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Uniqueness of time-independent electromagnetic fields

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As a comment on a recent paper by Steele, a more general uniqueness theorem for time-independent fields is mentioned.

Uniqueness theorems for time-independent electromagnetic fields in linear media are found in many textbooks. Recently, Steele[1] proved theorems of this kind for nonlinear isotropic media with positive differential permeability and permittivity. This result, however, is essentially contained in a more general theorem due to Graffi,[2] which will be mentioned here since Ref. 2 is not easily accessible. To facilitate applications, a somewhat extended form of Graffi's theorem will be given.

Consider a vector field $\mathbf{Y}$ that depends upon another vector field $\mathbf{X}$. Let R denote a simply connected region bounded by closed surfaces $S_1, S_2, \cdots, S_m$; S_m is the external boundary, absent when R is unbounded, and $\mathbf{n}$ denotes the outward normal unit vector. We introduce the following assumptions:

(i) $\mathbf{X}_1 \neq \mathbf{X}_2$ implies $\mathbf{Y}(\mathbf{X}_1) \neq \mathbf{Y}(\mathbf{X}_2)$ and $(\mathbf{X}_1 - \mathbf{X}_2) \cdot (\mathbf{Y}(\mathbf{X}_1) - \mathbf{Y}(\mathbf{X}_2)) > 0$;

(ii) In R, $\nabla \times \mathbf{X}$, and $\nabla \cdot \mathbf{Y}$ are prescribed;

(iii) If R is unbounded, $|\mathbf{X}|$ and $|\mathbf{Y}|$ vanish as r^{-2} when $r \to \infty$, r denotes the distance from some fixed point;

(iv) For each i $(=1, 2, \cdots, m)$ *either* $\mathbf{n} \cdot \mathbf{Y}$ *or* $\mathbf{n} \times \mathbf{X}$ on S_i is prescribed;

(v) If for $i < m$, $\mathbf{n} \times \mathbf{X}$ on S_i is prescribed, then *either* $\int_{S_i} \mathbf{n} \cdot \mathbf{Y}\, dS$ *or* $\int_{C_i} \mathbf{X} \cdot d\mathbf{s}$ is prescribed, where C_i is a curve in R from a point on S_i to a fixed point Q on S_m if R is bounded, otherwise to infinity.

Then the vector field $\mathbf{X}$, and thus $\mathbf{Y}$, is uniquely determined.

To prove this uniqueness theorem we assume (as usual) the existence of two solutions, $\mathbf{X}_1$ and $\mathbf{X}_2$, introduce the difference fields

$$\mathbf{X}_d = \mathbf{X}_1 - \mathbf{X}_2,$$

$$\mathbf{Y}_d = \mathbf{Y}_1 - \mathbf{Y}_2 = \mathbf{Y}(\mathbf{X}_1) - \mathbf{Y}(\mathbf{X}_2),$$

and consider the integral

$$I = \int_R \mathbf{X}_d \cdot \mathbf{Y}_d\, dv.$$

It follows from (ii) that $\nabla \times \mathbf{X} = 0$ in R; we can thus write $\mathbf{X}_d = -\nabla U$ with $U(P) = \int_P^Q \mathbf{X}_d \cdot d\mathbf{s}$. Moreover, (ii) implies $\nabla \cdot \mathbf{Y}_d = 0$. Hence, by the divergence theorem [and (iii) when applicable] we obtain

$$I = \int_R -(\nabla U) \cdot \mathbf{Y}_d\, dv = -\int_R \nabla \cdot (\mathbf{Y}_d U)\, dv = -\sum_{i=1}^{m} I_i,$$

where

$$I_i = \int_{S_i} \mathbf{n} \cdot (\mathbf{Y}_d U)\, dS.$$

We now utilize (iv). Either $\mathbf{n} \cdot \mathbf{Y}_d = 0$, which yields $I_i = 0$, or $\mathbf{n} \times \mathbf{X}_d = 0$, which means that U is a constant U_i on S_i, and thus

$$I_i = U_i \int_{S_i} \mathbf{n} \cdot \mathbf{Y}_d\, dS.$$

Obviously $U_m = 0$. For $i < m$, we infer from (v) that either $U_i = 0$ or $\int_{S_i} \mathbf{n} \cdot \mathbf{Y}_d\, dS = 0$. In any case, $I_i = 0$.

It now follows that $I = 0$. This is, however, because of (i), impossible unless $\mathbf{X}_d = 0$ throughout R. Consequently, $\mathbf{X}$ is uniquely determined.

A similar theorem holds when R is multiply connected. In that case we must prescribe together with $\mathbf{n} \cdot \mathbf{Y}$ all independent circulations $\oint \mathbf{X} \cdot d\mathbf{s}$ associated with the S_i in question (e.g., one circultation if S_i is the surface of a toroid). Then U becomes one valued and the proof just given remains valid.

The applications of the theorem to classical cases are straightforward. For electrostatic fields we can prescribe the charge density in the dielectric part of space and the potential *or* charge of each conductor; for stationary magnetic fields the current density is prescribable. In either case, condition (i) defines the class of materials to which the theorem is applicable. Clearly, condition (i) is satisfied when $\mathbf{X}$ and $\mathbf{Y}$ are parallel and $\mathbf{Y}$ is an increasing function of $\mathbf{X}$. The problem considered by Steele[1] is thus included as a particular case.

[1] C.W. Steele, J. Appl. Phys. **44**, 3790 (1973).
[2] D. Graffi, in *Scritti matematici in onore di Filippo Sibirani* (Zuffi, Bologna, 1956), p. 143.

