Vibro-Impact Mechanics

Analytical, Numerical and Experimental Investigations

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Abstract (English)

Vibro-impact phenomena are characterized by the occurrence of intermittent collisions between objects or internal machine components whose movement is constrained by physical barriers, such as stops and clearances. Vibro-impacts can enhance the effectiveness of hammering, riveting, and drilling applications, but are an undesirable phenomenon for loose joints, heat exchangers, and rotating machines, producing noise, wear and failure. Its applicability and relevance drive the continuous development of different model formulations and analytical/numerical solution schemes intended to reproduce specific experimental results or particular dynamic behavior.

This Ph.D. thesis presents a detailed discussion of the most common model formulations for vibro-impact problems: contact force models and the coefficient of restitution, (CoR). Common ground comparisons are made by correlating analytical and numerical model predictions to experimental results. Different impact situations are tested experimentally, from hard unilateral impacts with tightened, neutral or loose gaps, to bilateral soft collisions. The experimental setup consists of a lumped mass cantilever beam with base-excitation provided by an electrodynamic shaker.

To obtain qualitative insight, the experimental setup is represented by a single degree of freedom (SDOF) model, facilitating the use of different analytical tools like harmonic linearization, pointwise mapping, and averaging. The validity of approximated solutions is tested using numerical simulations. For hard impacts, non-smooth transformations are combined with standard averaging and also used as a pre and post-processing step for numerical simulations.

The frequency response of the experimental setup with bilateral soft impacts around its fundamental linear resonance is replicated using three contact force models consisting of a piecewise linear function, a high order power function, and a combination of the first two, combining their advantages. The influence of model parameters on the frequency response is also investigated numerically.

Experiments involving hard unilateral impacts reveal the existence of complex behavior, which could not be replicated by the SDOF model with CoR. However, it is still possible to confirm theoretical predictions about the influence of gap width on the location of the resonant peak. The effects of deviations of the gap width’s nominal value on the response amplitude are also investigated.
Abstract

Equivalence relations between piecewise linear impact forces and the coefficient of restitution are presented and compared numerically. The transition from soft to hard impacts is investigated experimentally by changing the contact stiffness with different helical springs.

Comparison of analytical, numerical and experimental results revealed the wider application range of impact force modeling with respect to the CoR. Still, both model approaches are able to provide qualitative insights on some important characteristics of experimental vibro-impact systems.
Vibrostød-fænomener er karakteriseret ved gentagne kollisioner mellem objekter eller interne maskinkomponenter hvis bevægelser begrænses af fysiske barrierer såsom stopklodser og klaringer. Sådanne sammenstød kan forbedre effektiviteten af hammer-slag, nagling og boring, men er uønskede fænomener i løse samlinger, varmevekslere og roterende maskineri, hvor de giver anledning til støj, slidtage og nedbrud. Områdets anvendelighed og relevans driver den fortsatte udvikling af forskellige modeller og analytiske/numeriske løsningsmetoder med det formål at reproducerere specifikke eksperimentelle resultater eller en bestemt dynamisk opførsel.


For at opnå kvalitativ indsigt modelleres forsøgstopstillingen med en frihedsgrad (SDOF), hvilket muliggør anvendelsen af forskellige analytiske værktøjer såsom harmonisk linearisering, punktvis afbildning og midling. Validiteten af de tilnærmelsesvisle løsninger undersøges ved hjælp af numerisk simulering. For hårde sammenstød kombineres diskontinuerte transformationer med klassisk midling og dette bruges ligeledes til pre- og postprocessering af numeriske simuleringer.

Forsøgstopstillingens frekvensrespons med bløde tosidige kollisioner omkring systemets første lineære resonansfrekvens, er eftervist ved brug af tre kontaktkraftsmodeller bestående af en stykvist lineær funktion, en potensfunktion af høj orden og en kombination af de to førstnævnte som forener deres fordele. Modelparametrene indflydelse på frekvensresponset undersøges ydermere numerisk.

Ekspentifier med hårde ensidige kollisioner afslører en kompleks opførsel som ikke har kunne eftervises af SDOF modellen med CoR. Det har dog været muligt at bekræfte teoretiske forudsigelser omkring klaringens indflydelse på resonanstoppens placering. Betydningen af afvigelser fra klaringens nominelle værdi på resonansamplituden er også undersøgt.
Ækvivalensrelationer mellem stykvist lineære kontaktkræfter og restitutionskoefficienten præsenteres og sammenlignes numerisk. Overgangen fra bløde til hårde sammenstød undersøges eksperimentelt ved at ændre kontaktstivheden med forskellige spiralfjedre.

Sammenligning af analytiske, numeriske og eksperimentelle resultater afslører et bredere anvendelsesområde for kontaktkraftmodellering med hensyn til CoR. Begge modelleringstilgange er fortsat i stand til at give kvalitativ indsigt i visse vigtige karakteristika ved eksperimentelle vibrostød systemer.
Preface

This thesis is submitted as partial fulfillment of the requirements for obtaining the Danish Ph.D. degree. The research was carried at the Section of Solid Mechanics (FAM) of the Department of Mechanical Engineering of the Technical University of Denmark, (DTU Mekanik) from July 2014 to June 2018 under the supervision of Professor Ilmar Ferreira Santos and co-supervised by Associate Professor Jon Juel Thomsen. The work was funded by DTU Mekanik and by the Brazilian federal government through the Coordination for Improvement of Higher Education Personnel (CAPES) under the scholarship program Science without Borders (SwB) with process number: 99999.013736/2013-02.

I would like to express my gratitude towards both my supervisors. To Ilmar, for accepting me as one of his Ph.D. students and providing me with the technical and emotional support needed to complete this journey. And to Jon for always having time for unscheduled meetings and for his accurate comments that increased the level of this work. A special thank you is directed to Cesar A. L. L. Fonseca and Alejandro M. Tejada for the good time spent in Denmark and for the conversations about technical and general matters. I also would like to say thanks to current and previous fellow Ph.D. students, office colleagues and other people that I met during my time at DTU: Bo B. Nielsen, Nikolaj A. D-Hansen, Sebastian von Osmanski, Søren Enemark, Alejandro C. Varela, Jorge A. G. Salazar, Fabian G. P. Vásquez, Jonas S. Lauridsen, André K. Sekunda, Andreas J. Voigt, Martin S. Nielsen, Vladislav S. Sorokin, Si M. Sah, Marco Túlio S. Alves, Hannibal T. Overgaard and Niels Steenfeldt. This project could not have been completed without the help of FAM’s technical staff, the secretaries, and technicians at the workshop and on the instrumentation group.

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# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>Resumé</td>
<td>iii</td>
</tr>
<tr>
<td>Preface</td>
<td>v</td>
</tr>
<tr>
<td>Contents</td>
<td>vi</td>
</tr>
<tr>
<td><strong>1 Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Literature Overview</td>
<td>2</td>
</tr>
<tr>
<td>1.2.1 Model formulation</td>
<td>2</td>
</tr>
<tr>
<td>1.2.2 Analytical Treatment</td>
<td>3</td>
</tr>
<tr>
<td>1.2.3 Numerical Simulations</td>
<td>3</td>
</tr>
<tr>
<td>1.2.4 Experimental Investigations</td>
<td>4</td>
</tr>
<tr>
<td>1.3 Contributions</td>
<td>6</td>
</tr>
<tr>
<td><strong>2 Overview</strong></td>
<td>9</td>
</tr>
<tr>
<td>2.1 Experimental setup [A1–A3]</td>
<td>9</td>
</tr>
<tr>
<td>2.1.1 Limitations</td>
<td>15</td>
</tr>
<tr>
<td>2.2 Modeling</td>
<td>17</td>
</tr>
<tr>
<td>2.2.1 Coefficient of restitution [A2]</td>
<td>18</td>
</tr>
<tr>
<td>2.2.2 Contact force models [A1]</td>
<td>21</td>
</tr>
<tr>
<td>2.2.3 Equivalence relations [A3]</td>
<td>29</td>
</tr>
<tr>
<td>2.3 Frequency domain comparisons [A1–A3]</td>
<td>32</td>
</tr>
<tr>
<td><strong>3 Conclusions</strong></td>
<td>37</td>
</tr>
<tr>
<td>References</td>
<td>39</td>
</tr>
</tbody>
</table>
### Appended articles

--- 
Errata .............................................. 51  
Abstract ........................................... 52  
A1.1 Introduction ................................... 52  
A1.2 Experimental setup and procedure ............... 54  
A1.3 Equations of motion ............................ 58  
A1.4 Impact force models ............................ 60  
A1.4.1 Kelvin-Voigt model .......................... 60  
A1.4.2 Power-law impact force model ................ 62  
A1.4.3 Modified Kelvin-Voigt model ................. 64  
A1.5 Approximate analysis of time periodic response ...... 66  
A1.5.1 Harmonic Linearization ....................... 66  
A1.5.2 Averaging ................................... 67  
A1.5.3 Application to specific impact force models ...... 68  
A1.6 Model tuning and analysis ....................... 70  
A1.6.1 Linear range ................................ 70  
A1.6.2 Power function exponents ................. 70  
A1.6.3 Model validation ............................ 73  
A1.6.4 Parameter analysis ........................... 73  
A1.7 Conclusions .................................. 76

[A2] Unilateral Vibro-Impact Systems — Experimental Observations against Theoretical Predictions based on the Coefficient of Restitution  
--- 
Abstract .............................................. 80  
A2.1 Introduction ................................... 80  
A2.2 Engineering modeling ............................ 83  
A2.3 Numerical Analysis .............................. 84  
A2.3.1 Direct Numerical Simulation ................. 85  
A2.3.2 Numerical Simulation using Zhuravlev’s Transformation .... 86  
A2.3.3 Numerical Simulation using Ivanov’s Transformation .... 88  
A2.4 Analytical Techniques ........................... 90  
A2.4.1 Averaging and Zhuravlev’s transformation .... 90  
A2.4.2 Pointwise mapping ........................... 94  
A2.5 Experimental setup .............................. 98  
A2.5.1 Coefficient of restitution .................... 107  
A2.5.2 Platform dynamics ........................... 110  
A2.6 Model for the experimental setup ............... 112  
A2.6.1 Linear parameter estimation ................ 113  
A2.7 Comparing Experimental and Numerical results ...... 114  
A2.8 Conclusions .................................. 115
[A3] On the influence of contact stiffness in unilateral vibro-impact  
Abstract .............................................................................................................. 120
A3.1 Introduction .................................................................................................. 120
A3.2 One phenomenon and two modeling approaches ........................................ 121
    A3.2.1 Equivalence relations ........................................................................... 122
    A3.2.2 Parameter Analysis .............................................................................. 123
A3.3 Analytical treatment ..................................................................................... 127
    A3.3.1 Harmonic Linearization ....................................................................... 127
    A3.3.2 Averaging with Zhuravlev transformation .............................................. 128
A3.4 Experimental Setup and its Mathematical Model ......................................... 129
A3.5 Experimental Observations .......................................................................... 132
A3.6 Conclusions ................................................................................................ 136

C  Extended Abstract for ENOC 2017 ................................................................. 139
Chapter 1

Introduction

1.1 Motivation

Impacts are one of the most common phenomena in nature, being present in activities like hammering a nail, skipping a stone or dribbling a basketball. Impacts have an important role on various equipment, from toys to engineering processes and applications: rotor-stator rubbing, loose joints, backlash in gear pairs and machines for hammering, forging, riveting, drilling, concrete breaking, impact printers, pile drivers and so on.

For some machines contacts between certain parts can be a problem or a way to enhance its capabilities. Variations on the nominal dimensions of manufactured parts are subjected to geometric dimensioning and tolerancing specifications, leading to loose/tight fittings which affect machine operation. For instance, contacts in loose gear pairs produce wear and noise, while rotor-stator contact could cause machine failure. Similar problems occur due to friction in tightened pairs with relative movement. Alternatively, periodic contact forces are fundamental to automatic forging/riveting machines and intermittent impacts can improve performance in drilling applications.

Either way, to maximize or mitigate vibro-impact behavior in different applications it is necessary to understand its working principles and characteristics. With this purpose, scientists from different backgrounds have been studying the behavior of colliding bodies since Isaac Newton in the XVII century. And still today, different formulations, analytical/numerical solutions, and empirical observations are reported in conferences and peer-reviewed journals, with much of the research efforts being directed to numerical analysis and model development.

Despite well-succeed efforts to produce comprehensive perspectives on vibro-impact mechanics in review articles and monographs, experiment-based common-ground comparisons of the most common model formulations for vibro-impact phenomena are necessary. This Ph.D. thesis aims at responding to this demand with a critical assessment of the most common modeling and analysis techniques for vibro-impact mechanics, comparing them between each other and with experimental observations.
1.2 Literature Overview

While impact mechanics is generally concerned with single collision systems, e.g. car crash, vibro-impact (VI) mechanics studies problems with recurring collisions, such as the ones in loose gear pairs. Besides the natural frequencies of the colliding pair that are excited at each impact, the frequency of impacts is also important to study the stability, magnitude and frequency content of VI oscillations. This Ph.D. thesis is dedicated to VI dynamics and readers interested in impact mechanics should consult [1–3].

There are several monographs dedicated to VI mechanics. Some of them emphasize physical applications, e.g. [4–6] and others prioritize mathematical aspects [7, 8]. Discussions on numerical methods for non-smooth models can be found on [9]. An extensive overview of VI mechanics containing more than a thousand references can be found on [10].

The next paragraphs are intended to present a brief overview of VI mechanics, emphasizing its most common modeling approaches together with relevant analytical, numerical and experimental results.

1.2.1 Model formulation

Contact forces and the coefficient of restitution (CoR) are the most typical ways of representing impacts. The choice between them should take into account contact characteristics, like its penetration depth and duration, being the main challenge for modeling such systems. In general, force models are more indicated for applications where contact duration and deformation are not negligible compared to the period and amplitude of oscillations, otherwise, both models should produce similar results as shown in [11] through numerical simulations.

Impact forces can be represented in several ways, with piecewise linear functions [5], Hertzian contacts [12, 13] and power-law terms [14] being the most common choices. In general, those representations demand knowledge on the contact duration, which can be difficult to obtain in some applications, but they also provide useful information on structural stresses.

The piecewise linear formulation considers the contact as an association of the colliding bodies and its dynamic properties (damping and stiffness). Despite its direct interpretation, this approach produces physically inaccurate outputs, which propelled the search for other representations, such as the Hunt and Crossley model [14], where the dissipative force is a power of both displacement and velocity, and the CoR is a model parameter. This model started an ongoing discussion on how to represent the damping force during collisions, leading to sophisticated mathematical expressions involving the CoR, the pre-contact velocity and rational powers of the contact deformation [15, 16].

Due to its simple formulation and physical interpretation, the coefficient of restitution is widely used to model vibro-impact [4, 5]. In this representation, the contact duration is negligible with respect to the period of oscillations excited by the impacts and the colliding elements are considered to be rigid bodies. The contact dynamics is neglected and only the negative ratio between pre and post-impact velocities is considered.
1.2 Literature Overview

(kinematic formulation). With this approach, it is not possible to obtain any information on structural stresses. The CoR is treated as a constant parameter in most applications despite empirical evidence of its dependence on the pre-contact velocity [17, 18].

1.2.2 Analytical Treatment

The existence of analytical solutions for nonlinear models is restricted to low-dimensional systems in some special cases, in certain parameter range, involving sophisticated mathematical manipulations. Despite all of those limitations, closed-form solutions can provide qualitative insight on the properties of more complex problems related to the original perturbed model.

There are different analytical tools to study vibro-impact problems. For piecewise linear systems, one of the first approaches was to connect contact and non-contact linear solutions using the impact condition [4]. This approach, known as stitching or pointwise mapping, has been used by several researchers for systems with a single [19, 20] and many [21, 22] degrees of freedom. In this approach one obtains a discrete map of impact instants and velocities, enabling analysis on the stability of periodic oscillations and bifurcations. However, to obtain results in time-domain it is necessary to use root-finding algorithms to find the collision times.

Babitsky [5] and his collaborators use periodic Green functions to obtain analytical responses of systems under intermittent impulsive excitation [23], piecewise-linear systems [24] and structures with discontinuities [25, 26]. The applicability of both pointwise mapping and periodic Green functions is restricted to problems where impacts are the only nonlinearity.

Techniques for models with weak nonlinearities between collisions like Harmonic Linearization [5], Standard Averaging [27, 28], and the Lindstedt-Poincaré method [29, 30] can be applied to VI models with weak impacts. For systems with strong collisions, one can use discontinuous transformations of time [31] or state [32, 33] variables, to reduce or even eliminate model non-smooth terms, enabling the use of standard perturbation methods.

Thomsen and Fidlin [34] used unfolding coordinate substitutions to bring vibro-impact systems to a weak form, suitable for the application of extended averaging. Gendelman [35, 36] analyzed the behavior of a linear oscillator with a VI energy sink using the method of Multiple Scales and a zigzag function, commonly used in non-smooth time transformations. Experimental evidence of the feasibility of this approach is reported in [37]. Alternatively, Newman and Makarenkov [38] combined standard averaging with Poincaré map to analyze the stability of resonant oscillations in a single degree of freedom, SDOF, model with unilateral impacts and near elastic CoR. Most of the analytical results reported in this subsection were accompanied by numerical simulations.

1.2.3 Numerical Simulations

The most common tool to obtain approximated outputs from nonlinear models nowadays is the numerical simulation, which is considered a typical intermediate step in the
design process. This approach does not demand any prior assumption about the model and can be used to test the effect of different formulations on model response or to confront analytical/experimental results.

For VI mechanics, the most common approach is to combine standard ODE solvers with event handling subroutines. Alternatively, one can replace non-smooth impact forces with smooth alternatives, enabling the use of standard ODE solvers without event handling subroutines [39]. In this case, the smooth representation should be properly tuned. For an overview of the most common ways of smoothening non-smooth nonlinearities see [40]. Application of discontinuous transformations to the equations of motion before its simulation as done in [41], can decrease the velocity jumps due to impact, reducing simulation time. There are also several numerical integration algorithms designed specifically for non-smooth models [9, 42–44], but their comparison is out of the scope of this work, which will use MATLAB’s® built-in ODE solvers with the event location option.

1.2.4 Experimental Investigations

Empirical observations are fundamental to the development of science because they can reveal phenomena not found in the theoretical/numerical analysis. Engineers and scientists have used experimental setups with different goals: to show novel phenomena, (in)validate theoretical/numerical predictions/assumptions, or to understand processes through data interpretation.

