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Antennas on Circular Cylinders

H. LOTTRUP KNUDSEN†

Summary—On the basis of the results obtained by Silver and Saunders [4] for the field radiated from an arbitrary slot in a perfectly conducting circular cylinder, expressions have been derived for the field radiated by a narrow helical slot, with an arbitrary aperture field distribution, in a circular cylinder. The cases of a standing wave and a progressing wave aperture field are given particular consideration. It has also been indicated how a finite width of the slot can be taken into consideration. The results for the helical slot have been used for calculating the field radiated from a *U*-shaped slot antenna in a circular cylinder.

By a procedure similar to the one used by Silver and Saunders, expressions have been derived for the field radiated from an arbitrary surface current distribution on a cylinder surface coaxial with a perfectly conducting cylinder. The cases where the space between the two cylindrical surfaces have the same characteristic constants and different constants are treated separately.

Extensive numerical computations of the field radiated from the slot antennas described here are being carried out, but no numerical results are yet available.

Introduction

DURING the last ten or fifteen years, there has been a great interest in antennas on or near circular cylinders. In particular, a large number of papers have dealt with the radiation from slots in such cylinders. In 1958 the author [1] gave a survey of the results of this work. The report containing this survey was made for the Martin Company, Littleton, Colo., by the P.E.C. Corporation, Boulder, Colo., and published by this corporation. This report contains references to the literature, working formulas for the radiated fields, and a compilation of graphs of radiated fields, many of these graphs published for the first time there. No derivations of the formulas are given; in collecting the material for this report, a bibliography published by Wait [2] in 1957 was found very helpful.

Wait [3] has published a very useful monograph in which a large amount of material is compiled regarding the fields radiated from slots in circular cylinders and other types of cylinders; derivations of the formulas are given in the necessary detail, and the meaning of the formulas are in many cases made clear by graphs. Much of the material in the book is due to his own investigations.

The present paper deals with the field radiated from various types of antennas on or near circular cylinders. The work described is a minor part of some work which the author undertook for the P.E.C. Corporation on a contract with the Martin Company from September, 1958, until June, 1959.

Among the many excellent papers published during the last decade on the field radiated from slot antennas in circular cylinders one paper, by Silver and Saunders [4], is outstanding in this author's opinion. Their paper has served, or could have served, as a model for any paper dealing with the field radiated from a slot, with a prescribed aperture field in a circular cylinder having a diameter of the same order of magnitude as the wavelength. It has also served as the basis for all our slot antenna computations and as a model for the computations made here regarding the field radiated from wire antennas near circular cylinders. We shall, therefore, start by stating the result of Silver and Saunders in a form suitable for our own use.

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Arbitrary Slot in Circular Cylinder

For convenient reference, we shall state here the formula derived by Silver and Saunders for the field radiated by an arbitrary slot in an infinitely long circular cylinder. Consider a circular cylinder with radius a as shown in Fig. 1. A rectangular coordinate system

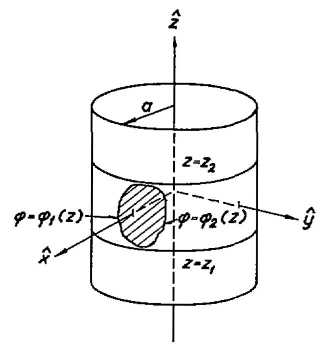

Fig. 1—Arbitrary slot in circular cylinder.

(x, y, z) is introduced with the axis of the cylinder as the z axis, and we further introduce a cylindrical coordinate system (ρ, ϕ, z) and a spherical coordinate system (r, θ, ϕ) related in the conventional way with the rectangular coordinate system. On the surface of the cylinder is a slot of arbitrary shape; the slot is situated in the region between $z=z_1$, and $z=z_2$ $(z_1<z_2)$, and it is bounded by the curves $\phi=\phi_1(z)$ and $\phi=\phi_2(z)$.

It is assumed that the tangential component of the electric field strength in the slot is described by

$$E_\phi = \begin{cases} F_1(\phi, z) & \text{in the aperture,} \\ 0 & \text{outside the aperture,} \end{cases}$$

$$E_z = \begin{cases} F_2(\phi, z) & \text{in the aperture,} \\ 0 & \text{outside the aperture,} \end{cases}$$

where $F_1(\phi, z)$ and $F_2(\phi, z)$ are prescribed functions of ϕ and z.

Here, and in what follows, we use as a time factor $e^{j\omega t}$ and define the wave number $k=2\pi/\lambda$ where λ is the wavelength. The components of the electric field strength in the far zone field may then be expressed as

$$E_r = 0,$$

$$E_\theta = -\frac{e^{-jkr}}{2\pi^2 r}\sum_{n=-\infty}^{\infty}\frac{j^{n+1}e^{-jn\phi}}{\sin\theta H_n^{(2)\prime}(ka\sin\theta)} I_n^{(2)},$$

$$E_\phi = \frac{e^{-jkr}}{2\pi^2 r}\sum_{n=-\infty}^{\infty}\frac{j^n e^{-jn\phi}}{H_n^{(2)}(ka\sin\theta)}\left[I_n^{(1)} + \frac{n\cos\theta}{ka\sin^2\theta} I_n^{(2)}\right],$$

where

$$I_n^{(\kappa)} = \int_{z_1}^{z_2} d\xi e^{jk\xi\cos\theta}\int_{\phi_1(\xi)}^{\phi_2(\xi)} F_\kappa(\beta,\xi)e^{jn\beta}d\beta \qquad \kappa = 1 \text{ and } 2.$$

