



## Stochastic control of traffic patterns

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*Published in:*  
Networks and Heterogeneous Media

*Link to article, DOI:*  
[10.3934/nhm.2013.8.261](https://doi.org/10.3934/nhm.2013.8.261)

*Publication date:*  
2013

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Gaididei, Y. B., Gorria, C., Berkemer, R., Christiansen, P. L., Kawamoto, A., Sørensen, M. P., & Starke, J. (2013). Stochastic control of traffic patterns. *Networks and Heterogeneous Media*, 8(1), 261-273. DOI: 10.3934/nhm.2013.8.261

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## STOCHASTIC CONTROL OF TRAFFIC PATTERNS

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ABSTRACT. A stochastic modulation of the safety distance can reduce traffic jams. It is found that the effect of random modulation on congestive flow formation depends on the spatial correlation of the noise. Jam creation is suppressed for highly correlated noise. The results demonstrate the advantage of heterogeneous performance of the drivers in time as well as individually. This opens the possibility for the construction of technical tools to control traffic jam formation.

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2010 *Mathematics Subject Classification*. Primary: 60H10, 34F05, 90B20; Secondary: 35Q84, 35B36, 34E05, 42A10.

*Key words and phrases*. Traffic model, optimal velocity model, discrete system, stochastic system, spatial and temporal colored noise, Ornstein-Uhlenbeck process, Fokker-Planck equation.

This paper is dedicated to Hiroshi Matano's 60th birthday.

**1. Introduction and model.** Traffic of people, goods and information is one of the major issues of our civilization. Traffic flow dynamics can be considered as a particular example of collective non-equilibrium behavior of asymmetrically coupled nonlinear elements. Many collective phenomena such as non-equilibrium phase transitions, pattern formations and bifurcations are inherent features of such models. Prominent examples are traffic flow on single lane highways and road networks which can be found in [31, 10, 21, 12, 13]. Further examples are rings of coupled biological oscillators [29, 28], intersegmental coordination of the neural networks responsible for generation of bipedal locomotion [4, 24], electronic circuits [11], multiple robotic systems [23], etc. The collective character of traffic flow is due to vehicle-vehicle correlation effects originating from interaction of drivers to avoid colliding with other moving vehicles and pedestrians [31, 10, 21, 12, 13]. Also, vertebrate locomotion can be described as a collective behavior of asymmetrically coupled neuronal oscillators, and different types of motor patterns and locomotion are due to generation of stable traveling waves [4].

Two approaches are commonly used in microscopic modeling of traffic dynamics: discrete and continuous ones. In the discrete approach the real traffic is considered discrete in time and space in terms of cellular automata (see e.g. a review paper [20]). By using Monte-Carlo simulations in the framework of stochastic discrete automaton model a spontaneous emergence of traffic jams was demonstrated in [22]. The continuous approach describes the microscopic traffic dynamics in terms of time-continuous differential-difference equations. In our paper we will use a continuous car-following model, which is the so-called optimal velocity (OV) model introduced in [2]. In the framework of this model the vehicles are ordered by their time-dependent position,  $s_n(t)$ , such that  $s_n(t) < s_{n+1}(t)$ . Each driver controls the acceleration to reduce the difference between the car velocity  $\dot{s}_n$  and an optimal velocity  $F(s_{n+1}(t) - s_n(t) - h_n)$  which depends on the headway  $s_{n+1}(t) - s_n(t)$ , i.e. the distance to the vehicle in front. Here  $h_n$  denotes the safety distance between the  $n$ -th car and the car which is in front of it. The safety distance is defined as the distance between cars for which the optimal velocity is equal to zero:  $F(0) = 0$ . The optimal velocity  $F(u)$  is assumed to be expressed as a sigmoidal function of the distance between cars. The sigmoidal function is motivated by the fact that small values of  $F$  are required for small deviations of distances to the front car, i.e. the headways  $s_{n+1}(t) - s_n(t)$  from the safety distance  $h_n$ . Furthermore,  $F$  should exhibit a monotoneous growth behaviour and a saturation for large distances. Then the differential equation of the model reads

$$\tau \ddot{s}_n + \dot{s}_n = F(s_{n+1}(t) - s_n(t) - h_n) + v, \quad n = 1, \dots, N, \quad (1)$$

where  $\tau$  is a reaction time,  $v$  is a common constant velocity of cars which drive with a distance equal to the safety distance  $h_n$  and  $N$  is the number of cars. An interesting artificial system which has many properties similar to the OV traffic model was proposed quite recently in [26] where an inanimate system composed of camphor boats in an annular water channel was studied and several modes of collective motion were observed. The boats move on the water and interact with each other through the concentration of the camphor molecules on the water surface. In this sense the camphor boat system provides an experimental realization of a general chemotactic model of oscillators considered in [30].