Models with 1-DOF are often used to describe the experimental VI behavior of continuous structures [45–47], discrete cart-on-track systems [48–50] and impact pendulums [51–53]. Cart-on-track setups eliminate the possibility of modal interactions that are common to occur in nonlinear multi-DOF systems such as pendulums and structures [48]. However, beam-setups are simpler and easier to manufacture [46]. Impacting pendulums with large oscillation angle gives rise to additional complex behavior.

Todd and Virgin used a cart-on-hill setup to analyze the influence of gap location on the natural frequency [54] and global dynamics of an SDOF oscillator with unilateral impacts and friction. Numerical and experimental observations are shown to agree quantitatively and qualitatively. Savi et al. [49] used a smoothed model to reproduce the experimental behavior of a cart-on-rail setup with soft unilateral impacts, obtaining quantitative/qualitative agreement between numerical and experimental results. Virgin et al. [50] use a cart-on-rail setup to replicate the dynamics of a pinball machine, describing unilateral collisions with a modified CoR rule including impulsive energy input and obtaining qualitative agreement between empirical and numerical diagrams.

The dynamics of a pendulum with bilateral hard constraints in normal and inverted configurations is analyzed experimentally by Moore and Shaw [51] considering small angles. Experimental observations for period-$n$ vibrations show qualitative compliance with theoretical predictions made by Shaw and Rand [55] using pointwise mapping. Bayly and Virgin [52] present an experimental/numerical investigation of the unilateral impact dynamics of a pendulum considering large oscillations. The results of both investigations presented qualitative and quantitative agreement.
Some of the predictions made by Shaw and his collaborators [19, 56, 57] for an SDOF model with unilateral soft and hard impacts using pointwise mapping and numerical simulations were empirically evaluated by Moon and Shaw [58] and Shaw [45], both using a base-excited cantilever beam (CB) with single-sided constraint setup and SDOF model with piecewise linear elastic impacts. Their modeling results were able to show qualitative agreement with different experimental behaviors. In [58] chaotic vibrations were the main focus while in [45] sub-harmonics and period-doubling were the main concern.

Stensson and Nordmark [59] show empirical evidence of theoretical results obtained by Nordmark [60, 61] for an SDOF system with unilateral low-velocity impacts and CoR using Poincaré mapping. This model was able to recreate strange attractors found on the experimental setup, which consists of a CB with a lumped mass under intermittent unilateral collisions. Fang and Wickert [62] use an identical setup and modeling, obtaining reasonable agreement between experimental vibrations and results from a nonlinear recurrence relation of post-impact position and velocity.

Bishop et al. [63] reproduced the experimental frequency-response of a unilateral VI CB setup around its fundamental frequency using an SDOF model with CoR impacts, achieving reasonable similarity. Wiercigroch and Sin [46] presented an experimental study on soft bilateral VI by using a mass supported by a pair of springs and two CBs as soft constraints. With this setup, it was possible to change the clearance width and both contact and non-contact stiffness, besides forcing amplitude and frequency. Comparison of numerical and experimental phase-planes and bifurcation diagrams showed reasonable agreement.

Balachandran [64] modeled the VI dynamics of an experimental setup similar to [62], using CoR and an impact force loading proportional to the contact acceleration. Comparison between numerical simulations and experiments shows that impact duration is critical for explaining differences on the bifurcation diagrams for ascending and descending forcing frequency sweeps. Soliman et al [65] propose an energy harvesting device given by a base-excited CB with single and bilateral soft constraints setup, which is modeled as an SDOF model with piecewise linear dissipative and elastic contact force. The approximated frequency response of this model around its natural frequency is obtained using Averaging, presenting reasonable concordance with empirical observations and numerical simulations. Elmegård et al. [47] proposed a piecewise linear SDOF model for a base-excited CB with lumped mass and bilateral impacts, presenting qualitative agreement with experimental observations from [66] and predicting the existence of an isola, which was reported experimentally in [67].

Multi-DOF models are also used to describe experimental VI oscillations in cantilever beams. For instance, Wagg and Bishop [68] use a 4-DOF model to reproduce observations from a forced CB with unilateral impacts setup around its second natural frequency, obtaining qualitative agreement. Model dimensionality is investigated numerically, showing little qualitative differences between 3 and 4-DOF models in the time domain. Dick et al. [69] use a piecewise linear elastic and dissipative contact force to describe the dynamics of a base-excited CB setup with unilateral constraint and additional geometric nonlinearities. Its numerical model provides results quantitative similar
1.3 Contributions

to experimental observations.

Andreaus [70] presents a 3-DOF model for a forced CB setup with unilateral impacts taking into account elastic and dissipative piecewise linear contact forces and the excitation source (electrodynamics shaker) dynamics. This modeling approach provides good agreement with experimental measurements of VI oscillations and impact force in the frequency domain. The dimensionality of vibro-impact beams has been studied experimentally by [71, 72] using correlation dimension and proper orthogonal decomposition, respectively.

1.3 Contributions

Roughly speaking, most of the works discussed above focused on obtaining accord- dance between specific numerical/analytical results and experimental findings. That motivated the preparation of this Ph.D. thesis, whose main objective and originality is to compare the most common model formulations and solution methods for different VI configurations against empirical observations. Due to space and time limitations, only the most common modeling techniques will be examined. In this work, the emphasis is given to period-one oscillations, despite the existence of more complex behavior in VI models and experiments.

The main contributions of this Ph.D. thesis are discussed in Chapter 2 which consists of a concise overview of the appended articles [A1–A3] written by the author of this thesis in collaboration with his supervisors. The intent of this chapter is to summarize the research spread on those papers in a unified framework. After that, the overall conclusions can be found in Chapter 3.

The appended papers can be read independently at the end of this work and their status, as of the writing of this thesis is the following: [A1] was published on the Journal of Sound and Vibration, [A2] was submitted for publication in the same journal in May 2018, being under peer-review since then, and [A3] is under preparation and has not been submitted to a peer-reviewed journal yet. An overview of each article is given below:

[A1]: The main contribution of this paper is to compare the results from three impact force models against experimental observations of soft bilateral VI. Harmonic Linearization and Averaging are used to obtain frequency response curves for each force model, which are fitted on empirical data to obtain numeric values for model parameters. All force models are able to replicate experimental behavior, with the third one being the most accurate. The performance of this model can be explained by its formulation, which combines the common piecewise-linear function with a nonlinear damping term similar to the one proposed by Hunt and Crossley [14]. The other models are given by a piecewise linear function of elastic and dissipative forces; and a smooth power-law function of displacement and velocity.
[A2]: The objective and main originality of this paper is to present a careful assessment of the most common techniques for analysis of unilateral VI problems using the coefficient of restitution, together with the presentation of empirical data for VI oscillations under different excitation levels and gap configurations. Despite experimental challenges, it was possible to link laboratory observations with predictions made in the literature. The influence of gap deviations on the response amplitude during an experimental frequency sweep is examined. Empirical evidence of the CoRs sensitivity to pre-contact velocity is shown. Under certain circumstances, it is possible to obtain qualitative compliance between theory and experiments.

[A3]: This article investigates the influence of contact stiffness in unilateral vibroimpact. Soft and hard impacts are represented by piecewise linear contact forces and the coefficient of restitution, respectively. The applicability conditions of each model formulation are discussed and equivalence relations between model parameters are derived, revealing the influence of different factors on the CoR, such as contact stiffness, damping, pre-impact velocity and gap width. The role of contact stiffness is investigated experimentally with the use of helical springs, showing that the occurrence of complex behavior decreases with spring stiffness. Predictions of contact duration present reasonable agreement with experimental observations, while the measured CoR was smaller than its theoretic estimation. The curvature of bent resonant peaks on experiments can be reasonably approximated using the contact stiffness of each spring.
Chapter

2

Overview

This chapter contains a general account of the main contributions of the appended articles [A1–A3] that can be read independently at the end of this manuscript. The overall goal of those papers is to use experimental observations to test different aspects and assumptions used to model VI mechanical systems. For that, vibro-impacts were divided as soft or hard impacts with unilateral or bilateral constraints. In [A1], the different formulations for contact force models (soft impacts) and bilateral constraints are discussed, while common techniques (numerical and analytical) to investigate unilateral vibro-impact systems using the coefficient of restitution (hard impact) are discussed in [A2]. After that, the role of the contact stiffness in hard and soft unilateral vibro-impact models is investigated on [A3].

The experimental setup, its limitations, and deviations from common mathematical assumptions is described in Section 2.1. Section 2.2 presents a SDOF model for the experimental setup together with the description and analysis of different model formulations used to describe soft and hard impacts. Then experimental and theoretical results are compared in Section 2.3.

2.1 Experimental setup [A1–A3]

The different impact configurations described above were physically implemented using the experimental setup shown in Figure 2.1, which consists of a cantilever beam with attached mass which is mounted on a platform (2) that is connected to an electrodynamic shaker (1). The oscillations of both mass and platform are measured by displacement sensors (3). Stops can be placed in one or both sides of the beam/mass to constrain its oscillations. Different impact configurations can be achieved, according to the location of stops and its distance to the beam/mass.

Placing stops around the beam, as shown in Figure 2.2(a), does not directly restrict mass oscillations, which can reach amplitudes far beyond the necessary to touch the stops due to beam’s flexibility, in which case the beam’s deflection pattern does not resemble the one of a non-impacting cantilever beam anymore. So, hard collisions between beam and stops produce soft impacts on the mass. Similar results could be obtained by placing
springs around the mass, as shown in Figure 2.2(c). In this case, the stiffness of the spring and the cantilever beam form a parallel association during contact.

Hard collisions can be obtained by positioning stops around the mass, as shown in Figure 2.2(b), where three gap configurations are possible: tightened, loose or neutral. A downside of placing stops or springs around the lumped mass is that it makes difficult to observe unstable states on the experimental frequency-response obtained with the use of control-based continuation [66, 67] and electromagnetic actuators.

The setup’s main natural frequencies were identified using experimental modal analysis. Figure 2.3 shows the setup’s frequency response for an impact excitation at 50 mm from the beam’s clamped end, where one can identify four natural frequencies: 3.25 Hz, 8 Hz, 67 Hz and 98.75 Hz obtained with a resolution of 0.25 Hz. At 3.25 Hz only the platform oscillates. The beam’s first and second bending resonances occur at 8 Hz and 98.75 Hz respectively, with a torsional resonance at 67 Hz.

The experiments in this Ph.D. thesis were made on the neighborhood of the beam’s first bending resonance $f_n = 7.6$ Hz or around twice that for unilateral vibro-impacts. When the beam is vibrating at its first bending mode, the influence of higher modes can be neglected as shown in Figure 2.4, where the amplitude of higher harmonics ($\geq 2\Omega$) is negligible if compared to the amplitude of the first harmonic, $1\Omega$. Similarly, observing the harmonic decomposition of the unilateral vibro-impact response around $2f_n$, shown in Figure 2.5, one can see that the amplitude of higher harmonics is negligible if compared to the amplitudes of the static and first harmonic, $0\Omega$ and $1\Omega$, respectively.

In [A3], the role of the contact stiffness on the response of VI systems is investigated experimentally. This parameter could be modified by placing the constraints at different locations of the beam as done in [A1, see Figure 2.2(a)]. In this case, one should derive an validate an expression for the stiffness variation with location. Also, that could
generate non-impacting vibrations with permanent contact for tightened configuration. Thus, the choice of helical springs to modify the contact stiffness. Their properties can be found in Table 2.1.

One can estimate the value of the beam’s structural stiffness using the lumped mass $m = 0.21$ kg and the measured linear natural frequency, $f_n = 7.6$ Hz, obtaining $K_S = 0.61$ kN m$^{-1}$. The contact natural frequency has been predicted using the rate of each spring $k_i$ and the beam’s lumped mass, $m$. With that, one can estimate the contact duration as half of the contact natural period according to [11, 19]. The coefficient of restitution is estimated according to an expression derived by [19] which will be discussed in Section 2.2.3. The values for $R$ are close to unity because they were calculated without taking the contact damping ratio into account.

Measured and estimated values of the contact duration and coefficient of restitution are presented in Table 2.2 for different frequency ranges. There one can see an agreement
2.1 Experimental setup [A1–A3]

![Graph](image)

Figure 2.3: Frequency response: (a) amplitude, (b) coherence. Taken from [A2].

Table 2.1: Properties of helical springs. Taken from [A3].

<table>
<thead>
<tr>
<th>Spring n.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire cross-sec. dimens., [mm]</td>
<td>$d = 6$</td>
<td>$b = 5$; $h = 3$</td>
<td>$d = 3$</td>
<td>$d = 2$</td>
</tr>
<tr>
<td>Coil diameter, $D$ [mm]</td>
<td>25</td>
<td>24</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>Number of turns, $N$ [-]</td>
<td>6</td>
<td>10</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>Shear modulus, $G$ [GPa]</td>
<td>80</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density, $\rho$ [kg m$^{-3}$]</td>
<td></td>
<td></td>
<td></td>
<td>7800</td>
</tr>
<tr>
<td>Spring rate, $k_i$ [kN m$^{-1}$]</td>
<td>138.2</td>
<td>21.9</td>
<td>8.6</td>
<td>1.5</td>
</tr>
<tr>
<td>Contact natural frequency, [Hz]</td>
<td>129</td>
<td>51</td>
<td>32</td>
<td>14</td>
</tr>
<tr>
<td>Natural frequency ratio, $\omega_R$ [-]</td>
<td>17</td>
<td>6.8</td>
<td>4.2</td>
<td>1.8</td>
</tr>
<tr>
<td>Contact duration, $\tau_C$ [ms]</td>
<td>3.9</td>
<td>9.7</td>
<td>15.5</td>
<td>36.7</td>
</tr>
<tr>
<td>$\tau_C/T_n$, [%]</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>28</td>
</tr>
<tr>
<td>$R_1$, [-]</td>
<td>0.998</td>
<td>0.996</td>
<td>0.993</td>
<td>0.985</td>
</tr>
</tbody>
</table>

between predictions and measurements for the contact duration on springs $k_1$ to $k_3$, together with variations according to the forcing frequency for $R$ in all springs. Only the softer springs $k_3$ and $k_4$ got close to their measured values.
2.1 Experimental setup [A1–A3]

Figure 2.4: (a) Experimental frequency response and its (b) harmonic decomposition around $f_n$. Taken from [A1].
2.1 Experimental setup [A1–A3]

Figure 2.5: Harmonic decomposition of period one orbits around $2f_n$ for different gap configurations: (a) pre-stressed $-0.1$ mm, (b) neutral $0$ mm and (c) loose $0.1$ mm. Taken from [A2].

Table 2.2: Comparison of contact duration and CoR. Taken from [A3].

<table>
<thead>
<tr>
<th>Spring n.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_C$, [ms] (estimated)</td>
<td>3.9</td>
<td>9.7</td>
<td>15.5</td>
<td>36.7</td>
</tr>
<tr>
<td>$\tau_C$, [ms] (meas. $\approx f_n$)</td>
<td>3.3</td>
<td>10.2</td>
<td>7.4</td>
<td>26.2</td>
</tr>
<tr>
<td>$\tau_C$, [ms] (meas. $\approx 2f_n$)</td>
<td>2</td>
<td>11.4</td>
<td>14.7</td>
<td>27</td>
</tr>
<tr>
<td>$R_1$ [-] (estimated)</td>
<td>0.998</td>
<td>0.996</td>
<td>0.993</td>
<td>0.985</td>
</tr>
<tr>
<td>$R$ [-] (meas. $\approx f_n$)</td>
<td>0.77</td>
<td>0.72</td>
<td>0.91</td>
<td>0.97</td>
</tr>
<tr>
<td>$R$ [-] (meas. $\approx 2f_n$)</td>
<td>0.71</td>
<td>0.88</td>
<td>0.94</td>
<td>0.92</td>
</tr>
</tbody>
</table>
2.1 Experimental setup

2.1.1 Limitations

The experimental setup in this Ph.D. thesis is used as a testbed for vibro-impact oscillations and its measurements are used to test and validate qualitative insights obtained from simple models about the behavior of general vibro-impact systems, which is the goal of this Ph.D. thesis. Thus, it is necessary to have a broad understanding of the experimental setup, knowing its limitations and evaluating the obstacles they present in order to achieve the objectives defined.

The attached mass has a considerable moment of inertia that can produce rotational vibrations due to eccentric impacts. However, observing the frequency spectrum for unilateral vibro-impact oscillations, shown in Figure 2.6(b), one can identify only harmonics of the forcing frequency, indicating that the higher natural frequencies identified using modal analysis are not relevant for the vibro-impact response around $2f_n$. That and the results from Figures 2.4 and 2.5 discussed previously, justifies modeling the beam-mass setup as a SDOF system where the cantilever beam behaves as a massless leaf spring. Otherwise, one would need a model considering the influence of high-order modes, making it difficult to fulfill the goal mentioned above.

![Figure 2.6: (a) Time series, (b) Frequency spectrum (c) phase-portrait. Taken from [A2].](image)

There are deviations on the value of the gap width due to wear on the constraints that can also slide on the platform due to the intermittent collisions. Figure 2.7 shows the influence of gap variation on the response amplitude, such as the pattern similarity for $|U(L_S)|$ and $\Delta G$ around 16 Hz on Figure 2.7(a, b) and the discontinuity of the FRC and the sudden increase of $\Delta G$ around 15 Hz on Figure 2.7(c, d). In those cases, the actual gap width was measured as the maximum vibration amplitude.

It is not possible to ensure a constant excitation level for different forcing frequencies because the shaker is not feedback controlled. Comparing the platform’s response around
2.1 Experimental setup [A1–A3]

Figure 2.7: Period-1 frequency response (a, c, e) and gap variation (b, d, f) around $2f_n$. Taken from [A2].

Once and twice resonance in Figure 2.8, one can see that this element attenuates the vibrations transmitted to the beam around $2f_n$. That motivated the use of electric signal sent to the shaker as the system’s excitation source.

Figure 2.8: Platform response for $\tilde{V} = 0.1$ V around 1× and 2× resonance $f_n = 7.6$ Hz. Taken from [A2].

