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The compound pendulum in intermediate laboratories and demonstrations

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A student laboratory course on the motion of the compound pendulum is described. The course is suited for physics and engineering students in their first year and requires a background in mechanics and mathematics corresponding to a one-semester course in these disciplines. The purpose of this course is to present a system to the students which can be approached experimentally and to some extent theoretically using elementary methods, and hence allow the students to practice their newly acquired knowledge. All the experimental results presented here are obtained by the students using a simple and readily fabricated version of the compound pendulum.

I. INTRODUCTION

The problem of describing rotating bodies within the framework of classical mechanics usually presents difficulties to the physics or engineering student in his freshman year. Although the description is simple and straightforward the student is often unfamiliar with concepts as torque, angular momentum, and moment of inertia. What he needs in order to familiarize himself with these concepts is to get the opportunity to gain his own experience. We have designed a laboratory course where the students, by combining experiments with a theoretical approach, discover how the various quantities are interrelated in determining the motion of the system.

There are several reasons why it was decided to consider the compound pendulum in this laboratory course. First, the equation of motion in the gravitational field, with damping and externally applied torque included, cannot be solved analytically, but it is quite easy to examine the system by experimental methods. Second, many systems of great practical importance are described by the same nonlinear, second-order differential equation, e.g., the Josephson junction (the compound pendulum as a mechanical analog to the Josephson junction has been discussed in this journal earlier[1,2]), the phase-locked loop,[3] and the synchronous motor.[4] This makes it valuable to acquire experience as to how such a system reacts to parameter changes. Third, the physical pendulum used here is a very simple system which is readily fabricated in any mechanical workshop, and yet it possesses an almost inexhaustible variety of modes of operation.

The laboratory course, the contents of which are described in detail below, was aimed at students of engineering in their freshman year and followed one semester of courses in classical mechanics and in mathematics. A total number of 22 students, divided into four groups, participated. Each team worked independently with its own experimental setup. The students worked full time for three weeks, and it is estimated that they spent half their time on experiments and the other half on theory, data analysis, etc.

II. EXPERIMENTAL EQUIPMENT

The compound pendulum is shown in Fig. 1. The pendulum stand carries a rigid stainless-steel shaft (diam 3 mm, length 110 mm) mounted in virtually frictionless ball bearings. On the shaft are mounted two aluminum disks (diam 170 mm, thickness 1.25 mm) and two pendulums (weight 8 g, length 70 mm). The angle between the pendulum arms and their lengths are adjustable.

One of the two aluminum discs may be dismounted, thus allowing a change of the total moment of inertia. The other is an indispensable part of the system, because it, in addition to being a second "fly-wheel," has four auxiliary functions:

(1) The pendulum is driven by an external torque which is applied by blowing compressed air (from the laboratory supply) tangentially on to a sheet of sandpaper glued to the disk. A water-filled manometer (length ≈ 1000 mm) connected as indicated in Fig. 1 was used to monitor the air pressure and hence the applied torque.

(2) The motion of the pendulum is damped due to the eddy currents induced in the aluminum disk by means of an electromagnet (4000 turns of 0.7-mm diam copper wire on a 10-mm diameter soft iron core).

(3) When the pendulum oscillates, the deflections is measured by means of a fixed pointer and a sheet of polar graph paper glued to the disk. The period of oscillation is in this case determined by means of a stop watch.

(4) Finally, when the pendulum rotates the angular velocity is determined as follows: two 1 mm holes about 20 mm apart are drilled near the periphery of the disc (shown in Fig. 1). A small light bulb is mounted in front of the disc and a photodiode connected to a pulse shaping circuit is mounted behind the disc. When the disc rotates, the time average of the angular velocity is determined from the time lapse between each set of double pulses from the photodiode, whereas the instantaneous angular velocity is determined from the time lapse between the individual pulses of a pair.

III. EXPERIMENTAL AND THEORETICAL APPROACH

The pendulum equation is

$$I\ddot{\theta} + k\dot{\theta} + D\sin\theta = H, \tag{1}$$

where θ is the angle measured from vertical down (the dot means taking the derivative with respect to time), I is the moment of inertia, k is the damping coefficient, D is the maximum gravitational torque, and H the applied external torque. The solutions fall naturally into two categories corresponding to the time average of the angular velocity

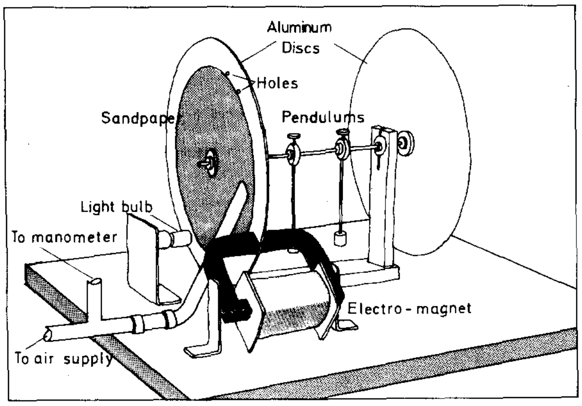

Fig. 1. The compound pendulum.