Most of the literature on traffic flow is devoted to the situation where drivers are responsible for keeping a safe distance from neighboring vehicles. However,

quite recently there appeared studies on traffic flows with vehicles which possess a control system trying to keep a desired headway: vehicles with adaptive cruise control (ACC) [18, 19, 14, 3]. Several microscopic models have been proposed to model the ACC vehicles [18, 16, 5]. It was shown in [16] that traffic jams are suppressed for ACC vehicles modeled by linear dynamical equations with a delayed-feedback control. Simulations of merging flows near an onramp and for random sequences of ACC vehicles and manual vehicles were carried out in [5]. In [6] the effect of deterministic high frequency time-modulation of the safety distance was investigated. It was found that the car propagation was facilitated when the mean distance between cars in the congestive traffic was less than the safety distance and hindered when the neighbouring cars in the flow were well separated. In the present paper, we demonstrate that in asymmetrically coupled systems a *spatiotemporal stochastic modulation* can effectively control traffic jam formation. To this end we study the optimal velocity model (1) with the safety distance in the form

$$h_n(t) = h + \nu_n(t) \quad (2)$$

where  $h$  is an equilibrium safety distance and  $\nu_n(t)$  describes a random variation of the safety distance. The safety distance enters nonlinearly into the optimal velocity model (1). As a consequence it is impossible to use white noise to model this variation, and hence we will use a colored noise process [9]. The nonlinear character of the optimal velocity function  $F(s_{n+1} - s_n - h - \nu_n(t))$  requires to calculate quantities like  $(\nu_n(t))^j$  with  $j = 2, 3, \dots$ . These quantities have no mathematical meaning in the case when the  $\nu_n(t)$  is a Gaussian white noise and therefore the safety distance stochastic modulation has to be modeled as a colored noise. It is also necessary to take into account that the behavior of different drivers (in other words, cars with different indices  $n$ ) may be stochastically correlated. This suggests that we use a minimalist model in the form of a stochastic Ornstein-Uhlenbeck process with spatio-temporal correlation properties given by

$$\langle \nu_n(t) \rangle = 0 \quad \text{and} \quad \langle \nu_n(t) \nu_{n'}(t') \rangle = D^2 e^{-\alpha|n-n'|} \frac{1}{\epsilon} \exp\left\{-\frac{|t-t'|}{\epsilon}\right\}. \quad (3)$$

The parameter  $D$  characterizes the intensity of the noise,  $\epsilon$  characterizes its correlation in time and the parameter  $\alpha$  is the inverse of the correlation length between cars which describes how many cars in the traffic flow behave in a stochastically correlated way. The Ornstein-Uhlenbeck process (3) may be obtained by solving the following set of stochastic equations in the limit  $N \rightarrow \infty$

$$\begin{aligned} \epsilon \dot{\nu}_n + \nu_n &= \zeta_n, \\ e^\alpha \zeta_n - \zeta_{n+1} &= 2D \sqrt{\sinh \alpha \tanh\left(\frac{N}{2} \alpha\right)} \xi_n(t), \end{aligned} \quad (4)$$

where  $\xi_n(t)$  is a white noise term with the properties  $\langle \xi_n(t) \rangle = 0$ ,  $\langle \xi_n(t) \xi_{n'}(t') \rangle = \delta_{n n'} \delta(t - t')$  as described in [9].

For the described sigmoidal shaped optimal velocity function we choose the specific form  $F(x) = \tanh x$  and consider the case where the cars are moving along the closed curve of the length  $L$  and impose the periodic boundary conditions:  $s_{n+N} = s_n + L$ . Letting  $s_n(t) = x_n(t) + n\ell$  with  $\ell = L/N$  being a mean distance between elements, we studied the following set of stochastic differential equations

by analytical approximation and numerically

$$\tau \ddot{x}_n + \dot{x}_n = \tanh \left( x_{n+1}(t) - x_n(t) + \delta - \nu_n(t) \right), \quad n = 1, \dots, N, \quad (5)$$

where  $\delta = \ell - h$  denotes the mismatch between the safety distance  $h$ , and the mean distance between neighboring cars  $\ell$ .