In carrying out the above integration, it is convenient to be able to use any orthogonal coordinates (u_1, u_2) on the cylinder surface which fit the geometry of the slot. For this purpose, we redefine the components of the aperture field $\overline{F}$ by

$$E_\phi = \begin{cases} F_1(u_1, u_2) & \text{in the aperture,} \\ 0 & \text{outside the aperture,}\end{cases}$$

$$E_z = \begin{cases} F_2(u_1, u_2) & \text{in the aperture,} \\ 0 & \text{outside the aperture.}\end{cases}$$

An element of length dl on the cylinder surface $r=a$ may now be expressed by

$$dl = \sqrt{(h_1du_1)^2 + (h_2du_2)^2},$$

where $h_1=h_1(u_1, u_2)$ and $h_2=h_2(u_1, u_2)$ are functions of the coordinates u_1 and u_2. We further express ξ and β as functions of u_1 and u_2,

$$\xi = \xi(u_1, u_2),$$

$$\beta = \beta(u_1, u_2).$$

Often the tangential electric field $\overline{F}(u_1, u_2)$ in the aperture is expressed as a certain constant voltage V_0 divided by a certain constant length w and multiplied by a normalized dimension-free field distribution function $\overline{f}(u_1, u_2)$

$$\overline{F}(u_1, u_2) = \frac{V_0}{w}\overline{f}(u_1, u_2).$$

It may be convenient to further express the far zone field $\overline{E}(r, \theta, \phi)$ in the following way

$$\overline{E}(r,\theta,\phi) = V_0\frac{e^{-jkr}}{r}\bar{e}(\theta,\phi).$$

The normalized electric field strength $\bar{e}(\theta, \phi)$, which does not depend upon r, has the components

$$e_r = 0,$$

$$e_\theta = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{j^{n+1}e^{-jn\phi}}{\sin\theta H_n^{(2)}(ka\sin\theta)} i_n^{(2)},$$

$$e_\phi = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{j^n e^{-jn\phi}}{H_n^{(2)\prime}(ka\sin\theta)}\left[i_n^{(1)} + \frac{n\cos\theta}{ka\sin^2\theta} i_n^{(2)}\right],$$

where $i_n^{(\kappa)}$ are dimension-free quantities given by

$$i_n^{(\kappa)} = \frac{1}{aw}\iint f_\kappa(u_1, u_2) \cdot e^{j[k\xi(u_1,u_2)\cos\theta + n\beta(u_1,u_2)]}h_1(u_1, u_2)h_2(u_1, u_2)du_1, du_2$$

$$\kappa = 1 \text{ and } 2.$$

This is a convenient formulation of Silver and Saunders' results for many practical purposes.

Helical Slot in Circular Cylinder

Arbitrary Aperture Field

Although there is a considerable practical interest attached to the use of inclined slots in a circular cylinder, to the author's knowledge no analytical expression for, and no numerical computation of, the field radiated from an inclined slot in a circular cylinder have been published so far. The content of this section is a revised edition of part of a Master's thesis by Pedersen [5]; the thesis work was done under the author's direction in 1958.

With reference to Fig. 2, let us first consider an in-

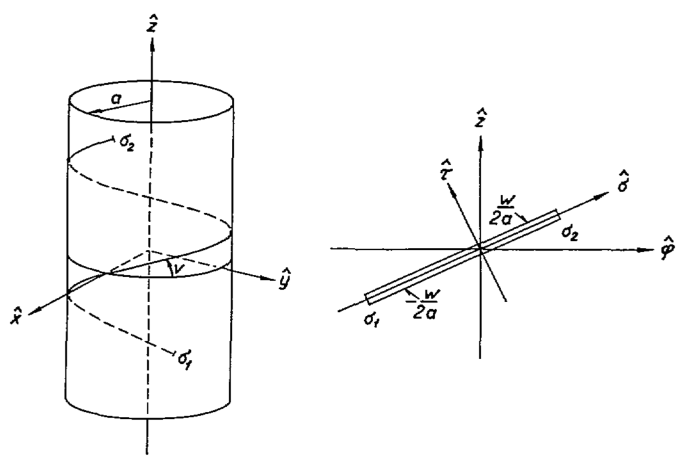

Fig. 2—Helical slot in circular cylinder.

finitely narrow, helical slot in an infinitely long, circular cylinder, and let us assume that the field in the slot is prescribed. The radius of the cylinder is called a. The centerline of the slot is assumed to be a piece of a helix with its midpoint passing through the point $(x, y, z) = (a, 0, 0)$ as shown in Fig. 2. The angle which the helix forms with a plane transverse to the axis is called v. We introduce a coordinate σ along the slot so that $s=a\sigma$ is the length of arc in the σ direction. The ends of the slot are assumed to be situated at

$$\sigma = \pm \sigma_1 = \pm \frac{s_1}{2a}.$$

The tangential component of the aperture field is assumed to be transverse to the slot. Denoting by $\hat{\tau}$ a unit vector pointing in such a direction transverse to the slot that $\hat{\sigma}\times\hat{\tau}$ is a unit vector pointing normally outwards from the cylinder surface, we may express the tangential aperture field by

$$\overline{F} = \frac{V_0 f(\sigma)}{w}\hat{\tau},$$

where V_0 has the dimension of voltage and w the dimension of length, whereby $f(\sigma)$ becomes dimension-free. We may think of w as the width of the slot and of V_0 as a reference voltage across the slot.

Using the general formulas for the field radiated from an arbitrary slot in a circular cylinder given in the last section, we may now obtain the following expressions for the components of the normalized electric field strength radiated from the inclined slot

$$e_\theta = -\frac{\cos v}{2\pi^2 \sin\theta}\sum_{n=-\infty}^{\infty}\frac{j^{n+1}e^{-jn\phi}j_n}{H_n^{(2)}(ka\sin\theta)},$$

$$e_\phi = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{j^n e^{-jn\phi}j_n}{H_n^{(2)\prime}(ka\sin\theta)}\left[-\sin v + \frac{n\cos v\cos\theta}{ka\sin^2\theta}\right],$$

where

$$j_n = \int_{-\sigma_1}^{\sigma_1} f(\sigma)e^{jK_n\sigma}d\sigma$$

with

$$K_n = ka\sin v\cos\theta + n\cos v.$$

When the aperture field is given, *i.e.*, when $f(\sigma)$ is known, the integral j_n may be computed and the field found from the above formulas.