Most of the setup’s limitations were revealed using experiments for the setup with unilateral hard vibro-impacts around twice the beam’s resonance, indicating that this configuration is more sensitive to those limitations. However, deviations of the gap width
2.2 Modeling

and excitation level can also occur for soft vibro-impact experiments around resonance, but they do not seem to be as relevant as they are for unilateral hard vibro-impacts around $2f_n$.

2.2 Modeling

One can write the setup’s equation of motion using Bernoulli-Euler theory, obtaining:

$$
\left(\rho A + m\delta(X - L_m)\right)\ddot{U} + DEI\dot{U}''' + EIU''' + f_c(U, \dot{U})\delta(X - L_C) = \left(\rho A + m\delta(X - L_m)\right)\ddot{\Omega}^2\gamma V \sin(\Omega \tilde{t}),
$$

$$
U(0, \tilde{t}) = U'(0, \tilde{t}) = U''(L, \tilde{t}) = U'''(L, \tilde{t}) = 0,
$$

where $(\cdot)' \equiv \partial(\cdot)/\partial \tilde{t}$, $(\cdot)' \equiv \partial(\cdot)/\partial X$, $\rho$, $D$, $E$, $A$ and $I$ represent the beam’s density, damping proportional to stiffness, elasticity modulus, cross-sectional area, and cross-section area moment of inertia, respectively, $m$ is the attached mass, $\delta(\cdot)$ is Dirac’s delta function, $V$ is the electric amplitude in volts, and $\gamma$ is the unit conversion gain in m/V. The impact force is represented by $f_c(U, \dot{U})$, which will be discussed in detail in Section 2.2.2. For modeling impacts using the coefficient of restitution, one should remove $f_c(U, \dot{U}) \equiv 0$ and add the following impact rule:

$$
U(L_m, \tilde{t}_-) = \Delta_C \Rightarrow \begin{cases} 
U(L_m, \tilde{t}_+) = U(L_m, \tilde{t}_-), \\
\dot{U}(L_m, \tilde{t}_+) = -R \dot{U}(L_m, \tilde{t}_-),
\end{cases}
$$

where $R$ is the coefficient of restitution, $\tilde{t}_-$ and $\tilde{t}_+$ represent instants before and after impact, respectively and $U_m = U(L_m, \tilde{t})$ is the transverse displacement of the attached mass.

One can turn (2.1) into a SDOF model by following the steps taken in [A1], which are summarized below:

- Normalize transversal displacement and axial position: For bilateral impacts, $U$ was normalized using the measured gap width, while for unilateral impacts the beam’s length $L$ was the normalizing factor. In both cases the axial position $X$ was normalized by the beam’s length;
- Assume that the attached mass is much larger than the beams: $m \gg \rho AL$;
- Perform spatial discretization:

$$
u(x, \tilde{t}) = \sum_{i=1}^{N} \phi_i(x) q_i(\tilde{t}),
$$

where the assumed modes $\phi_i(x)$ satisfy all essential boundary conditions and $q_i(\tilde{t})$ are modal coordinates;
2.2 Modeling

- Neglect higher modes: \( N = 1 \);
- Normalize time using the beam’s first bending resonance: \( t = \omega_n \tilde{t} \);

Obtaining:

\[
\ddot{q} + 2\beta_S \dot{q} + q + f_C(q, \dot{q}) = \Omega^2 \gamma \tilde{V} \sin(\Omega \tilde{t}),
\]

where \( f_C(q, \dot{q}) \) is the normalized impact force. Alternatively:

\[
\ddot{q} + 2\beta_S \dot{q} + q = \gamma \tilde{V} \Omega^2 \sin(\Omega \tilde{t}), \quad \text{if} \quad q < \Delta,
\]

\[
\dot{q}(t_+) = -R\dot{q}(t_-) \quad \text{and} \quad q(t_+) = q(t_-) \quad \text{if} \quad q(t) = \Delta,
\]

for the model using the coefficient of restitution \( R \). \( t_+ \) and \( t_- \) are the time instants after and before impact, respectively. Other model parameters are the structural damping ratio \( \beta_S \), the normalized forcing frequency \( \Omega = \tilde{\Omega}/\omega_n \) and the normalized unit conversion gain. In both cases, model and experimental results can be related to each other by multiplying the modal coordinate \( q(\omega_n \tilde{t}) \) by the normalizing factors for each model, \( L \) for CoR modeling and \( \Delta_G \) for contact force model. Schematic descriptions for both model approaches can be seen in Figure 2.9, where \( q(t) = y(t) - b(t) \) is the relative displacement between the inertial coordinate \( y(t) \) and the base excitation \( b(t) = \gamma \tilde{V} \sin(\Omega \tilde{t}) \).

![Figure 2.9: Single-degree of freedom system modeled with (a) contact force \( f_C(q, \dot{q}) \) and (b) coefficient of restitution, \( R \).](image)

### 2.2.1 Coefficient of restitution [A2]

In this formulation, the contact duration and deformation are considered to be small if compared to the system’s oscillating period and amplitude. In this case, the post-contact velocity is a fraction of the velocity before contact. The simplicity of this model made it very popular in literature. At the same time, the abrupt change of velocity makes the analysis of systems with such a model more complex from both theoretical and numerical standpoints.

This limitation leads to the development of specific coordinate transformations which remove or decrease the effects caused by the velocity jump. For systems with bilateral constraints, one can use sawtooth transformations of time and displacement variables [73]. For unilateral constraints, there are two common options, one developed by Zhuravlev [32] and another one due to Ivanov [33].
Zhuravlev’s transformation consists of unfolding the physically constrained variable $q(t)$ into an unconstrained variable $z(t)$ which is symmetric with respect to the constraint location, as shown in Figure 2.10. Mathematically, this transformation is given by:

\begin{align}
q(t) &= \Delta - |z(t)| \\
\dot{q} &= -\dot{z} \text{sgn}(z) \quad (2.8) \\
\ddot{q} &= -\ddot{z} \text{sgn}(z) \quad (2.9)
\end{align}

Substituting into (2.6):

\[ \dot{z} + 2\beta \dot{z} + z = \text{sgn}(z) \left( \Delta - \Omega^2 \gamma \tilde{V} \sin(\Omega t) \right), \quad \text{if } z \neq 0. \quad (2.11) \]

Knowing that $z_+ z_- < 0$, the impact condition, given by (2.7), can be rewritten as:

\begin{align}
\dot{z}_+ &= R \dot{z}_-, \quad \text{if } z = 0, \quad (2.12) \\
\dot{z}_+ - \dot{z}_- &= -(1 - R) \dot{z}_-. \quad (2.13)
\end{align}

The last expression shows that the impacts create a velocity jump on the unfolded variable which is smaller than that on the problem’s original coordinate: $\dot{q}_+ - \dot{q}_- = -(1 + R)\dot{q}_-$, thus decreasing the velocity discontinuity.

Ivanov’s transformation [33] can be seen as an expanded version of Zhuravlev substitution, where the velocity coordinate is transformed as well. A disadvantage of Ivanov’s transformation, if compared to Zhuravlev’s, is that there is no physical interpretation for the new variable. An advantage is that the energy losses due to impacts are inserted.
2.2 Modeling

directly on the transformation, eliminating the need for an impact condition. For elastic impacts, this substitution reduces to Zhuravlev’s. The use of those transformations as a pre-processing step to numerical simulations can decrease computation considerably.

One can obtain qualitative insights on the solutions of vibro-impact systems with CoR by applying standard averaging to the model in Zhuravlev coordinates (2.11) and (2.13) admitting near elastic impacts \( R \approx 1 \) or using pointwise mapping to the model on its original coordinates (2.5). While averaging is only valid for a small range around the system’s resonance, it provides an estimative for the amplitude and stability of period-1 oscillations on this region, the pointwise mapping method is useful to analyze the stability of period-\( n \) orbits without parameter limitations.

Using averaging and Zhuravlev’s transformation [32], one obtains the following expression which can be used to estimate the location of the resonance peak according to the gap width:

\[
Q^* = -\frac{2\sigma_k \Delta}{\pi^2 (\sigma_k^2 + \beta_k^2)}.
\] (2.14)

As the amplitude \( Q \) is positive by definition, (2.14) implies that \( \sigma_k \Delta < 0 \). So, as pointed out in [32], if the system is tightened (\( \Delta < 0 \)), stable solutions should be seen after crossing resonance, where the frequency detuning parameter \( \sigma_k \) is positive. Similarly, if the gap is positive, one should expect stable solutions before crossing resonances. This variation according to the gap width can be observed in the experimental data shown in 2.11, where one can see a concentration of period-1 orbits before and after twice the system’s fundamental resonance (2\( f_n \approx 15.5 \) Hz) for loose (\( \Delta > 0 \)) and pre-stressed (\( \Delta < 0 \)) configurations respectively. This figure also shows period-2 oscillations for the red square dots between 15 Hz to 15.5 Hz and more complex behavior in the blank regions.

2.2.1.1 Measuring \( R \)

The experimental estimations for the CoR can be seen in Figure 2.12, where one can see that \( R \) seems to grow with the forcing frequency in determinate ranges, like between 16 Hz and 16.5 Hz in (a) and around 15.5 Hz in (c). It also decreases with the pre-contact velocity in (f) and increases for \( U^- \) greater than 30 mm/s in (d).

The uncertainty observed on the measurements for \( R \) shown in Figure 2.12 can be explained by some experimental limitations such as:

- Deviations on the gap width during operation, see Figure 2.7;
- Inherent mounting errors make it difficult to ensure identical impact conditions when the constraint location is changed;
- The velocity is not measured directly, being obtained through central finite differences from noise-contaminated displacement measurements;
Figure 2.11: Periodicity for different gap configurations: (a) pre-stressed \(-0.1\) mm, (b) neutral \(0\) mm and (c) loose \(0.1\) mm. Taken from [A2].

Figure 2.12: Coefficient of restitution for tight \(\Delta G \approx -0.1\) mm (a, b), neutral \(\Delta G \approx 0\) mm (c, d) and loose \(\Delta G \approx 0.1\) mm (e, f) gap configurations as a function of forcing frequency (a, c, e) and pre-impact velocity (b, d, f). Taken from [A2].

2.2.2 Contact force models [A1]

Different from the CoR formulation, contact force modeling assume collisions with non-zero duration and deformation, which can be big or small if compared to the ampli-
2.2 Modeling

tude and period of oscillations, depending on the parameter choice. In this section, the advantages and limitation of common impact force models will be presented.

The models are represented schematically in Figure 2.13. The piecewise linear model (a) is one of the most common ways of modeling impacting systems, the power-law model (b) can be considered as a generic version of the contact models derived by Hunt and Crossley [14], and the last model (c) is a combination of the first two, being called the modified piecewise linear model.

![Figure 2.13: Single-degree of freedom system modeled with different contact force models: (a) piecewise linear model, (b) power-law model and (c) modified piecewise linear model.](image)

### 2.2.2.1 Piecewise linear model

Generally speaking, the bending stiffness of a beam (leaf spring) is associated with its deformation pattern. So, when the beam in Figure 2.2(a) touches the constraint or when the attached mass in Figure 2.2(c) touches the helical spring, in both cases the elastic characteristics of the system are modified. This change of properties can be defined as:

\[
f_C(q, q) = \phi_C^2 \begin{cases} 
0, & \text{if } |q| \leq \Delta, \\
\omega_R^2(q - \Delta) + 2\beta_C\omega_Rq, & \text{if } q \geq \Delta, \\
\omega_R^2(q + \Delta) + 2\beta_C\omega_Rq, & \text{if } q \leq -\Delta,
\end{cases}
\]  

(2.15)

where \(\omega_R\) is the ratio between contact and non-contact natural frequencies and \(\beta_C\) is the contacting damping ratio. \(\phi_C\) is a geometric factor accounting for the distance between contact and measurement points. Thus, the impact force is attenuated if the measurement location is bigger than the contact location \((L_S > L_C)\), and amplified otherwise.

The popularity of this model comes from its straightforward physical interpretation. Also, it is possible to use the impact condition \(|q| \geq \Delta\) to estimate the contact duration for single periodic sinusoidal oscillations \(q = Q \sin \varphi\) in terms of the grazing angle \(\varphi_0 = \arcsin(\Delta/Q)\) as shown in Figure 2.14(b) for \(\Delta = 1\). Contour lines in Figure 2.14(c) show the results from this estimation against normalized experimental data, indicating that contact duration grows together with the first-order modal amplitude \(Q\) and normalized forcing frequency \(\Omega\). To obtain values in seconds one should multiply the values of \(\tau\) in Figure 2.14(c) by \(1/\omega_n \approx 21 \times 10^{-3}\) s.

A disadvantage of this formulation is that it can lead to situations which do not make sense from the physical point of view. As shown in Figure 2.14(a), the impact
2.2 Modeling

Figure 2.14: (a): Elastic and dissipative components of the piecewise linear impact force model. (b): Time series of the elastic component of the piecewise linear impact force model together with $q = Q \sin(\varphi)$. (c): Normalized experimental data (red x’s) and curves of constant normalized contact duration (blue lines), the numbers on each curve represent the normalized contact duration. $\phi_C = 0.241$. The values of the model parameters can be found in Table 2.3. Taken from [A1].

The force becomes negative at the end of the contact, as if the moving impacting element is being pulled towards the constraint again. There are two ways of avoiding this situation, one is to neglect the contact dissipation ($\beta_C \equiv 0$), in which case the piecewise linear force would be non-negative for $|q| \geq \Delta$, growing smoothly from zero. Another one is to add another condition to the impact force, preventing it to be negative, leading to:

$$\dot{q} \geq \frac{\omega_R}{-2\beta_C}(q - \Delta),$$  \hspace{1cm} (2.16)

$$\alpha = \arctan(2\beta_C/\omega_R),$$  \hspace{1cm} (2.17)

where $\alpha$ is the angle between both impact conditions on the phase plane, shown in Figure 2.15. Analyzing the expression for $\alpha$, one can see that both conditions are similar (i.e. $\alpha \approx 0$) for negligible damping ($\beta_C \approx 0$) or high contact stiffness ($\omega_R \to \infty$). For the contact parameters from [A1], one has $\alpha \approx 4^\circ$, showing that both conditions are similar. Also, this condition does not help to solve the discontinuity problem of the impact force at the beginning of contact.
2.2 Modeling

The impact condition ($|q| \geq \Delta$) makes the piecewise linear model discontinuous at the beginning and at the end of the contact, which can lead to numerical problems. That, together with the occurrence of negative contact forces on the piecewise linear
model, motivated the development of alternative contact formulations, such as power-law models. Considering the ratio between displacement and contact width $U/\Delta$, one can define the normalized contact force using the following smooth function:

$$f_c(q, \dot{q}) = \omega_R^2 q^{2n-1} \phi_C + 2\beta_C \omega_R q^{2p} \dot{q} \phi_C^2,$$

where the gap has been normalized to unity, $n \geq 1$ and $p \geq 0$ are integer exponents for the elastic and dissipative terms, respectively. $\beta_C$, $\phi_C$, and $\omega_R$ were defined previously for the piecewise linear model. However, it is not possible to attach a physical meaning to those parameters due to different model definitions. The exception to that is $\phi_C$, which in this case affects elastic and dissipative forces in different ways. Observing the output of power-law model in Figure 2.17, one can notice its smoothness if compared to Figure 2.14(a). One can have negative contact forces if the model is purely dissipative.

![Figure 2.17: Elastic and dissipative components of Power-law impact force model for $q = Q \sin(\phi)$. $\phi_C = 0.241$. The values of the model parameters can be found in Table 2.3. Taken from [A1].](image-url)

In Figure 2.17, one can see that although $f_c(q, \dot{q})$ is negligible in the region $|q| < 1$, it grows fast as $|q| \to 1$, being active also on the non-contact region. That is not necessarily a problem since errors in the measurement of the gap width can shift the ratio $U/\Delta$ away from unity. The choice of model parameters also has an influence on the contributions of the power-law contact force on the non-contacting region.

From the numerical point of view, this model can lead to stiff ODE’s if the exponents $n$ and $p$ are high, losing its advantage of being easier to simulate if compared to the piecewise linear model.

The results from parameter variation for the power-law model can be seen in Figure 2.18. There, one can see that parameters associated with the elastic component of the impact force govern the peak curvature and length, affecting the non-impacting region, $Q < 1$. The response amplitude is controlled by parameters associated with the dissipative component of the impact force.
Figure 2.18: Parameter analysis of Power-law model varying (a) $\omega_R$, (b) $\beta_C$, (c) $n$ and (d) $p$. The arrows indicate how the frequency response changes as the value of each parameter increases. Taken from [A1]

### 2.2.2.3 Modified piecewise linear model

The power-law model solves the discontinuity problems presented by the dissipative piecewise linear model, but its parameters do not have any physical meaning. This situation can be overcome by combining the advantages of the previous contact models.

As discussed at the end on Section 2.2.2.1, most of the problems from the piecewise linear model came from its dissipative term, so one could modify this component by multiplying it by a power of the displacement, leading to:

$$f_C(q, \dot{q}) = \phi_C^2 \begin{cases} 
0, & \text{if } |q| \leq 1, \\
(\omega_R^2 - 1)(q - 1) + 2\beta_C\omega_R\dot{q}(q - 1)^2 p, & \text{if } q \geq 1, \\
(\omega_R^2 - 1)(q + 1) + 2\beta_C\omega_R\dot{q}(q + 1)^2 p, & \text{if } q \leq -1,
\end{cases} \quad (2.19)$$

where the gap has been normalized to unity, with $p$, $\beta_C$, $\omega_R$, and $\phi_C$ being the same as before and $\omega_R$ and $\phi_C$ keeping its straightforward physical meaning. The term multiplying the damping term can be interpreted as a mechanical impact strain. This model can be seen as a particular case of Hunt and Crossley’s model [14], which originally assumed the elastic term to be a power function as well.
2.2 Modeling

The advantages of the combined model can be seen by realizing that the model given by (2.19) does not present any influence on the non-contacting phase, nor presents sign changes or jumps at the beginning and end of the contact. That only occurs if elastic forces are absent.

![Figure 2.19: Elastic and dissipative components of the piecewise linear model with power-law damping for \( q = Q \sin(\varphi) \). \( \phi_C = 0.241 \). The values of the model parameters can be found in Table 2.3. Taken from [A1].](image)

From the analytical standpoint, a disadvantage of this model is that it leads to a cumbersome expression for the harmonic linearization damping coefficient, without a general term depending on \( Q \) and \( p \).

