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Letter

Trinhammer, Ole Lynnerup

Published in:
Europhysics Letters

Link to article, DOI:
10.1209/0295-5075/124/31001

Publication date:
2018

Document Version
Publisher’s PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
https://doi.org/10.1209/0295-5075/124/31001

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To cite this article: Ole L. Trinhammer 2018 EPL 124 31001

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On Cabibbo angle from theory

Ole L. Trinhammer

Department of Physics, Technical University of Denmark - Fysikvej bld 307, DK-2800 Kongens Lyngby, Denmark, EU

received 25 August 2018; accepted in final form 29 October 2018; published online 3 December 2018

PACS 12.15.Ff - Quark and lepton masses and mixing
PACS 12.40.Yx - Hadron mass models and calculations
PACS 12.90.+b - Miscellaneous theoretical ideas and models

Abstract – We find an expression for the Cabibbo angle from quark flavour generators of the first two generations. The flavour generators operate on the toroidal components in an intrinsic dynamics for colour degrees of freedom. The generators have led to parton distributions for $u$ and $d$ valence quarks of the proton that compare well with those derived from experiment. The present result $0.974996...$ for the cosine of the Cabibbo angle compares rather well with the experimentally established value for the up-down quark mixing element $0.97420^{+0.00021}_{-0.00021}$ of the Cabibbo-Kobayashi-Maskawa matrix.

Introduction. – In 1963 Nicola Cabibbo introduced a connection between strangeness-conserving and strangeness-changing processes expressed in cosine and sine factors of a certain angle [1]. Cabibbo remarked as a consequence that the vector coupling constant in beta decay is not the full Fermi constant $G_F$, but contains a factor $\cos \theta_C$ where $\theta_C$ has come to be known as the Cabibbo angle. This was a first sign of slight differences in the description of purely leptonic decay and leptonic decays of hadrons. Cabibbo’s description was taken up by Glashow, Iliopoulos and Maiani in 1970 with the introduction of the charm quark [2]. A few years later Kobayashi and Maskawa introduced a three-generation quark mixing [3] whose elements have come to be understood as mutual quark coupling strengths. For instance the original cosine factor is now understood to express a coupling between a $u$ and a $d$ quark flavour. We shall introduce in the present work, quark flavour generators from which it is possible to derive an expression for the Cabibbo angle.

In the Standard Model of elementary particles [4], the strong interactions are described by quantum chromodynamics where quarks in strong interaction mass eigenstates can change colour by emission or absorption of gluon interaction quanta. The weak interactions are described by the electroweak theory (quantum flavourdynamics), where quarks in weak interaction eigenstates can change flavour by emission or absorption of intermediate vector bosons, $W^\pm$. Figure 1 shows the situation in a schematic form. Note that the colour transformations do not change the flavours. Note also that the flavours come in three generations. The generations can be mixed by flavour transformations but not by colour transformations.

The mixing between generations is a consequence of the mixing between flavour mass eigenstates and flavour interaction states. The mass eigenstates $u, c, t, d, s, b$ constitute the base for the strong interactions, the (weak) interaction eigenstates $u', c', t', d', s', b'$ constitute the base for the weak interactions. The connection between the two base sets is usually described by the Cabibbo-Kobayashi-Maskawa matrix

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

such that the $u$-type flavours are chosen to be the same for both interactions, whereas the $d$-type flavours are transformed, i.e.,

$$\begin{pmatrix} u' \\ c' \\ t' \end{pmatrix} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}.$$ 

1See the caption of fig. 2 for a reservation on this when higher-order processes are taken into account.
angle $\theta_C = \theta_{12}$ whereby the third generation is decoupled and we have the Cabibbo mixing matrix for the first two generations [5],

$$V_C = \begin{pmatrix}
cos\theta_C & \sin\theta_C \\
-\sin\theta_C & \cos\theta_C
\end{pmatrix}.$$  

(4)

So far the parameters in $V_{\text{CKM}}$ —and thus in the approximate $V_C$— have had to be settled by experiment.

**Flavour generators.** — In the present work we investigate a simple model which combines colour and flavour degrees of freedom to determine the value of the Cabibbo angle from a theoretical point of view. The model is not complete but it does open a road into a possible structure behind the Standard Model. The road opens from an intrinsic relation between flavour and colour degrees of freedom from which it has been possible to derive $u$ and $d$ quark parton distribution functions for a proton state by the use of flavour generators $T_u$ and $T_d$ acting on the toroidal degrees of freedom in an intrinsic dynamics for baryons like the neutron and the proton [6], see fig. 2.