So far, we have considered only infinitely narrow slots. It is desirable also to have formulas available for the field radiated from an inclined slot of finite width. Considering an inclined slot with a finite width w, as shown in Fig. 3, and assuming that the tangential aperture field is transverse to the longitudinal direction of the slot and that the field does not vary in the transverse direction, we find that the formulas for the radiated fields given above should be replaced by

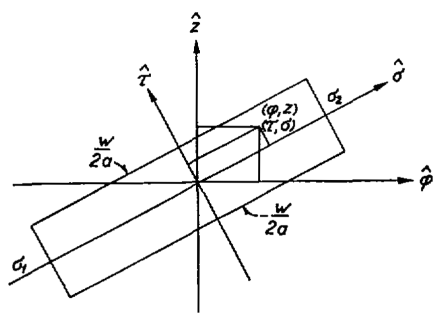

Fig. 3—Inclined slot with finite width.

$$e_\theta = -\frac{\cos v}{2\pi^2\sin\theta}\sum_{n=-\infty}^{\infty}\frac{j^{n+1}e^{-jn\phi}j_n}{H_n^{(2)}(ka\sin\theta)}P_n,$$

$$e_\phi = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{j^n e^{-jn\phi}j_n}{H_n^{(2)\prime}(ka\sin\theta)}\left[-\sin v + \frac{n\cos v\cos\theta}{ka\sin^2\theta}\right]P_n,$$

where

$$P_n = \frac{\sin\dfrac{H_n w}{2a}}{\dfrac{H_n w}{2a}}$$

with

$$H_n = ka\cos v\cos\theta - n\sin v.$$

The present formulas differ from the corresponding expressions for an infinitely narrow, inclined slot with the same field distribution only by the factor P_n, which is present here, but absent in the formulas for the infinitely narrow slot. If we let the width, w, of the slot tend towards zero, we find

$$\lim_{w\to 0} P_n = 1.$$

The expressions for the components e_θ and e_ϕ, derived here, then converge towards the expressions for e_θ and e_ϕ for an infinitely narrow, inclined slot, as they should.

In the following, only the expressions for an infinitely narrow slot will be stated.

Sinusoidal Aperture Field

One use of the above expressions is for obtaining an approximate expression for the field radiated from a plane, inclined slot with a sinusoidal aperture field.

A plane slot in a circular cylinder actually is part of an ellipse. However, when the slot is short, as compared to the circumference of the cylinder, it may be well approximated by part of a helix. The theory given in the last section may, therefore, be used in finding the field radiated from such a slot. The length of the slot is called l, and the center of the slot is assumed to be situated at $(x, y, z) = (a, 0, 0)$. A sinusoidal field distribution in the slot will then be expressed by the normalized field distribution function

$$f(\sigma) = \cos\frac{\pi a}{l}\sigma.$$

Inserting this function in the above formula for the integral j_n, we obtain

$$j_n = \frac{\dfrac{2\pi a}{l}\cos K_n\dfrac{l}{2a}}{\left(\dfrac{\pi a}{l}\right)^2 - K_n^2},$$

where K_n is given above. This expression for j_n should be inserted in the above expressions for the θ- and ξ-components of the normalized electric field strength $\bar{e}$.

Setting $v=0$ in the formulas described above, we obtain the well-known formulas for the field radiated from a circumferential slot. On the other hand, setting $v=90°$ in the above mentioned formulas, we obtain expressions which are identical with the well-known formulas for the axial slot found in the literature, except for the fact that the two formulas have opposite signs. This difference in sign is due to the circumstance that here the positive direction for the field in the aperture is chosen as the direction of $\hat{\sigma}$ (which in the case of the axial slots becomes $\hat{\sigma}=-\hat{\phi}$) whereas in most papers dealing with the axial slot, it is customary to choose the direction of $\hat{\phi}$ as the positive direction for the aperture field.

In the above investigation, we have assumed the inclined slot to be helical instead of being an ellipse as it would be if the slot were formed by cutting the cylinder with a plane. The distance Δ between the end points of the actual elliptical slot and the helical slot (see Fig. 4) is approximately

$$\Delta = a \sin v(\phi_m - \sin \phi_m),$$

where

$$\phi_m = \frac{l \cos v}{2a}.$$

For the special case of a half wavelength slot, the relative distance ϵ between the helix and the ellipse at their end points, defined as the distance Δ divided by the length l of the slot, has been plotted in Fig. 5. It is seen from this figure that when the diameter of the cylinder is one wavelength or more, the maximum relative deviation ϵ between the actual and the approximating slot is only a few tenths of one per cent.

From the above expressions for the components of the field radiated by an inclined slot with a sinusoidal aperture field distribution, we may obtain the following symmetry relations

$$e_\theta(\pi - \theta, -\phi) = e_\theta(\theta, \phi),$$

$$e_\phi(\pi - \theta, -\phi) = e_\phi(\theta, \phi).$$

Since these symmetry relations could also have been established directly by inspection of the geometrical properties of the slot and its excitation, the derivation of these relations furnishes a partial check of the above expressions. The preceding symmetry relations seem generally to exhaust the symmetry properties of the field radiated from an inclined slot with a symmetric excitation. However, in special cases further symmetry relations may exist. For a circumferential slot ($v=0$), for example, we have further

$$e_\theta(\theta, -\phi) = e_\theta(\theta, \phi),$$

$$e_\phi(\theta, -\phi) = - e_\phi(\theta, \phi).$$

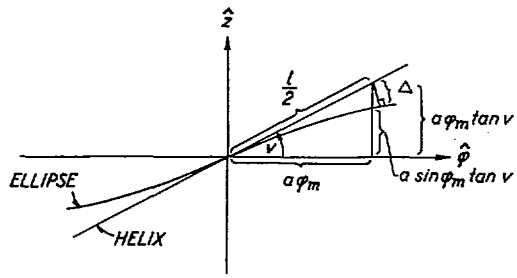

Fig. 4—Relative deviation of helical slot from plane, elliptical slot.