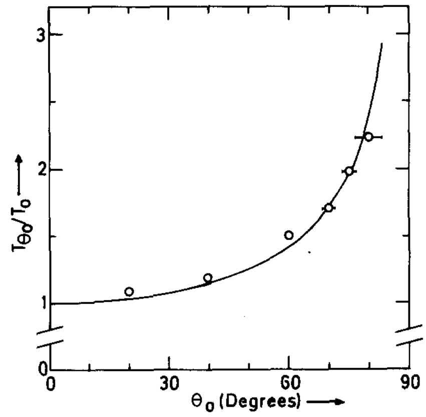

Fig. 3. Small amplitude oscillations T_{θ_0}/T_0 as a function of the equilibrium angle θ_0. Full curve $T_{\theta_0}/T_0 = (\cos\theta_0)^{-1/2}$.

being zero or nonzero. We will use the terms oscillating and rotating solutions, respectively.

The pendulum equation, Eq. (1), has no known general analytical solution. However, during the course the students did solve the equation in a number of limiting cases as theoretical exercises, and compared the predictions to the corresponding experimental results.

In the following, the contents of the course will be presented. The general approach was that the experimental program and the theoretical problems were formulated in a few survey lectures at regular intervals, which also served the purpose of setting the pace.

A. Oscillating solutions

1. Small amplitude oscillations. $k \simeq 0$, $H = 0$

The period of oscillation, T_0 was measured (average of 10 periods) for various values of I (1 and 2 disks) and D (different pendulum lengths). Following careful measurements of geometrical dimensions and masses of the various parts of the pendulum, the period of oscillation was calculated from

$$T_0 = 2\pi\,(I/D)^{1/2} \tag{2}$$

and the calculations were compared to the experimental results. Agreement was usually within a few percent.

2. Large amplitude oscillations. $k \simeq 0$, $H = 0$

Using a particular set of values for I and D the period of oscillation T_{θ_s}, for large amplitude, undamped oscillations was measured as a function of the starting angle, θ_s. A set of experimental values for the quantity T_{θ_s}/T_0 is shown in Fig. 2 as a function of θ_s. For large amplitudes the period of oscillation can be expressed as[5]

$$T_{\theta_s} = 2\pi\,(I/D)^{1/2}\left\{1 + \frac{1}{4}a^2 + \frac{9}{64}a^4 + \cdots\right\}, \tag{3}$$

where $a = \sin(\theta_s/2)$.

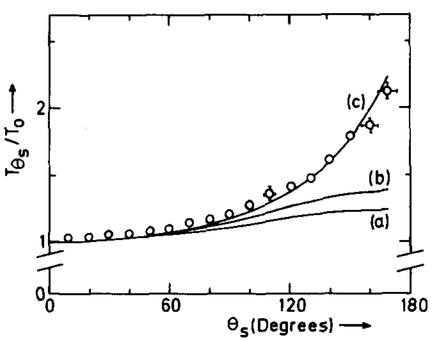

Fig. 2. Large amplitude oscillations T_{θ_s}/T_0 as a function of starting angle θ_s. (a) Equation (3) to second order in a; (b) Equation (3) to fourth order in a; and (c) elliptic integral Eq. (4).

The full expression for T_{θ_s} is given in terms of the complete elliptic integral of first king $K(m)$ as

$$\frac{T_{\theta_s}}{T_0} = \frac{2}{\pi}K(m) = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{(1 - m\sin^2\phi)^{1/2}}. \tag{4}$$

Here, $m = \sin^2(\theta_s/2)$. The elliptic integral is tabulated in Ref. 6. The theoretical results are also shown in Fig. 2.

3. Small amplitude oscillations. $k \simeq 0$, $H \neq 0$

The period of oscillation T_{θ_0} for small amplitude, undamped oscillations around an equilibrium angle $\theta_0 \neq 0$ was measured as a function of an applied constant torque, $H < D$. The results are shown in Fig. 3, where T_{θ_0}/T_0 is plotted as a function θ_0.