We shall use the same flavour generators here and shall include also a strangeness generator, $T_s$ to have all the three flavours below the charm threshold,

$$T_u = \begin{pmatrix}2 & 0 & 0 \\3 & 0 & 0 \\0 & -1 & 0\end{pmatrix}, \quad T_d = \begin{pmatrix}0 & 0 & 0 \\3 & 0 & 0 \\0 & 0 & -1\end{pmatrix},$$

(5)

$$T_s = \begin{pmatrix}-1 & 0 & 0 \\0 & 0 & 0 \\0 & 0 & 0\end{pmatrix}.$$  

**Flavour in colour.** — The physical origin of the connection between flavour and colour that we use in the present work and used in [6] to generate the parton distribution functions in fig. 2 is deeply rooted in an underlying model for baryon structure.

We assume baryons to be stationary states of a Hamiltonian on the configuration space $U(3),$

$$\frac{\hbar c}{a} \left[-\frac{1}{2}\Delta + \frac{1}{2}\text{Tr} \chi^2 \right] \Psi(u) = \mathcal{E} \Psi(u),$$

(6)

where the length scale $a$ is determined from the classical electron radius $r_e = e^2/(4\pi\epsilon_0 mu_e c^2)$ [4] as $\pi a = r_e$ [6] and the configuration variable $u \in U(3)$ contains nine dynamical angular variables $\theta_j, \alpha_j, \beta_j$

$$u = e^{i\chi} = e^{i(\theta_j T_j + \alpha_j S_j + \beta_j M_j)/h}, \quad j = 1, 2, 3.$$  

(7)

0.836 $\pm$ 0.015, \(p = 0.122 \pm 0.018\gamma = 0.355 \pm 0.012\) in [4] we find a clear hierarchical structure $\theta_2 \sim 12.98\circ, \theta_3 = 2.42\circ, \theta_1 = 0.21\circ, \delta = 71.03\circ$ with only a minute angular admixture (but large phase) of the third generation into the first.

The “flavour in colour” relation is not an algebraic relation between quantum numbers. Instead it implies flavour degrees of freedom to be considered as specifically generated vector fields on an intrinsic three-dimensional torus.

\footnote{From the Wolfenstein parameters $\lambda = 0.22453 \pm 0.00044, A =}$
The potential $\frac{1}{2} \text{Tr} \chi^2$ is periodic and only depends on the eigenangles $\theta_j$ (see footnote 4). With a suitable parametrization of the Laplacian [12],[5]

$$\Delta = \sum_{j=1}^{3} \frac{1}{f^2} \frac{\partial^2}{\partial \theta_j^2} f^2 \frac{\partial}{\partial \theta_j} - \sum_{i<j, k \neq j} (S_i^2 + M_k^2)/h^2 \cdot \frac{8 \sin^2 \frac{1}{2} (\theta_i - \theta_j)}{8} \cdot \frac{8 \sin^2 \frac{1}{2} (\theta_i - \theta_j)}{8}$$

(8)  

the off-diagonal degrees of freedom can therefore be integrated out to get

$$\frac{h c}{a} \left[ -\frac{1}{2} \sum_{j=1}^{3} \frac{\partial^2}{\partial \theta_j^2} + W \right] R(\theta_1, \theta_2, \theta_3) = \mathcal{E} R(\theta_1, \theta_2, \theta_3),$$

(9)  

where $R$ is antisymmetric in the three angles. The total potential $W(\theta_1, \theta_2, \theta_3)$ [6] contains the centrifugal term from the Laplacian (8) and depends on hypercharge and isospin through the value of $S_i^2 + M_k^2$.

From the measure-scaled toroidal wave function $R$, we can generate colour fields $c_j$ by use of the exterior derivative expanded on the torus forms $d\theta_j$ that are conjugate to the left-invariant coordinate fields $\partial_j = u_i T_j$ generated by $T_j$, i.e., $d\theta_j(\partial_j) = \delta_{ij}$. Thus, we have

$$dR = c_j d\theta_j$$

(10)  

and can read off colour components

$$c_j(u) = dR_{u=\exp(iT_j)}(iT_j)$$

(11)  

along tracks on the torus, generated by any combination $T = k_1 T_1 + k_2 T_2 + k_3 T_3$ of generators. In particular up and down flavour tracks leading to the parton distribution functions in fig. 2 for the proton correspond to applying $T_u = \frac{2}{3} T_1 - T_3$ and $T_d = -\frac{1}{2} T_1 - T_3$ from (5) on an approximate expression for $R$. The fractional coefficients—which correspond to the quark electrical charges—enter because $U(3)$ and $SU(3)$ only share off-diagonal generators whereas the diagonal generators are different. Generators $X_j^3$ of $U(3)$ may be defined from creation and annihilation operators $a_i^T$ and $a_j$, $i, j = 1, 2, 3$ for the 3-dimensional harmonic oscillator, see pp. 71 and 221 in [13],

$$X_j^3 = a_i^T a_j \sim E_{ij}.$$  

(12)  