The effect of contact damping and natural frequency ratios are the same for both versions of the piecewise linear force, see Figures 2.16 and 2.20. However, the amplitude increases for higher values of \( p \), see Figure 2.20(c), which is different from what was observed for the power-law model in Figure 2.18(d). This can be explained by the definition of the dissipative force which is proportional to \((q \pm 1)^{2p}\), and, for lower exponents, is stronger in the neighborhood of impact, \( Q \approx 1 \). As \( p \) increases the dissipative term becomes negligible for low impacting amplitudes.

2.2.2.4 Model tuning

The parameters from the different contact models were obtained by fitting the frequency-amplitude relations with the harmonic linearization coefficients to normalized experimental data using a nonlinear least-square solver. The data were normalized with respect to the grazing amplitude \( \Delta_G \approx 1.6 \text{ mm} \) and fundamental linear natural frequency \( f_n \approx 7.6 \text{ Hz} \).

Models with power-law terms were fitted by varying \( \beta_C \) and \( \omega_R \) while keeping the exponents \( 1 \leq n \leq 10 \) and/or \( 0 \leq p \leq 10 \) constant. From that, model parameters were chosen as those whose model response was closer to experimental observations with the smaller integer exponent. The values of the fitted model parameters can be seen in Table 2.3.
2.2 Modeling

Figure 2.20: Parameter analysis of Modified Kelvin-Voigt model varying (a) $\omega_R$, (b) $\beta_C$ and (c) $p$. The arrows indicate how the frequency response changes as the value of each parameter increase. Taken from [A1].

Table 2.3: Values of the parameters from impact force models. Adapted from [A1].

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>$\omega_R$</th>
<th>$\beta_C$</th>
<th>$n$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piecewise linear</td>
<td></td>
<td>7.278</td>
<td>0.258</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Power-law</td>
<td></td>
<td>0.715</td>
<td>0.659</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Modified piecewise linear</td>
<td></td>
<td>7.306</td>
<td>4.636</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

The fitting process revealed some insights into the role of $n$ and $p$ on the frequency response, see Figure 2.21. For instance, by increasing $p$ the power-law model response gets closer to the experimental upper folding point, see Figure 2.21(b). In this case, the fitted $\omega_R$ keeps almost constant, while $\beta_C$ decreases. By increasing $n$, as shown in Figure 2.21a, the model response gets closer to both the upper and lower folding points, going further away from the upper one if $n$ grows too much. Also, increasing $n$ the model response approaches the curvature of the bent peak. In this case, the fitted $\beta_C$ goes up, while $\omega_R$ goes down. This approach to the lower folding point and
peak curvature for growing exponents can be explained by the fact that for small $n$ and $p$ the power-law force is not negligible in the non-impacting region. For the piecewise linear model with power-law damping, Figure A1.8(c) shows that there are no significant changes for $1 \leq p \leq 5$, with the parameter $\omega_R$ remaining constant while $\beta_C$ grows.

2.2.3 Equivalence relations [A3]

From the contact force models described in Section 2.2.2, the piecewise linear model is the one with the most straightforward interpretation and popularity despite its physical
2.2 Modeling

inconsistencies. That makes this model a suitable candidate to represent soft impacts and obtain equivalence relations with the coefficient of restitution model. Those relations can be used to investigate the role of contact force parameters on the coefficient of restitution and contact duration. All of them are based on the equation of motion (2.5) in contact configuration, i.e. (2.15) for \( q \geq \Delta \):

\[
\ddot{q} + 2\beta S \dot{q} + q + \omega_R^2 (q - \Delta) + 2\beta C \omega_R \dot{q} = \gamma \tilde{V} \Omega^2 \sin(\Omega t),
\]

\[
\ddot{q} + 2\xi w \dot{q} + w^2 q = P + \gamma \tilde{V} \Omega^2 \sin(\Omega t), \quad \text{if} \quad q \geq \Delta,
\end{equation}

where \( w, \xi, \) and \( P \) are the equivalent natural frequency, damping ratio and step forcing for contact configuration, respectively. The initial conditions are the gap width and pre-contact velocity.

Shaw ad Holmes \[19\] obtained an expression for \( R_1 \) by solving (2.20) and assuming a high natural frequency ratio and that the contact duration is half of the contact period, obtaining:

\[
R_1 = 1 - \pi \xi.
\] (2.22)

Applying an energy-based analysis on the unforced (2.20) (\( \tilde{V} \equiv 0 \)), following the steps from Blazejczyk-Okolewska et al. \[11\], and considering the contact duration to be half of the damped natural period, one obtains a cumbersome expression for \( R_2 \) which depends on the gap width and pre-contact velocity. Considering the gap width to be null in \( R_2 \), one obtain another expression which is also velocity independent and given by:

\[
R_3 = \sqrt{\frac{\exp \left( \frac{-2\pi \xi}{\sqrt{1-\xi^2}} \right) - 2\beta S}{1 - 2\beta S}}
\] (2.23)

This expression can be further simplified by considering a small equivalent damping ratio \( \xi \):

\[
R_4 = 1 - \frac{\pi \xi}{1 - 2\beta S} + O(\xi^2),
\] (2.24)

which becomes equivalent to (2.22) for \( \beta S \ll 1 \). Another relation can be obtained by evaluating the derivative of the unforced solution of (2.20) at half of its damped natural period, obtaining:

\[
R_5 = \exp \left( \frac{-\pi \xi}{\sqrt{1-\xi^2}} \right).
\] (2.25)

One can obtain \( R_1 \) from this expression by considering small damping ratio \( \xi \).

Figure 2.22 shows the relations between the different expressions obtained. All of them depend indirectly on the natural frequency ratio \( \omega_R \) and on the contact damping
ratio \( \beta_C \) through the equivalent damping ratio \( \xi \). The expression for \( R_2 \) depends on the pre-contact velocity, which is supported by empirical evidence, e.g. [18, A2] and Figure 2.12. However, \( R_2 \) also depends on the gap width, a relationship which has no precedent in literature, at least to the author’s knowledge. This gap-dependence can occur due to the specific coordinate system used, or it can be related to the pre-contact velocity because when either one is set to zero the other one becomes irrelevant, as shown for \( R_3 \).

\[
\begin{align*}
R_2 & \xrightarrow{\Delta = 0} R_5 \xrightarrow{\xi << 1} R_4 \xrightarrow{\beta_S << 1} R_1 \\
R_6 & \xrightarrow{\xi << 1} R_7
\end{align*}
\]

Figure 2.22: Flowchart showing the relations between the expressions for \( R_i \).

Figure 2.23 shows how the different expressions for \( R_i \) behave according to the contact parameters. There, one can see that \( R_i \) is mostly influenced by the contact damping, while the contact duration is governed by the natural frequency ratio. So, to satisfy the main assumption of CoR modeling of impacts with small duration one needs a high \( \omega_R \). From the expressions derived previously, only (2.22) from Shaw and Holmes [19] explicitly demanded that.

![Figure 2.23](image)

Figure 2.23: Effect of contact parameters on the coefficient of restitution \( R_i \), with \( i \) equal to: (a) 1 (2.22), (b) 2, (c) 3 (2.23), (d) 4 (2.24) and (e) 5 (2.25). (f) Influence on the contact duration. \( \beta_S = 15 \times 10^{-3}, \Delta = 0.5 \times 10^{-3} \) and \( \dot{q}_L = 0.01 \). Taken from [A3].

Figure 2.24 shows the results from numerical simulations using the different expressions for \( R \) against results from a piecewise linear model. The first column (a-e), shows the results for \( \beta_C = 0.03, \omega_R = 150 \) and \( \tau_C = 0.02 \), while on the second column (f-j) the
parameters are $10\beta_C$, $\omega_R/10$ and $\tau_C = 0.22$. Despite the different contact durations, all expressions for $R_i$ are able to reproduce the contact force response. For higher $\tau_C$, this can be explained by the small contact deformation due to high damping.

![Figure 2.24: Comparison of time responses for the piecewise linear model (red lines) against coefficient of restitution (dashed blue lines). (a, e) $\beta_C = 0.03$, $\omega_R = 150$ and $R_1 - 5 = 0.9$, (f, j) $10\beta_C$ and $\omega_R/10$. (f) $R_1 = 0.06$, (g) $R_2 = 0.38$, (h) $R_3 = 0.33$, (i) $R_4 = 0.03$, (j) $R_5 = 0.37$. Taken from [A3].](image)

2.3 Frequency domain comparisons [A1–A3]

The validity of the assumptions made in [A1] for bilateral soft impacts is tested by comparing numerical and analytical predictions for each of the contact force models given by (2.15), (2.18) and (2.19) against experimental observations as shown in Figure 2.25. There, one can see that all of the contact force models produce results which are reasonably close to experimental observations, indicating that the assumptions made are appropriate.

Comparing the contact models against each other, one can see that the results from the modified piecewise linear model are able to reproduce the location of both folding points more precisely than the other models. The results from the power-law model are further away from both folding points and the results from the standard piecewise linear model are able to reproduce only the location of the lower folding point.

For systems with hard impacts and unilateral constraints [A2], literature [19, 74] predicts the occurrence of vibro-impact resonances on the neighborhood of even multiples of the linear natural frequency. Due to the platform’s attenuating characteristic discussed in Section 2.1.1, the forcing amplitude was decreased in order to represent experimental
2.3 Frequency domain comparisons [A1–A3]

Figure 2.25: Comparison of frequency responses obtained using (a) the piecewise linear model (b) the power-law model and (c) the piecewise linear model with power-law damping. $V = 0.6 \text{ V}$. The values of the model parameters can be found in Table 2.3. Taken from [A1].

conditions, and $R$ is defined as its mean value for each gap configuration. Observing experimental and numerical results in Figure 2.26 one can see a reasonable agreement for neutral configuration (b), while for loose configuration (c) agreement occurs for results after twice resonance ($\Omega \geq 2$).

In [A3], the applicability conditions of piecewise linear contact force and CoR modeling for different levels of contact stiffness is tested by comparing experimental and numerical results. The contact stiffness and coefficient of restitution are given in Table 2.1. The results around resonance are presented in Figure 2.27, showing that the simulations using the CoR failed to reproduce the bent-peak inclination only for the softest spring $k_4$. Results around $2f_n$ are shown in Figure 2.28, where the results from piecewise linear contact force are closer to experimental data than the results from CoR modeling, whose closest agreement occurs for the hardest spring $k_1$.

An evidence of the setup limitations Section 2.1.1, in which is not possible to ensure a constant excitation level for different forcing ranges is the fact that numerical predictions for both models are closer to the experimental data around $f_n$ than for the data around $2f_n$, see Figures 2.27 and 2.28.
2.3 Frequency domain comparisons [A1–A3]

Figure 2.26: Experimental and numerical frequency responses. (a) $\Delta G \approx -0.1$ mm, (b) $\Delta G \approx 0$ mm and (c) $\Delta G \approx 0.1$ mm. Taken from [A2].

Figure 2.27: Comparison between experimental, numerical and analytical frequency responses for different springs around resonance. (a) $k_1$, (b) $k_2$, (c) $k_3$, (d) $k_4$. Taken from [A3].
2.3 Frequency domain comparisons [A1–A3]

Figure 2.28: Comparison between experimental, numerical and analytical frequency responses for different springs around twice resonance. (a) $k_1$, (b) $k_2$, (c) $k_3$, (d) $k_4$. Taken from [A3].
Conclusions

Experimental observations for different impact scenarios have been used to test analytical and numerical predictions made using the most common formulations for vibro-impact problems. The advantages and disadvantages of contact force models and the coefficient of restitution were discussed using analytical and numerical tools, obtaining overall agreement with experiments despite limitations of the test setup. Connections between the different formulations were presented and compared against each other numerically.

In [A1], three contact force models were used to replicate the frequency response of the experimental setup with soft bilateral contacts. Impact forces were expressed by a piecewise linear model, a power law model and a combination of those two. Despite its physical inaccuracies, the piecewise linear representation has a clear interpretation and is able to represent experimental observations with a reasonable quantitative agreement.

The power-law model can be seen as a generic version of Hunt and Crossley’s model [14]. It is a smooth model whose output grows very fast in the neighborhood of impact. Its major weakness is the lack of physical meaning for its parameters. This formulation was able to represent the general shape of the frequency curve, but not its folding points. The strengths of both models were combined in a third one, where the dissipative term of the piecewise linear model was replaced by a power of the contact strain. This removed the piecewise linear model’s physical inaccuracy and slightly increased quantitative agreement with experiments, but it also added considerable complexity to the model.

In [A2], the coefficient of restitution was used to model the setup’s experimental response with unilateral hard impacts around twice its natural frequency, $2f_n$, where literature [19, 74] predicts resonant behavior. The usage of Zhuravlev and Ivanov’s transformation, motivated by [34, 41], was able to reduce velocity jumps due to impact, reducing computation time and allowing application of the averaging method.

The influence of impact scenario on the resonance peak location was confirmed experimentally. For loose configuration, most of the period-one solutions were found below $2f_n$ and the opposite for the tightened configuration. It was also possible to correlate the effects of fluctuations on the gap width on the response amplitude during a frequency sweep. Despite experimental challenges, like the presence of complex behavior and high-
3 Conclusions

frequency content on the response, it was possible to obtain the qualitative agreement of experimental and numerical results.

Connections between the coefficient of restitution and piecewise linear contact force model were explored in [A3]. Parameter analysis showed that the CoR’s value is mostly influenced by the contact damping, while the impact stiffness controls the contact duration. Numerical analysis showed that it is possible to obtain reasonably equivalent results for small contact duration, also shown by [11], or for small impact deformation.

Four helical springs were inserted in the experimental setup to obtain contacts with different characteristics. The presence of complex behavior decreased with the spring stiffness, as predicted by [11]. Measurement of the contact duration agreed with theoretical estimation for all springs. The spring rates were used to estimate the CoR using Shaw and Holmes assumption [19]. In this case, theoretical approximations exceeded measured values and only the estimations for softer springs got closer to measured values. Frequency domain comparisons showed the flexibility of the piecewise linear impact force over the CoR, whose closest approximations occurred for the hardest springs. On the other hand, predictions made using the piecewise linear formulation were able to reproduce the frequency responses for all springs, especially around the setup’s natural frequency.

The analysis carried out in this Ph.D. thesis showed the main approaches used to model vibro-impact problems. Their strengths and weaknesses were applied to different contact configurations, revealing the wider application range of impact force formulations at the cost of higher complexity, delving into the contact dynamics, which is completely ignored by the CoR. Still, the results obtained by both models using a simple single degree of freedom system are able to capture some important properties of vibro-impact systems, producing qualitative insight into their dynamic behavior.
References


References


References


References


Appended articles
Validation of Vibro-Impact Force Models by Numerical Simulation, Perturbation Methods and Experiments

The following article was published online in the Journal of Sound and Vibration in September 2017. An initial version of its main results was presented in an oral session of the 9th European Nonlinear Dynamics Conference, ENOC 2017, held in Hungary. The extended abstract submitted to this conference can be found in Appendix C.

The article’s highlights are:

• Comparison of three different impact force models.
• Simple analytical frequency-amplitude relations are obtained and experimentally validated for the impact force models considered.
• Parameter analysis of the impact force models.

Errata

There is a mistake on the definition of the elastic term of the Kelvin-Voigt model (standard and modified). So, the following substitutions were made:

1. $K_C$ instead of $K_C - K_S$ in (A1.12) and (A1.18);
2. $\omega_R^2$ instead of $\omega_R^2 - 1$ in (A1.13), (A1.19) and (A1.36)

This modification does not produce noticeable changes on the numeric value of $\omega_R$ in Table A1.1 because of the difference between $\omega_R^2$ and $\omega_R^2 - 1$ is small if compared with its absolute value.
Abstract

The frequency response of a single degree of freedom vibro-impact oscillator is analyzed using Harmonic Linearization, Averaging and Numeric Simulation, considering three different impact force models: one given by a piecewise-linear function (Kelvin-Voigt model), another by a high-order power function, and a third one combining the advantages of the other two. Experimental validation is carried out using control-based continuation to obtain the experimental frequency response, including its unstable branch.

Keywords: vibro-impact dynamics; impact force models; Kelvin-Voigt impact model; Power-law impact model; vibro-impacting beam; lumped mass beam; experimental validation; control-based continuation;

A1.1 Introduction

Generally speaking, the modeling of vibro-impact systems considers two distinct situations: with and without contact. Hence, one can solve each case separately, connecting them using the contact condition [4]. In [19, 21], the impact condition of piecewise linear systems is used to obtain discrete maps, enabling investigations of bifurcations and stability of periodic motions. However, while it is straightforward to attach solutions of linear systems, the same cannot be said when additional nonlinearities are also present between impacts.

The contact between two bodies can be modeled as a temporary association of the stiffness and damping properties of the colliding bodies, each one modeled as a linear system. Despite its simplicity, this approach has some limitations, mainly the non-zero values of the impact force on the initial and final parts of the contact phase due to dissipative forces. This motivated the development of other contact models such as the one suggested by Hunt and Crossley [14], who derived the damping coefficient as a power function of the impact deformation. This solved the physical inaccuracy of the linear model and lead to an ongoing discussion about the appropriate form of the nonlinear dissipative term [15, 16]. Most of the expressions for the dissipative term combine the velocity immediately before impact and the coefficient of restitution with a rational power of the contact deformation, leading to complicated expressions whose analysis is possible only through numerical simulation.

One of the oldest techniques to model impacting systems is to use a coefficient of restitution (CoR) to relate the velocities before and after impact. This classic approach has been used in many applications [34, 35, 50, 63, 75]. For instance, Bishop et al. [63] used a single degree of freedom (SDOF) oscillator with a CoR rule to reproduce the experimental behavior of an impacting cantilever beam around its first natural frequency. Also, the coefficient of restitution, together with a power-law elastic force, can be used to obtain various nonlinear contact force models, such as the one mentioned previously by Hunt and Crossley [14]. Despite its popularity, it should be pointed out that the CoR
is not an intrinsic property of the material, depending on the impact velocity [76]. Also, using a simple CoR kinematic rule does not give any direct information about contact forces or stresses.

From the numerical point of view, one can replace discontinuities by smooth equivalent functions and use standard techniques to solve the smoothed model as done by Savi et al. in [49], where the impact condition is smoothed, or by Elmegård et al. [47], who applied numeric continuation to a smoothed model of a lumped-mass impacting beam. In these cases, while it is safer to use standard ODE solvers provided by accredited sources, it is also necessary to properly tune smooth approximations of discontinuous functions; see [40] for a discussion on the effect of smoothing functions on the frequency response of oscillators with clearance. There are also several numerical integration algorithms designed for non-smooth systems [9, 42, 43], but their comparison is out of the scope of this work.

