1-steps faster $\Rightarrow$ show bound for 2-steps.
Maximum Matchings

A matching in an undirected graph is a subset of pairwise disjoint edges; aim: find a maximum matching (solvable in poly-time)

Simple example: path of odd length

Maximum matching with more than half of edges

Suboptimal matching

Concept: augmenting path
- Alternating between edges being inside and outside the matching
- Starting and ending at “free” nodes not incident on matching
- Flipping all choices along the path improves matching

Example: whole graph is augmenting path

Interesting: how simple EAs find augmenting paths

Maximum Matchings: Upper Bound

Fitness function \( f : \{0,1\}^\# \text{edges} \rightarrow \mathbb{R} \):
- one bit for each edge, value 1 iff edge chosen
- value for legal matchings: size of matching
- otherwise penalty leading to empty matching

Example: path with \( n + 1 \) nodes, \( n \) edges: bit string selects edges

Theorem

The expected time until (1+1) EA finds a maximum matching on a path of \( n \) edges is \( O(n^4) \).
Maximum Matchings: Upper Bound (Ctnd.)

Proof idea for $O(n^4)$ bound
- Consider the level of second-best matchings.
- Fitness value does not change (walk on plateau).
- If “free” edge: chance to flip one bit! $\rightarrow$ probability $\Theta(1/n)$.
- Else steps flipping two bits $\rightarrow$ probability $\Theta(1/n^2)$.
- Shorten or lengthen augmenting path
- At length 1, chance to flip the free edge!

Length changes according to a fair random walk
$\rightarrow$ equal probability for lengthenings and shortenings

Fair Random Walk

Scenario: fair random walk
- Initially, player $A$ and $B$ both have $\frac{n}{3}$ USD
- Repeat: flip a coin
- If heads: $A$ pays 1 USD to $B$, tails: other way round
- Until one of the players is ruined.

How long does the game take in expectation?

Theorem:
Fair random walk on $\{0, \ldots, n\}$ takes in expectation $O(n^2)$ steps.

Maximum Matchings: Lower Bound

Worst-case graph $G_{h, \ell}$

Augmenting path can get shorter but is more likely to get longer.
(unsafe random walk)

Theorem
For $h \geq 3$, (1+1) EA has exponential expected optimization time $2^{\Omega(\ell)}$ on $G_{h, \ell}$.

Proof requires analysis of negative drift (simplified drift theorem).
Maximum Matching: Approximations

**Insight:** do not hope for exact solutions but for approximations

For maximization problems: solution with value $a$ is called $(1 + \varepsilon)$-approximation if $\frac{\text{OPT}}{a} \leq 1 + \varepsilon$, where OPT optimal value.

**Theorem**

For $\varepsilon > 0$, $(1+1)$ EA finds a $(1 + \varepsilon)$-approximation of a maximum matching in expected time $O(m^2/\varepsilon^2)$ ($m$ number of edges).

**Proof idea:** If current solution worse than $(1 + \varepsilon)$-approximate, there is a “short” augmenting path (length $\leq 2/\varepsilon + 1$); flip it in one go.

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All-pairs-shortest-path (APSP) problem

**Definition:** Given a connected directed graph $G = (V, E)$, $|V| = n$ and $|E| = m$, and a function $w: E \rightarrow \mathbb{N}$ which assigns positive integer weights to the edges.

Compute from each vertex $v_i \in V$ a shortest path (path of minimal weight) to every other vertex $v_j \in V \setminus \{v_i\}$.

**Representation:**

Individuals are paths between two particular vertices $v_i$ and $v_j$.

**Initial Population:** $P := \{I_{u,v} = (u, v)| (u, v) \in E\}$
Mutation:

Pick individual \( I_{u,v} \) uniformly at random

\( E^-(u) \): incoming edges of \( u \)
\( E^+(v) \): outgoing edges of \( v \)

Pick uniformly at random an edge \( e = (x,y) \in E^-(u) \cup E^+(v) \)

Add \( e \) to \( u \)

New individual \( I'_{s,t} \)

Lemma:

Let \( \ell \geq \log n \). The expected time until has found all shortest paths with at most \( \ell \) edges is \( O(n^3 \ell) \).

Proof idea:

Consider two vertices \( u \) and \( v \), \( u \neq v \).