The dynamics of a closed loop of asymmetrically coupled nonlinear elements based on Eqs. (1) with equal safety distances  $h_n = h$  was considered in [7, 8]. It was shown that there are two distinctive regimes of oscillatory behavior of one-way nonlinearly coupled elements depending on the reaction time and the strength of the coupling. In the subcritical regime where the reaction time is shorter than the critical one,  $\tau < \tau_c \equiv \cosh^2(\delta)/2$ , a spatially uniform stationary state is stable. In terms of traffic dynamics this corresponds to a free flow, i.e. to a flow without jams where all cars move with the same velocity and keep the same distance  $\ell$  between nearest neighbors, i.e.,  $s_n(t) = n\ell + (\tanh(\delta) + v)t$ . For  $\tau > \tau_c$  the free flow is unstable due to a Hopf bifurcation, leading to spontaneous formation of periodic wave trains of jam and rarefaction regions. Figure 1 shows a typical car distribution for the case where the reaction time  $\tau$  is slightly above a critical value. A congested

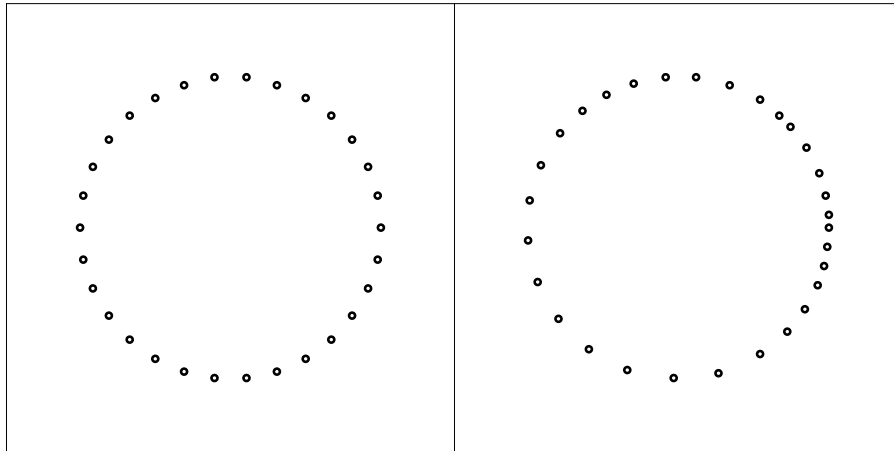


FIGURE 1. The car distribution for  $t = 4 \cdot 10^4$  along a circular road: free flow (left) for  $\tau = 0.48 < \tau_c = 0.5$  and congested flow (right) for  $\tau = 0.52$ . Left: Noise level  $D = 0$  is used. Right: The influence of the stochastic modulation with noise level  $D = 0.25$  of the safety distance is clearly visible for cars in the region of the traffic jam. Parameters  $N = 30$ ,  $h = \ell = 1$ .

region (jam cluster) moves with a constant velocity in the direction opposite to the common velocity of the cars. The first experimental evidence that the emergence of a traffic jam is a bifurcation phenomenon where stochasticity plays an important role was presented in [27] and shows that tiny fluctuations grow larger so that the homogeneous movement cannot be maintained.

**2. Variance of car distribution.** As the first step, we analyze how the stochastic modulation (2) influences a car distribution on the circular road. It is useful to

introduce quantities which characterize integral features of the flow. These are moments of the car distribution. The second moment

$$M_2 = \frac{1}{N} \sum_n u_n^2(t) \quad (6)$$

with  $u_n(t) = s_{n+1}(t) - s_n(t) - \ell$ , gives a mean square deviation of the headway  $s_{n+1}(t) - s_n(t)$  (or a variance of headway) from the free flow solution. The variance vanishes ( $M_2 = 0$ ) for the free (uniform) flow. In the flow with  $M_2 \neq 0$  some of vehicles are closer to each other than the mean distance  $\ell L/N$  other are more separated. In other words, in flows with  $M_2 \neq 0$  show a traffic jam. The set of stochastic differential equations in (5) and (4) is solved numerically by the use of an implicit Euler method. The integration of the stochastic term is done by using the strong Taylor scheme of first order described in [15]. The results of these calculations are presented in Fig. 2, where the second moment of the car distribution  $M_2$  is depicted as a function of the noise intensity  $D$  for different values of the inverse correlation length  $\alpha$ . The figure shows an average of 65 simulations with different series of random numbers for each intensity of the noise. The most interesting feature of this figure is that it is possible to see how the qualitative behaviour of the system changes when the correlation length decreases. For  $\alpha = 0.1$  and  $\alpha = 0.00002$  the interacting mode approach compares very well with the full scale numerical simulation. However, for  $\alpha = 0.1$  the quantitative comparison is only good for larger values of  $D$  which demonstrates that the interactive mode approach is only valid for  $\alpha$  considerably smaller than 0.1.

When all cars change their safety distance randomly in time but synchronously in space, that is, identical safety distance for all cars, which is described by  $\alpha = 0$ , the second moment is a monotonically decreasing function of the noise intensity  $D$ . This does not model individual drivers behaviour but describes a situation where all cars are guided automatically with the identical safety distance. In other words, this means that such type of safety distance modulation inhibits the jam creation and that in the presence of strong noise the car distribution is closer to the free flow type than in the weak noise case. For finite values of the inverse correlation length  $\alpha$  this function becomes non-monotonic: weak safety distance modulation plays a constructive role in the free flow creation, however, further increasing the noise intensity destroys this tendency and the flow becomes less ordered.