Assuming, in this case, a solution to Eq. (1) of the form

$$\theta(t) = \theta_0 + \theta_1(t), \quad (\theta_1 \ll 1),$$

one finds after a first order expansion in the small quantity, θ_1, that $\theta_0 = \sin^{-1}(H/D)$ and that θ_1 has a period of oscillation given by

$$T_{\theta_0}/T_0 = (\cos\theta_0)^{-1/2}. \tag{5}$$

This dependence is also shown in Fig. 3.

4. Calibration of the external torque, H

In Sect. III B which deals with the rotating pendulum, it becomes necessary to know the actual value of the external torque H. That is, the relation between the air pressure—measured by the water manometer—and the exerted torque must be established. This calibration is most conveniently made using the results already obtained in Sec. III A 3. It follows directly from Eq. (1) that for $\dot{\theta} \equiv 0$

$$H = D\sin\theta_0. \tag{6}$$

Hence, a measurement of the equilibrium angle θ_0 corresponding to a particular air pressure p permits H to be determined from Eq. (6). Experimentally, it is found that H is directly proportional to p.

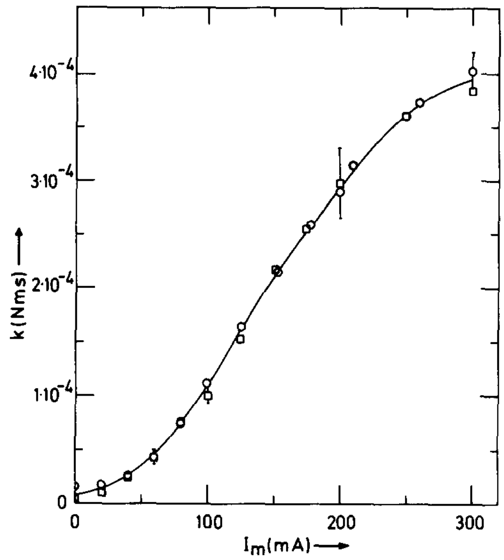

Fig. 4. Damping constant k as a function of magnet current I_m determined from damped harmonic oscillations, Eq. (7) (squares) and uniform rotation, Eq. (9) (circles). A smooth curve has been fitted to the data.

5. Small amplitude oscillations. $k \neq 0$, $H = 0$

The damping coefficient k was determined as a function of the current applied to the electromagnet I_m by measuring the decay rate of small amplitude, damped oscillations. In this limit the pendulum performs damped harmonic oscillations given by the expression $\theta(t) = \theta_s \exp(-k/2I) \cos(2t/T_1 + \phi)$, where T_1 is the period of oscillation and ϕ is a phase angle. Introducing δ, where

$$\delta = (1/n) \ln [\theta(t)/\theta(t + nT_1)], \quad (7)$$

the following relation is readily derived

$$k = 2I\delta/T_1. \quad (8)$$

From Eq. (8) and measured values of the quantities, δ and T_1, the damping coefficient is determined. The obtained

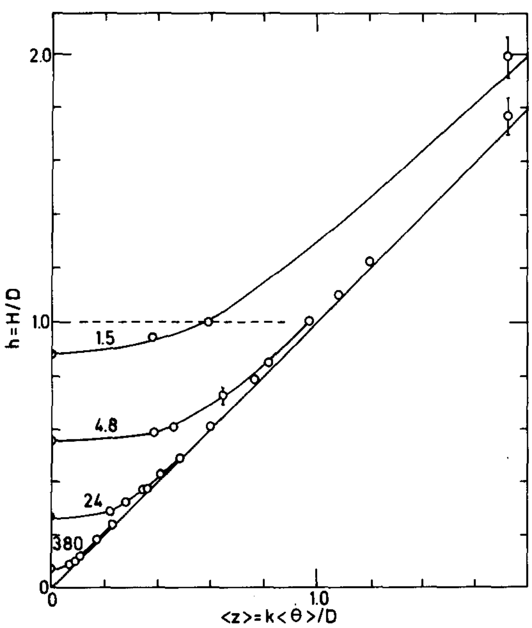

Fig. 5. Normalized average angular velocity $\langle z \rangle$ as a function of normalized torque h for different values of $\beta = DI/k^2$. Smooth curves have been fitted to the data.