Let \( \gamma := (v^1 = u, v^2, \ldots, v^{\ell+1} = v) \) be a shortest path from \( u \) to \( v \) consisting of \( \ell' \), \( \ell' \leq \ell \), edges in \( G \).

the sub-path \( \gamma' = (v^1 = u, v^2, \ldots, v^j) \) is a shortest path from \( u \) to \( v^j \).

Mutation-based EA

Steady State EA

1. Set \( P = \{ I_{u,v} = (u,v) \mid (u,v) \in E \} \).
2. Choose an individual \( I_{x,y} \in P \) uniformly at random.
3. Mutate \( I_{x,y} \) to obtain an individual \( I'_{s,t} \).
4. If there is no individual \( I_{s,t} \in P \), \( P = P \cup \{ I'_{s,t} \} \),
   else if \( f(I'_{s,t}) \leq f(I_{s,t}) \), \( P = (P \cup \{ I'_{s,t} \}) \setminus \{ I_{s,t} \} \)
5. Repeat Steps 2–4 forever.

Population size is upper bounded \( n^2 \)
(for each pair of vertices at most one path)

- Pick shortest path from \( u \) to \( v_j \) and append edge \( (v_j, v_{j+1}) \)
- Shortest path from \( u \) to \( v_{j+1} \)

- Probability to pick \( I_{u,v_j} \) is at least \( 1/n^2 \)
- Probability to append right edge is at least \( 1/(2n) \)
- Success with probability at least \( p = 1/(2n^3) \)
- At most \( l \) successes needed to obtain shortest path from \( u \) to \( v \)
Consider typical run consisting of $T = cn^3$ steps.

What is the probability that the shortest path from $u$ to $v$ has been obtained?

We need at most $l$ successes, where a success happens in each step with probability at least $p = 1/(2n^3)$

Define for each step $i$ a random variable $X_i$.

- $X_i = 1$ if step $i$ is a success
- $X_i = 0$ if step $i$ is not a success

Shortest paths have length at most $n-1$.

Set $l = n-1$

**Theorem**

The expected optimization time of Steady State EA for the APSP problem is $O(n^4)$.

**Remark:**

There are instances where the expected optimization of $(\mu + 1)$-EA is $\Omega(n^4)$

**Question:**

Can crossover help to achieve a better expected optimization time?

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### Analysis

For a run consisting of $T=cn^3l$ steps, consider the probability of success for each step.

Let $X_i$ be a binary random variable indicating whether step $i$ is a success.

- $X_i = 1$ with probability $p = 1/(2n^3)$
- $X_i = 0$ otherwise

Define $X = \sum_{i=1}^{T} X_i$ as the total number of successful steps.

The probability of achieving at least $\ell$ successes is given by

$$P(X \geq \ell) \geq e^{-T/(16n^3)}$$

**Crossover**

Pick two individuals $I_{u,v}$ and $I_{s,t}$ from population uniformly at random.

If $v = s$, then $v = t$. Otherwise, $v = s$ and $u = t$.

---

Probability for failure of at least one pair of vertices at most:

$$\Phi = n^2 \cdot e^{-c\ell/16}$$

Large enough $\ell$ and $\delta \geq \log n$:

No failure in any path with probability at least $\alpha = 1 - n^2 \cdot e^{-c\ell/16} = 1 - o(1)$

Holds for any phase of $T$ steps

Expected time upper bound by $T/\alpha = O(n^3 \ell)$
Steady State GA

1. Set \( P = \{ I_{u,v} = (u, v) \mid (u, v) \in E \} \).
2. Choose \( r \in [0, 1] \) uniformly at random.
3. If \( r \leq p_c \), choose two individuals \( I_{x,y} \in P \) and \( I_{x',y'} \in P \) uniformly at random and perform crossover to obtain an individual \( I'_{x,y} \),
else choose an individual \( I_{x,y} \in P \) uniformly at random and mutate \( I_{x,y} \) to obtain an individual \( I'_{x,y} \).
4. If \( I'_{x,y} \) is a path from \( s \) to \( t \) then
   - If there is no individual \( I_{s,t} \in P \), \( P = P \cup \{ I'_{s,t} \} \).
   - Else if \( f(I'_{s,t}) \leq f(I_{s,t}) \), \( P = (P \cup \{ I'_{s,t} \}) \setminus \{ I_{s,t} \} \).
5. Repeat Steps 2–4 forever.

\( p_c \) is a constant.