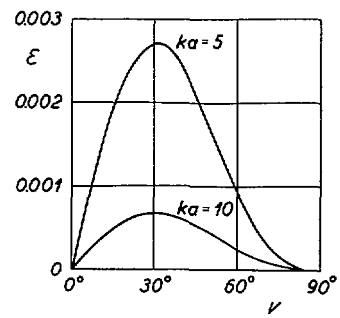

Fig. 5—Relative deviation of helical slot from plane, elliptical slot.

For an axial slot ($v=90°$), we have the following additional symmetry relation

$$e_\phi(\theta, -\phi) = e_\phi(\theta, \phi).$$

All of the previous symmetry relations also hold for the approximate field expressions obtained from the exact expressions by replacing the summation from $-\infty$ to ∞ by a summation from $-N$ to N. The above symmetry relations may therefore be used as a partial check in carrying out the numerical computations.

Progressing Aperture Wave Field

It has been suggested that the field radiated from a helical slot in a circular cylinder with a progressing wave in the aperture be investigated. We assume that the normalized aperture field distribution $f(\sigma)$ is a progressing wave given by

$$f(\sigma) = e^{-ipka\sigma},$$

where p is the quotient between the velocity of light and the velocity of the aperture wave field. Considering only a helical slot with an integral number, N, of turns, we obtain, by using the general expression for the radiated field given above, the following expression for the normalized field $\bar{e}(\theta, \phi)$ radiated from the slot

$$\bar{e}(\theta, \phi) = G(\theta)\bar{e}^*(\theta, \phi - (N - 1)\pi),$$

where $G(\theta)$ is the array characteristic of an array of N single turns, and where $\bar{e}^*(\theta, \phi)$ is the field radiated from a single turn of such a helical slot. The components of the normalized electric field strength $\bar{e}^*(\theta, \phi)$ radiated from a single turn are given by

$$e_\theta^*(\theta, \phi) = -\frac{\sin \pi\mu}{\pi^2 \sin \theta} \sum_{n=-\infty}^{\infty} \frac{e^{j[n((3\pi/2)-\phi)+\pi/2]}}{(\mu + n)H_n^{(2)}(ka \sin \theta)},$$

$$e_\phi^*(\theta, \phi) = \frac{\sin \pi\mu}{\pi^2 \cos v} \sum_{n=-\infty}^{\infty} \frac{e^{jn((3\pi/2)-\phi)}}{(\mu + n)H_n^{(2)\prime}(ka \sin \theta)} \cdot \left[-\sin v + \frac{n \cos v \cos \theta}{ka \sin^2 \theta} \right],$$

where $\mu = ka$ ($\tan v \cos \theta - p/\cos v$).

It should be noted that the above relation between the field $\bar{e}$ from a helical slot with N turns and the field $\bar{e}^*$ from a single turn of this slot connects $\bar{e}(\theta, \phi)$, not with $\bar{e}^*(\theta, \phi)$, but with $\bar{e}^*(\theta, \phi - (N-1)\pi)$. This is due to the fact that helical slots with an even number of turns are oriented differently in the coordinate system (x, y, z) from the helical slot with an odd number of turns, in that the end-points of a helical slot with an even number of turns are situated on the line $x = a$, $y = 0$, whereas the end-points of a helical slot with an odd number of turns are situated on the line $x = -a$, $y = 0$. This is illustrated in Fig. 6 for the cases of $N = 1$ and $N = 2$.

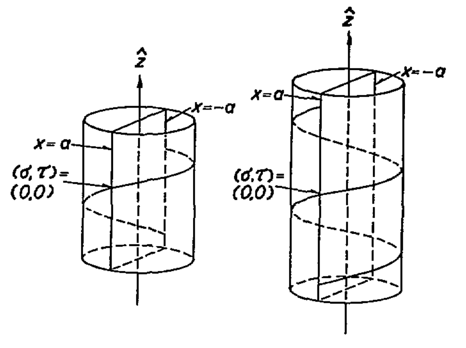

Fig. 6—Helical slot with (a) an odd and (b) an even number of turns.

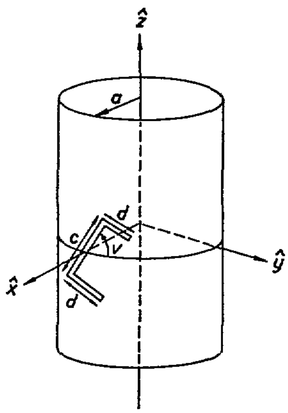

Fig. 7—U-shaped slot in circular cylinder.

U-Shaped Slot in Circular Cylinder

One type of antenna that seems to be generally used as a flush-mounted, radiating element in a circular cylinder is a U-shaped slot. In this section, expressions shall be derived for such an antenna under the assumption that the slot is infinitely narrow, and that it is made up of parts of three helical slots. Only the case of a symmetrical slot shall be considered here.