To gain better insight into the mechanism of the noise induced transitions in asymmetrically coupled nonlinear elements, we have developed an analytical approach which is based on an interacting mode approach to the dynamics of the traffic flow which was proposed recently in [8]. For the sake of simplicity we restrict ourselves to the case when  $\delta = 0$ , i.e. the mean safety distance  $h$  coincides with the mean distance  $\ell = L/N$  between cars in the flow. By subtracting Eqs. (5) pairwise, one can obtain equations for the functions  $u_n(t)$  in the form

$$\tau \ddot{u}_n + \dot{u}_n = \tanh(u_{n+1} - \nu_{n+1}(t)) - \tanh(u_n - \nu_n(t)). \quad (7)$$

We assume that the safety distance modulation is weak:  $\langle \nu_n^2(t) \rangle < 1$  and the deviation of the headway from the mean distance is small  $|u_n| < 1$ . We approximate the optimal velocity function by a Taylor polynomial and by treating higher order stochastic terms in a mean-field approximation, we obtain that Eqs. (7) can be

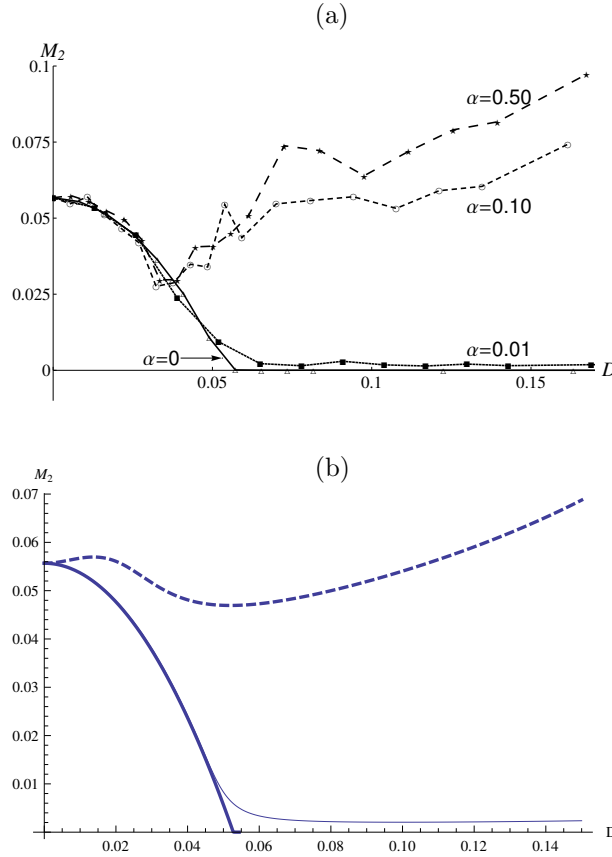


FIGURE 2. The second moment of car distribution  $M_2$  vs the noise intensity  $D$  for different values of the correlation length  $1/\alpha$ . The results shown in (a) are obtained from the full-scale numerical simulations. (b) gives the second moment of car distribution  $M_2$  vs the noise intensity  $D$  for three values of the inverse correlation length  $\alpha = 0.00002$  (solid line),  $\alpha = 0.01$  (thin solid line) and  $\alpha = 0.1$  (dashed line) obtained by the interacting mode approach. Parameters  $N = 30$ ,  $L = 30$ ,  $h = 1$ ,  $\epsilon = 0.1$ ,  $\tau = 0.52$ .

presented approximately as

$$\tau \ddot{u}_n + \dot{u}_n = C (u_{n+1} - u_n) - \frac{1}{3} (u_{n+1}^3 - u_n^3) + \nu_{n+1}(t) - \nu_n(t) \quad (8)$$

with  $C := 1 - \langle \nu_n^2(t) \rangle = 1 - D^2/\epsilon$ . Equations for the Fourier components

$$\hat{u}_j(t) = \frac{1}{N} \sum_{n=1}^N u_n(t) \exp \left\{ i \frac{2\pi j n}{N} \right\} \quad (9)$$

of the functions  $u_n(t)$  can be written in the form

$$\tau \ddot{\hat{u}}_j + \dot{\hat{u}}_j = \gamma_j \left( C \hat{u}_j - \frac{1}{3} \sum_{j_1, j_2, j_3} \hat{u}_{j_1} \hat{u}_{j_2} \hat{u}_{j_3} \delta_{j, j_1+j_2+j_3} \right) + \gamma_j \hat{\nu}_j(t), \quad (10)$$

