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Locating a circle on the plane using the minimax criterion

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Abstract

We consider the problem of locating a circle with respect to existing facilities on the plane, such that the largest weighted distance between the circle and the facilities is minimized. The problem properties are analyzed, and a solution procedure proposed.

1 Introduction

In this paper, we examine the problem of locating a circle on the plane in order to minimize the maximum distance between the circle and a set of fixed points. We consider two cases:
Model 1: The radius of the circle is given. The problem requires finding the optimal center of the circle.
Model 2: The radius of the circle is allowed to vary. The problem requires finding the optimal center and radius of the circle.

The second model was treated by Drezner et al. [4], who suggested the use of this model to determine the out-of-roundness of a machined part. The fixed points would represent actual measurements along the circumference of the part. The authors also provide an approximate solution method assuming that the part is only slightly out-of-round.

Model 1, on the other hand, appears to be new. However, examining further the machining context suggests an important application. The diameter of the part is specified by a nominal value and an acceptable tolerance on this value. Verifying that the part has been machined to this specification is normally done separately by measuring the diameter at various spots around the circumference using, for example, high-precision vernier calipers. This procedure does not guarantee that the diameter passes through a common center point each time. By using model 1, with radius equal to the corresponding nominal value for the part, we are able to take the earlier circumferential measurements for out-of-roundness to determine simultaneously (and more accurately) if the part meets the design specification on its diameter.

In this paper, we also generalize models 1 and 2 to the case of weighted distances, which may apply when the fixed points are considered not to be of equal importance. For example, if the fixed points represent population centers and the circle a ring road connecting them, the weights could reflect the different populations. Circular routes are examined in Pearce [8] and Suzuki [11].

The problem of locating a circle with variable radius, minimizing the maximal distance to a set of existing facilities has a nice geometric interpretation: We are looking for an annulus with minimal width covering all given points. This problem has been studied in computational geometry.

Rivlin [9] showed that the minimium width annulus of n points is either the width of the convex hull of the n points or must have two points on the inner circle and two points on the outer circle of the annulus. The same result was obtained by Drezner et al. [4]. In the former case the radius of the circle is infinite such that the circle becomes a line, in the latter case it is a circle with finite radius.

Consequently, the center of a minimax circle is either a vertex of the (nearest neighbour) Voronoi diagram or of the farthest neighbour Voronoi diagram or lies at an intersection point of both diagrams. Ebara et al. [6] use this result to give an algorithm. Agarwal and Sharir [2] present a randomized algorithm solving the minimax circle problem in $\mathrm{O}(n^{\frac{3}{2}+\epsilon})$ expected time for $\epsilon > 0$. Special cases have been considered in de Berg et al. [3] and in Duncan et al. [5].

There also exist approximation approaches for the minimax circle problem. As noted above, Drezner et al. [4] propose an approximate solution method for the slightly out-of-round case. Agarwal et al. [1] present an $\mathrm{O}(n \log n)$ approach with approximation factor 2. They also have an algorithm with approximation factor $1 + \epsilon$ but running time $\mathrm{O}(n \log n + \frac{n}{\epsilon^2})$.

2 Notation

We use the following notation.

The number of existing facilities is n, and facility j is located at $A_j = (a_j, b_j)$ with associated positive weight w_j, for $j = 1, \ldots, n$. The existing facility locations are called the fixed points.

The circle to be located, C, is determined by its center, $X = (x, y)$, and its radius, r. We use the shortcut $C = (X, r)$.

The Euclidean distance between the center and facility j is denoted by $d(X, A_j)$, for $j = 1, \ldots, n$. The shortest Euclidean distance between the circle and facility j, $d_j(C)$, is $r - d(X, A_j)$, if the facility is inside the circle, and $d(X, A_j) - r$, if it is outside. (If the facility is on the circle, the distance is 0, and both expressions apply). So we have $d_j(C) = |d(X, A_j) - r|$ in general, for $j = 1, \ldots, n$.

For any two fixed points, A_i, A_j, let the straight line segment between them be denoted by $[A_i, A_j]$. The bisector of this segment (i.e., the locus of points with equal distance to A_i and A_j) is called B_{ij}.

