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A SINGULAR VALUE SENSITIVITY APPROACH TO
ROBUST EIGENSTRUCTURE ASSIGNMENT

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Abstract

A design technique for improving the feedback properties of multivariable state feedback systems designed using eigenstructure assignment is presented. Based on a singular value analysis of the feedback properties a design parameter adjustment procedure is outlined. This procedure allows for the preservation of important properties of the initial design, by selecting only a subset of the design parameters for adjustment.

The computational requirements are modest and straight-forward to implement with standard software. An example illustrating the method is included.

1. Introduction

Recent developments have showed that eigenstructure assignment in multivariable systems is well-suited for obtaining specified I/O-response properties [1-5]. Unfortunately this ability is not followed by any significant robustness guarantees comparable to LQ-guarantees. Here the robustness measure is the minimal singular value of the return difference or inverse return difference matrix [6-8]. In practice it may therefore be necessary to "recover" acceptable robustness properties like the multivariable stability margins. In this "recovery" procedure it is important that significant properties of the initial design are preserved.

In this paper an approach to this recovery problem is presented. The recovery information used is the singular value sensitivities of the robustness measures taken at the frequencies where robustness should be improved. These sensitivities indicate the directions the design parameters should be changed towards, to ensure the robustness improvement. By selecting only a subset of the design parameters for adjustment it is possible to preserve significant aspects of the initial design.

The adjustment of the design parameters can be performed by formal optimization methods or by direct tuning.

Earlier approaches to robustness improvement using eigenstructure assignment have focused on a time-domain characterization of the robustness problem, and have aimed towards improving the spectral condition number of the closed-loop eigenstructure [16, and references herein]. The approach in this paper differs from these approaches since the robustness problem is viewed from a frequency-domain perspective.

Other applications of singular value sensitivities for robustness improvement based on other design methodologies may be found in refs. [10, 12].

The outline of the paper is as follows. In section two - eigenstructure assignment and the singular value - based robustness concepts are shortly reviewed, in section three the singular value sensitivities are derived. In section four the adjustment procedure is outlined and the computational aspects discussed. The method is illustrated by an example in section five

and a few summarizing remarks follow in section 6.

2. The problem statement

Consider the finite-dimensional linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (2-1)$$

where A, B are constant matrices of appropriate dimensions. B is assumed to be of full rank. (A, B) is a controllable pair.

It is desired to design a full state feedback control law

$$u = -Kx + r, \quad r \in \mathbb{R}^m \quad (2-2)$$

In the context of eigenstructure assignment the extra freedom inherent in multivariable state feedback (beyond eigenvalue placement) is utilized to place parts of the closed-loop eigenvectors.

2.1 Eigenstructure assignment

The extra freedom of selecting eigenvectors may be summarized as [1]:

Let $\Lambda = \{\lambda_i\}_{i=1}^n$ be a self-conjugate set of distinct complex numbers containing no open-loop eigenvalues. There exists a real matrix K such that v_i is an eigenvector of A-BK iff.

- a) $\{v_i\}_{i=1}^n$ are linearly independent in \mathbb{C}^n and
 $v_i = \bar{v}_j$ if $\lambda_i = \bar{\lambda}_j$
- b) $\begin{bmatrix} v_i \\ t_i \end{bmatrix} \in \text{Ker} [\lambda_i I - A, -B]$ (2-3)
- c) $K = -[t_1 \dots t_n] \cdot [v_1 \dots v_n]^{-1} = -TV^{-1}$

The design parameters of this approach are the vectors t_i and the set Λ . V denotes the closed-loop eigenvector-matrix.

Several approaches to utilize this result in I/O-response design have been derived [1-5].

Unfortunately the resulting designs have no significant guaranteed robustness properties, when the robustness measure is the minimum singular value of a return difference or inverse return difference matrix [6-8]. If the resulting robustness properties evaluated over an appropriate frequency - interval is unsatisfactory, some method for "recovering" acceptable robustness levels is necessary. If possible this method should preserve as many of the desirable I/O-response properties of the initial design as possible.