where  $\gamma_j = \exp\{i \frac{2\pi j}{N}\} - 1$  and  $\delta_{j,j'}$  is the Kronecker delta. Note that the quantities  $u_n$  are real valued and therefore the Fourier amplitudes  $\hat{u}_j$  satisfy the condition  $\hat{u}_j = \hat{u}_{-j}^*$ . The first term in the right-hand-side of Eqs. (10) describe noise effects on the traffic dynamics in a mean-field approximation while the last term takes fluctuation effects into account. The Fourier amplitudes  $\hat{\nu}_j(t)$  of the stochastic function  $\nu_n(t)$  satisfy the correlation properties

$$\langle \hat{\nu}_j(t) \rangle = 0, \quad \langle \hat{\nu}_j(t) \hat{\nu}_j(t') \rangle = \delta_{j,-j'} \frac{D^2}{\epsilon} \exp\left\{-\frac{|t-t'|}{\epsilon}\right\} \frac{1}{N} \frac{\sinh \alpha \tanh\left(\frac{N}{2} \alpha\right)}{\cosh \alpha - \cos\left(\frac{2\pi j}{N}\right)}. \quad (11)$$

In the mean-field approach the time evolution of the modes  $\hat{u}_j$  in the linear approximation is given by  $e^{z_j^\pm t} z_j^\pm \equiv \pm i\omega_j + \lambda_j^\pm = \left(-1 + \sqrt{1 + 4\tau\gamma_j}\right)/2\tau$ . The free flow (all  $\hat{u}_j = 0$ ) is linearly unstable ( $\lambda_j^+ > 0$ ) with respect to the Fourier modes with  $|j| \leq j_c$ , where the critical index  $j_c$  is given by the equation  $2C\tau \cos^2(\pi j_c/N) = 1$ . The modes with  $|j| > j_c$  are linearly stable. Eqs. (10) for  $|j| \geq 2$  represent a system of damped weakly nonlinear oscillators. Near the threshold and for weak noise the amplitudes of the Fourier modes  $\hat{u}_j$  are small. Therefore in Eqs. (10) for  $|j| \geq 2$  one can neglect nonlinear terms and write them approximately in the form of stochastically driven linear oscillators:

$$\tau \ddot{\hat{u}}_j + \dot{\hat{u}}_j = \gamma_j \left(C \hat{u}_j + \hat{\nu}_j(t)\right). \quad (12)$$

An equation for the linearly unstable mode  $j = 1$  is

$$\tau \ddot{\hat{u}}_1 + \dot{\hat{u}}_1 = \gamma_1 \left(C - |\hat{u}_1|^2\right) \hat{u}_1 + \gamma_1 \hat{\nu}_1(t) - \frac{1}{3} \sum_{j_1, j_2, |j_1| \geq 2, |j_2| \geq 2} \hat{u}_{j_1} \hat{u}_{j_2} \hat{u}_{1-j_1-j_2}. \quad (13)$$

Here the last term describes an influence of the linearly stable modes on the dynamics of the linearly unstable one. For the sake of simplicity we will consider this influence in the mean field approximation and by replacing the last term in Eq. (13) with

$$\frac{1}{3} \sum_{j_1, j_2, |j_1| \geq 2, |j_2| \geq 2} \hat{u}_{j_1} \hat{u}_{j_2} \hat{u}_{1-j_1-j_2} \approx 2 \hat{u}_1 \sum_{j=2}^{N/2} \langle |\hat{u}_j(t)|^2 \rangle,$$

one can write approximately instead of Eq. (13)

$$\tau \ddot{\hat{u}}_1 + \dot{\hat{u}}_1 = \gamma_1 \left(\bar{C} - |\hat{u}_1|^2\right) \hat{u}_1 + \gamma_1 \hat{\nu}_1(t), \quad (14)$$

where

$$\bar{C} = C - 2 \sum_{j=2}^{N/2} \langle |\hat{u}_j(t)|^2 \rangle. \quad (15)$$

By separating contributions from the linear Fourier modes and the nonlinear one, the second moment  $M_2$  can be expressed as a sum

$$M_2(t) = M_{lin}(t) + M_{nl}(t),$$

$$M_{lin}(t) = 2 \sum_{j=2}^{N/2} \langle |\hat{u}_j(t)|^2 \rangle, \quad M_{nl} = 2 \langle |\hat{u}_1(t)|^2 \rangle. \quad (16)$$