3 Circle of fixed radius

In this section we consider the case where the radius of the circle has a fixed value, $r = r_0$.

In the out-of-roundness problem, each point measured on the circumference of a given part has the same importance. This is referred to as the unweighted case; each fixed point is assigned a unit weight. In a more general context, such as the location of a circular road or irrigation pipe, the fixed points may be assigned different weights reflecting their relative importance. This weighted case will also be investigated.

3.1 Unweighted case

We consider the following problem:
minimize $g(C) = \max_{j=1,\ldots,n}\{d_j(C)\}$,
or equivalently,
Problem FU: minimize $g(X) = \max_{j=1,\ldots,n}\{|d(X, A_j) - r_0|\}$.

It is useful to note the two limiting cases contained in this model; namely, the location of a point facility, for which $r_0 = 0$, and a linear facility where $r_0 \to \infty$. These two cases have been studied extensively in the facility location literature, e.g., see Love et al. [7]. The problem studied by us provides a link or bridge between them.

In general, the objective function g is neither convex nor concave in X, and may contain several local minima. If A_j is outside the circle, $d_j(C)$ is locally convex; if it is inside, $d_j(C)$ is locally concave. Thus $g(X)$ becomes the maximum of a set of convex and concave functions. The subset of points inside (or outside) the circle also changes with X, resulting in nondifferentiable frontiers of the objective function. This may further complicate the solution process. In one limiting case, $r_0 \to \infty$, $g(X)$ is also not convex. By contrast, the other limiting case, $r_0 = 0$, supports a convex function $g(X) = \max_j\{d(X, A_j)\}$, and the solution to this problem is readily obtained.

Let X_p denote the solution of the associated minimax problem with $r_0 = 0$. It is well known that in the optimal solution there are at least two critical points that are at the same distance, $r_{\max}$, furthest from X_p. Let $r_{\min}$ then denote the distance to the closest A_j from X_p. We obtain the following initial result.

Property 1
If $r_0 \in [0, r_m]$, where $r_m = (r_{\min} + r_{\max})/2$, then X_p is the optimal solution of Problem FU.

Proof:
For any $X \neq X_p$, it is clear that $\max_j\{d(X, A_j)\} > \max_j\{d(X_p, A_j)\} = r_{\max}$, and thus $g(X) > r_{\max} - r_0$. But $g(X_p) = r_{\max} - r_0$, because $r_{\max} - r_0 \geq r_0 - r_{\min}$, so X_p must be the unique optimal solution. $\diamond$

On the other hand, if $r_0 > r_m$, X_p will generally not be the optimal solution, and further analysis will be required.

Let $d_{ij} = d(A_i, A_j)$ for all pairs (i, j), and $d_{\max} = \max_{i<j}\{d_{ij}\}$. Also let $C^* = (X^*, r_0)$ denote an optimal solution of Problem FU, with objective value $g^* = g(X^*)$, and X_m be the mid-point of the line segment between A_r and A_s, where (r, s) is any pair such that $d_{\max} = d_{rs}$. Analogous to the minimax point facility problem, there are two possibilities to consider, as shown in the following two results.

Property 2
If $g(X_m) = d_{\max}/2 - r_0$, then $X_m = X^*$ and $g(X_m) = g^*$.

Proof:
Consider any $X \neq X_m$. We have
$g(X) \geq \max\{d(X, A_r), d(X, A_s)\} - r_0 > d_{rs}/2 - r_0 = d_{\max}/2 - r_0$.
It follows that X_m is the unique optimal solution in this case. $\diamond$

Property 3
If $g(X_m) > d_{\max}/2 - r_0$, then at least three extreme points, A_i, A_j, A_k exist such that $d_i(C^*) = d_j(C^*) = d_k(C^*) = g^*$; that is, the subset of points at the maximum distance from an optimal circle must have cardinality of at least three.

Proof:
Suppose there is only one extreme point A_i. Then move the center X a small distance from X^* along the straight line through A_i and X^* in the direction away from A_i, if this point is inside the circle, or towards A_i, if it is outside, to obtain an immediate contradiction on the optimality of X^*.