3.2 Design parameters λ_i

If p is selected as a distinct closed-loop eigenvalue λ_i the following results apply.

$$\begin{aligned} \frac{\partial(I+G)}{\partial\lambda_i} \Big|_{\omega_e} &= \frac{\partial K}{\partial\lambda_i} \psi(j\omega_e) \\ \frac{\partial K}{\partial\lambda_i} &= -K \frac{\partial V}{\partial\lambda_i} V^{-1} \\ \frac{\partial V}{\partial\lambda_i} &= \left[\begin{array}{ccc} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{array} \right] \end{aligned} \quad (3-9)$$

← i →

Since $v_i = \phi(\lambda_i) B t_i$ then

$$\begin{aligned} \frac{\partial v_i}{\partial\lambda_i} &= \frac{\partial\phi(\lambda_i)}{\partial\lambda_i} B t_i \\ \frac{\partial\phi(\lambda_i)}{\partial\lambda_i} &= -\phi(\lambda_i) \frac{\partial\phi^{-1}(\lambda_i)}{\partial\lambda_i} \phi(\lambda_i) = -\phi(\lambda_i)^2 \\ \frac{\partial v_i}{\partial\lambda_i} &= -\phi(\lambda_i) v_i \end{aligned} \quad (3-10)$$

and thus

$$\frac{\partial K}{\partial\lambda_i} = K \phi(\lambda_i) v_i w_i^T \quad (3-11)$$

where w_i^T is the i 'th left eigenvector of $A-BK$ and

$$\frac{\partial\sigma}{\partial\lambda_i} \Big|_{\omega_e} = \text{Re}[\underline{u}^H K \phi(\lambda_i) v_i w_i^T \psi(j\omega_e) \underline{z}] \quad (3-12)$$

The gradient of $\sigma(I+G)$ with respect to Λ is

$$\begin{aligned} \nabla_{\Lambda} K &= \left[\frac{\partial K}{\partial\lambda_1}, \dots, \frac{\partial K}{\partial\lambda_n} \right] \\ \nabla_{\Lambda} \sigma \Big|_{\omega_e} &= \text{Re}[\underline{u}^H \cdot \nabla_{\Lambda} K \cdot \underline{z}_{\psi B}] \end{aligned} \quad (3-13)$$

The results of sections 3.1 and 3.2 concerning singular value gradients may be summarized as:

Proposition

Let σ be a distinct minimal singular value of $M = I+K(j\omega_e I-A)^{-1}B$, where $K = K(T,\Lambda)$ and M is analytic in the elements of T and Λ . (A, B) is a controllable pair, and let \underline{u} and \underline{z} denote the left and right singular vectors associated with σ . Λ is a set of distinct complex conjugate numbers which does not contain any members of the spectrum of A . Then the gradients of σ with respect to the design parameters T and Λ evaluated at $s = j\omega_e$ are

$$\nabla_T \sigma(I+K\phi B) = \text{Re}[\underline{u}_B \cdot \nabla_T K \cdot \underline{z}_{\psi B}] \quad (3-14)$$

$$\nabla_{\Lambda} \sigma(I+K\phi B) = \text{Re}[\underline{u}^H \cdot \nabla_{\Lambda} K \cdot \underline{z}_{\psi B}] \quad (3-15)$$

Remark 1. If σ is not a distinct singular value the concept of generalized gradients can be applied. This requires the Gateaux differentials, as discussed in [9].

Remark 2. When the elements t_{ij} and λ_i are complex the corresponding expressions $\nabla_T K$ and $\nabla_{\Lambda} K$ are given in appendix B.

Remark 3. Gradients according to other loop-breaking points and other robustness measures are given in appendix A.

Remark 4. The inclusion of multiple eigenvalues in Λ does not pose any principal problem, but it is computationally tedious. Anyway this is not justified from robustness considerations [16].

Remark 5. Open-loop eigenvalues of A and uncontrollable eigenvalues can be included in the set Λ . This requires slightly modified methods, as discussed in [17].

In section 4 the design adjustment procedure is outlined.