Here $E_{ij}$ is a $3 \times 3$ matrix representation used on (21) with the $ij$th element equal to 1 and all other elements 0. This set is equivalent to our set of $T_i, S_i/\hbar, M_i/\hbar$, e.g.,

\[a^3 \]  

This follows from the fact that the eigenvalues of $u$ are not changed by conjugation and that the trace is invariant under conjugation: Any $u = e^{i x} \chi^3$ with eigenvalues $e^{i \theta_1}, e^{i \theta_2}, e^{i \theta_3}$ can be diagonalized via conjugation with a particularly chosen $v \in U(3)$ such that $v^{-1} u v = e^{i \theta_j} T_j \equiv e^{i \theta_j}$. Now $e^{i x} \chi^3 = 1 + i x + \frac{1}{2} (i x)^2$ and we have $u^{-1} v u = 1 + v^{-1} i x v + v^{-1} (\frac{1}{2} (i x)^2 v + \cdots = 1 + i (v^{-1} i x v) + \frac{1}{2} v^{-1} (i x v - i x v) + \cdots = e^{v^{-1} i x v} = e^{i x} \chi^3$. From the cyclic property of the trace, $\text{Tr} A^{-1} B A = Tr B$, it follows that $Tr \chi^2 = Tr \xi^2 = \sum_{j=1}^{3} \theta_j^2$, with $-\pi \leq \theta_j \leq \pi$ for the shortest geodesic. See also [11].

Measure-scaling “Jacobian” $J = \prod_{j=1}^{3} 2 \sin \frac{1}{2} (\theta_j - \theta_j)$.
respectively. Thus, left invariance of the coordinate fields

\[ A_j = X_j - \frac{1}{3} \delta^3_j X, \quad X = X_j. \]  

(13)

It is the factor 1/3 that makes the fractional charges appear

in the generators (5) if one assumes the common Gell-

Mann-Nakano-Nishijima formula

\[ Q = I_3 + \frac{Y}{2}, \quad Q = A_1, \]
\[ Y = -A^3_3, \quad I_3 = \frac{1}{2} (A_1 - A_2) \]  

(14)

and for \( U(3) \) define charge and hypercharge operators as

\[ Q' = X_1^1 \sim T_1, \quad Y' = -X_3^3 \sim -T_3. \]  

(15)

Thus, eq. (5) mixes quark charges \( c_q = \frac{2}{3}, \frac{1}{3} \) with bary-

onic hypercharges \( y = 1, 0 \).

To sum up the interpretation: The colour degrees of freedom are contained in the torus of the \( U(3) \) configuration space and the flavour structure follows from the Laplacian on this configuration space. The interconnection between the two originates from the assumed Hamiltonian in (6) (see footnote 6). The model thus contains both the exact colour \( su_c(3) \) symmetry and the broken flavour \( su_f(3) \); the first reflecting the \( U(3) \) Lie group configuration space, the second reflecting the \( u(3) \) algebra of the generators from which the Laplacian (8) is con-

structed [12]. It can be shown [9,16] that the exterior derivative of the wave function generates colour quark fields, the \( c_j \)'s in (11), and gluon fields that transform under \( su_f(3) \) as the fundamental and adjoint representations, respectively. Thus, left invariance of the coordinate fields used for the intrinsic configuration space leads to local gauge invariance for the fields generated in the space-time frame of the laboratory space.

**Cabibbo angle from strange decay.** – Let us consider a strangeness-changing, non-leptonic decay

\[ \Lambda \rightarrow p + \pi^-. \]  

(16)

At tree level, this decay is described by the Feynman dia-

gram in fig. 3. We follow the notation conventions of [5]. In the standard description for a two-generation model one inserts a factor \( \cos \theta_C \) in the upper vertex and \( \sin \theta_C \) in the lower vertex to have the decay amplitude

\[ \mathcal{M} = \frac{g^2_W}{8(M_W)^2} [\bar{u}(3)\gamma^\mu(1 - \gamma^5)(\sin \theta_C)u(1)] \cdot [\bar{u}(4)\gamma_\mu(1 - \gamma^5)(\cos \theta_C)u(2)]. \]  

(17)

\[ g_W \]

Historically it grew out of work on lattice gauge theory with

Manton’s action [14] in a Kogut-Susskind Hamiltonian [15], which is

here completely reinterpreted [6].