In Fig. 7 is shown a symmetrical, U-shaped slot on the surface of a circular cylinder with radius a. The length of the central part of the slot is called c and the length of the outer parts of the slot d. A coordinate system (x, y, z) is introduced as shown in the figure, so that the midpoint of the slot is located at $(x, y, z) = (a, 0, 0)$. The slot is given a positive direction, and the angle which the central part of the slot so oriented forms with a plane transverse to the axis of the cylinder is called v. We introduce a dimension-free coordinate σ along the slot with $\sigma = 0$ corresponding to one end-point of the slot, and so defined that $s = a\sigma$ is the length of the arc along the slot. The aperture field in the slot is assumed to be everywhere transverse to the longitudinal direction of the slot. The component $F_\tau(\sigma)$ of the aperture field in the transverse direction, counted positive from the "inside" of the slot towards the "outside," is assumed to be

$$F_\tau(\sigma) = \frac{V_0}{w} \sin q\sigma,$$

where

$$q = \frac{\pi a}{c + 2d}.$$

By using the general formulas given above for the field radiated from an arbitrary slot with an arbitrary aperture field distribution, we then obtain the following expressions for the components of the normalized electric field strength of the field radiated from the U-shaped slot with a sinusoidal aperture field distribution,

$$e_\theta = -\frac{1}{2\pi^2 \sin \theta} \sum_{n=-\infty}^{\infty} \frac{j^{n+1} e^{-jn\phi}}{H_n^{(2)}(ka \sin \theta)} B_n,$$

$$e_\phi = \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \frac{j^n e^{-jn\phi}}{H_n^{(2)\prime}(ka \sin \theta)} \left[A_n + \frac{n \cos \theta}{ka \sin^2 \theta} B_n \right],$$

where

$$A_n = j_n^* \cos v - j_n^{(2)} \sin v,$$

$$B_n = j_n^* \sin v + j_n^{(2)} \cos v,$$

with

$$j_n^* = \frac{2\sin\left(K_n^{(2)}\frac{c}{2a}\right)}{q^2-(K_n^{(1)})^2}\left\{q\sin\left(K_n^{(1)}\frac{d}{a}\right) - K_n^{(1)}\cos\left(q\frac{c}{2a}\right) + jq\left[\cos\left(K_n^{(1)}\frac{d}{a}\right)-\sin\left(q\frac{c}{2a}\right)\right]\right\},$$

$$j_n^{(2)} = \frac{2}{q^2-(K_n^{(2)})^2}\left\{q\cos\left(K_n^{(2)}\frac{c}{2a}\right)\sin\left(q\frac{c}{2a}\right) - K_n^{(2)}\sin\left(K_n^{(2)}\frac{c}{2a}\right)\cos\left(q\frac{c}{2a}\right)\right\},$$

$$K_n^{(1)} = ka\cos v\sin\theta - n\sin v,$$

$$K_n^{(2)} = ka\sin v\cos\theta + n\cos v.$$

A partial check of the above formulas may be obtained by applying them to the case of $d=0$, in which case the U-shaped slot degenerates into an ordinary inclined slot with the length c. It may readily be verified that in this case the above formulas simplify to the formulas obtained previously for a simple inclined slot.

Arbitrary Current Distribution on Circular Cylindrical Surface Coaxial with Circular, Conducting Cylinder

The author has been asked to solve the problem of computing the field radiated from a helical wire, with a progressing current wave, coaxial with a conducting, circular cylinder. This problem may be considered a particular case of the more general problem of finding the field radiated from an arbitrary, given distribution of surface current on a circular cylinder coaxial with the conducting, circular cylinder. Since the latter, more general problem may be solved with only a little more work than required for solving the particular problem of a helical current distribution, and since the solution of the general problem may also have practical applications to other problems, we shall here solve this problem.

Consider a perfectly conducting cylinder with radius a and a coaxial, circular cylindrical surface with radius b as shown in Fig. 8. In formulating the problem, we shall use cylindrical coordinates (ρ, ϕ, z) with the axis of the cylinder as the z axis, whereas for expressing the far zone field, we shall use spherical coordinates (r, θ, ϕ), these coordinates being related to the cylindrical coordinates in the usual way. On the cylindrical surface with radius b, a distribution of electric current is assumed to exist. The surface current density is assumed to be equal to zero outside a finite region (the aperture), the circumference of which is given by the curves $\phi=\phi_1(z)$ and $\phi=\phi_2(z)$, these curves being confined to

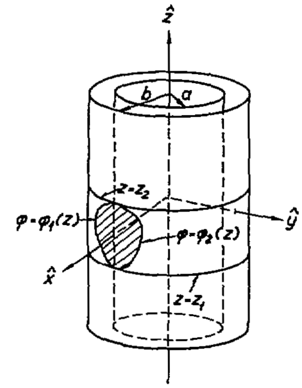

Fig. 8—Surface current distribution on circular cylindrical surface coaxial with conducting, circular cylinder.

the region $z_1 \leqq z \leqq z_2$. The surface current density $\overline{K}(\phi, z) = K_\phi(\phi, z)\hat{\phi} + K_z(\phi, z)\hat{z}$ is given by

$$K_\phi = \begin{cases} G_1(\theta, \phi) & \text{in the aperture,} \\ 0 & \text{outside the aperture,} \end{cases}$$

$$K_z = \begin{cases} G_2(\theta, \phi) & \text{in the aperture,} \\ 0 & \text{outside the aperture,} \end{cases}$$

where $G_1(\theta, \phi)$ and $G_2(\theta, \phi)$ are prescribed functions. The problem is to find the field that satisfies Maxwell's equations in the regions $a<\rho<b$ and $b<\rho<\infty$, the boundary conditions at the metallic surface $\rho=a$, the saltus conditions at the surface $\rho=b$, and the radiation conditions at infinity (*i.e.*, for $r\to\infty$). The boundary condition at $r=a$ is expressed by

$$\left.\begin{aligned} E_\phi &= 0 \\ E_z &= 0 \end{aligned}\right\} \quad \text{for } \rho = a,$$

and the saltus condition at $\rho=b$ is expressed by

$$\left.\begin{aligned} -H_{1\phi} + H_{2\phi} &= K_z \\ H_{1z} - H_{2z} &= K_\phi \\ \text{and} \quad E_{1\phi} - E_{2\phi} &= 0 \\ E_{1z} - E_{2z} &= 0 \end{aligned}\right\} \quad \text{for } \rho = b,$$

where index 1 refers to the inside and index 2 to the outside of the surface $r=b$.