4.0 Design parameter adjustment procedure

The gradients $\nabla_T \sigma$ and $\nabla_{\Lambda} \sigma$ indicates the directions the design parameters should be adjusted towards to improve the robustness characteristics. In many situations it is not acceptable to adjust all design parameters since some of these correspond to desirable characteristics. In eigenstructure assignment some vectors t_i and the associated eigenvalues λ_i might have been chosen to obtain certain dominant response characteristics and certain "mode-mixing" i.e. a desirable I/O-response, whereas other vectors t_i and λ_i are less important/dominant from the I/O-point of view. These design parameters should then be chosen for adjustment in order to obtain good robustness properties.

If the measure $\sigma(I+G)$ evaluated of some frequency ω_e is too small, then the design parameters selected for adjustment (adjustment parameters) should be re-selected as

$$\begin{aligned} t_{ij} &= t_{ij}^0 + \epsilon_{ij} \frac{\partial\sigma}{\partial t_{ij}} & \epsilon_{ij} &\geq 0 \\ \lambda_i &= \lambda_i^0 + \epsilon_i \frac{\partial\sigma}{\partial\lambda_i} & \epsilon_i &\geq 0 \end{aligned} \quad (4-1)$$

where ϵ_{ij} and ϵ_i are adjustment parameters which are selected by the designer. Since only first order sensitivities are considered the parameters ϵ_{ij} and ϵ_i should be chosen carefully to guarantee an improvement of $\sigma(I+G)$.

If $\sigma(I+G)$ and possibly also other singular value quantities should be adjusted at several frequencies, and furthermore some quantities must not decrease at other frequencies, the problem of finding feasible solutions Δt_{ij} and $\Delta\lambda_i$ is less transparent. In this case the problem may be formulated as a linear programming problem as discussed in [10], or as a formal optimization problem.

One such formulation is based on the index [12]

$$J = \sum_{i=1}^N \left[\max\{0, \sigma_D(j\omega_i) - \sigma(j\omega_i)\} \right]^2 \quad (4-2)$$

where N is the total number of frequencies selected for adjustment and $\sigma_D(j\omega_i)$ is the desired level at $\omega = \omega_i$.

By using the gradients developed in §3.1 and 3.2 the optimization can be performed.

In [12] a constant level $\underline{\sigma}_D$ is used to improve the stability margins. If $\underline{\sigma}_D(j\omega_i)$ is instead frequency - dependent the optimization can be applied to multi-variable loop-shaping [11], where $\underline{\sigma}_D(j\omega_i)$ indicates the desired loop-shape.

The design procedure described above may be summarized in the following design principle

- Step 1. Do the initial design with some eigenspace technique.
- Step 2. Evaluate the stability robustness and the performance. If these are not acceptable go to step 3.
- Step 3. Select the frequencies where robustness should be improved, or should not be allowed to change. Evaluate the gradients $\nabla_{T\underline{\sigma}}$ and $\nabla_{\Lambda\underline{\sigma}}$ at the frequencies where improvements are desired and for the relevant robustness quantities.
- Step 4. Select the design parameters which are allowed to be adjusted and select the new set of adjustment parameters, T_{new} and Λ_{new} either by direct adjustment or with some optimization scheme. Evaluate the new controller K. Go to step 2.

The method does not provide any guarantees concerning achievable singular value plots. The final controller can be very dependent on the nominal controller.

Notice that the design objectives are only obtainable if they are in accordance with fundamental performance limitations [6, 7].

4.1 Computational and numerical issues

The computational requirements for evaluation of the singular value gradients are modest, since the formulas involved are straight-forward to evaluate. All necessary computations may be performed using standard numerically well-proven software like MATLAB [14]. MATLAB also provides a relevant media for the interaction between the designer and the design software.

If the parameter adjustment is performed by some optimization scheme, standard software also exists for this task [10, 12].

If the singular values of a matrix are clustered the calculation of the singular vectors is ill-conditioned. Techniques for coping with this problem are considered in [9]. Efficient calculation of transfer function matrices is considered in [15].