Vibro-impact oscillators can also be analyzed using common perturbation techniques, such as Harmonic Linearization [77, 78], Averaging [65, 79, 80], and the Lindstedt-Poincaré method [30]. Besides their widespread use with nonlinear problems, these methods assume weak and smooth nonlinearities, which are not reasonable assumptions for general impacting systems. This limitation motivated the development of non-smooth transformations with respect to state [32, 33] and time [73] variables. By these, one can remove non-smooth terms from a certain model, or make them small, allowing the usage of common perturbation methods. An example of such combination can be found in [34] where non-smooth transformations were used to weaken a near-elastic kinematic impact condition, enabling subsequent application of extended averaging.

In some cases, the applicability of the theoretical/numerical tools mentioned above is accompanied by physical experiments. In [58] the experimental chaotic behavior of a base-excited cantilever beam with one-sided stop is qualitatively compared with results from numeric simulations of a piecewise linear oscillator. In [63] the experimental frequency response of a forced cantilever beam with a unilateral constraint is compared with the one obtained by numeric simulation of a SDOF oscillator with a coefficient of restitution, showing reasonable accuracy. The Averaging method has been extensively used to obtain analytical frequency-amplitude expressions for piecewise linear oscillators with one [65, 79] and two [80] degrees of freedom.

In [49] the nonlinear dynamics of a mass-spring system with discontinuous stiffness and damping was analyzed experimentally and numerically by smoothing the impact condition. A similar numerical-experimental analysis was performed by Aguiar and Weber [81], focusing on the behavior of the impact force. Using control-based continuation Bureau et al. [66] obtained experimental frequency responses of a cantilever beam with lumped mass and bilateral constraints. A single-DOF numerical model for this system was proposed by Elmegård et al. [47], who used the experimental data from [66] to validate the model and predict the existence of an isola, which was later identified experimentally by Bureau et al. [67].

From the overview presented above, one can realize that besides the match between results from particular numerical/analytical techniques with experimental findings there are little efforts on comparing the different paradigms used to model and analyze vibro-
A1.2 Experimental setup and procedure

The objective and main originality of the present work is to compare different impact force models using analytical, numerical and experimental techniques. The frequency response of a SDOF vibro-impact oscillator is analyzed using Harmonic Linearization and Averaging, considering three different impact force models: one given by the Kelvin-Voigt model (piecewise-linear function), another using a power-law function and a third one combining the strengths of the first two. Experimentally, control-based continuation [66, 67] is used to obtain frequency responses of an impacting beam, including its unstable branch. Numerical simulations are used to validate the simple analytic approximations obtained by perturbation methods.

Despite the ability of the mentioned models to produce different qualitative behaviors such as quasiperiodicity and chaos, the analysis presented here is restricted to single-periodic oscillations only.

As the main contribution of this manuscript is to compare different impact force formulations, only the most common models are considered. That is the case of the Kelvin-Voigt model, which is widely used despite its inaccuracies [15, 19, 80]. The Power-law model can be viewed as a generic version of the compliant force models first derived by Hunt and Crossley [14] and further developed by others [15, 82].

A1.2 Experimental setup and procedure

The experimental setup has been described previously in [66, 67] and is shown in Figure A1.1. In Figure A1.1a,b an electrodynamic shaker (1) [B&K® 4808] is used to apply a harmonic excitation to a platform (2), containing a cantilever beam and a pair of symmetrically located stops (5) to restrain the lateral movement of the beam. The shaker is driven by a power amplifier [B&K® 2712] and is connected to the platform by a stinger. The impacting beam can be seen in detail in Figure A1.1c,d, where two DC holding electromagnetic actuators (6) [Magnet-Schultz® G MH X 030] are placed on each side of the lumped mass (4) to execute control-based continuation. The displacement of both platform and lumped mass are measured by two laser sensors (3) [OMRON® ZXL-D40]. Due to the electromagnetic actuators around the lumped mass, its displacement is measured below the mass location, Figure A1.1d. A dSPACE® DS1104 R&D controller board is used to perform data acquisition and control-based continuation of the experimental setup.

The experimental frequency response of the impacting beam, shown in Figure A1.2a, was obtained by the authors, who repeated some of the experiments done by [66, 67] using control-based continuation. In this model-free approach, the equilibrium states are found by a predictor-corrector algorithm. A non-invasive proportional-derivative controller is used to stabilize the system through its unstable branch (lower part of the bent peak) using the electromagnetic actuators (6).

Figure A1.2b illustrates the harmonic decomposition of the experimental frequency response from its static component, represented by $\tilde{\Omega}$, up to five times the excitation frequency $\tilde{\Omega}$. In this figure one can see that the impacting regime increases the ampli-
A1.2 Experimental setup and procedure

Figure A1.1: Experimental setup (a,c) and its schematic representation (b,d).
tude of higher harmonics but they are still negligible if compared with the fundamental harmonic, $1\Omega$.

In this work, the experimental observations are restricted to the neighborhood of the system’s fundamental linear natural frequency, measured as $f_n = 7.6\,\text{Hz}$, which is much smaller than its second linear natural frequency, which is around $200\,\text{Hz}$. Also, the lumped mass ($m = 0.2\,\text{kg}$) is approximately 8 times heavier than the beam’s mass, being the dominant inertial element. In addition to that, the lumped mass is located at a reasonable distance from the stops, which can be considered as rigid supports, causing inelastic impacts.

Based on the above, one can simplify the mathematical modeling of the impacting beam shown in Figure A1.1 in the neighborhood of its fundamental linear natural frequency, by using a SDOF model, where the lumped mass dynamics dominate the oscillations in comparison to the flexible beam alone.
Figure A1.2: (a) Experimental frequency response and its (b) harmonic decomposition
A1.3 Equations of motion

As the measurement and impact locations $L_S$ and $L_C$ are not coincident and due to beam deflection, the measured grazing amplitude $\Delta_G$ will differ from the gap width $\Delta_C$. This difference can be seen in Figure A1.3 for a cantilever beam oscillating at its first mode. The deflection is caused by the inertial force $F = -m\ddot{U}(L_m, \tilde{t})$. The measured grazing amplitude will be used to normalize the equations of motion later on and to define impact force models in Section A1.4.

The electrodynamic shaker is not feedback controlled, being not able to provide constant excitation amplitude on the investigated frequency range. Also, the displacement of the sub-system shaker-platform around resonance is affected by a nonlinear coupling with the impacting beam as mentioned in [66]. Alternatively, one could consider the single harmonic electric signal fed into the shaker’s power amplifier as the system’s external excitation, since it has constant amplitude (in volts). This electric signal excites the shaker-platform subsystem, which in turn excites the impacting beam.

So, the base excitation can be written as:

$$\tilde{b}(\tilde{t}) = \tilde{\gamma}\tilde{V}\sin(\tilde{\Omega}\tilde{t}),$$

where $\tilde{V}$ is the electric amplitude in V and $\tilde{\gamma}$ is the unit conversion gain in m/V. In the schematic representation of the forced impacting beam in Figure A1.1d, $L_C$, $L_m$, $L_S$ and $L$ represent the axial location of the stops, lumped mass, displacement sensor and beam length, respectively, $\Delta_C$ represents the gap width. Using Bernoulli-Euler beam theory,

![Figure A1.3: Deflection of a cantilever beam.](image-url)
A1.3 Equations of motion

one can write its equation of motion for transverse displacement \( U(X, \tilde{t}) \) as [47]:

\[
\left( \rho A + m \delta(X - L_m) \right) \ddot{U} + DEIU''' + EIU''' + \tilde{f}_C(U, \dot{U}) \delta(X - L_C) = \\
\left( \rho A + m \delta(X - L_m) \right) \tilde{\Omega}^2 \gamma \tilde{V} \sin(\tilde{\Omega} \tilde{t}),
\]

\[
U(0, \tilde{t}) = U''(0, \tilde{t}) = U'''(L, \tilde{t}) = 0,
\]

\[\text{(A1.2)}\]

where \((\cdot)' \equiv \partial(\cdot)/\partial X\), \(\rho\), \(E\), \(A\) and \(I\) represents the beam’s density, elasticity modulus, cross-sectional area and cross-section area moment of inertia, respectively, \(m\) is the lumped mass and \(\delta(\cdot)\) is Dirac’s delta function. The function \(\tilde{f}_C(U, \dot{U})\) represents the transverse impact force, resulting from the contact between beam and stops. This function can be modeled by several means from which three will be addressed in Section A1.4.

The present model is similar to the one proposed by Elmegård et al. [47], which did not define an impact force explicitly but used a piecewise relation for the spatial discretization of the equations of motion.

One way of inserting the stiffness proportional damping coefficient \(D\) in the Bernoulli-Euler model is to assume the beam’s material to have a viscoelastic behavior, described by Kelvin and Voigt’s (KV) model. Assuming that lumped mass is much bigger than the beam’s mass, i.e. \(m \gg \rho AL\), and defining \(U = u \Delta_G\) and \(X = xL\) as normalized transverse displacement and axial position, one arrives at:

\[
\delta(x - x_m) \ddot{u} + (Du''' + u''')EI/(mL^3) + \\
+ \tilde{f}_C(u \Delta_G, \dot{u} \Delta_G) \delta(x - x_C)/(m \Delta_G) = \delta(x - x_m) \tilde{V}(\dot{\gamma}/\Delta_G) \tilde{\Omega}^2 \sin(\tilde{\Omega} \tilde{t}),
\]

\[\text{(A1.4)}\]

where now \((\cdot)' \equiv \partial(\cdot)/\partial x\), \(x_C = L_C/L\), and \(x_m = L_m/L\). This system can be spatially discretized by defining:

\[
u(x, \tilde{t}) = \sum_{i=1}^{N} \phi_i(x) q_i(\tilde{t}),
\]

\[\text{(A1.5)}\]

with the mode shapes \(\phi_i(x)\) satisfying all essential boundary conditions and \(q_i(\tilde{t})\) as the modal coordinates. When the discretized version of (A1.4) is driven around its first resonance the high order modes \((i \geq 2)\) are expected to have low influence because they are increasingly damped by the stiffness proportional damping. The weak significance of higher modes for the experimental setup was already illustrated by Figure A1.2b. So, a single-DOF discretization of (A1.4) appears to be an appropriate model for the experimental setup around its first resonance. Using standard Galerkin approximation this model is obtained as:

\[
\ddot{q} + (D \dot{q} + q) \omega_n^2 + \tilde{f}_C(q, \dot{q}) = \tilde{\Omega}^2 \gamma \tilde{V} \sin(\tilde{\Omega} \tilde{t}),
\]

\[\text{(A1.6)}\]
where:

\[ \omega_n^2 = K_S/M_S, \quad M_S = m\phi(x_m)^2, \quad (A1.7) \]
\[ K_S = \frac{EI}{L^3} \int_0^1 \left(\phi''(x)\right)^2 dx, \quad \gamma = \frac{\tilde{\gamma}}{\phi(x_m)\Delta_G}; \quad (A1.8) \]
\[ \tilde{f}_C(q, \dot{q}) = \frac{\phi(x_C)}{M_S\Delta_G} \tilde{f}_C\left(\phi(x_C)q\Delta_G, \phi(x_C)\dot{q}\Delta_G\right), \quad (A1.9) \]

are the squared fundamental linear natural frequency, equivalent structural mass and stiffness coefficient, and normalized unit conversion gain and impact force, respectively. Additionally, \( \phi(\cdot) \) is the static deformation pattern of a cantilever beam loaded at \( x = x_m \).

Defining the structural damping ratio as \( \beta_S = \frac{1}{2}D\omega_n \), substituting it into (A1.4) and normalizing time \( \tilde{t} \) using the system’s fundamental linear natural frequency \( \omega_n \), one arrives at:

\[ \ddot{q} + 2\beta_S\dot{q} + q + f_C(q, \dot{q}) = \Omega^2\gamma \tilde{V} \sin(\Omega\tilde{t}), \quad (A1.10) \]

where:

\[ t = \omega_n\tilde{t}, \quad \Omega = \Omega/\omega_n \quad \text{and} \quad f_C(q, \dot{q}) = \tilde{f}_C(q, \omega_n\dot{q})/\omega_n^2 \quad (A1.11) \]

are the normalized time, normalized forcing frequency, and nondimensional impact force, respectively. Also, from now on \( (\cdot) \equiv \partial(\cdot)/\partial t \).

The vibro-impact oscillations measured from the experimental setup can be related to (A1.10) by multiplying the model solution by the measured grazing amplitude and by the first mode shape at the measurement location, \( U(L_S, \tilde{t}) = q(\omega_n\tilde{t})\Delta_G\phi(L_S/L) \). This relation can be even simpler if the first mode shape is normalized with respect to the measurement location, \( L_S \). By doing so, the measured vibrations become proportional to the grazing amplitude, i.e., \( U(L_S, \tilde{t}) = q(\omega_n\tilde{t})\Delta_G \).

### A1.4 Impact force models

As mentioned in Section A1.1, there are several ways of modeling vibro-impact forces. In this work three impact force models will be considered. The piecewise-linear Kelvin-Voigt model, the Power-law function impact force and a combination of the two called the modified Kelvin-Voigt model. This section describes these models, showing their physical basis, advantages and limitations.

#### A1.4.1 Kelvin-Voigt model

The mass and stops are mounted in different locations of the beam, being not coincident \( (L_m \neq L_C) \), see Fig. A1.1c,d. Thus the vibrations of the lumped mass are not directly constrained by the stops, further deformations beyond those necessary to reach the stops. With first-mode beam oscillations, in a contact configuration only a
segment of the beam, from \( L_C \) to \( L_m \), suffers further deformation. This segment can be considered as a lumped mass beam in a pinned-free configuration, with its own dynamic properties, such as natural frequency and damping.

So one can model the contact configuration applying Kelvin-Voigt’s approach to the beam segment as well. This secondary system is inactive by default, being active only if the measured transverse mass displacement exceeds the measured grazing amplitude, \( \Delta_G \). For symmetric bilateral impacts:

\[
\tilde{f}_C(U, \dot{U}) = \begin{cases} 
0, & \text{if } |U_C| \leq \Delta_C, \\
K_C(U_C - \Delta_C) + D_C\dot{U}_C, & \text{if } U_C \geq \Delta_C, \\
K_C(U_C + \Delta_C) + D_C\dot{U}_C, & \text{if } U_C \leq -\Delta_C,
\end{cases}
\]

(A1.12)

where \( K_C \) and \( D_C \) are the stiffness and damping coefficients for the beam in the contact configuration and \( U_C = U(L_C, t) \) because the force it is being applied at \( L_C \). Its non-dimensional version can be obtained by applying the same steps taken from (A1.2) to (A1.10), giving:

\[
f_C(q, \dot{q}) = \begin{cases} 
0, & \text{if } |q| \leq 1, \\
\phi_C^2(\omega_R^2(q - 1) + 2\beta_C\omega_R\dot{q}), & \text{if } q \geq 1, \\
\phi_C^2(\omega_R^2(q + 1) + 2\beta_C\omega_R\dot{q}), & \text{if } q \leq -1,
\end{cases}
\]

(A1.13)

where:

\[
\beta_C = \frac{D_C}{2M_S\omega_C}, \quad \omega_C^2 = K_C/M_S, \quad \omega_R = \omega_C/\omega_n, \quad \phi_C = \phi(x_C) = \frac{\Delta_C}{\Delta_G}
\]

(A1.14)

are the contact damping ratio, the contact natural frequency, the natural frequency ratio and the cantilever beam mode shape evaluated at \( x_C \), which is equal to \( \Delta_C/\Delta_G \).

The factor \( \phi_C \) is purely geometric, accounting for the distance between impact and measurement points. For the experimental setup in Figure A1.1: \( \phi_C = 0.241 \). Comparing the locations of measurement and impact on the beam, see Figures A1.1 and A1.3, one can see that the contact point is closer to the beam’s clamped end than the measurement point, so for first mode oscillations, \( \phi_C < 1 \) attenuating the impact force. On the other hand, swapping impact and measurement positions would strengthen the impact force.

Despite its straightforward physical interpretation, Figure A1.4a illustrates that KV’s model dissipative term is discontinuous with respect to velocity. Also, the impact force changes its sign at the end of the contact, indicating that the impacting element is being pulled towards the stop again, a situation which does not make sense from the physical point of view.

Besides the drawbacks mentioned previously, the impact condition enables one to estimate the impact duration. From the contact intervals (where \( |q| \geq 1 \) shown in Figure A1.4b, one can define the normalized contact duration for a full oscillation as:

\[
\tau = 2(\pi - 2\phi_0)/\Omega,
\]

(A1.15)
A1.4 Impact force models

Figure A1.4: (a): Elastic and dissipative components of the piecewise linear impact force model ($\omega_R = 7.278$, $\beta_C = 0.258$ and $\phi_C = 0.241$). (b): Time series of elastic Kelvin-Voigt impact force together with $q = Q \sin(\varphi)$. (c): Normalized experimental data (red x’s) and curves of constant normalized contact duration (blue lines), the numbers on each curve represent the contact duration.

where $\varphi_0 = \arcsin(1/Q)$ is the grazing angle and $Q$ is the first-order modal amplitude, which heavily depends on the model parameters, $\omega_R$ and $\beta_C$. When using this model, one can estimate the normalized contact duration for any pair $(\Omega, Q)$ as shown in Figure A1.4c, where curves of constant $\tau$ are presented with normalized experimental data (crosses). The experimental data was taken from Figure A1.1c and normalized using the fundamental linear natural frequency and the measured grazing amplitude, given by $f_n \approx 7.6$ Hz and $\Delta_G \approx 1.6$ mm, respectively. As expected, the contact duration grows for higher amplitudes $Q$.