If the field set up by a Hertz dipole near a conducting circular cylinder were known, the solution to the problem formulated above could be obtained as a vectorial superposition of the fields set up by the infinitely many dipoles, into which the present current distribution may be decomposed. Now, the far zone field of a Hertz dipole near a conducting circular cylinder has been obtained by Carter [6], who first computed the current induced in a Hertz dipole near a cylinder by an incident plane wave and then used the reciprocity theorem. However, for the following three reasons, we shall here use a more direct approach, without making use of Carter's results.

1) It is inherent in Carter's derivation that his results can be used only for finding the far zone field. For some problems, it may be desirable to have formulas available for the field at any distance from the current distribution.
2) A certain amount of work is necessary for arriving at the desired result using Carter's formulas. It seems, therefore, more motivated to spend the energy in making a direct derivation of the formulas instead of using a method which depends upon the use of the reciprocity theorem.
3) As has been pointed out elsewhere by the author [7] and by Nielsen [8], Carter's paper, the excellence of which is undisputed, contains many trivial errors. It seems, therefore, preferable to use a different and more direct method for obtaining the desired formulas.

The method which will be used for solving the present problem is completely analogous to the method used by Silver and Saunders [4] in finding the field radiated from an arbitrary slot in a circular cylinder. The surface current density on the cylinder may be expressed by the following expansion

$$K_\kappa(\phi, z) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} e^{-jn\phi} \int_{-\infty}^{\infty} dh \int_{z_1}^{z_2} d\xi \int_{\phi_1(\xi)}^{\phi_2(\xi)} G_\kappa(\beta, \xi) e^{jn\beta} e^{-jh(z-\xi)} d\beta.$$

The field outside the metallic cylinder is expressed by superposition of basic sets of cylindrical waves (see for example Stratton [9]). We have to use different expansions in the region 1 (*i.e.*, for $a<\rho<b$) and in the region 2 (*i.e.*, for $b<\rho<\infty$). The field in the region $\alpha(\alpha=1$ or 2) may be expressed by

$$E_\nu^{(\alpha)}(\rho, \phi, z) = \int_{-\infty}^{\infty} \mathcal{E}_\nu^{(\alpha)}(\rho, \phi, z; h) dh,$$

$$H_\nu^{(\alpha)}(\rho, \phi, z) = \int_{-\infty}^{\infty} \mathcal{H}_\nu^{(\alpha)}(\rho, \phi, z; h) dh,$$

where

$$\alpha = 1 \text{ or } 2 \quad \text{and} \quad \nu = \rho, \phi \text{ or } z.$$

Region 1

$$\mathcal{E}_\rho^{(1)} = \left\{ \sum_{n=-\infty}^{\infty} \left[-jh\gamma(a_n J_n'(\gamma\rho) + b_n H_n^{(2)\prime}(\gamma\rho)) - \frac{n\omega\mu}{\rho}(c_n J_n(\gamma\rho) + d_n H_n^{(2)}(\gamma\rho)) \right] e^{-jn\phi} \right\} e^{-jhz}$$

and corresponding expressions for $\mathcal{E}_\phi^{(1)}$, $\mathcal{E}_z^{(1)}$, $\mathcal{H}_\rho^{(1)}$, $\mathcal{H}_\phi^{(1)}$ and $\mathcal{H}_z^{(1)}$,

Region 2

$$\mathcal{E}_\rho^{(2)} = \sum_{n=-\infty}^{\infty} \left[-jh\gamma e_n H_n^{(2)\prime}(\gamma\rho) - \frac{n\omega\mu}{\rho} f_n H_n^{(2)}(\gamma\rho) \right] e^{-jn\phi} \Bigg\} e^{-jhz},$$

and corresponding expressions for $\mathcal{E}_\phi^{(2)}$, $\mathcal{E}_z^{(2)}$, $\mathcal{H}_\rho^{(2)}$, $\mathcal{H}_\phi^{(2)}$ and $\mathcal{H}_z^{(2)}$, where

$$\gamma = \sqrt{k^2 - h^2},$$

and where the quantities a_n, b_n, c_n, d_n, e_n, and f_n are constants to be determined from the boundary conditions and the saltus conditions of the problem.

Inserting the previously determined expressions for the field components in the equations expressing the boundary conditions and the saltus conditions, we find

$$\begin{Bmatrix} A_n & B_n & C_n & D_n & 0 & 0 \\ E_n & F_n & 0 & 0 & 0 & 0 \\ G_n & H_n & J_n & K_n & L_n & M_n \\ N_n & O_n & 0 & 0 & P_n & 0 \\ Q_n & R_n & S_n & T_n & U_n & V_n \\ 0 & 0 & X_n & Y_n & 0 & Z_n \end{Bmatrix} \begin{Bmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \Phi_2 \\ \Phi_1 \end{Bmatrix},$$

where

$$A_n = -\frac{nh}{a} J_n(\gamma a),$$

. . . .

. . . .

$$R_n = j\frac{k^2\gamma}{\omega\mu} H_n^{(2)\prime}(\gamma b),$$

. . . .