The calculation of the vector space, which spans the subspace which v_i belongs to, can be numerically ill-conditioned. For such cases the calculation of the assignable eigenvectors must be based on [4, 16]

$$[\lambda_i I - A, -B] \begin{bmatrix} v_i \\ t_i \end{bmatrix} = 0 \quad (4-3)$$

The singular value decomposition of the first matrix in (4-3) is

$$[\lambda_i I - A, -B] = U \Sigma Z^H \quad (4-4)$$

$$[\lambda_i I - A, -B] \begin{bmatrix} Z_{11i} & Z_{12i} \\ Z_{21i} & Z_{22i} \end{bmatrix} = U \begin{bmatrix} \sigma_1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \sigma_n & 1 & 0 \end{bmatrix}$$

$\begin{matrix} \longleftrightarrow & \longleftrightarrow \\ n & m \end{matrix} \qquad \begin{matrix} \longleftrightarrow & \longleftrightarrow \\ n & m \end{matrix}$

clearly

$$\begin{bmatrix} v_i \\ t_i \end{bmatrix} = \begin{bmatrix} Z_{12i} \\ Z_{22i} \end{bmatrix} \alpha_i \quad (4-5)$$

Based on this expression gradients can be evaluated. Details of this is outlined in [17].

5. Example

Consider the system (defined in [13]).

$$A = \begin{bmatrix} 0 & 0.9945 & 0.1044 & 0 \\ 0 & -1.525 & 0.0678 & -30.02 \\ 0 & -0.0166 & -0.1502 & 5.159 \\ 0.035 & 0.0698 & -0.9992 & -0.0903 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 11.51 & 5.241 \\ 0.1894 & -1.968 \\ -0.003 & 0.135 \end{bmatrix}$$

The initial design parameters have been selected as

$$T_0 = \begin{bmatrix} 4 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\Lambda_0 = [-2.63 \pm j3.26, -3.44 \pm j1.60]$$

In Figs. 1-3 the "nominal design" curve represents $\underline{\sigma}(I+K\Psi(j\omega))$. It is desired to improve the stability margins without deteriorating the nominal low frequency performance against disturbances. Therefore two frequencies $\omega_1 = 0.1$ r/s and $\omega_2 = 10$ r/s are selected as tuning frequencies, where $\nabla_{T\underline{\sigma}}$ and $\nabla_{\Lambda\underline{\sigma}}$ are evaluated.

Three different sets of adjustment parameters are considered, and the associated robustness improvement is considered.

Case 1. All elements of T are subject to adjustments, whereas Λ should remain unchanged.

Case 2. All elements of Λ are subject to adjustments, whereas T should remain unchanged.

Case 3. λ_3 and λ_4 and t_3 and t_4 are subject to adjustments, whereas the other elements of Λ and T should remain unchanged.

The improvements in the three cases are shown in Figs. 1-3 in 4 steps. In all three cases the objectives are satisfied. The improvements in the multivariable stability margins (based on $\underline{\sigma}(I+G)$) are summarized in Figure 4. Notice that in case 3 the stability robustness is almost equal to the LQ-margins, in the other two cases the margins are not much below LQ-margins.

In case 2 the nominal performance is also increased significantly. In cases 1 and 3 the low frequency nominal performance is almost equal to the initial value. In all 3 cases the price for the improvements is an increase in bandwidth and a change in T_0 and Λ_0 . The final values of T and Λ are in the 3 cases.

$$T_1 = \begin{bmatrix} 4 & -0.6 & 1.2 & -1.6 \\ 1.8 & 0.6 & 0.6 & 0 \end{bmatrix} \quad \Lambda_1 = \Lambda_0$$

$$T_2 = T_0 \quad \Lambda_2 = [-2.3 \pm j3.0, -5 \pm j4.5]$$

$$T_3 = \begin{bmatrix} 4 & 1 & 1 & -1.3 \\ 1 & 0 & 0.5 & 0.4 \end{bmatrix}$$

$$\Lambda_3 = [-2.63 \pm j3.26, -4.8 \pm j2.9]$$

6. Summary

In this paper a technique for improving insufficient robustness characteristics of control systems designed with eigenstructure assignment is considered. The technique is based on singular value gradients evaluated at selected frequencies with respect to the design parameters. With this technique it is possible to select a subset of the design parameters for adjustments, while the remaining design parameters are not affected. This property implies that significant characteristics of the initial design remain unaffected, e.g. closed-loop eigenvalues, mode-mixing and dominant behaviour. The formulas for evaluating the gradients are computationally simple and they are easily implemented with standard software.