A1.4.2 Power-law impact force model

The drawbacks of Kelvin and Voigt’s model can be overcome by approximating the impact force by a power function. However, instead of considering powers of impact deformation ($U_C \pm \Delta_C$) as done by Hunt and Crossley [14], one can consider powers of the ratio between displacement and the gap width, $U_C/\Delta_C$. With this approach, one can define a smooth force impact which is weak in the region $|U_C| < \Delta_C$ but grows very
fast around the gap width, $\pm \Delta C$. Also, the participation of the Power-law impact force on the non-contact regime is not necessarily a problem; a similar effect can actually occur due to a small measurement error on the gap width, shifting the ratio $U_C/\Delta C$ a little bit away from 1, which is the value where the power function starts to grow faster. Also, the participation in the non-contact phase can be increased/decreased according to model parameters. For symmetric bilateral impacts, the elastic force should be an odd exponent power, while the dissipative force should have an even exponent term multiplying the velocity. So:

$$\tilde{f}_C(U, \dot{U}) = F \left( \frac{U_C}{\Delta C} \right)^{2n-1} + D_C \left( \frac{U_C}{\Delta C} \right)^{2p} \dot{U}_C,$$

(A1.16)

where $n \geq 1$ and $p \geq 0$ are integer exponents for the restoring and dissipative terms, respectively. As $U_S/\Delta G$ is non dimensional, $D_C$ has the same physical dimension of a linear viscous damping coefficient, while $F$ has dimensions of force. So, considering $F = K_C \Delta G$ and applying the same steps as from (A1.2) to (A1.10), one can obtain the non-dimensional Power-law impact force as:

$$f_C(q, \dot{q}) = \phi_C \omega_R^2 q^{2n-1} + 2\beta_C \omega_R q^{2p} \dot{q} \phi_C^2,$$

(A1.17)

where $\beta_C$, $\phi_C$ and $\omega_R$ were defined previously for the KV force model. However, it should be mentioned that despite having dynamic parameters with the same name and definition, the models are not equivalent. With the exception of $\phi_C$, one should not expect their parameters to have similar numeric values or physical meaning. For instance, the terms $\omega_R$ and $\beta_C$ have no physical meaning for the Power-law model. Also, notice that $\phi_C$ affects the restoring and dissipative parts of the impact force differently.

Comparing Figure A1.4a and Figure A1.5, the smoothness of the Power-law model is clearly seen. Instead of jumping to a nonzero value, the dissipative Power-law force grows smoothly. However, nonphysical sign changes similar to KV model are possible if the model is purely dissipative, $\omega_R \equiv 0$.

Finally, it is important to remember that numeric simulations involving the Power-law model become demanding for high $n$ and $p$, due to the stiffening of the ODE, losing the advantage of being easier to simulate than the KV model, which is given by a piecewise linear function.
A1.4 Impact force models

A1.4.3 Modified Kelvin-Voigt model

While the Power-law model solves the problems presented by Kelvin and Voigt’s model, it also removes most of the physical meaning of the parameters involved in the impact phenomenon. In order to obtain the expected behavior for the impact force while keeping part of the physical insight given by the KV approach, a combination of both models is presented.

As the dissipative term seems to be the problematic part of the KV model, one can multiply it by a power of the displacement. However, instead of using $U_C/\Delta_C$ as power term, one can add $\pm 1$ to it, obtaining $(U_C/\Delta_C \pm 1)$. In this way, it is assured that the damping term starts to grow smoothly from zero, without any discontinuity. Also, this term can be rewritten as $(U_C \pm \Delta_C)/\Delta_C$ which can be interpreted as a mechanical impact strain. Another option would be using the impact deformation $(U_C \pm \Delta_C)$ as power term, but this term is highly dependent on the gap width, which is very small and would demand an extremely high damping coefficient to compensate for it. So, the combined model is given by:

$$
\hat{f}_C(U, \dot{U}) = \begin{cases} 
0, & \text{if } |U_C| \leq \Delta_C, \\
K_C(U_C - \Delta_C) + D_C \dot{U}_C \left( \frac{U_C}{\Delta_C} - 1 \right)^{2p}, & \text{if } U_C \geq \Delta_C, \\
K_C(U_C + \Delta_C) + D_C \dot{U}_C \left( \frac{U_C}{\Delta_C} + 1 \right)^{2p}, & \text{if } U_C \leq -\Delta_C, 
\end{cases}
$$

(A1.18)

which leads to the original KV model if $p = 0$. The nondimensionalization of this
expression gives:

\[
f_C(q, \dot{q}) = \begin{cases} 
0, & \text{if } |q| \leq 1, \\
\phi_C^2 \left( \omega_R^2 (q - 1) + 2 \beta_C \omega_R \dot{q} (q - 1)^2 \right), & \text{if } q \geq 1, \\
\phi_C^2 \left( \omega_R^2 (q + 1) + 2 \beta_C \omega_R \dot{q} (q + 1)^2 \right), & \text{if } q \leq -1,
\end{cases}
\] (A1.19)

with \( \beta_C \) and \( \omega_R \) being the same as before and \( \omega_R \) keeping its straightforward physical meaning. This model can be seen as a particular case of Hunt and Crossley’s model [14], which originally assumed the elastic term to be a power function as well.

The advantages of mixing both models can be seen by comparing Figure A1.6 with Figure A1.4a and Figure A1.5. While the original KV model changes its sign during impact, leading to nonphysical situations, and the Power-law model is active also in the non-contact phase, the combined model has none of this undesired behaviors. Also, it is possible to obtain the normalized contact duration for a full oscillation using this model, shown in Figure A1.4c and given by (A1.15). However, all the three models considered in this work present nonphysical sign changes, when elastic forces are neglected.
A1.5 Approximate analysis of time periodic response

After the presentation of three different models for the impact force, it is necessary to solve the equation of motion, (A1.10), for each of them. That is the objective of this section, which employs two analytical techniques commonly used to obtain approximated solutions of nonlinear systems.

A1.5.1 Harmonic Linearization

The method of Harmonic Linearization assumes that the steady-state response of a nonlinear system, like (A1.10), can be well approximated by a mono-frequency solution, neglecting terms with higher harmonics. Then one can approximate model nonlinearities, like $f_C(q, \dot{q})$ in (A1.10), using a single harmonic Fourier series:

$$f_C(q(t), \dot{q}(t)) \approx a_1 \sin(\Omega t) + b_1 \cos(\Omega t),$$  \hspace{1cm} (A1.20)

where:

$$a_1 = \frac{2}{T} \int_0^T f_C(q(t), \dot{q}(t)) \sin(\Omega t) \, dt, \quad b_1 = \frac{2}{T} \int_0^T f_C(q(t), \dot{q}(t)) \cos(\Omega t) \, dt$$  \hspace{1cm} (A1.21)

are Fourier coefficients, $\Omega$ is the forcing frequency and $T$ is the forcing period. To obtain some physical insight, this approximation can be rearranged as a spring-dashpot system, $f_C(q, \dot{q}) \approx \kappa q + 2\sigma \dot{q}$, whose coefficients can be found by comparing the responses of its mechanical and Fourier approximations to a solution $q(t) = Q \sin(\Omega t + \theta)$, as follows:

$$f_C(q(t), \dot{q}(t)) \approx \kappa Q \sin(\Omega t + \theta) + 2\sigma \Omega Q \cos(\Omega t + \theta) \approx a_1 \sin(\Omega t) + b_1 \cos(\Omega t)$$  \hspace{1cm} (A1.22)

Defining $\varphi = \Omega t + \theta$, the harmonic linearization coefficients $\kappa$ and $\sigma$ can be defined as:

$$\kappa = \frac{1}{\pi Q} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \sin \varphi \, d\varphi,$$  \hspace{1cm} (A1.23)

$$\sigma = \frac{1}{2\pi Q \Omega} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \cos \varphi \, d\varphi.$$  \hspace{1cm} (A1.24)

Substituting the spring-dashpot approximation into (A1.10) leads to:

$$\ddot{q} + 2(\beta_S + \sigma)\dot{q} + (1 + \kappa)q = \Omega^2 \gamma \dot{V} \sin(\Omega t),$$  \hspace{1cm} (A1.25)

which corresponds to a standard mechanical oscillator with a pair of springs and dampers in parallel. Its stationary frequency response is given by $q(t) = Q \sin(\Omega t + \theta)$, where:

$$Q = \frac{\Omega^2 \gamma \dot{V}}{\sqrt{(\Omega^2 - 1 - \kappa)^2 + 4\Omega^2(\beta_S + \sigma)^2}},$$  \hspace{1cm} (A1.26)
A1.5 Approximate analysis of time periodic response

or:

\[ Q^2 \left( \left( \Omega^2 - 1 - \kappa \right)^2 + 4\Omega^2 (\beta_S + \sigma)^2 \right) = (\Omega^2 \gamma \tilde{\nu})^2. \] (A1.27)

The difference between the current case and a standard mechanical oscillator is that the harmonic linearization coefficients \( \kappa \) and \( \sigma \) depend on the response amplitude, \( Q \). Both coefficients will be defined in Subsection A1.5.3 for different models of the impact force.

A1.5.2 Averaging

Standard first-order averaging [83] implicitly uses the same assumption made for harmonic linearization, that higher harmonics can be neglected and the solution can be approximated by a single harmonic expression, \( q(t) \approx Q \sin \varphi, \varphi = \Omega t + \theta \). However standard averaging is restricted to weakly nonlinear systems only [28], i.e., systems with small forcing amplitude \( \tilde{V} \), damping \( \beta_S \) and nonlinearities \( f_C(q, \dot{q}) \). The equation of motion can be written as a weakly nonlinear system in the following way:

\[ \ddot{q} + q = \epsilon \left( \Omega^2 \gamma \tilde{\nu} \sin(\Omega t) - 2\beta_S \dot{q} - f_C(q, \dot{q}) \right) \] (A1.28)

where the parameter \( \epsilon \ll 1 \) indicates which terms are small in comparison with linear terms. Another feature of the standard averaging is that it allows both amplitude and phase to vary in time, i.e. \( Q = Q(t) \) and \( \theta = \theta(t) \). Defining the time-derivative of the solution as \( \dot{q}(t) \approx \Omega Q \cos \varphi \) implies that \( \dot{\theta}Q \cos \varphi = -\dot{Q} \sin \varphi \). Substituting these relations into (A1.28) results in a system of differential equations in terms of amplitude and phase:

\[ \Omega \dot{Q} = \frac{1}{2} Q (\Omega^2 - 1) \sin 2\varphi + \epsilon \left( \cos \varphi \left( \gamma \tilde{V} \Omega^2 \sin(\varphi - \theta) + f_C(q, \dot{q}) \right) - 2\Omega \beta_S Q \cos^2 \varphi \right) \] (A1.29)

\[ \Omega Q \dot{\theta} = -Q (\Omega^2 - 1) \sin^2 \varphi + \epsilon \left( \sin \varphi \left( f_C(q, \dot{q}) + \gamma \tilde{V} \Omega^2 \sin(\varphi - \theta) \right) + \beta_S \Omega Q \sin 2\varphi \right) \] (A1.30)

Almost all terms on the right-hand sides of these equations are multiplying \( \epsilon \) and therefore are small. Around the primary external resonance the term \( \Omega^2 - 1 \) is also small and so the whole right-hand side of these differential equations is small, which means that both \( Q \) and \( \theta \) vary slowly with time. In this case one can approximate (A1.29) and (A1.30) by their average over one period of oscillations, obtaining:

\[ 2\pi \Omega \dot{Q} = -\epsilon \left( 2\pi \Omega \beta_S Q + \pi \gamma \tilde{V} \Omega^2 \sin \theta + \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \cos \varphi \, d\varphi \right), \] (A1.31)

\[ 2\pi \Omega Q \dot{\theta} = -\epsilon \pi Q (\Omega^2 - 1) + \epsilon \left( \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \sin \varphi \, d\varphi - \pi \gamma \tilde{V} \Omega^2 \cos \theta \right), \] (A1.32)
where $\epsilon$ is multiplying the term $(\Omega^2 - 1)$ to explicitly indicate that this term is small. The integrals of the impact force can be replaced by the harmonic linearization coefficients $\kappa$, $\sigma$ given by (A1.23) and (A1.24). The equilibria of the resulting system correspond to the steady-state response of (A1.10), so:

\begin{align*}
\Omega^2 \gamma \tilde{V} \sin \theta + 2 \Omega Q (\beta_S + \sigma) &= 0, \quad (A1.33) \\
Q (1 + \kappa - \Omega^2) - \Omega^2 \gamma \tilde{V} \cos \theta &= 0. \quad (A1.34)
\end{align*}

Eliminating $\theta$ from the last expressions gives:

\begin{equation}
\left( \gamma \tilde{V} \Omega^2 \right)^2 = Q^2 \left( (\Omega^2 - 1 - \kappa)^2 + 4 \Omega^2 (\beta_S + \sigma)^2 \right), \quad (A1.35)
\end{equation}

which is identical to (A1.27). This same relation could be obtained using Multiple Scales or modified Lindstedt-Poincaré [30] methods, but Averaging was chosen due to its well developed mathematical basis and effectiveness for other vibro-impacting systems [65, 79]. In this context, despite its loose mathematical basis, Harmonic Linearization can be seen as a shortcut method, giving the same results for stationary solutions as Averaging, but bypassing its intermediate steps such as defining small terms, obtaining a system of differential equations for amplitude and phase and finding its equilibrium state.

**A1.5.3 Application to specific impact force models**

**A1.5.3.1 Kelvin-Voigt impact force**

As mentioned previously, Kelvin-Voigt’s impact force is not active during the whole period of oscillation. So, the limits of integration in (A1.23) and (A1.24) can be restricted to the contact period, i.e. for $\varphi \in [\varphi_0; \pi - \varphi_0]$ for $q \geq 1$ and $\varphi \in [\pi + \varphi_0; 2\pi - \varphi_0]$ for $q \leq -1$, as shown in Figure A1.4b. Considering $q \approx Q \sin \varphi$ and substituting (A1.13) into (A1.23) and (A1.24) leads to:

\begin{align*}
\kappa &= \phi_C^2 \omega_R^2 W(Q), \quad (A1.36) \\
\sigma &= \phi_C^2 \beta_C \omega_R W(Q), \quad (A1.37)
\end{align*}

where:

\begin{equation}
W(Q) = 1 - \frac{2}{\pi} \left( \varphi_0 + \frac{1}{2} \sin(2\varphi_0) \right), \quad \varphi_0 = \arcsin(1/Q), \quad (A1.38)
\end{equation}

is a function of the forcing amplitude $Q$. Despite its complicated formula, $W(Q)$ does not grow unbounded, being limited between 0 and 1 for $Q$ between 1 and infinity. Substituting $\kappa$ and $\sigma$ in the frequency response, (A1.27) leads to:

\begin{equation}
Q^2 \left( \left( \Omega^2 - 1 - \phi_C^2 \omega_R^2 W(Q) \right)^2 + 4 \Omega^2 (\beta_S + \phi_C^2 \beta_C \omega_R W(Q))^2 \right) - (\Omega^2 \gamma \tilde{V})^2 = 0. \quad (A1.39)
\end{equation}
As \( W(Q) \) varies between 0 and 1, it works as a tuning parameter between two linear oscillators, one given by (A1.10) and \( f_C \equiv 0 \), and another given by the same equation and \( f_C \equiv 0 \), but with natural frequency \( \omega_R \) instead of unity and damping \( \beta_S + \beta_C \omega_R \) instead of \( \beta_S \). The frequency response equations for these oscillators are given by respectively:

\[
Q^2 \left( (\Omega^2 - 1)^2 + 4\Omega^2 \beta_S^2 \right) - (\Omega^2 \gamma \dot{V})^2 = 0, \tag{A1.40}
\]

\[
Q^2 \left( (\Omega^2 - 1 - \phi_C^2 \omega_R^2)^2 + 4\Omega^2 (\beta_S + \phi_C^2 \beta_C \omega_R)^2 \right) - (\Omega^2 \gamma \dot{V})^2 = 0. \tag{A1.41}
\]

### A1.5.3.2 Power-law impact force

Substituting (A1.17) into (A1.23) and (A1.24) leads to:

\[
\kappa = 2\phi_C G_n \frac{\omega_R^2}{Q^2}, \tag{A1.42}
\]

\[
\sigma = \frac{\phi_C^2 \beta_C \omega_R G_p}{p + 1}, \tag{A1.43}
\]

where:

\[
G_j = \frac{Q^2 j \Gamma(j + 1/2)}{\sqrt{\pi} \Gamma(j + 1)}, \quad j = n, p \tag{A1.44}
\]

is a function of the exponent \( n \) or \( p \) and the steady-state amplitude \( Q \) and \( \Gamma \) denotes the Gamma function. As expected for a power-function term, \( G_j \) grows unbounded for \( j \to \infty \) and \( |Q| > 1 \). Substituting \( \kappa \) and \( \sigma \) in the frequency response equation, (A1.27) gives:

\[
Q^2 \left( (\Omega^2 - 1 - 2\phi_C G_n \omega_R^2/Q^2)^2 + 4\Omega^2 (\beta_S + \phi_C^2 \beta_C \omega_R G_p/(p + 1))^2 \right) - (\Omega^2 \gamma \dot{V})^2 = 0. \tag{A1.45}
\]

### A1.5.3.3 Modified Kelvin-Voigt impact model

The equivalent stiffness \( \kappa \) is given by (A1.36) and the equivalent damping is given by:

\[
\pi \sigma = \beta_C \omega_R \phi_C^2 \int_{\pi - \varphi_0}^{\varphi_0} \cos^2 \varphi (Q \sin \varphi - 1)^{2p} d\varphi + \beta_C \omega_R \phi_C^2 \int_{\varphi_0}^{2\pi - \varphi_0} \cos^2 \varphi (Q \sin \varphi + 1)^{2p} d\varphi, \tag{A1.46}
\]

whose integration limits are the same as the original Kelvin-Voigt model. When \( p = 0 \), \( \sigma \) is given by (A1.37), but there is no general formula for this integral as a function of \( Q \) and \( p \). As an example of its complexity, the formula for \( \sigma \) when \( p = 1 \) is shown below:

\[
\sigma = \frac{\beta_C \omega_R \phi_C^2}{4\pi} \left( (Q^2 + 4)(\pi - 2\varphi_0) - \frac{2}{3} \cos \varphi_0 (13Q + 2\sin \varphi_0) \right) \tag{A1.47}
\]
A1.6 Model tuning and analysis

The frequency response for each impact model can be obtained by substituting the harmonic linearization coefficients found in Subsection A1.5.3 into the generic amplitude-frequency relationship, (A1.27). Then one can use a nonlinear least squares solver to fit the frequency response equations of each model to experimental data, such as the one from Figure A1.2a. Again, the experimental data is normalized using the fundamental linear natural frequency, $f_n \approx 7.6$ Hz, and grazing amplitude $\Delta_G \approx 1.6$ mm.