. . . .

$$Z_n = -\gamma^2 H_n^{(2)}(\gamma b),$$

and where

$$\Phi_\kappa = \frac{1}{4\pi^2} \int_{z_1}^{z_2} d\xi \int_{\phi_1(\xi)}^{\phi_2(\xi)} G_\kappa(\beta, \xi) e^{j(n\beta + h\xi)} d\beta \qquad \kappa = 1 \text{ or } 2.$$

Solving the preceding system of six linear equations and inserting the expressions for the coefficients a_n to f_n so obtained in the expressions for the field components given before, we obtain the expressions for the components of the electric and the magnetic field strength in regions 1 and 2. As an example of the expressions for these twelve quantities, we here give the expression for $E_\rho^{(2)}$,

$$E_\rho^{(2)} = \frac{j\pi}{2}\int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}\left[\frac{h(nh\Phi_1 - \gamma^2 b\Phi_2)(-J_n(\gamma b)H_n^{(2)}(\gamma a) + J_n(\gamma a)H_n^{(2)}(\gamma b))H_n^{(2)\prime}(\gamma\rho)}{k\gamma H_n^{(2)}(\gamma a)} + \frac{nkb\Phi_1(-J_n'(\gamma b)H_n^{(2)\prime}(\gamma a) + J_n'(\gamma a)H_n^{(2)\prime}(\gamma b))H_n^{(2)}(\gamma\rho)}{\gamma\rho H_n^{(2)\prime}(\gamma a)}\right]e^{-jn\phi}\right\}e^{-jhz}dh,$$

where $\zeta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of free space. In the far zone field, those parts of the expressions for the field components in which ρ occurs in the denominator are insignificant as compared to the remaining parts of these expressions. Making use of this fact, we notice that each component of the electric field strength may be expressed as

$$E_\nu(\rho, \phi, z) = \int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}\left[s_\nu^* s_{\nu n}H_n^{(2)}(\gamma\rho) + t_\nu^* t_{\nu n}H_n^{(2)\prime}(\gamma\rho)\right]e^{-jn\phi}\right\}e^{-jhz}dh,$$

where

$$s_{\nu n} = s_{\nu n}^{(1)}\Phi_1 + s_{\nu n}^{(2)}\Phi_2,$$

$$t_{\nu n} = t_{\nu n}^{(1)}\Phi_1 + t_{\nu n}^{(2)}\Phi_2,$$

and where the index ν stands for one of the cylindrical coordinates ρ, ϕ and z. The symbols $s_{\nu n}$ and $t_{\nu n}$ stand for one of the coefficients e_n and f_n, and the coefficients s_ν^* and t_ν^* are determined from the expressions for the various components.

In the expression for the components in the far zone field, we now approximate the Hankel function and its derivative, both of argument $\gamma\rho$, with the first term of the asymptotic expansions. Introducing further spherical coordinates instead of cylindrical coordinates, we find

$$E_\nu(\rho, \phi, z) = \frac{1}{4\pi^2}e^{j(\pi/4)}\sqrt{\frac{2}{\pi r\sin\theta}} \cdot \sum_{n=-\infty}^{\infty} j^n e^{-jn\phi}\left\{\int_{z_1}^{z_2}d\xi\int_{\phi_1(\xi)}^{\phi_2(\xi)}G_1(\beta, \xi)e^{jn\beta}d\beta\int_{-\infty}^{\infty}u_1(h) \cdot e^{-jr(\gamma\sin\theta + h\cos\theta)}dh + \int_{z_1}^{z_2}d\xi\int_{\phi_1(\xi)}^{\phi_2(\xi)}G_2(\beta, \xi)e^{jn\beta}d\beta\int_{-\infty}^{\infty}u_2(h) \cdot e^{-jr(\gamma\sin\theta + h\cos\theta)}dh\right\},$$

where

$$u_\kappa(h) = \left[s_\nu^* s_{\nu n}^{(\kappa)} - jt_\nu^* t_{\nu n}^{(\kappa)}\right](k^2 - h^2)^{-1/4}e^{jh\xi} \quad \kappa = 1 \text{ or } 2.$$

For large values of r, *i.e.*, for points in the far zone field, approximate expressions for the above integrals with h as the variable may be obtained by using the saddle-point method. Silver and Saunders [4], in their paper on the field radiated by an arbitrary slot in a circular cylinder, have made a saddle-point evaluation of integrals of the above type. For large values of r, they hereby find

$$\int_{-\infty}^{\infty}u(h)e^{-jr(\gamma\sin\theta + h\cos\theta)}dh \cong \sqrt{\frac{2\pi k}{r}}\sin\theta u(k\cos\theta)e^{-jkr}e^{j(\pi/4)}.$$

Using this approximation in the previous expression for E_ν, we obtain the desired far zone expressions for the cylindrical components of the electric field strength. From these formulas, we finally obtain the following expressions for the spherical components of the electric field strength in the far zone field.

$$E_r = 0,$$

$$E_\theta = \frac{e^{-jkr}}{r}\frac{kb}{4\pi}\sum_{n=-\infty}^{\infty}\frac{j^{n+1}e^{-jn\phi}S_n(ka\sin\theta, kb\sin\theta)}{H_n^{(2)}(ka\sin\theta)} \cdot \left\{\frac{n\cos\theta}{kb\sin\theta}I_n^{(1)} - \sin\theta I_n^{(2)}\right\},$$

$$E_\phi = \frac{e^{-jkr}}{r}\frac{kb}{4\pi}\sum_{n=-\infty}^{\infty}\frac{j^n e^{-jn\phi}V_n(ka\sin\theta, kb\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)}I_n^{(1)},$$

where

$$I_n^{(\kappa)} = \int_{z_1}^{z_2}d\xi e^{jk\xi\cos\theta}\int_{\phi_1(\xi)}^{\phi_2(\xi)}G_\kappa(\beta, \xi)e^{jn\beta}d\beta \quad \kappa = 1 \text{ or } 2,$$