The example illustrates that significant improvements can be obtained while important nominal design properties are maintained.

The methods in this paper are easily extended to the adjustment of observer design parameters, if these are formulated in the eigenstructure assignment framework.

The limitations of this technique are that gradient techniques do not necessarily find the "best" solutions to a given problem, unless it is used carefully. Furthermore, it is assumed that the robustness specifications can be expressed in the singular value-based context. More complex problems require the structured singular value [11].

References

- [1] B.C. Moore: "On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment", IEEE Transact. on Aut. Control 21, p. 689-92, 1976.
- [2] A.N. Andry et. al: "Eigenstructure assignment in linear systems", IEEE Transact. Aerospace & Electr. Syst., vol. 19, p. 711-29, 1983.
- [3] T. Sadeghi et. al: "A control law for pole/variant zero placement", American Control Conf. 1983, p. 912-7.
- [4] T.B. Cunningham: "Eigenspace selection procedures for closed-loop response shaping with modal control", IEEE CDC 1980, p. 178-86.
- [5] P. Sogaard-Andersen et. al: "Eigenstructure and residuals in multivariable state feedback design", Int. J. Control, vol. 44, p. 427-39, 1986.
- [6] J.C. Doyle and G. Stein: "Multivariable feedback design: Concepts for a classical/modern synthesis", IEEE Transact. on Aut. Control 26, p. 4-26, 1981.
- [7] M.G. Safonov et. al: "Feedback properties of multivariable systems: The role and use of the return difference matrix", IEEE Transact. on Aut. Control 26, p. 47-65, 1981.
- [8] N.A. Lehtomaki et. al: "Robustness results in linear quadratic gaussian based multivariable control designs", IEEE Transact. on Aut. Control 26, p. 75-92, 1981.

- [9] J.S. Freudenberg et. al: "Robustness analysis using singular value sensitivities", Int. J. Control, vol. 41, p. 1129-55, 1982.
- [10] R.S. Meeven and D.P. Looze: "Quadratic weight adjustment for the enhancement of feedback properties", American Control Conf. 1982, p. 996-1001.
- [11] G. Stein: "Beyond singular values and loop shapes". Rep. No. LIDS-P-1504, MIT Lab. of Information and Decision Systems, 1985.
- [12] J.R. Newson and V. Mukhopadhyay: "A multiloop robust-controller design study using singular value gradients", J. Guidance vol. 8, p. 514-9, 1985.
- [13] A.N. Andry et. al: "Robustness and eigenstructure in observer design", IEEE CDC 1983, p. 342-6.
- [14] C. Moler: "MATLAB - An interactive matrix laboratory", University of New Mexico, 1980.
- [15] A.J. Laub: "Efficient multivariable frequency response computations", IEEE Transact. on Aut. Control 26, p. 407-8, 1981.
- [16] J. Kautsky: "Robust pole assignment in linear state feedback", Int. J. Control, vol. 41, p. 1129-55, 1985.
- [17] P. Sogaard-Andersen, Ph.D. Thesis, Control Engineering Institute, DTH, Copenhagen, Denmark, 1986.
- [18] P.C. Hansen: "SVD - Theory and Applications", Ph.D. Thesis, Numerical Institute, DTH, Copenhagen, Denmark, 1984.

Appendix A

If the loop-breaking point is at the plant output, the corresponding robustness measure is

$$M = I + \psi(\omega_e)K = U\Sigma Z^H \quad (A-1)$$

and the corresponding gradients are

$$\frac{\partial \sigma}{\partial t_{ij}} = \text{Re} \left[\underline{u}^H \psi(j\omega_e) \frac{\partial K}{\partial t_{ij}} \underline{z} \right] \quad (A-2)$$

$$\frac{\partial \sigma}{\partial \lambda_i} = \text{Re} \left[\underline{u}^H \psi(j\omega_e) \frac{\partial K}{\partial \lambda_i} \underline{z} \right]$$