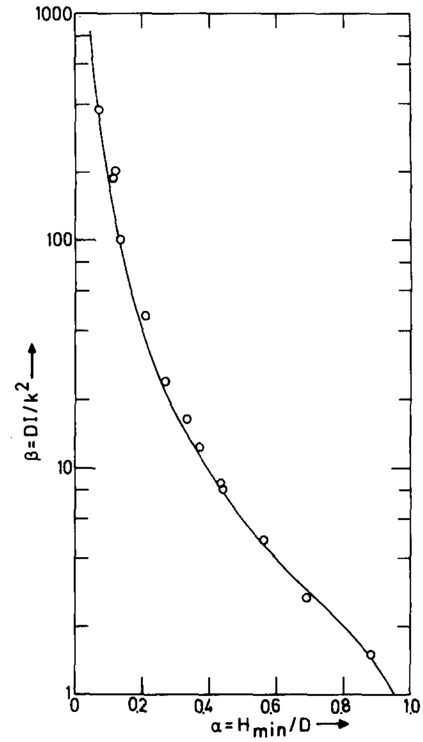

Fig. 6. The experimentally observed α-β relationship compared to the numerical calculation from Ref. 8. α is the minimum value of the normalized torque which can sustain rotation.

results are plotted as a function of the magnetization current in Fig. 4 (squares).

B. Rotating Solutions

1. Uniform rotation. $D = 0$, $H \neq 0$, $k \neq 0$

By removing the pendulum bobs or by placing the pendulums in opposition the condition $D = 0$ could be satisfied. For this case the steady state solution to Eq. (1) is

$$\dot{\theta} = H/k. \quad (9)$$

For a given value of the magnet current and a constant torque H, (determined as discussed in Sec. III A 4), the steady state angular velocity $\dot{\theta}$ was measured and k determined from Eq. (9). These results are also shown in Fig. 4 (circles).

As is evident from Fig. 4 the two methods of determining k are consistent within experimental error.[7]

2. Non-uniform rotation. $D \neq 0$, $H \neq 0$, $k \neq 0$

When the maximum gravitional torque D is nonzero, the rotation becomes nonuniform, i.e., the angular velocity $\dot{\theta}$ depends on the angle θ. Hence, the time average of the angular velocity $\langle\dot{\theta}\rangle$ for different combinations of the parameters I, D, and k was measured as a function of the applied torque H.

By introducing the substitution $z = k\dot{\theta}/D$ the pendulum equation (1) may be transformed into the two first-order differential equations

$$\frac{IDz}{k^2}\frac{dz}{d\theta} + z + \sin\theta = H/D \quad (10a)$$

and

$$z = k\dot{\theta}/D. \quad (10b)$$

Accordingly, in normalized units, the solution to Eqs. (10) may be expressed in terms of only two parameters, $\beta = ID/k^2$ and $h = H/D$. In Fig. 5 the measured normalized angular velocity $\langle z \rangle$ is plotted as a function of the normalized torque h with β as a parameter.

It is observed from Fig. 5 that two solutions for $\langle z \rangle$ exist for h between one and a minimum value α which depends only on the parameter β. Thus, starting from zero torque $\langle \dot{\theta} \rangle$ remains zero up to $H = D$, where $\langle \dot{\theta} \rangle$ will switch to a finite value. Upon decreasing the torque from there the pendulum will still rotate until H reaches the value $H_{\min} = \alpha D$, where the pendulum will come to a stop going through damped oscillations around the equilibrium position $\theta = \sin^{-1}(H_{\min}/D)$.

A computer calculation[8] of the relation between α and β is shown in Fig. 6 where typical experimental results obtained in this course are also plotted.

3. *Nonuniform rotation: $\dot{\theta}$ versus θ*

The nonuniformity of the rotation was determined by measuring the angular velocity $\dot{\theta}$ as a function of the angle θ. Figure 7 shows the experimental results with $\beta = 8.71$ and $h = 1.22$, 0.85, and 0.49, respectively. This experiment was made by measuring the time lapse between the individual pulses of the double pulse as described in Sec. II. The angle θ was changed by adjusting the angle between the pendulum arm and the double holes in the aluminum disk. (The $\dot{\theta}$ versus θ plane is often referred to as the phase plane.) A comparison with theory could be made in one of two ways: An approximate solution to Eq. (1) valid for small nonuniformity may conveniently be expressed as