Next suppose there are exactly two extreme points, A_i and A_j. If both of these points are outside the circle, they cannot be colinear with X^*, since this would imply $g^* \leq d_{\max}/2 - r_0$. Then move from X^* along the bisector of $[A_i, A_j]$ in the direction towards $[A_i, A_j]$ to obtain a contradiction once again. If both points are inside the circle, do the same, except we move away from $[A_i, A_j]$, to arrive at an identical conclusion.

Finally, if A_i is outside the circle and A_j is inside, consider three possible scenarios:

(i) A_i, A_j and X^* are noncolinear.
Then draw a ray from A_j through X^*, and select the image A'_j on the ray that is outside and equidistant to the circle. Move along the bisector of $[A_i, A'_j]$ towards A_i, A'_j.

(ii) A_i, A_j and X^* are colinear, and X^* is between A_i and A_j.
Move along the segment $[A_i, A_j]$ towards A_i.

(iii) A_i, A_j and X^* are colinear, and A_j is between A_i and X^*.
In this case, draw a circle of radius r_0 with center at the intersection of $[A_i, X^*]$ and C^*. Then move from X^* in either direction along this new circle.

In all three cases, the new center X gives a better solution than X^*, and hence, a contradiction.

We conclude that an optimal circle must have at least three extreme points. $\diamond$

In view of Property 3, we examine the following question: Given an arbitrary triplet (A_i, A_j, A_k), find all the points X, such that $C = (X, r_0)$ and $d_i(C) = d_j(C) = d_k(C)$. Denote the set of points thus obtained by S_{ijk}. We consider two possibilities:

(i) The three points are all located on the same side of the circle.
This case is straightforward, since it requires finding the unique intersection point of the bisectors of $[A_i, A_j]$ and $[A_j, A_k]$, given, respectively, by B_{ij} and B_{jk}. The details are left to the reader. Note that this case is not possible if A_i, A_j, A_k are colinear.

(ii) One point, say A_i, is on one side of the circle, while the other two points, A_j, A_k, are on the other side.
This time $X \in S_{ijk}$ lies on the bisector B_{jk}, and it also satisfies the equation, $d(X, A_i) + d(X, A_j) = 2r_0$. By rotating and translating the coordinate system so that the x-axis coincides with B_{jk}, and the origin with the mid-point of $[A_j, A_k]$, the equation may be written as $\sqrt{x^2 + (d_{jk}/2)^2} + \sqrt{(a-x)^2 + b^2} = 2r_0$, where

$A_i = (a,b)$, $A_j = (0, d_{jk}/2)$, and $X = (x,0)$ in the new coordinate system. In turn this simplifies to a quadratic equation,
$4(a^2 - 4r_0^2)x^2 + 4a(d_{jk}^2/4 - s^2 + 4r_0^2)x + (s^2 - 4r_0^2)^2$
$+ (d_{jk}^2 - 8s^2 - 32r_0^2)d_{jk}^2/16 = 0$, where $s = \sqrt{a^2+b^2}$.
The quadratic equation is easily solved, yielding two real roots, one, or none, depending on whether $\min_{X \in B_{jk}}\{d(X,A_i) + d(X,A_j)\}$ is less than, equal to, or greater than $2r_0$, respectively.

We need to consider separately the possibility that the single point on one side of the circle is A_i, A_j, or A_k. Combining cases (i) and (ii), it follows that the number of points in S_{ijk} is at most $1 + 2 \cdot 3 = 7$. We see that determining S_{ijk} is easily done in constant time.

A straightforward approach to solving Problem FU, based on the insight from Property 3, would be to enumerate all triplets A_i, A_j, A_k, determining the set S_{ijk} for each one, and retaining the best candidate point from all these sets. That is, given that Property 3 applies, $X^* \in \bigcup_{i=1}^{n-2} \bigcup_{j=i+1}^{n-1} \bigcup_{k=j+1}^{n} S_{ijk}$. To verify the objective value of a candidate requires $O(n)$ operations, and there are $O(n^3)$ candidates. Thus, the overall complexity based on total enumeration would be $O(n^4)$.

The following solution procedure uses an implicit enumeration scheme. The basic assumption is that most of the fixed points can be eliminated implicitly as candidate extreme points in the final solution. The goal then is to identify in a sequence those A_j that will most likely produce an optimal solution. We believe this procedure will be effective, for example, in solving exactly the out-of-roundness problem investigated by Drezner et al. [4].