A1.6.1 Linear range

Before defining appropriate values for the impact force model parameters, one should deal with the conversion gain $\gamma$ and structural damping ratio $\beta_S$, which are necessary to model both impacting and non-impacting vibrations. Figure A1.7 illustrates the normalized linear experimental frequency response together with its model approximations for $\gamma = 0.195$ $V^{-1}$ and $\beta_S = 14 \times 10^{-3}$. Despite been able to reproduce the resonance peak accurately, model predictions deteriorate outside the peak neighbourhood. That can be explained by the unmodeled dynamics of the subsystem shaker-platform.

For vibro-impact oscillations the excitation amplitude was chosen as $\bar{V} = 0.6$ V in order to maximize the nonlinear frequency range without saturating the displacement sensors.

A1.6.2 Power function exponents

The fitting of models with power function was made considering pairs of $1 \leq n \leq 10$ and $0 \leq p \leq 10$ as constant while varying $\omega_R$ and $\beta_C$. This revealed some insights into the role of the power exponents on the frequency response, see Figure A1.8. For
Table A1.1: Fitted model parameters.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\omega_R$</th>
<th>$\beta_C$</th>
<th>$n$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kelvin-Voigt</td>
<td>7.278</td>
<td>0.258</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Power-law</td>
<td>0.715</td>
<td>0.659</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Mod. Kelvin-Voigt</td>
<td>7.306</td>
<td>4.636</td>
<td>—</td>
<td>1</td>
</tr>
</tbody>
</table>

instance, by increasing $p$ the Power-law model response gets closer to the experimental upper folding point (Figure A1.8b). In this case, the fitted $\omega_R$ keeps almost constant, while $\beta_C$ decreases. By increasing $n$, as shown in Figure A1.8a, the model response gets closer to both the upper and lower folding points, going further away from the upper one if $n$ grows too much. Also, by increasing $n$ the model response approaches the curvature of the bent peak. In this case, the fitted $\beta_C$ goes up, while $\omega_R$ goes down. This approach to the lower folding point and peak curvature for growing exponents can be explained by the fact that for small $n$ and $p$ the Power-law force is not negligible in the non-impacting region. For the MKV model, Figure A1.8c shows that there are no significant changes for $1 \leq p \leq 5$, with the parameter $\omega_R$ remaining constant while $\beta_C$ grows.

Based on this analysis, the model parameters are chosen as the ones which led their models’ frequency response closer to experimental data with the lower exponent. For the MKV model, the exponent $p = 1$ is chosen as the one whose response is closer to experimental data. The fitted model parameters are shown in Table A1.1 were obtained by applying the same reasoning to all models. In this table one can see that the MKV model has lower exponent coefficient than the Power-law model. That can be explained by the fact that both KV models are only defined on contact areas, while the power function is defined also on the non-contact region and demands high order exponents to be negligible in this area.
Figure A1.8: Comparison of power exponents on frequency response of the Power-law model (a) varying $n$ and $p = 3$, (b) varying $p$ and $n = 4$, and (c) varying $p$ for the Modified Kelvin-Voigt model. $\tilde{V} = 0.6\,\text{V}$. The arrows indicate how the frequency response changes as the value of each parameter increase.
A1.6.3 Model validation

To check the validity of the assumptions made, the fitted values shown in Table A1.1 are used to obtain the frequency response of the system by numeric simulation of the single-DOF model from (A1.10) together with the force models from (A1.13), (A1.17) and (A1.19) using a standard MATLAB® ODE solver ode45() together with the option ‘Events’ to handle the transition between impacting and non-impacting regimes when necessary.

Looking at the results from the perturbation methods, numeric simulations and experiments shown in Figure A1.9 one can see that all of the mentioned mathematical models give results reasonably close to experimental observations. Thus, the assumptions made appear adequate. Also, one can see that the modified Kelvin-Voigt model is able to predict the experimental behavior more precisely, capturing both fold points, while Kelvin-Voigt’s classic model is further away from the upper folding point and the Power-law model is further away from both folding points.

![Figure A1.9](image)

Figure A1.9: Comparison of frequency responses obtained using (a) the Kelvin-Voigt model (b) the Power-law model and (c) the Modified Kelvin-Voigt model. $V = 0.6\, V$.

A1.6.4 Parameter analysis

After finding appropriate parameter values for the impact force models, one can analyze the individual effects of each parameter on the frequency response. That can
expand one’s knowledge about the different models, being used to design other vibro-impact devices, which enhance or mitigate certain effects presented here such as the maximum response amplitude or the length and curvature of the bent peak.

![Parameter analysis of Kelvin-Voigt's model varying (a) $\omega_R$ and (b) $\beta_C$. The arrows indicate how the frequency response changes as the value of each parameter increase.](image)

Figure A1.10: Parameter analysis of Kelvin-Voigt’s model varying (a) $\omega_R$ and (b) $\beta_C$. The arrows indicate how the frequency response changes as the value of each parameter increase.

Starting with Kelvin-Voigt’s model, shown in Figure A1.10, one can see that as expected, the peak curvature is very sensitive to $\omega_R$, with the peak length (hysteresis region) growing as $\omega_R$ increases. Also, by increasing the contact damping one can decrease the oscillation amplitude.

Similarly, for the Power-law impact model, shown in Figure A1.11, one can see that the parameters related to the elastic force control the peak curvature and length, while the ones related to dissipative forces control the response amplitude. Increasing $\beta_C$ and $p$ causes the response amplitude to decrease. While increasing $\omega_R$ moves both folding points to the right, increasing $n$ moves the upper folding point to the right and the lower one to the left. Also, only the restoring force parameters have an influence on the non-impacting region of the frequency response.

The influence of the natural frequency ratio and contact damping ratio are the same for both versions of Kelvin-Voigt’s model, see Figures A1.10 and A1.12a,b. However, different from the Power-law model, the amplitude increases for higher values of $p$, see Figure A1.12c. This happens because the dissipative force is proportional to $(q\pm 1)^{2p}$, which, for lower exponents, is stronger in the neighborhood of impact, $Q \approx 1$. As $p$ increases the dissipative term becomes negligible for low impacting amplitudes. Also, it is worth to point out that despite the modified Kelvin-Voigt model being equivalent to its standard version for $p = 0$, the frequency response on Figure A1.12c is not equivalent to the one in Figure A1.9a for the standard Kelvin-Voigt model because the contact damping has different numerical values.
Figure A1.11: Parameter analysis of Power-law model varying (a) $\omega_R$, (b) $\beta_C$, (c) $n$ and (d) $p$. The arrows indicate how the frequency response changes as the value of each parameter increase.

Figure A1.12: Parameter analysis of Modified Kelvin-Voigt model varying (a) $\omega_R$, (b) $\beta_C$ and (c) $p$. The arrows indicate how the frequency response changes as the value of each parameter increase.
A1.7  Conclusions

The experimental behavior of an impacting forced lumped mass cantilever beam around its first resonance was modeled as a single degree of freedom oscillator. Harmonic decomposition of the frequency response of the experimental setup showed that higher harmonic components were negligible if compared with the fundamental one. The SDOF model was analyzed using Averaging and Harmonic Linearization to obtain frequency-amplitude relations. Numeric values for model parameters were obtained by fitting the nonlinear frequency response relationship to the experimental frequency response obtained using control-based continuation. After choosing the most appropriate model parameters, their role on the frequency response was also analyzed.

For all the impact models discussed in the present work, it was found that length and curvature of the bent peak are governed by the parameter $\omega_R$, defined as the natural frequency ratio, while the contact damping ratio controls the amplitude of vibration. For the Power-law model, while the exponent of the elastic term has an influence on both impacting and non-impacting regions of the frequency response, the damping exponent affects only the amplitude of oscillation of the impacting region. Also, the exponent of the dissipative term has opposite effects on MKV and power function models.

Experimental observations could be reproduced by all of the impact force models under analysis, with the modified Kelvin-Voigt model describing the experimental frequency response more accurately, predicting both fold points, while the other models fail to predict the upper fold point. The reason for that relies on the modified KV model being a combination of the other models, mixing their positive characteristics to obtain a physically more accurate model. Nevertheless, numeric simulations of the MKV model are more demanding than the continuous Power-law model, due to its non-smoothness with respect to displacement and velocity. Also, the lack of a general compact formula for the equivalent damping term, which has to be obtained for every power exponent, gives rise to complicated expressions.

The experimental setup could be modified in order to analyze systems whose mass displacement is directly limited by the stops, becoming a test bed for experimental validation of kinematic impact models together with nonsmooth transformations [34, 84], thus, making it possible to compare also to kinematically based vibro-impact models in terms of ease of use, applicability and reliability.

The discussion presented in this work can be applied to the analysis of other mechanical systems with non-rigid constraints and whose steady-state oscillations are single-periodic. However, for different parameter configurations, the considered models can produce other qualitative behaviors, such as quasiperiodicity and chaos, not discussed here.

Acknowledgments

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Unilateral Vibro-Impact Systems — Experimental Observations against Theoretical Predictions based on the Coefficient of Restitution

The following article was submitted for publication in the Journal of Sound and Vibration in May 2018 and remains under review as of the writing of this Ph.D. thesis. Some of its results will be presented at the 4th International Conference on Vibro-Impact systems and Systems with Non-Smooth Interactions, ICOVIS 2018, to be held in Kassel, Germany from July 30th to August 3rd, 2018.

The article’s highlights are:

• Reproduce empirical observations to a certain level, validating analytical predictions;

• Discussion of common techniques for analysis of unilateral vibro-impact problems;

• Presentation of experimental data for 3 gap configurations: loose, neutral and tight;

• Dependence of the coefficient of restitution to forcing frequency and pre-contact velocity is shown empirically;

• Discussion on model tuning for unilateral vibro-impact experiments
A2.1 Introduction

Abstract

The vibro-impact response of a single-degree of freedom model with the coefficient of restitution is analyzed using pointwise mapping and a standard averaging combined with non-smooth transformations. Experimental data are taken from a cantilever beam with attached mass and unilateral constraint submitted to different gap configurations and levels of excitation. Numerical simulations are used to reproduce empirical observations to a certain extent and validate theoretical predictions. Investigations on the coefficient of restitution show its dependence on the forcing frequency and pre-contact velocity. The effect of gap variations due to sliding of the constraint during frequency sweep is analyzed experimentally.

Keywords: vibro-impact dynamics; unilateral impact; kinematic impact; coefficient of restitution; vibro-impacting beam; beam with attached mass;

A2.1 Introduction

Impacts are very common in engineering practice, being necessary for some applications and avoided in others. For instance, while the interaction between different structural components is essential for some processes, like hammering and riveting, the contact between rotor and stator is a problem. Thus, impacts are a widely studied subject in mechanics and applied mathematics.

Roughly speaking, there are two main ways of modeling impacts: using an impact force function or a coefficient of restitution, CoR. Impact forces are very common in literature, with piecewise linear functions of the elastic term being the most common choice [19, 21, 25, 45, 58, 77, 85–88]. Other works regard stiffness and damping as piecewise linear [69, 70, 80, 81]. Physical inaccuracies on those models motivated the development of formulations where the impact force is a power of both displacement and velocity [14, 15, 82], incorporating the CoR as a parameter. An advantage of using the force approach is that it provides information on structural stresses. However, it also demands knowledge on the contact duration, which can be difficult to obtain.

The coefficient of restitution, given by the negative ratio between velocities after and before impact against an immovable stop is another very common way of modeling impacts [34, 35, 59, 62, 63, 68, 75, 89–91], due to its straightforward physical interpretation and formulation. In this approach the impacting elements are considered to be rigid, the contact duration is supposed to be negligible and there is no way to estimate contact forces. While the CoR is considered a constant parameter in most applications, there is evidence of its dependence on the impact velocity [18].

The impact’s nature should be taken into account when choosing the model formulation. For instance, force modeling is more appropriate when contact duration and deformation are not negligible, otherwise, both models should produce similar results, as shown in [11] through numerical simulation.
A2.1 Introduction

For problems where impacts are the only nonlinearity, one can use Pointwise mapping [4, 92] to access some model properties, such as stability of solutions and bifurcation points. Other techniques such as Harmonic Linearization [5], Averaging [79], and the Lindstedt-Poincaré method [30] can be used to solve problems with weak smooth and non-smooth nonlinearities, providing information about the stability of solutions and steady-state vibration amplitude. For general impacting systems (not weak), discontinuous transformations [73] of time [31] and state variables [32][33] can be used to eliminate or weaken non-smooth terms. After that, standard analytical techniques such as Averaging [34, 74] and Multiple Scales [35] can be applied.

Discontinuous transformations can be subdivided according to impact characteristics, such as unilateral/bilateral or elastic/inelastic contacts. While Zhuravlev’s [32] and Ivanov’s [33] transformations are used only for unilateral vibro-impact problems, the non-smooth temporal substitutions proposed by Pilipchuk [31, 73] is applied to bilateral vibro-impact problems with elastic collisions. Recently, this transformation was generalized to inelastic impact problems [84]. For near-elastic vibro-impact problems, Thomsen and Fidlin [34] use Zhuravlev transformation for unilateral constraints and a sawtooth function of the displacement variable for bilateral impacts. Czolczynski et. al. [93] uses a modified Peterka method to analyze the dynamics of a base excited SDOF system, obtaining regions of stability for period-$n$ oscillations and agreement with numerical results.

It is common to use standard ODE solvers together with event handling subroutines to perform numerical simulations of impact vibrations. An alternative is to use smoothed versions of the discontinuities on impact force models directly with common ODE solvers [39, 47, 49]. Grace et. al. [41] uses non-smooth coordinate transformations as a preprocessing step to numerical simulations of vibro-impacts systems to weaken the kinematic impact rule. The development of numerical time-integration algorithms for non-smooth models is an active field of research [9, 43, 44] whose discussion is out of the scope of this work.

There are many experimental works which use low order models to describe vibro-impacts on continuous structures [45, 58, 59, 62–64, 68–70]. Moon and Shaw [58] used a piecewise linear single degree of freedom, SDOF, model to reproduce the chaotic behavior of an experimental base excited cantilever beam with unilateral constraint setup, obtaining qualitative agreement. Shaw and Holmes [19] obtain an expression for the stability and bifurcation of periodic orbits in an impacting SDOF model using CoR and piecewise linear impact force, applying pointwise mapping analysis. Those predictions were evaluated by Shaw [45] using an experimental base-excited cantilever beam with unilateral constraint setup, obtaining reasonable quantitative and qualitative agreement.

Stensson and Nordmark [59] use a attached mass cantilever beam driven by intermittent impacts setup to illustrate geometric features particular to attractors of impacting systems. The experiment was modeled with a SDOF model with CoR impact and solved numerically, being able to reproduce the experimental findings. Fang and Wickert [62] model a similar setup using a SDOF model with CoR impact. The system is analyzed using pointwise mapping and numerical simulations. The same modeling was used by Bishop et al. [63] to reproduce the experimental frequency response of a base-excited
Wagg and Bishop [68] present a technique which enables the usage of many degrees of freedom to model continuous structures with CoR impact, using a base-excited cantilever beam with unilateral constraint setup as a testbed for the presented technique. Andreaus et al. [70] use a 3-DOF piecewise linear model to simulate the experimental behavior of a attached mass harmonically forced cantilever beam with one-sided constraints. The dimensionality of vibro-impact beams has been studied experimentally by [71, 72] using correlation dimension and proper orthogonal decomposition, respectively. Czolczynski et. al. [94], presents numerical comparisons between low-dimensional models (1 and 2 DOFs) against a high-order finite element model of a base-excited cantilever beam with attached mass and unilateral Hertzian impacts.

Experimental bifurcation diagrams for a setup similar to [62] were obtained by Balachandran [64]. The structural impacts were modeled in two ways, with infinitesimal (CoR) and finite contact duration (force). Dick et al. [69], studies grazing bifurcations using a multi-DOF model with piecewise linear contact to model a base-excited cantilever beam with unilateral constraint setup.

There are also many experimental works dealing with unilateral impacts on discrete SDOF systems, e.g. [46, 48–50, 95]. Todd and Virgin [48] used a cart rolling in an energy well track experiment to study the influence of impact location on the cart dynamics. Using a SDOF model with amplitude-dependent natural frequency, they were able to reproduce experimental observations. Savi et al. [49] used a piecewise-linear model to mimic the dynamics of a cart sliding on a straight horizontal rail and hitting a support. As pointed out by [48], those setups eliminate the possibility of modal interactions that can affect the structural impact response. However, the manufacturing of those experiments is more complex, requiring more parts.

The discussion above shows that, under certain conditions, the coefficient of restitution can be used to obtain reasonable agreement between theory and experiments. On the other hand, the motivation and main originality of the present work is to analyse the limitations of the coefficient of restitution model, using experimental observations from a simple vibro-impact setup, investigating the applicability conditions of the CoR model approach. In the present work, this is done through the experimental analysis of a unilateral vibro-impact setup with different gap widths (pre-stressed, neutral and loose) and excitation levels and its subsequent correlation with predictions made in the literature using numerical and analytical tools. The effect on the magnitude response of gap deviation during an experimental frequency sweep is discussed. Limitations on the excitation source are discussed. Agreement between theory and experiments is obtained under certain circumstances despite experimental challenges. Usage of discontinuous transformations as a preprocessing step for numerical simulation of vibro-impact problems is considered.

The manuscript is organized as follows: Section A2.2 discusses the model for a SDOF base-excited mechanical oscillator under single-sided impacts described by a coefficient of restitution. The numerical simulation of this model is discussed in Section A2.3, where the computation time is reduced by using non-smooth coordinate transformations as a preprocessing step. Then, standard averaging and pointwise mapping are employed.
in Section A2.4 to obtain approximated period-one solutions and investigate their stability. This section concludes with a comparison between the results from numerical and analytical techniques. A brief description of the experimental setup and its mathematical model are presented in Sections A1.3 and A2.5 respectively. Numerical and analytical predictions are compared with empirical data in Section A2.7. Finally, conclusions and future perspectives are presented in Section A2.8.