Theorem:
The expected optimization time of Steady State GA is \( O(n^{3.5} \sqrt{\log n}) \).

Mutation and \( \ell^* := \sqrt{n \log n} \)

All shortest path of length at most \( \ell^* \) edges are obtained.

Show: Longer paths are obtained by crossover within the stated time bound.

Analysis Crossover

Long paths by crossover:

Assumption: All shortest paths with at most \( \ell^* \) edges have already been obtained.

Assume that all shortest paths of length \( k \leq \ell^* \) have been obtained.

What is the expected time to obtain all shortest paths of length at most \( 3k/2 \)?

Analysis Crossover

Consider pair of vertices \( x \) and \( y \) for which a shortest path of \( r \), \( k < r \leq 3k/2 \), edges exists.

There are \( 2k-r \) pairs of shortest paths of length at most \( k \) that can be joined to obtain shortest path from \( x \) to \( y \).

Probability for one specific pair: at least \( 1/n^4 \)
At least \( 2k+1-r \) possible pairs: probability at least \( (2k+1-r)/n^4 \geq k/(2n^4) \)
At most \( n^2 \) shortest paths of length \( r \), \( k < r \leq 3k/2 \)
Time to collect all paths \( O(n^4 \log n/ k) \)
(similar to Coupon Collectors Theorem)
Analysis Crossover

Sum up over the different values of \( k \), namely

\[
\sqrt{n \log n}, c \cdot \sqrt{n \log n}, c^2 \cdot \sqrt{n \log n}, \ldots, c^{\log \left( \frac{n}{\sqrt{n \log n}} \right)} \cdot \sqrt{n \log n},
\]

where \( c = 3/2 \).

Expected Optimization

\[
\sum_{s=0}^{\log \left( \frac{n}{\sqrt{n \log n}} \right)} \left( O \left( \frac{n^4 \log n}{\sqrt{n \log n}} \right) c^{-s} \right) = O(n^{3.5} \sqrt{\log n}) \sum_{s=0}^{\infty} c^{-s} = O(n^{3.5} \sqrt{\log n})
\]

Makespan Scheduling

What about NP-hard problems? \( \rightarrow \) Study approximation quality

Makespan scheduling on 2 machines:
- \( n \) objects with weights/processing times \( w_1, \ldots, w_n \)
- 2 machines (bins)
- Minimize the total weight of fuller bin = makespan.

Formally, find \( I \subseteq \{1, \ldots, n\} \) minimizing

\[
\max \left\{ \sum_{i \in I} w_i, \sum_{i \notin I} w_i \right\}.
\]

Sometimes also called the Partition problem. This is an “easy” NP-hard problem, good approximations possible.
Types of Results

- Worst-case results
- Success probabilities and approximations
- An average-case analysis
- A parameterized analysis

Sufficient Conditions for Progress

Abbreviate $S := w_1 + \cdots + w_n \Rightarrow$ perfect partition has cost $\frac{S}{2}$.

Suppose we know
- $s^* = \text{size of smallest object in the fuller bin}$,
- $f(x) > \frac{S}{2} + \frac{s^*}{2}$ for the current search point $x$

then the solution is improvable by a single-bit flip.

If $f(x) < \frac{S}{2} + \frac{s^*}{2}$, no improvements can be guaranteed.

Lemma

If smallest object in fuller bin is always bounded by $s^*$ then (1+1) EA and RLS reach $f$-value $\leq \frac{S}{2} + \frac{s^*}{2}$ in expected $O(n^2)$ steps.
Sufficient Conditions for Progress

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Suppose we know
- $s^* =$ size of smallest object in the fuller bin,
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Lemma

If smallest object in fuller bin is always bounded by $s^*$ then $(1+1)$ EA and RLS reach $f$-value $\leq \frac{S}{2} + \frac{s^*}{2}$ in expected $O(n^2)$ steps.

Worst-Case Results

Theorem

On any instance to the makespan scheduling problem, the $(1+1)$ EA and RLS reach a solution with approximation ratio $\frac{4}{3}$ in expected time $O(n^2)$.

Use study of object sizes and previous lemma.

Theorem

There is an instance $W^*_\varepsilon = \{w_1, \ldots, w_n\}$ such that the $(1+1)$ EA and RLS need with prob. $\Omega(1)$ at least $n^{\Omega(n)}$ steps to find a solution with a better ratio than $4/3 - \varepsilon$.