If the inverse return difference matrix is used as robustness measure and the loops are broken at the input then

$$M = I + (K\psi(j\omega_e))^{-1} = U\Sigma Z^H$$

$$\frac{\partial \sigma}{\partial t_{ij}} = -\text{Re} \left[U_1^H \frac{\partial K}{\partial t_{ij}} \psi(j\omega_e) Z_1 \right] \quad (A-3)$$

$$\frac{\partial \sigma}{\partial \lambda_i} = -\text{Re} \left[U_1^H \frac{\partial K}{\partial \lambda_i} \psi(j\omega_e) Z_1 \right]$$

$$U_1^H = \underline{u}^H (K\psi(j\omega_e))^{-1}, \quad Z_1 = (K\psi(j\omega_e))^{-1} \underline{z}$$

If $n = m$ results for the inverse return difference with the loop broken at the output are

$$M = I + (\psi(j\omega_e)K)^{-1} = U\Sigma Z^H$$

$$\frac{\partial \sigma}{\partial t_{ij}} = -\text{Re} \left[U_2^H \frac{\partial K}{\partial t_{ij}} Z_2 \right] \quad (A-4)$$

$$\frac{\partial \sigma}{\partial \lambda_i} = -\text{Re} \left[U_2^H \frac{\partial K}{\partial \lambda_i} Z_2 \right]$$

$$U_2^H = \underline{u}^H K^{-1}, \quad Z_2 = (\psi(j\omega_e)K)^{-1} \underline{z}$$

Appendix B

When some parameters λ_i and t_{ij} are complex numbers, expressions for the gradients are given below. The formula (3-2) requires the parameter p to be real [9, 12],

this is accomplished by using the real and imaginary parts of λ_i and t_{ij} separately. The associated eigenspace is [1] (subscripts R and I denotes real and imaginary part of)

$$\begin{aligned} v &= [v_1 \dots v_{iR} \quad v_{iI} \dots v_n] \\ T &= [t_1 \dots t_{iR} \quad t_{iI} \dots t_n] \\ v_{iR} &= \phi_{iR}(\lambda_i) B t_{iR} - \phi_{iI}(\lambda_i) B t_{iI} \\ v_{iI} &= \phi_{iI}(\lambda_i) B t_{iR} + \phi_{iR}(\lambda_i) B t_{iI} \\ \phi_{iR}(\lambda_i) &= \phi_{iR} = \text{Re} \phi(\lambda_i), \quad \phi_{iI}(\lambda_i) = \phi_{iI} = \text{Im}(\lambda_i) \end{aligned} \quad (B-1)$$

Then

$$\begin{aligned} M &= I + K \psi(j\omega_e) = U \Sigma Z^H \\ \frac{\partial \sigma}{\partial t_{ijR}} &= \text{Re} \left[u^H \frac{\partial K}{\partial t_{ijR}} \psi(j\omega_e) z \right] \\ \frac{\partial \sigma}{\partial t_{ijI}} &= \text{Re} \left[u^H \frac{\partial K}{\partial t_{ijI}} \psi(j\omega_e) z \right] \\ \frac{\partial K}{\partial t_{ijR}} &= - (e_j + K \phi_{iR} b_j) w_{iR}^T - K \phi_{iI} b_j w_{iI}^T \\ \frac{\partial K}{\partial t_{ijI}} &= K \phi_{iI} b_j w_{iR}^T - (e_j + K \phi_{iI} b_j) w_{iI}^T \end{aligned} \quad (B-2)$$

$$\begin{aligned} \frac{\partial \sigma}{\partial \lambda_{iR}} &= \text{Re} \left[u^H \frac{\partial K}{\partial \lambda_{iR}} \psi(j\omega_e) z \right] \\ \frac{\partial \sigma}{\partial \lambda_{iI}} &= \text{Re} \left[u^H \frac{\partial K}{\partial \lambda_{iI}} \psi(j\omega_e) z \right] \\ \frac{\partial K}{\partial \lambda_{iR}} &= -K \left[\frac{\partial v_{iR}}{\partial \lambda_{iR}} w_{iR}^T + \frac{\partial v_{iI}}{\partial \lambda_{iR}} w_{iI}^T \right] \\ \frac{\partial K}{\partial \lambda_{iI}} &= -K \left[\frac{\partial v_{iR}}{\partial \lambda_{iI}} w_{iR}^T + \frac{\partial v_{iI}}{\partial \lambda_{iI}} w_{iI}^T \right] \end{aligned} \quad (B-3)$$