![Fig. 3: Feynman diagram for \( \Lambda \) decay. An \( s \) quark with \( u, d \) as spectators in the \( \Lambda \) baryon transforms to a \( u \) quark by emission of an intermediate gauge boson, \( W^- \) which creates a \( d \bar{u} \) pair. Leaving the scene, one observes a baryon, the proton \( p \) and a meson, the negative pion, \( \pi^- \). We discuss the factor \( \sin \theta_C \) in the lower vertex (17).](image)

Here the integration over the \( W \) propagator

\[ \frac{-i(g_{\mu\nu} - q_\mu q_\nu/(M_W)^2)}{q^2 - (M_W)^2} \]  

(18)

has been carried out under the condition that the four-

momentum exchange \( q = p_1 - p_3 \) fulfills \( q^2 \ll (M_W)^2 \),

where \( M_W \) is the \( W \) mass. Further, \( g_{\mu\nu} \) is the electroweak coupling constant \( g^2_W = e^2/\sin^2 \theta_W = 4\pi \alpha/\sin^2 \theta_W \) with \( \theta_W \) the electroweak mixing angle and \( u \) and \( v \) are particle and anti-particle spinors, respectively. Unitarity of the Cabibbo matrix (4) is automatically fulfilled by the parametrization via a single angle \( \theta_C \),

\[ \cos \theta_C = \sqrt{1 - \sin^2 \theta_C}. \]  

(19)

We want to derive \( \sin \theta_C \). The lower vertex in fig. 3 in-
volve quarks of strange and up flavours, respectively. We can generate these flavours according to (11) from acting on colour components in the following way:

\[ c_j = T_f c_j, \quad f = u, d, s; \quad j = r, b, g. \]  

(20)

Here the colour states are [5]

\[ c_r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]  

(21)

Thus, we include colour in the incoming and outgoing quark states and sum over colour

\[ \sum_{j=r,b,g} \left[ \bar{u}(3)c_j(3)\gamma^\mu(1 - \gamma^5)c_j(1)u(1) \right], \]  

(22)

Note that \( c_j(u) \) in (11) are vector fields on \( U(3) \), whereas \( c_r, c_b, c_g \) are algebraic state vectors of specific colour. Therefore, \( c_3(u), c_2(u), c_1(u) \) are equivalent to, but not identical to \( c_r, c_b, c_g \). However, application of \( T_f \) on both sets gives the required connec-
tion between the exterior algebraic description and the intrinsic dy-
namics in the colour degrees of freedom.
with $c_1(1) = T_s c_j$ and $c_j(3) = T_u c_j$. The Dirac matrices $\gamma^\mu$ and $\gamma^5$ operate on the spinors independently of the operation of the flavour generators on the colour components. We can therefore factorize the colour algebra from the spinor algebra to get

$$
\sum_{j=r,b,g} \left[ \tilde{u}(3) (T_u c_j)^\dagger \gamma^\mu (1 - \gamma^5) T_s c_j u(1) \right] =
\left[ \tilde{u}(3) \gamma^\mu (1 - \gamma^5) u(1) \right] \sum_{j=r,b,g} c_j T_u^j T_s c_j
= \left[ \tilde{u}(3) \gamma^\mu (1 - \gamma^5) u(1) \right] \text{Tr} T_u^1 T_s.
$$

(23)

Note that there are no gluon propagators here, so this is not a strong interaction effect as such. Comparing the last expression to the standard expression in (17) we find the Cabibbo angle determined by

$$
\sin \theta_C = \text{Tr} T_u^1 T_s = -\frac{2}{9}.
$$

(24)

This yields

$$
\cos \theta_C = \frac{1}{9} \sqrt{\frac{1}{77}} = 0.974996 \ldots
$$

which is opposite in sign to $\sin \theta_C$, whereas our model would read

$$
\sum_{j=r,b,g} \left[ \tilde{u}(3) (T_u c_j)^\dagger \gamma^\mu (1 - \gamma^5) T_s c_j u(1) \right] =
\left[ \tilde{u}(3) \gamma^\mu (1 - \gamma^5) u(1) \right] \sum_{j=r,b,g} c_j T_u^j T_s c_j
= \left[ \tilde{u}(3) \gamma^\mu (1 - \gamma^5) u(1) \right] \text{Tr} T_u^1 T_s.
$$