Worst-Case Instance

Instance $W^*_\varepsilon = \{w_1, \ldots, w_n\}$ is defined by $w_1 := w_2 := \frac{1}{3} - \frac{\varepsilon}{2}$ (big objects) and $w_i := \frac{1}{3} + \frac{\varepsilon}{2}$ for $3 \leq i \leq n$, $\varepsilon$ very small constant; $n$ even

Sum is 1; there is a perfect partition.

But if one bin with big and one bin with small objects: value $\frac{2}{3} - \frac{\varepsilon}{2}$.

Move a big object in the emptier bin $\Rightarrow$ value $(\frac{1}{3} + \frac{\varepsilon}{2}) + (\frac{2}{3} - \frac{\varepsilon}{2}) = \frac{2}{3} + \frac{\varepsilon}{2}$!

Need to move $\geq \varepsilon n$ small objects at once for improvement: very unlikely.

With constant probability in this situation, $n^{\Omega(n)}$ needed to escape.
Worst Case – PRAS by Parallelism

Previous result shows: success dependent on big objects

**Theorem**
On any instance, the $(1+1)$ EA and RLS with prob. $\geq 2^{-c\frac{1}{\varepsilon} \ln(1/\varepsilon)}$ find a $(1 + \varepsilon)$-approximation within $O(n \ln(1/\varepsilon))$ steps.

- $2^{O(\frac{1}{\varepsilon} \ln(1/\varepsilon))}$ parallel runs find a $(1 + \varepsilon)$-approximation with prob. $\geq 3/4$ in $O(n \ln(1/\varepsilon))$ parallel steps.
- Parallel runs form a polynomial-time randomized approximation scheme (PRAS)!

Worst Case – PRAS by Parallelism (Proof Idea)

Set $s := \lceil \frac{n}{2} \rceil$
Assuming $w_1 \geq \cdots \geq w_n$, we have $w_i \leq \varepsilon \frac{s}{2}$ for $i \geq s$.

analyze probability of distributing
- large objects in an optimal way,
- small objects greedily $\Rightarrow$ error $\leq \varepsilon \frac{s}{2}$,

Random search rediscovers algorithmic idea of early algorithms.

Average-Case Analyses

Models: each weight drawn independently at random, namely
- uniformly from the interval $[0, 1]$.
- exponentially distributed with parameter 1 (i.e., $\text{Prob}(X \geq t) = e^{-t}$ for $t \geq 0$).

Approximation ratio no longer meaningful, we investigate: discrepancy $=$ absolute difference between weights of bins.

How close to discrepancy 0 do we come?

Makespan Scheduling – Known Average-Case Results

**Deterministic, problem-specific heuristic LPT**
Sort weights decreasingly, put every object into currently emptier bin.

Known for both random models:
LPT creates a solution with discrepancy $O((\log n)/n)$.

What discrepancy do the $(1+1)$ EA and RLS reach in poly-time?
**Average-Case Analysis of the (1+1) EA**

**Theorem**
In both models, the (1+1) EA reaches discrepancy $O((\log n)/n)$ after $O(n^{c+4}\log^2 n)$ steps with probability $1 - O(1/n^c)$. Almost the same result as for LPT!

Proof exploits order statistics:
If $X(i)$ ($i$-th largest) in fuller bin, $X(i+1)$ in emptier one, and discrepancy $> 2(X(i) - X(i+1)) > 0$, then objects can be swapped; discrepancy falls
Consider such "difference objects".

$W. h. p. X(i) - X(i+1) = O((\log n)/n)$
(for $i = \Omega(n)$).

**Value of Optimal Solution**
Recall approximation result: decent chance to distribute $k$ big jobs optimally if $k$ small.

Since $w_1 \geq \cdots \geq w_n$, already $w_k \leq S/k$.

Consequence: optimal distribution of first $k$ objects $\rightarrow$ can reach makespan $S/2 + S/k$ by greedily treating the other objects.

**Theorem**
$(1+1)$ EA and RLS find solution of makespan $\leq S/2 + S/k$ with probability $\Omega((2k)^{-ek})$ in time $O(n \log k)$. Multistarts have success probability $\geq 1/2$ after $O(2^{(e+1)k}ek^k n \log k)$ evaluations.

$2^{(e+1)k}ek^k \log k$ does not depend on $n \rightarrow$ a randomized FPT-algorithm.