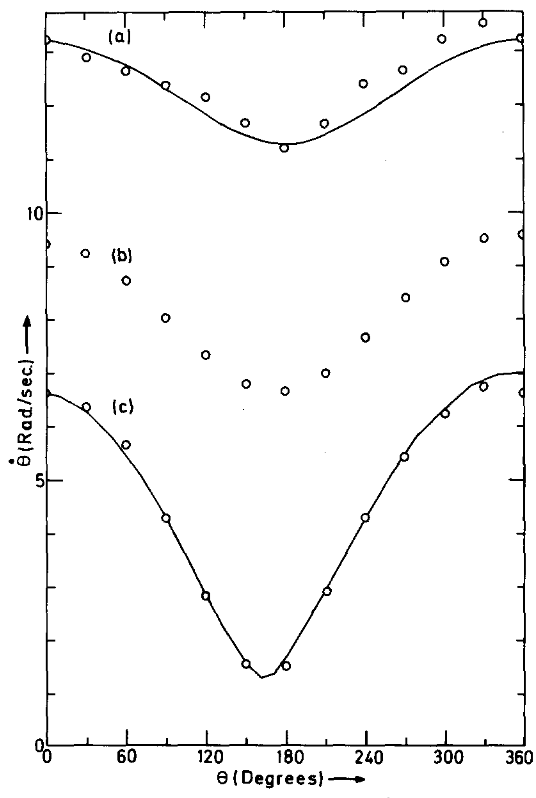

Fig. 7. Experimental and theoretical results for $\dot{\theta}$ as a function of θ with $I = 3.11 \times 10^{-4}$ kg m^2, $D = 36.3 \times 10^{-4}$ N m, $k = 3.6 \times 10^{-4}$ N m/s, and $\beta = 8.71$. (a) $h = 1.22$, (b) $h = 0.85$, and (c) $h = 0.49$. Upper curve; Eq. (11). Lower curve; graphical integration of Eq. (12).

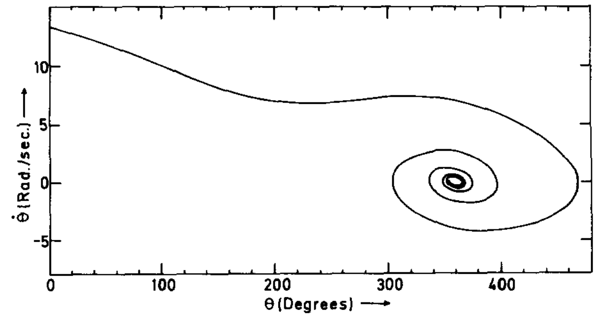

Fig. 8. An example of a computer calculation of the pendulum decay in the phase plane [Eq. (12)].

$$\begin{aligned} \dot{\theta} &= \frac{H}{k} + \frac{Dk}{HI}\cos\left(\frac{H}{k}t\right) \\ \theta &= \frac{Ht}{k} + \frac{Dk^2}{HI}\sin\left(\frac{H}{k}t\right), \end{aligned} \tag{11}$$

derived under the assumptions $Dk^2/IH^2 \ll 1$ and $k^2/HI \ll 1$. Considering the time t as a parameter, the $\dot{\theta}$ versus θ curve could be calculated. An example is shown as the upper curve of Fig. 7.

Another approach was to rearrange Eq. (10a) to give

$$\frac{dz}{d\theta} = \frac{h - \sin\theta}{\beta z} - \frac{1}{\beta} \tag{12}$$

and perform the integration graphically. Starting with a line element $(\theta, z) = [\theta_0, z(\theta_0)]$ the slope is calculated from Eq. (12) and the tangent is constructed. A new line element on the tangent is chosen and the procedure is repeated a sufficient number of times to produce a reasonable fit. An example is shown as the lower curve of Fig. 7.

If the laboratory has easy access to a computer facility, a different approach would be to solve Eq. (12) numerically. This permits the calculation of arbitrary trajectories in the phase plane. The result of such a calculation describing the decay of the pendulum is shown in Fig. 8.

CONCLUSION

The rather extensive program described here provides the students with a good opportunity to study a single system in great detail and it also brings a welcomed variety to the mainly theoretical courses in the early stages of their studies. Furthermore, the system we have studied above is conceptually simple, and yet the behavior is sufficiently complex to supply many challenging problems which cover a broad spectrum of recently acquired knowledge. In addition, new concepts such as elliptic integrals, nonlinear differential equations, phase plane, hysteresis, etc. are introduced. Finally, judging from the enthusiasm and the amazing pace which the students maintained throughout the course and from the quality of the reports they wrote,[9] we believe that the course presented here meets a demand for experimental work closely related to the textbook material.

ACKNOWLEDGMENT

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