It is straightforward to calculate  $\langle |\hat{u}_j(t)|^2 \rangle$  from Eqs. (12) and by considering the case where the correlation time  $\epsilon$  is short ( $\epsilon \ll \tau$ ), to obtain that for  $t \rightarrow \infty$  the linear part  $M_{lin}$  is determined by the expression

$$M_{lin} = \frac{D^2 \sinh \alpha \tanh \left( \frac{N}{2} \alpha \right)}{N C \tau} \left\{ \frac{1}{\cosh \alpha - \cos \beta} \left[ \frac{2 \coth \alpha - N \coth \left( \alpha N/2 \right)}{2 \sinh \alpha} + \frac{2 \cot \beta - N \cot \left( \beta N/2 \right)}{2 \sin \beta} \right] - \frac{1}{\left( \cosh \alpha - \cos(2\pi/N) \right) \left( \cos \beta - \cos(2\pi/N) \right)} \right\}, \quad (17)$$

where  $\beta = \arccos \left( \frac{1-C\tau}{C} \right)$ . As it is seen from Eq. (17) the linear part of the second moment  $M_{lin}$  is a monotonically increasing function of the noise intensity  $D$ .

To calculate the nonlinear part  $M_{nl}$  it is convenient to use polar coordinates  $u_1(t) = R(t) e^{i\varphi(t)}$ . In polar coordinates the linearly unstable mode is governed by the equations

$$\tau \left( \ddot{R} - R \dot{\varphi}^2 \right) + \dot{R} = -2 \sin^2 \left( \frac{\pi}{N} \right) \left( \bar{C} - R^2 \right) R + \text{Re}f(t), \quad (18)$$

$$\tau \left( R \ddot{\varphi} + 2 \dot{R} \dot{\varphi} \right) + R \dot{\varphi} = \sin \left( \frac{2\pi}{N} \right) \left( \bar{C} - R^2 \right) R + \text{Im}f(t), \quad (19)$$

where  $f(t) = \gamma_1 \hat{v}_1(t) e^{-i\varphi}$ . Taking into account that the deterministic dynamics of the amplitude  $R(t)$  is slow:  $\lambda_1^+ \tau < 1$ , one can solve approximately Eq. (19) and obtain that

$$\begin{aligned} \dot{\varphi} &\approx \sin \left( \frac{2\pi}{N} \right) \left( \bar{C} - R^2 \right) + \frac{1}{R} \mu(t), \\ \mu(t) &\equiv \frac{1}{\tau} \int_0^t dt_1 e^{-\frac{t-t_1}{\tau}} \text{Im}f(t_1). \end{aligned} \quad (20)$$

By inserting Eq. (20) into Eq. (18) and neglecting fluctuations of the quadratic noise  $\mu^2(t)$ , as they are small compared with the fluctuations of the linear noise term  $\mu(t)$ , we obtain that the dynamics of the amplitude of the linearly unstable mode is described by the equation

$$\tau \ddot{R} + \dot{R} = F(R) + \text{Re}f(t) + 2\tau \sin \left( \frac{2\pi}{N} \right) \left( \bar{C} - R^2 \right) \mu(t). \quad (21)$$

Here,  $F(R)$  is an effective force defined by the relation

$$F(R) = |\gamma_1|^2 \tau \cos^2 \left( \frac{\pi}{N} \right) \left( \bar{C} - R^2 \right) \left( \delta - R^2 \right) R + \frac{\tau}{R} \langle \mu^2(t) \rangle, \quad (22)$$

where

$$\delta = \bar{C} - \frac{1}{2\tau} \sec^2 \left( \frac{\pi}{N} \right) \equiv 1 - \frac{1}{2\tau} \sec^2 \left( \frac{\pi}{N} \right) - \frac{D^2}{\epsilon} - M_{lin} \quad (23)$$

is a square stationary amplitude of the linearly unstable mode in the mean field approach.

Being interested in the behavior near the threshold of instability and assuming that the noise is weak, one can apply the limit  $\delta \rightarrow 0$ , and  $D \rightarrow 0$  so that

$D/\delta = \text{const.}$  In this case Eq. (21) reduces to an effective Langevin equation for a stochastically driven nonlinear oscillator

$$\frac{dR}{d\bar{t}} = -\frac{\partial U}{\partial R} + z(\bar{t}), \quad (24)$$

where  $\bar{t} = |\gamma_1|^2 t$  is a rescaled time, the function

$$U(R) = \frac{1}{4} \tau \cos^2\left(\frac{\pi}{N}\right) (R^2 - \delta)^2 - \frac{\tau}{\epsilon + \tau} T \ln R \quad (25)$$

defines an effective potential, and  $z(t)$  is an effective white noise with correlation properties of the form

$$\langle z(\bar{t}) \rangle = 0, \quad \langle z(\bar{t}) z(\bar{t}') \rangle = 2 \left[ 1 + 4\tau^2 \sin^2\left(\frac{2\pi}{N}\right) \right] T \delta(\bar{t} - \bar{t}') \quad (26)$$

using the notation

$$T = \frac{1}{N} \frac{2D^2 \sinh \alpha \tanh\left(\frac{N}{2} \alpha\right)}{\cosh \alpha - \cos\left(\frac{2\pi}{N}\right)}. \quad (27)$$