**A Parameterized Analysis**

Have seen: problem is hard for (1+1) EA/RLS in the worst case, but not so hard on average.

What parameters make the problem hard?

**Definition**
A problem is fixed-parameter tractable (FPT) if there is a problem parameter $k$ such that it can be solved in time $f(k) \cdot \text{poly}(n)$, where $f(k)$ does not depend on $n$.

Intuition: for small $k$, we have an efficient algorithm.

Considered parameters (Sutton and Neumann, 2012):
- Value of optimal solution
- No. jobs on fuller machine in optimal solution
- Unbalance of optimal solution

**No. Objects on Fuller Machine**
Suppose: optimal solution puts only $k$ objects on fuller machine.

Notion: $k$ is called critical path size.

Intuition:
- Good chance of putting $k$ objects on same machine if $k$ small,
- other objects can be moved greedily.

**Theorem**
For critical path size $k$, multistart RLS finds optimum in $O(2^k (en)^{ek} n \log n)$ evaluations with probability $\geq 1/2$.

Due to term $n^{ek}$, result is somewhat weaker than FPT (a so-called XP-algorithm). Still, for constant $k$ polynomial.

Remark: with (1+1)-EA, get an additional $\log w_1$-term.
Unbalance of Optimal Solution

Consider discrepancy of optimum $\Delta^* := 2(\text{OPT} - S/2)$.

Question/decision problem: Is $w_k \geq \Delta^* \geq w_{k+1}$?

Observation: If $\Delta^* \geq w_{k+1}$, optimal solution will put $w_{k+1}, \ldots, w_n$ on emptier machine. Crucial to distribute first $k$ objects optimally.

Theorem

Multistart RLS with biased mutation (touches objects $w_1, \ldots, w_k$ with prob. $1/(kn)$ each) solves decision problem in $O(2^k n^3 \log n)$ evaluations with probability $\geq 1/2$.

Again, a randomized FPT-algorithm.

The Problem

The Vertex Cover Problem:
Given an undirected graph $G=(V,E)$.

Find a minimum subset of vertices such that each edge is covered at least once.
NP-hard, several 2-approximation algorithms.
Simple single-objective evolutionary algorithms fail!!!
**Evolutionary Algorithm**

**Representation:** Bitstrings of length \( n \)

- Minimize fitness function:
  - \( f_1(x) = (|x|_1, |U(x)|) \)
  - \( f_1(x) = (2, 2) \)
  - \( f_2(x) = (|x|_1, LP(x)) \)
  - \( f_2(x) = (2, 1) \)

- \( U(x) \): Edges not covered by \( x \)
- \( G(x) = G(V, U(x)) \)
- \( LP(x) \): value of LP applied to \( G(x) \)

**Multi-Objective Approach:**
Treat the different objectives in the same way

- Keep trade-offs of the two criteria

**Evolutionary Algorithm**

Two mutation operations:
1. Standard bit mutation with probability \( 1/n \)
2. Mutation probability \( 1/2 \) for vertices adjacent to edges of \( U(x) \).
   Otherwise mutation probability \( 1/n \).
   Decide uniformly at random which operator to use in next iteration

**Multi-Objective Approach:**
Treat the different objectives in the same way

- Empty set included in the population

- \(|x|_1| \)
- \(|U(x)| \)
What can we say about these solutions?

$|x|_1 \uparrow$

$\log n$-approximation (Friedrich, Hebbinghaus, He, N., Witt (2010))

Approach can be generalized to the SetCover Problem
(best possible approximation in polynomial time)

Kernelization in expected polynomial time
- Subset of a minimum vertex cover
- $G(x)$ has maximum degree at most $OPT$
- $G(x)$ has at most $OPT + OPT^2$
  non-isolated vertices

Optimal solution
Expected time $g(OPT) \cdot poly(n)$
Fixed parameter evolutionary algorithm

$|U(x)|$

Linear Programming

Combination with Linear Programming
- LP-relaxation is half integral, i.e.

$x_i \in \{0, 1/2, 1\}, 1 \leq i \leq n$

Theorem (Nemhauser, Trotter (1975)):
Let $x^*$ be an optimal solution of the LP. Then there is a minimum vertex cover
that contains all vertices $v_i$ where $x_i^* = 1$.

Lemma:
All search points $x$ with $LP(x) = LP(0^n) - |x|_1$ are Pareto optimal.
They can be extended to minimum vertex cover by selecting additional
vertices.