(27)

This substitutes $\cos \theta_C$ by $\text{Tr} T_u^1 T_s$ which yields $\frac{2}{9}$ —somewhat off the value of $V_{ud}$. A radical solution to this discrepancy is to absorb $\text{Tr} T_u^1 T_s$ in a redefined coupling strength

$$
g^2 \rightarrow g_0^2 = g^2 \text{Tr} T_u^1 T_s
$$

with $g_0^2 = e^2 / \sin \theta_W$ as an a priori coupling strength for weak interactions and then to keep only $\cos \theta_C$ in the spinor brackets such that $V_C$ remains unitary by virtue of (19). From (17) we would then conclude

$$
\cos^2 \theta_W = \text{Tr} T_u^1 T_s = \frac{7}{9} = 0.777 \ldots
$$

(29)

which—as it should—compares rather well with $\frac{7}{9}$.

\[ \frac{m_W}{m_Z} = \left( \frac{80.379(12) \text{ GeV}}{91.1876(21) \text{ GeV}} \right)^2 = 0.7771(3). \]

(30)

**Charm and GIM-mechanism.** — Let us try to enlarge the model to charm quarks. We take the following charm flavour generator to supplement the set in (5)

$$
T_c = \begin{pmatrix}
\frac{2}{3} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

(31)

Just like $T_u$, neither will $T_c$ generate any partons in the protonic state we referred to in fig. 2. We consider the $d \rightarrow c$ vertex analogous to the $s \rightarrow u$ vertex in fig. 3,

$$
d \rightarrow c + W^-, \quad \frac{-i g_{V}}{2 \sqrt{2}} \gamma^\mu (1 - \gamma^5) V_{cd}, \quad V_{cd} \approx -\sin \theta_C.
$$

(32)

Using the same idea as in (23) we would conclude a Cabibbo factor

$$
-\sin \theta_C = \text{Tr} T_c^1 T_c = \frac{2}{9}.
$$

(33)

It is the same value of the trace as in (23) but this time should compare to $-\sin \theta_C$ which is opposite in sign to the relation $\sin \theta_C = -2/9$ in (24). Can this immediate inconsistency be remedied? Yes, we can exploit the freedom of choice of an overall phase factor in $V_{CKM}$. Instead of the standard $V_{CKM}$ we may use

$$
U_{CKM} = -i V_{CKM}.
$$

(34)

As $V_{CKM}$ is unitary ($V^\dagger = V^{-1}$), so is $U_{CKM}$,

$$
U^\dagger U = (-i V)^\dagger (-i V) = V^\dagger i(-i) V = 1, \quad U^\dagger = U^{-1}.
$$

(35)

With $U_{CKM}$ in stead of $V_{CKM}$ we get the following vertex Cabibbo factors for the lower vertex in fig. 3 to compare with a similar generation-changing charm decay:

$$
\begin{align*}
s & \rightarrow u + W^-, \quad U_{us} = -i V_{us} \approx -i \sin \theta_C, \\
c & \rightarrow d + W^+, \quad U_{cd}^* = (-i V_{cd})^* \approx -i \sin \theta_C.
\end{align*}
$$

(36)

In the second decay we used the vertex factor rule in the right part of fig. 5. Thus, the immediate problem of opposite signs on $\sin \theta_C$ in (23) and (33) is eliminated, albeit at the price of a common non-zero phase, $i = e^{i \pi/2}$ —a phase which we would like to understand better.
The reader may remember in the back of his or her mind that the opposite signs on the sin $\theta_C$ in the Cabibbo matrix (4) are the basis of the standard exposition of the GIM-mechanism (1) – note that the $i, j$ indices are reversed in the left figure. See p. 439 and p. 350 in ref. [5].

As a consequence, the decay $K^0 \rightarrow \mu^+ + \mu^-$ is heavily suppressed.

**Conclusion.** – We have relaxed on the independent treatment of the electroweak and strong interactions. We used a relation between flavour generators and colour components to find a possible physical origin of the Cabibbo angle. The relation is supported by previously derived parton distribution functions from the same set of flavour generators that are here related to the Cabibbo angle. The model needs improvement to include mixing into the third generation.

***

I appreciated the anonymous referee queries into the physical background of flavour in colour. I thank MOGENS STIBIUS JENSEN and HENRIK GEORG BOHR for helpful discussions. I thank the Technical University of Denmark for an inspiring working environment.

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