and where we have introduced the functions

$$S_n(x, y) = J_n(x)H_n^{(2)}(y) - H_n^{(2)}(x)J_n(y),$$

$$V_n(x, y) = J_n'(x)H_n^{(2)\prime}(y) - H_n^{(2)\prime}(x)J_n'(y).$$

In carrying out this integration, it is convenient to be able to use any orthogonal coordinates (u_1, u_2) on the cylinder surface which fit the geometry of the aperture. For this purpose, we redefine the components G_κ of the surface current distribution $\overline{G}$ on the cylindrical surface $r = b$ by

$$K_\phi = \begin{cases} G_1(u_1, u_2) & \text{in the "aperture,"} \\ 0 & \text{outside the "aperture,"} \end{cases}$$

$$K_z = \begin{cases} G_2(u_1, u_2) & \text{in the "aperture,"} \\ 0 & \text{outside the "aperture."} \end{cases}$$

An element of length dl on the cylinder surface $r=b$ may now be expressed by

$$dl = \sqrt{(h_1du_1)^2 + (h_2du_2)^2},$$

where $h_1 = h_1(u_1, u_2)$ and $h_2 = h_2(u_1, u_2)$ are functions of the coordinates u_1 and u_2. We further express β and ξ as functions of the coordinates u_1 and u_2,

$$\beta = \beta(u_1, u_2),$$

$$\xi = \xi(u_1, u_2).$$

Often the surface current density $\overline{G}(u_1, u_2)$ in the aperture is expressed as a certain constant current I_0 divided by a certain constant length w and multiplied by a normalized, dimension-free surface current distribution function $\bar{g}(u_1, u_2)$,

$$\overline{G}(u_1, u_2) = \frac{I_0}{w} \bar{g}(u_1, u_2).$$

We also introduce a normalized, dimension-free electric field strength $\bar{e}(\theta, \phi)$ in the far zone field defined by the equation

$$\overline{E}(r, \theta, \phi) = \zeta I_0 \frac{e^{-jkr}}{r} \bar{e}(\theta, \phi),$$

where, as usual, $\zeta = \sqrt{\mu/\epsilon}$ is the intrinsic impedan of the free space. We then have

$$e_\theta = \frac{kb \sin\theta}{4\pi} \sum_{n=-\infty}^{\infty} \frac{j^{n+1}e^{-jn\phi}S_n(ka \sin\theta, kb \sin\theta)}{H_n^{(2)}(ka \sin\theta)} \cdot \left[\frac{n \cos\theta}{kb \sin^2\theta} i_n^{(1)} - i_n^{(2)}\right],$$

$$e_\phi = \frac{kb}{4\pi} \sum_{n=-\infty}^{\infty} \frac{j^n e^{-jn\phi} V_n(ka \sin\theta, kb \sin\theta)}{H_n^{(2)\prime}(ka \sin\theta)} i_n^{(1)},$$

where

$$i_n^{(\kappa)} = \frac{1}{bw} \iint g_\kappa(u_1, u_2) e^{j[k\xi(u_1,u_2) \cos\theta + n\beta(u_1,u_2)]} \cdot h_1(u_1, u_2) h_2(u_1, u_2) du_1 du_2.$$

It appears from the preceding expressions that when the surface current density is purely axial, *i.e.*, when $g_1(\phi, z) = 0$, then the radiated field will have no component in the ϕ-direction, *i.e.*, $E_\phi = 0$. On the other hand, the formulas also show that when the surface current density is purely circumferential, *i.e.*, when $g_2(\phi, z) = 0$, then the radiated field will have a θ component and a ϕ component that are both nonvanishing except in certain directions, *i.e.*, in this case the radiated fields will in general be elliptically polarized.

An obvious way of establishing a current distribution on a circular cylindrical surface coaxial with a circular conducting cylinder is to coat the conducting cylinder with a dielectric or permeable material and then to establish the current distribution on the outside surface of this coating. The expression for the field radiated from the current distribution on this structure has been obtained by an analysis similar to the one made above for the special case with no coating present. However, since the resulting expressions are so complicated that their practical utility may be questioned, they are omitted here.

Conclusion

It has been the object of this paper to obtain an analytical expression for the field radiated from slots in and from wire antennas near conducting, circular cylinders under the assumption that the field in the slots or the current in the wires are given. All the work done in this report on the field radiated from various types of slots in circular cylinders is based on the paper by Silver and Saunders [4] regarding the field radiated from an arbitrary slot in a circular cylinder; this paper has further served as a model for all the work done here on the field radiated from wire antennas near circular cylinders. The derivation of the formulas in this report has, therefore, in principle been a fairly straightforward matter. However, due to the complexity of some of the physical problems solved here, the derivations have in several cases become rather involved.

The expressions for the far zone field in all the cases treated in this paper are series expressions containing cylinder functions. In general, very little information regarding the radiated field can be obtained by inspection of the formulas. For obtaining the desired information, numerical computations must be carried out. For cylinders having a diameter of the same order of magnitude as the wavelength, most of the field expressions derived in this report are well suited for numerical computations. Due to the complexity of the formulas and to the large amount of parameter values for which it will in general be desirable to have numerical computations carried out, the use of an electronic computer is strongly indicated.

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Addendum

When presenting this paper at the Toronto Symposium, the author was informed by Dr. George Sinclair that part of the material given here on the inclined slot with a sine-shaped aperture field is contained in an unpublished report by Sinclair, "The Distant Field of an Arbitrary Slot Antenna in a Cylinder," Antenna Laboratory, The Ohio State University, Columbus, 1950. This report also contains some numerical results, which will be very valuable as a partial check of the extensive electronic computer computations we are undertaking.

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