Can we also say something about approximations?

$|x|_1 \uparrow$

$|x|_1 \leq (1 + \epsilon)OPT$

Kernelization in expected polynomial time
- Subset of a minimum vertex cover
- $G(x)$ has at most $2OPT$ non-isolated vertices

Optimal solution
Expected time $O(4^{OPT} \cdot poly(n))$
Fixed parameter evolutionary algorithm

$|LP(x)|$
Representation and Mutation

Representation: Permutation of the n cities

For example: (3, 4, 1, 2, 5)

Inversion (inv) as mutation operator:
- Select i,j from {1, ...n} uniformly at random and invert the part from position i to position j.
- Inv(2,5) applied to (3, 4, 1, 2, 5) yields (3, 5, 2, 1, 4)

Euclidean TSP

Given n points in the plane and Euclidean distances between the cities.

Find a shortest tour that visits each city exactly once and return to the origin.

NP-hard, PTAS, FPT when number of inner points is the parameter.

(1+1) EA

\[ x \leftarrow \text{a random permutation of } [n]. \]

\[ \text{repeat } \text{ forever} \]

\[ y \leftarrow \text{MUTATE}(x) \]

\[ \text{if } f(y) < f(x) \text{ then } x \leftarrow y \]

Mutation:

(1+1) EA: k random inversion, k chosen according to 1+Pois(1)
There may be an exponential number of inversions to end up in a local optimum if points are in arbitrary positions (Englert et al., 2007).

We assume that the set $V$ is angle bounded. $V$ is angle-bounded by $\epsilon > 0$ if for any three points $u, v, w \in V$, $0 < \theta < \pi - \epsilon$ where $\theta$ denotes the angle formed by the line from $u$ to $v$ and the line from $v$ to $w$.

If $V$ is angle-bounded then we get a lower bound on an improvement depending on $\epsilon$.

**Assumptions:**
- $d_{\text{max}}$: Maximum distance between any two points
- $d_{\text{min}}$: Minimum distance between any two points
- $V$ is angle-bounded by $\epsilon$

Whenever the current tour is not intersection-free, we can guarantee a certain progress.

**Lemma:**
Let $x$ be a permutation such that is not intersection-free. Let $y$ be the permutation constructed from an inversion on $x$ that replaces two intersecting edges with two non-intersecting edges. Then, $f(x) - f(y) > 2d_{\text{min}} \left( \frac{1 - \cos(\epsilon)}{\cos(\epsilon)} \right)$.
Tours

A tour $x$ is either

• Intersection free
• Non intersection free

Intersection free tour are good. The points on the convex hull are already in the right order (Quintas and Supnick, 1965).

Claim: We do not spend too much time on non intersection free tours.

Parameterized Result

Lemma:
Suppose $V$ has $k$ inner points and $x$ is an intersection-free tour on $V$. Then there is a sequence of at most $2k$ inversions that transforms $x$ into an optimal permutation.

Theorem:
Let $V$ be a set of points quantized on an $m \times m$ and $k$ be the number of inner points. Then the expected optimisation time of the (1+1)-EA on $V$ is $O(n^3m^5) + O(n^k(2k - 1)!)$.

Time spend on intersecting tours

Lemma:
Let $(x^{(1)}, x^{(2)}, \ldots, x^{(t)}, \ldots)$ denote the sequence of permutations generated by the (1+1)-EA. Let $\alpha$ be an indicator variable defined on permutations of $[n]$ as

$$\alpha(x) = \begin{cases} 1 & x \text{ contains intersections;} \\ 0 & \text{otherwise.} \end{cases}$$

Then $E \left( \sum_{t=1}^{\infty} \alpha(x^{(t)}) \right) = O \left( n^3 \left( \frac{d_{\max}}{d_{\min}} - 1 \right) \left( \cos(\frac{\pi}{2n}) \right)^{1.0} \right)$.

For an $m \times m$ grid:
For points on an $m \times m$ grid this bound becomes $O(n^3m^5)$.

Summary and Conclusions

- Runtime analysis of RSHs in combinatorial optimization
- Starting from toy problems to real problems
- Insight into working principles using runtime analysis
- General-purpose algorithms successful for wide range of problems
- Interesting, general techniques
- Runtime analysis of new approaches possible
- An exciting research direction.

Thank you!
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