$$\begin{aligned} \frac{\partial v_{iR}}{\partial \lambda_{iR}} &= (\lambda_{iR} I - A)^{-1} \phi_{iR} B t_{iI} - 2 \phi_{iR} v_{iR} \\ \frac{\partial v_{iI}}{\partial \lambda_{iR}} &= (\lambda_{iR} I - A)^{-1} \phi_{iR} B t_{iI} - 2 \phi_{iR} v_{iI} \\ \frac{\partial v_{iR}}{\partial \lambda_{iI}} &= -\phi_{iI} B t_{iI} / \lambda_{iI} + 2 \phi_{iI} v_{iR} \\ \frac{\partial v_{iI}}{\partial \lambda_{iI}} &= -\phi_{iI} B t_{iR} / \lambda_{iI} + 2 \phi_{iI} v_{iI} \end{aligned}$$

Similar expressions can be found for other robustness measures.

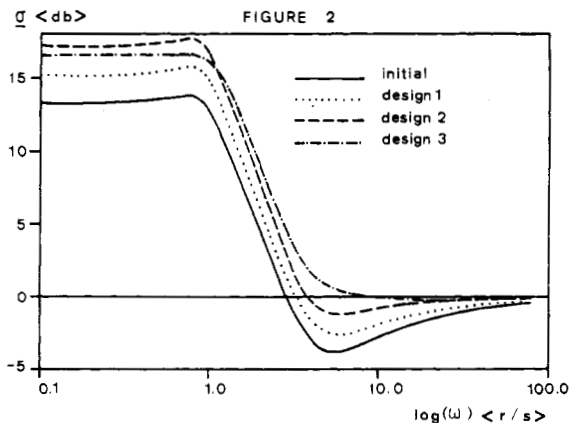
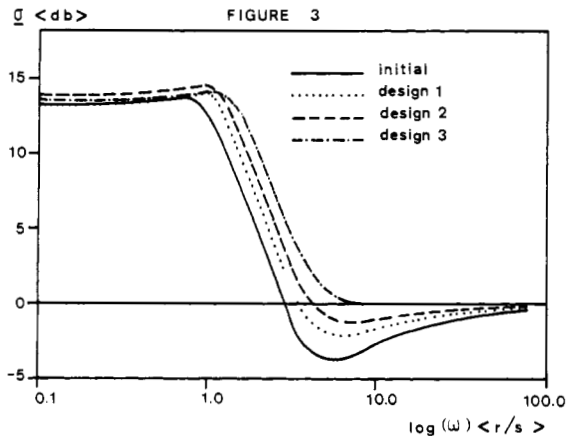
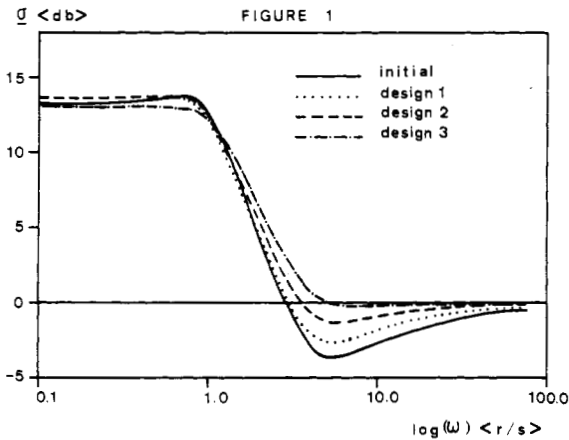


Figure 4 Gain margins Phase margins

	Gain margins	Phase margins
Initial design	0.61 < g_m < 2.8	37.6° > $ \theta_m $
Case 1 - final	0.51 < g_m < 33.33	57.6° > $ \theta_m $
Case 2 - final	0.51 < g_m < 33.33	57.6° > $ \theta_m $
Case 3 - final	0.5004 < g_m < 678.9	59.9° > $ \theta_m $
LQ-margins	0.5 < g_m < ∞	60° > $ \theta_m $