The next step is to study effects of stochastic modulations by using the formalism of the Fokker-Planck equation. To this end we introduce the probability distribution density

$$P(R; t) = \left\langle \delta(R - R(t)) \right\rangle. \quad (28)$$

The Fokker-Planck equation for the probability density (28) has the form

$$\partial_t P = \frac{\partial}{\partial R} \left( \frac{\partial U}{\partial R} P \right) + \left[ 1 + 4\tau^2 \sin^2\left(\frac{2\pi}{N}\right) \right] T \frac{\partial^2}{\partial R^2} P. \quad (29)$$

The nonlinear part of the second moment  $M_2$  has the form

$$M_{nl}(t) = 2 \langle R^2(t) \rangle = \int_0^\infty dR R^2 P(R; t). \quad (30)$$

A stationary probability density  $P_{st}(R) \equiv P(R; t) \Big|_{t \rightarrow \infty}$  can be found for the Fokker-Planck equation (29) by simple integration. It has the form

$$P_{st}(R) = \mathcal{N} \exp \left\{ -\frac{U(R)}{\left(1 + 4\tau^2 \sin^2(2\pi/N)\right) T} \right\}, \quad (31)$$

where  $\mathcal{N}$  is a normalization coefficient. It is not difficult to obtain a general expression for the nonlinear part  $M_{nl}$  for  $t \rightarrow \infty$  from Eqs. (30), (31) and (25). However, this expression becomes rather cumbersome. It is simplified significantly in the practically important case where the stochastic correlation time is short,  $\epsilon \ll \tau$ , and the number of cars is big such that  $4\tau^2 \sin^2(2\pi/N) \ll 1$ . In this case the nonlinear part of the second moment tends to the value given by the following expression

$$M_{nl} = 2\delta + \frac{4\sqrt{T}}{\sqrt{\pi\tau} \cos(\pi/N)} \frac{\exp\left[-\frac{\tau \cos^2(\pi/N)}{4T} \delta^2\right]}{1 + \text{sign}(\delta) \text{Erf}\left(\sqrt{\frac{\tau \cos^2(\pi/N)}{4T}} |\delta|\right)}, \quad (32)$$

where  $\text{Erf}(x)$  is the error function [1]. Here the first term represents a mean field contribution to the second moment and the second part is a result of fluctuations.

As it is seen from Eqs. (23) and (17) in the mean field approximation the role of the noise reduces to the shift of the threshold of the free flow instability and as a result in this approximation the second moment is a monotonically decreasing function of the noise intensity  $D$ . In contrast to this the second term in Eq. (32) similarly to  $M_{lin}$  monotonically increases with  $D$ .

The combination of these two tendencies provides a qualitative explanation of the non-monotonic second moment dependence  $M_2(D)$  observed in the numerical simulations (see Fig. 2). The linear mode contribution decreases when the corre-

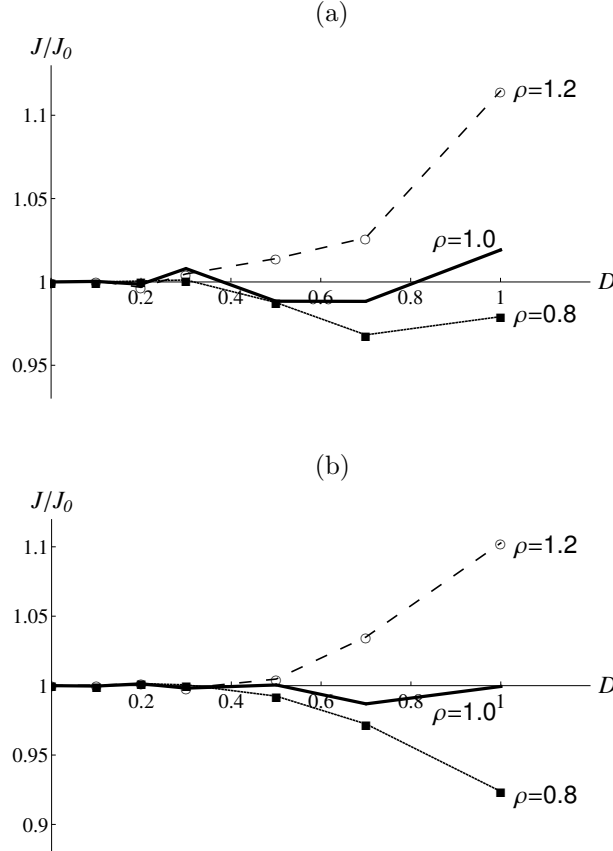


FIGURE 3. Numerical simulations of the normalized flux  $J/J_0$  ( $J_0$  is the flux in the unmodulated case) vs. the noise intensity  $D$  for three values of the car density  $\rho = 1.2$  (open circles),  $\rho = 1$  (solid line),  $\rho = 0.8$  (filled squares). (a) is for  $\alpha = 0.5$  and (b) is for  $\alpha = 5$ . Parameters  $N = 30$ ,  $\tau = 0.56$ ,  $h = 1$ ,  $\epsilon = 0.1$ .