Algorithm 1 (exact solution of Problem FU):

Step 1.
Determine the center X_p of the associated point facility minimax problem ($r_0 = 0$). If $\max_j\{d(X_p, A_j)\} - r_0 = g(X_p)$, stop ($X^* = X_p$). Otherwise, initialize the subset of fixed points that are candidate extreme points, $S = S_0$, where S_0 contains the points at maximum distance from X_p, and the current solution, $X_c = X_p$.

Step 2.
Determine the subset of extreme points, S_c, associated with the current solution X_c. Update $S \leftarrow S \bigcup S_c$.

Step 3.
Use total enumeration to solve Problem FU for the reduced set S. (Note that triplets (A_i, A_j, A_k) that have been evaluated in a previous iteration should not be evaluated again. Only new triplets are considered in each iteration.) Let X_c denote the new solution. (Retain all ties for verification in the next step.)

Step 4.
If the objective value at X_c for the original (full) problem is the same as for the

reduced problem, stop ($X^* = X_c$). Otherwise, return to step 2. $\diamond$

3.2 Weighted case

We now consider the generalized version of model 1 with weighted distances.
Problem FW: minimize $g(X) = \max_{j=1,\ldots,n}\{w_j|d(X,A_j) - r_0|\}$,
where the weights $w_j > 0,\ j = 1,\ldots,n$.

When all the weights are equal to unity, we observe in Property 1 that the solution of the minimax point facility problem is optimal over a range of values for r_0. Interestingly, this result does not extend to the weighted case. For example, suppose A_1 and A_2 are extreme points and X_p the solution of the associated problem with $r_0 = 0$. By increasing the radius a small amount ($r_0 = \delta$), we obtain $w_1(d(X_p,A_1) - \delta)$ and $w_2(d(X_p,A_2) - \delta)$ as the respective weighted distances to the circle of A_1 and A_2. But these distances are no longer necessarily equal, so that, in general, there is only one extreme point left, and the solution X_p for the perturbed problem cannot be optimal.

By contrast, Properties 2 and 3 are readily extended to the weighted case. Consider any pair (A_i, A_j) with $d_{ij} = d(A_i, A_j) > 2r_0$, and let X_{ij} be the (weighted mid-) point on the segment $[A_i, A_j]$ such that $C(X_{ij}, r_0)$ is equidistant (in the weighted sense) to A_i and A_j. We obtain $X_{ij} = A_i + t_{ij}(A_j - A_i)$, where $t_{ij} = (r_0 w_i + (d_{ij} - r_0)w_j)/(d_{ij}(w_i + w_j))$.
Let $g_{ij} = w_i(d(X_{ij}, A_i) - r_0) = w_j(d(X_{ij}, A_j) - r_0)$,
$g_L = \max\{g_{ij}; \forall\ i,j \ni\ d_{ij} > 2r_0\}$, and X_m be the weighted mid-point of the line segment between A_r and A_s, where (r,s) is any pair such that $g_{rs} = g_L$.

Property 2 (generalized to weighted case)
If $g(X_m) = g_L$, then $X_m = X^*$ and $g(X_m) = g^*$.

Proof:
Consider any $X \neq X_m$. We obtain
$g(X) \geq \max\{w_r(d(X,A_r) - r_0), w_s(d(X,A_s) - r_0)\} > g_L = g(X_m)$. $\diamond$

Property 3 (generalized to weighted case)
If $g(X_m) > g_L$, then at least three extreme points, A_i, A_j, A_k, exist such that $w_i d_i(C^*) = w_j d_j(C^*) = w_k d_k(C^*) = g^*$; that is, the subset of points at the maximum weighted distance from an optimal circle must have cardinality of at least three.