lation length  $1/\alpha$  increases. It explains why a weak spatially high correlated noise (small  $\alpha$  and  $D$ ) plays a constructive role in preventing a traffic jam formation.

**3. Flux of cars.** One of the important quantities which characterize the traffic dynamics is the flux of cars. The microscopic definition of the flux is [17], [6]

$$\hat{J}(x, t) = \sum_n \dot{s}_n(t) \delta(x - s_n(t)). \quad (33)$$

The mean flux is given by the expression

$$\hat{J}_m(t) = \frac{1}{L} \int_0^L \hat{J}(x, t) dx. \quad (34)$$

Introducing the mean density  $\rho = N/L$ , the macroscopic flux  $J$  can be defined as an average of the mean flux and in the framework of the stochastic optimal velocity model (5) with (3) can be expressed as

$$J = \rho \left[ v + \frac{1}{N} \sum_n \int_{-\infty}^{\infty} dz \tanh\left(\frac{1}{\rho} - h + z\right) g(z - u_n - \nu_n) \right], \quad (35)$$

where

$$g(z - u_n - \nu_n) = \frac{1}{T} \int_t^{t+T} dt' \left\langle \delta(z - u_n(t') - \nu_n(t')) \right\rangle, \quad T \rightarrow \infty \quad (36)$$

is an averaged probability density. By applying a cumulant expansion [9] for the probability density (36), in the lowest approximation one can write the average flux in the form

$$J = \rho \left[ v + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du e^{-u^2} \tanh\left(\frac{1}{\rho} - h + u\sqrt{2M}\right) \right], \quad (37)$$

where  $M = \left\langle \left( \nu_n(t) + u_n(t) \right)^2 \right\rangle \Big|_{t \rightarrow \infty}$  is the second cumulant. In the vicinity of the density  $\rho = 1$  the flux (37) can be presented as

$$J = \rho v + (1 - \rho h) \kappa(M),$$

$$\kappa(M) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} du e^{-u^2} \operatorname{sech}^2(\sqrt{2M} u). \quad (38)$$

Taking into account that  $\frac{d\kappa}{dM} < 0$  and that  $M \sim D^2$ , one can conclude from Eq. (38) that the presence of noise facilitates the traffic with a high density of cars ( $h\rho > 1$ ) and hinders in the opposite case.

To verify these results we solve numerically the stochastic equations (5), (4) and calculate the average flux for several values of the car density  $\rho$  and the inverse correlation length  $\alpha$ . The results of these numerical studies presented in Fig. 3 are in full agreement with our analytical predictions.

**4. Summary and conclusion.** In the present paper, we investigate the influence of noise on the safety distance in a following-the-leader model which opens up also the possibility of controlling vehicular traffic. We considered the optimal velocity model of traffic flow with a stochastically modulated safety distance. We studied the problem both analytically and numerically. We have found that the effect of the random modulation of the safety distance on congested flow formation depends on the spatial correlation of the noise: when all cars change their safety distance randomly but synchronously the jam creation is suppressed for all values of the noise intensity; for finite values of the correlation length only a weak noise plays a constructive role in the free flow creation, however, further increasing the noise intensity makes the car flow less ordered. The random safety distance modulation

also influences the flux of cars. Our simulations and analytical considerations show that the flux is an increasing function of the noise intensity for dense traffic flows and a decreasing function in the opposite case. In other words, the flux on a highway with dense traffic is augmented by random individual choice of the safety distance by each driver (i.e. for each car as well as by fluctuations over time of this choice).

**Acknowledgments.** We acknowledge financial support from the Danish Center for Applied Mathematics and Mechanics (DCAMM). Yu. Gaididei acknowledges a Guest Professorship funded by a grant from Civilingeniør Frederik Christiansens almennyttige fond and a fellowship provided by FUNDACION IKERBASQUE, as well as partial financial support from the special program of NAS of Ukraine. He is thankful to the Department of Mathematics and the Department of Physics, Technical University of Denmark, Kongens Lyngby, and to the Department of Applied Mathematics and Statistics, University of the Basque Country, Bilbao, for their hospitality.

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