Proof:
Analogous to the unweighted case. However, when examining extreme points A_i

and A_j on the same side of the circle, select them such that $w_i \geq w_j$ ($d_i(C^*) \leq d_j(C^*)$). If A_i and A_j are both outside the circle, draw a ray from X^* through A_i, locate the point A'_i on this ray that is equidistant to the circle as A_j and use the bisector of $[A'_i, A_j]$. Similarly, if A_i and A_j are both inside, use the bisector of $[A_i, A'_j]$, where A'_j is on the ray from X^* through A_j and equidistant to the circle as A_i. When examining A_i on the outside, A_j on the inside, and A_i, A_j, X^* noncolinear, select the image A'_j at the same distance to the circle as A_i. ◇

The preceding results imply that a similar solution approach may be used as in the unweighted case. However, the subproblem for an arbitrary triplet (A_i, A_j, A_k) unfortunately does not appear to have a closed-form solution as before. When solving for the subset S of candidate extreme points, we therefore propose a mathematical programming approach based on a reformulation of the problem as follows:

minimize K
subject to $K \geq w_i^2(r_0 - d(X, A_i))^2, \ \forall \ i \in S.$

The current solution may be used as the starting point in each iteration for the updated subset S. However, since the feasible region is nonconvex, only a total enumeration, such as branch-and-bound, can guarantee convergence to an optimal solution.

4 Circle of variable radius

In this version of the problem, the radius of the circle is unknown. Drezner et al. [4] examine the unweighted case in the context of the out-of-roundness problem. They show that an optimal solution must have at least two extreme points on each side of the circle. Assuming that the part is only slightly out-of-round, they then develop an approximate method for solving the problem. In this section, we examine both the unweighted and weighted cases, and outline an exact solution method applicable to any set of fixed points.

4.1 Unweighted case

Consider Problem VU:
minimize $g(X, r) = \max_{j=1,2,\ldots,n}\{|d(X, A_j) - r|\}$.
We now use the results obtained for the fixed radius case to derive an alternate proof of the main result in Drezner et al. [4]. Whereas their proof relies on some elementary calculus, ours is based on a more intuitive geometric approach.

Property 4
Let $C^* = (X^*, r^*)$ denote an optimal solution to Problem VU, and S be the

associated set of extreme points. Then at least two extreme points are located on each side of C^*, and hence, $|S| \geq 4$.

Proof:
Suppose there are exactly two extreme points. Then C^* cannot be the limiting case of a straight line ($r \to \infty$), since it is known for this case that $|S| \geq 3$ (see, e.g., Schöbel [10]). By properties 2 and 3, both extreme points must be located outside the circle and on the same line through X^*. Then hold $X = X^*$, and increase r a small amount, to arrive at an obvious contradiction. We conclude that $|S| \geq 3$.

Clearly, not all the extreme points can be on the same side of C^*. Suppose exactly one of them, say A_i, is on one side, and all the others are on the other side. For the limiting case, $r \to \infty$, draw a ray from A_i perpendicular to the straight line, and let P denote the intersection of the two lines. We can always find a circle of finite radius, $r = d(X, P)$, with the following properties: the center X is a point on the perpendicular, there are at most two extreme points, one being A_i, and the objective value is unchanged. But this new solution cannot be optimal (see properties 2 and 3), and hence, the original straight line cannot be optimal either.

Now suppose the circle has finite radius and A_i is the only extreme point on the inside. If we displace the center X^* a small distance δ in the diametrically opposite direction of A_i, and increase the radius by δ, ($r = r^* + \delta$), a similar contradiction is arrived at. (If A_i is outside the circle, reverse the direction, and decrease the radius by δ.)

It follows that an optimal solution must have at least two extreme points on each side of the circle, and $|S| \geq 4$. $\diamond$

The following two corollaries are directly obtained from property 4, and provide an interesting link between the minimax line and the minimax circle.

Corollary 1
Suppose a minimax line is found with exactly three extreme points (one on one side, two on the other). Then the minimax circle must have a finite radius, and $g(X^*, r^*)$ is smaller than the objective value of the minimax line.

Corollary 2
If all foursomes A_i, A_j, A_k, A_ℓ have intersecting bisector pairs $(B_{ij}, B_{k\ell})$, $(B_{ik}, B_{j\ell})$ and $(B_{i\ell}, B_{jk})$, the optimal solution to Problem VU cannot be a straight line; i.e., r^* must be finite.

We now present an exact procedure for solving Problem VU. We use the same basic idea as in the fixed radius case, that is, candidate points are identified and examined in a sequence that is expected to restrict the search to a small subset of the fixed points. Again, we believe this approach may prove effective for the slightly out-of-round problem investigated by Drezner et al. [4]. The conditions in Corollary 2 are assumed to hold, so that the possibility of a straight line is avoided. (A small modification is otherwise required).

Algorithm 2 (exact solution of Problem VU):

Step 1 (Initialization).
Determine the center X_p of the associated point facility minimax problem ($r = 0$). Initialize the subset of fixed points that are candidate extreme points, $S = \emptyset$, the current center, $X_c = X_p$, and the current objective value, $g_c = \infty$.

Step 2.
Set $r_{\max} = \max_j\{d(X_c, A_j)\}$, $r_{\min} = \min_j\{d(X_c, A_j)\}$, and $r_c = (r_{\max} + r_{\min})/2$. Form the subset of extreme points, S_c, associated with the current solution (X_c, r_c), and update $S \leftarrow S \bigcup S_c$.

Step 3.
Repeat for each new pair of bisectors $(B_{ij}, B_{k\ell})$:
determine the unique intersection point, $X_{ijk\ell}$;
set $r_{ijk\ell} = (d(A_i, X_{ijk\ell}) + d(A_k, X_{ijk\ell}))/2$;
if $g(X_{ijk\ell}, r_{ijk\ell}) < g(X_c, r_c)$ update $(X_c, r_c) \leftarrow (X_{ijk\ell}, r_{ijk\ell})$;
if $\max_j\{d_j(X_{ijk\ell}, r_{ijk\ell})\} = |r_{ijk\ell} - d(A_i, X_{ijk\ell})|$, retain $C(X_{ijk\ell}, r_{ijk\ell})$ as a candidate solution.

Step 4 (Optimality test).
If there are no candidate solutions from step 3, return to step 2;
otherwise, choose the best candidate solution as the final solution. $\diamond$

4.2 Weighted case

Consider the general form of model 2 with weighted distances.
Problem VW: minimize $g(X, r) = \max_{j=1,2,\ldots,n}\{w_j|d(X, A_j) - r|\}$,
where, once again, the weights $w_j > 0,\ j = 1, 2, \ldots, n$.

The problem with all weights equal to unity has received considerable attention in the literature, as noted in the Introduction; however, the more general Problem VW appears to be new. This may be due to the limited applications perceived for the weighted case. Also, this version is more difficult to solve.

Fortunately, the main result readily extends to the weighted case:

Property 4 (generalized to weighted case)
Let $C^* = (X^*, r^*)$ denote an optimal solution to Problem VW, and S be the associated set of extreme points with maximum weighted distance to C^*. Then, $|S| \geq 4$, with at least two extreme points located on each side of C^*.

Proof:
Direct extension of the unweighted case. $\diamond$

Corollary 1 also extends directly to the weighted problem, but not so for Corollary 2. Consider any pair A_i, A_j and let L_{ij} denote the line passing through this pair of points. Using the law of similar triangles, we may argue that all lines equidistant to A_i and A_j (in the weighted sense) must pass through the same vertex V_{ij} on L_{ij}, where V_{ij} is readily obtained. (If $w_i = w_j$, then V_{ij} moves out to infinity along L_{ij}). This observation leads directly to an extended version of Corollary 2.

Corollary 2 (generalized to weighted case)
If, for all foursomes, A_i, A_j, A_k, A_ℓ, the line passing through V_{ij} and $V_{k\ell}$, or V_{ik} and $V_{j\ell}$, or $V_{i\ell}$ and V_{jk}, is not equidistant (in the weighted sense) to all four points, the optimal solution to Problem VW cannot be a straight line; i.e., r^* must be finite.

Property 4 implies that a similar procedure may be used as in Algorithm 2 to solve Problem VW. However, once again, the subproblems in the weighted case do not appear to have a closed-form solution. We therefore recommend that a mathematical programming approach be used in each iteration based on the following reformulation:

minimize K
subject to $K \geq w_i^2(r - d(X, A_i))^2, \; \forall \, i \in S.$

Again, a global optimization approach would be required to guarantee convergence to an optimal solution.

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