A2.2 Engineering modeling

Based on the review presented in Section A2.1 for vibro-impact phenomena, Figure A2.1 shows a flowchart for modeling and analysis of general physical systems. Using signal processing it is possible to record some features of the empirical behavior of some equipment, whose mathematical model is obtained with the aid of physical laws and assumptions, in the form of a differential equation in most cases. Differential equations can be solved either analytically or numerically. While approximated numerical solutions can always be obtained through discretization of the equation’s independent variables, the existence of closed-form analytical solutions depends on the equation’s mathematical properties. For nonlinear models, one can establish certain mathematical assumptions, and then obtain an approximated analytical solution under strict conditions.

Experimental, numerical and analytical results are discussed in this work. In order to produce a concise discussion, focused on the vibro-impact behavior, the single-degree of freedom model shown in Figure A2.2 will be used to illustrate the application of different analytical techniques to non-smooth systems. This is a nondimensional model, time-normalized using its linear natural frequency, and its equation of motion is given by:

\[ \ddot{q} + 2\beta \dot{q} + q = \Omega^2 B \sin(\Omega t), \quad \text{if} \quad q < \Delta, \]

where \( \beta, \Omega, B \) and \( \Delta \) are the damping ratio, forcing frequency, forcing amplitude and gap width, respectively. \( q(t) = y(t) - b(t) \) is the relative displacement between the

![Figure A2.1: Modeling diagram.](image-url)
inertial coordinates $y(t)$ and $b(t) = B \sin(\Omega t)$. The impact condition is:

$$q(t_-) = \Delta \Rightarrow \begin{cases} q(t_+) = q(t_-), \\ \dot{q}(t_+) = -R \dot{q}(t_-), \end{cases}$$  \hspace{1cm} (A2.2)

where $R$ is the coefficient of restitution and $t_+$ and $t_-$ are the time instants after and before impact, respectively. From now on terms such as $q(t_\pm)$ will be represented by $q_\pm$. The impact condition, (A2.2), can be rewritten as:

$$\dot{q}_+ - \dot{q}_- = -(1 + R) \dot{q}_-$$  \hspace{1cm} (A2.3)

which shows that the impacts create a velocity jump which has at least the same order of magnitude as the pre-contact velocity for $R = 0$ or twice that for elastic impacts, $R = 1$.

![Figure A2.2: Base-excited SDOF system with unilateral constraint. $b(t) = B \sin(\Omega t)$](image)

Model parameter values are $\beta = 15 \times 10^{-3}$ and $B = 0.5 \times 10^{-3}$, unless otherwise specified. The forcing frequency varies in the range $\Omega = 1.5$ to $2.5$. Numeric values for gap width and the coefficient of restitution will be given later on.

**A2.3 Numerical Analysis**

Numerical simulation is the most common tool to obtain approximate solutions for arbitrary nonlinear models. While (A2.1) is continuous, the impact condition, given by (A2.2), produces a discontinuity which prevents the direct use of classical numerical schemes, which should be integrated with auxiliary schemes to produce reasonable
A2.3 Numerical Analysis

Results. One of the most common approaches is to use an event-handling scheme to identify the occurrence of impacts, using the post-contact state as initial conditions to the ODE solver. Alternatively, one can use coordinate transformations to reduce or eliminate the velocity jump due to impacts. In this section, both event-handling techniques and coordinate transformations will be used separately or combined and their results will be compared. Results from analytical techniques will be added to the comparison in Section A2.4.

Time-integration of the model equations was performed using the built-in MATLAB® ODE solver `ode23` with relative and absolute error tolerances of $\text{RelTol} = 1.0\text{e}^{-4}$ and $\text{AbsTol} = 1.0\text{e}^{-8}$. The option ‘events’ is used to handle the transition between impacting and non-impacting regimes when necessary.

A2.3.1 Direct Numerical Simulation

![Figure A2.3: Results from direct numerical simulation: time series for (a) position and (b) velocity, and (c) phase-portrait.](image)

The results from direct numerical time-integration of (A2.1) and (A2.2) using the nominal parameters for $R = 0.9$ and $\Delta = 0$ are shown in Figure A2.3. The stop location is represented by a straight dotted black line and impact states are marked with red circles.
Non-smooth model components such as (A2.2) are the main barrier to overcome when dealing with discontinuous systems. There are many works devoted to bypassing this problem from both numerical \[9, 43, 44, 96\] and analytical \[10, 34, 73\] perspectives. A combined version of both approaches is presented in \[41\], which applies non-smooth coordinate transformations to a specific problem in order to weaken discontinuous terms and then performs numerical simulation on the transformed problem. This approach is briefly discussed in the next two sub-sections.

### A2.3.2 Numerical Simulation using Zhuravlev’s Transformation

Zhuravlev’s transformation \[32\], was developed specifically for unilateral vibro-impact problems. It consists of unfolding the displacement \(q(t)\) limited by a unilateral constraint at \(q = \Delta\) into an unconstrained variable \(z(t)\), shifting the barrier to \(z = 0\) as shown in Figure A2.4. Notice that \(z\) is smooth and change its sign at the points where \(q\) is non-smooth. Mathematically, the transformation is given by \[32\]:

\[
q(t) = \Delta - |z(t)|, \tag{A2.4}
\]

where \(z(t)\) is called unfolded variable. The first and second time-derivatives of (A2.4) during non-impact times \((z \neq 0)\) are:

\[
\dot{q} = -\dot{z} \text{sgn}(z), \tag{A2.5}
\]

\[
\ddot{q} = -\ddot{z} \text{sgn}(z), \tag{A2.6}
\]

where \text{sgn} is the sign function. Substituting (A2.4) and its derivatives on (A2.1), one obtains:

\[
\ddot{z} + 2\beta \dot{z} + z = \text{sgn}(z)\left(\Delta - \Omega^2 B \sin(\Omega t)\right), \quad \text{if} \quad z \neq 0. \tag{A2.7}
\]

Knowing that \(z_+z_- < 0\), the impact condition, given by (A2.2), can be rewritten as:

\[
\dot{z}_+ = R\dot{z}_-, \quad \text{if} \quad z = 0, \tag{A2.8}
\]

\[
\dot{z}_+ - \dot{z}_- = -(1 - R)\dot{z}_-. \tag{A2.9}
\]

The last expression shows that the impacts create a velocity jump on the unfolded variable proportional to \(1 - R\) which is small for near elastic impacts, \(R \approx 1\). Although this jump gets bigger for \(R \to 0\) it is always smaller than the one obtained from the original coordinates of the problem, given by (A2.3). One can avoid the use of the impact condition, (A2.8) by inserting the velocity jump due to impacts directly into (A2.7) using Dirac’s delta function, which becomes \[41\]:

\[
\ddot{z} + 2\beta \dot{z} + z + (1 - R)\dot{z}\delta_-(z) = \text{sgn}(z)\left(\Delta - \Omega^2 B \sin(\Omega t)\right), \tag{A2.10}
\]

where, according to \[41\], \(\delta_-(\cdot)\) is a modified version of Dirac delta function, whose singularity occurs on the left of zero, instead of at zero. In this case, the resulting model
still contains a singularity, which is now multiplied by a small factor $1 - R$, if $R \approx 1$. This can be done using the model original coordinates as well, see [35, 84]. In this case, one has to interpret the model in terms of the distribution theory, due to the presence of a summation of classical Dirac delta functions $(1 + R)\dot{q}\delta(t - t_j)$ instead of $(1 - R)\dot{z}\delta_-(z)$, where $t_j$ represents the $j$-th impact instant.

From a numerical perspective, there are two ways to handle the term $\delta_-(z)$ in (A2.10): replace it by a finite smooth approximation, or neglect it under the assumption that $R \approx 1$ and $1 - R \approx 0$. The first alternative leads to additional model parameters to describe the smoothed function, while the second one eliminates the possibility of examining the influence of small variations of the coefficient of restitution on the model response. Alternatively, one can use (A2.7) with an event handling algorithm for (A2.8), whose smaller velocity jump (compared to (A2.8)) should decrease computation efforts. After simulation, the model original coordinates can be obtained using (A2.4) and (A2.5).

Despite the persistence of the impact rule on the transformed model, the velocity jump caused by the impacts has been decreased. That can be observed in Figure A2.5(b) which shows the velocity and its transformed counterpart. A minor problem of this approach, which could not be solved by varying the ODE’s solver properties, is that the original velocity presents an additional jump immediately after impact, see the bottom part of Figure A2.5(b,c).
Figure A2.5: Numerical simulation using Zhuravlev transformation showing physical and transformed coordinates. Position and velocity in time(a, b) and phase-portrait (c).

A2.3.3 Numerical Simulation using Ivanov’s Transformation

Ivanov’s transformation can be seen as an extension of Zhuravlev’s substitution, which is also restricted to models with unilateral constraint. Ivanov [33] proposed an additional transformation for the velocity term, which is not directly related to the displacement variable:

\[ q(t) = \Delta - |z(t)|, \quad (A2.11) \]
\[ \dot{q}(t) = -K_N s(t) \text{sgn}(z(t)), \quad (A2.12) \]
\[ K_N = 1 - K \frac{\text{sgn}(z(t)s(t))}{\frac{1}{1+R}}, \quad (A2.13) \]

A downside of this approach is the lack of physical interpretation for the new variable \( s(t) \). This change of coordinates reduces to Zhuravlev’s if \( R \approx 1 \), in which case \( s = \dot{z} \). While Zhuravlev’s transformation decreases the velocity jump during impacts, the term \( K_N \) in Ivanov’s coordinate conversion inserts the energy losses related to the collisions directly in the transformation, eliminating the need for an impact condition.
Writing (A2.1) on state-space form and substituting (A2.11) and (A2.12), leads to:

\[ \dot{z} = K_N s, \]  
\[ \dot{s} = -2\beta s + \frac{1}{K_N} \left( (\Delta - \Omega^2 B \sin(\Omega t)) \text{sgn}(z) - z \right). \]  

(A2.14)  
(A2.15)

One can obtain (A2.7) from (A2.14) and (A2.15), by realizing that \( K_N \) is constant in each quadrant of the \((z, s)\)-plane, \( z, s \neq 0 \). However, this removes the energy loss due to impacts from the model. Thus, Ivanov transformed systems are usually described as a set of first-order differential equations.

The steady-state response of (A2.14) and (A2.15) using numerical simulation can be seen in Figure A2.6. The problem’s original coordinates are obtained after simulation using (A2.11) and (A2.12). While \( s \) is symmetric about the time-axis, \( \dot{q} \) it is not, see Figure A2.6(c). The displacement variables \( q \) and \( z \) are coincident for \( q \leq \Delta \). Due to the elimination of the impact condition, the impact states are not explicitly recorded during simulation.

The results obtained from the three simulation approaches discussed in this section can be seen in Figure A2.7. Observing the phase-portrait Figure A2.7(a) one can see that the results are overlapped, except during contact, where Zhuravlev’s response (red dashed line) has an additional velocity jump after impact and is not parallel to the \( \Delta \)-line. In the frequency domain Figure A2.7(b), the three methods produce similar results, being overlapped. Using Ivanov and Zhuravlev transformations prior to numeric integration reduces the simulation time by 80 and 70 % respectively.

Figure A2.6: Numerical simulation using Ivanov transformation showing physical and transformed coordinates. Position and velocity in time(a, b) and phase-portrait (c).
A2.4 Analytical Techniques

A2.4.1 Averaging and Zhuravlev’s transformation

To obtain approximate steady-state periodic solutions for the vibro-impacting oscillator given by (A2.1) and (A2.2) one can combine Zhuravlev’s transformation (A2.4) with standard averaging. This section briefly describes this process, which is detailed discussed in [32, 34, 74].

The coordinate transformed system is the same as (A2.7) and (A2.8) and is shown again here to facilitate reading:

\[
\ddot{z} + 2\beta\dot{z} + z = \text{sgn}(z) \left( \Delta - B\Omega^2 \sin(\Omega t) \right), \quad \text{if } z \neq 0 \tag{A2.16}
\]

\[
\dot{z}_+ - \dot{z}_- = -(1 - R)\dot{z}_-, \quad \text{if } z = 0 \tag{A2.17}
\]

Admitting that the impacts are almost elastic \((R \approx 1)\) and that the damping ratio \(\beta\), gap width \(\Delta\) and forcing amplitude \(B\) are small, one can apply the Van der Pol transformation \((z = Q\sin(\varphi(t))\) and \(\dot{z} = Q\cos(\varphi(t)), \ Q = Q(t) > 0\) in (A2.16) and (A2.17). After some mathematical manipulations, one can show that single-sided vibro-impact systems such as (A2.16) and (A2.17) are under resonance when excited at even multiples of their fundamental natural frequency, \(\Omega \approx 2k\), for \(k = 1, 2, 3, \ldots\). Also, one can obtain the following quadratic polynomial in \(Q\), representing the frequency-amplitude

Figure A2.7: Results obtained using direct numerical simulation and together with Zhuravlev and Ivanov transformations. (a) Phase-portrait for \(\Omega = 2\) (b) and frequency response.
relationship, [74]:

\[ P(Q) = g_2 Q^2 + g_1 Q + g_0 = 0 \]  \hspace{1cm} (A2.18)

where:

\[ g_2 = \pi^2(\sigma_k^2 + \beta_k^2), \quad g_1 = 4\pi \Delta \sigma_k, \]  \hspace{1cm} (A2.19)

\[ g_0 = 4\left(\Delta^2 - (B_k \Omega^2)^2\right), \quad B_k = \frac{B}{4k^2 - 1} \]  \hspace{1cm} (A2.20)

\[ \sigma_k = \frac{\Omega}{2k} - 1, \quad \beta_k = \beta_{\text{eff}}/2, \quad \beta_{\text{eff}} = \beta + \frac{1 - R}{\pi}. \]  \hspace{1cm} (A2.21)

The detuning parameter \( \sigma_k \) indicates how close the forcing frequency \( \Omega \) is to the resonances \( 2k \). The term \( \beta_{\text{eff}} \) represents the system’s effective damping, which is a combination of \( \beta \) and \( R \). The phase is given by:

\[ \varphi(t) = \frac{\Omega t}{2k} + \theta, \]  \hspace{1cm} (A2.22)

\[ \theta = \arctan\left(\frac{\pi \sigma_k Q + 2\Delta}{-\pi \beta_k Q}\right) \frac{1}{2k}. \]  \hspace{1cm} (A2.23)

As \( g_2 \) is positive, \( P(Q) \) has a unique global minimum point at:

\[ Q^* = -\frac{g_1}{2g_2} = -\frac{2\sigma_k \Delta}{\pi^2(\sigma_k^2 + \beta_k^2)}. \]  \hspace{1cm} (A2.24)

As the amplitude \( Q \) is positive by definition, (A2.24) implies that \( \sigma_k \Delta < 0 \). So, as pointed out in [32], if the system is tightened (\( \Delta < 0 \)), stable solutions should be seen after crossing resonance, where \( \sigma_k > 0 \). Similarly, if the gap is positive, one should expect stable solutions before crossing resonances. One can show that \( Q^* \) is also a lower stability boundary for the amplitude variable through eigenvalue analysis of the amplitude-phase \( (Q, \theta) \) system of ODE’s generated during averaging analysis [74].

A stability boundary can be defined in terms of the gap width \( \Delta \) as well. Analyzing the discriminant of (A2.18), which is given by:

\[ D[P(Q)] = g_1^2 - 4g_0g_2 = h_2\Delta^2 + h_0, \]  \hspace{1cm} (A2.25)

with:

\[ h_2 = -(4\pi \beta_k)^2, \quad h_0 = (4\pi B_k \Omega^2)^2(\sigma_k^2 + \beta_k^2). \]  \hspace{1cm} (A2.26)

To obtain real-valued amplitude solutions, \( D[P(Q)] \) should be positive, therefore the gap’s absolute value \( |\Delta| \) cannot exceed the following boundary value:

\[ \Delta_{\text{MAX}} = \sqrt{\frac{h_0}{-h_2}} = B_k \Omega^2 \sqrt{\left(\frac{\sigma_k}{\beta_k}\right)^2 + 1}. \]  \hspace{1cm} (A2.27)
As $B_k$ is positive, $\Delta_{MAX}$ is also positive, having a point of minimum at resonance $\Omega = 2k$. The advantage of defining the stability boundary in terms of the gap instead of amplitude is that while $Q$ is a value to be found, $\Delta$ is a design parameter in some applications. Finally, one could obtain a stability boundary using the forcing frequency $\Omega$ and (A2.25), but the resulting $\Omega$-polynomial will be of fourth degree, implying more complex expressions.

The equivalence between stability boundaries for amplitude and gap width is shown in Figure A2.8(b,c) for $\Delta = 0.7 \times 10^{-3}$. In Figure A2.8(a), forward and backward numeric sweeps are able to follow only the upper branch of the frequency-response curve obtained using averaging, indicating the stability of this branch and agreeing with the stability boundary. Based on (A2.4), the amplitude $Q$ is obtained by subtracting the minimum steady state value from $\Delta$, i.e., $Q = \Delta - \min(q_{ss})$, where $q_{ss}$ is the steady-state response.

![Figure A2.8: (a) Frequency response curve and minimum amplitude for $\beta = 15 \times 10^{-3}$ and $\Delta = 0.7 \times 10^{-3}$, (b) zoom on (a) and (c) gap boundary value.](image)

Despite being not possible to obtain expressions like (A2.24) and (A2.27) for more complex systems, it is still possible to use the parameter relations to get qualitative insight about the stability boundaries. For instance, the presence of $\beta_k$ on the denominator of (A2.24) and (A2.27), indicates that decreasing the effective damping increases both stability thresholds. At first, this sound counter-intuitive since adding damping usually improves a system’s stability characteristics, but as lower damping increases the response amplitude, it seems reasonable that its stability boundary expands as well. The influence of damping on frequency response can be seen by comparing Figures A2.8 and A2.9 for $\beta = 15 \times 10^{-3}$ and $1 \times 10^{-3}$ respectively. As expected both amplitude and its stability boundaries increase for smaller damping.
The influence of $\Delta$ and $R$ on the frequency response is shown in Figure A2.10. In Figure A2.10(a), one can see that the location of the peak of amplitude response moves from after $\Omega = 2$ to before this value as the gap width $\Delta$ passes from tight (negative) to loose (positive gap) configuration. The maximum amplitude is also proportional to the coefficient of restitution, Figure A2.10(b), due to the decrease in energy dissipation due to impacts for higher $R$.

One of the main conditions for using the results discussed in this section is that the impacts are almost elastic ($R \approx 1$). The extent of this condition is analyzed in Figure A2.11 for different values of $R$, where one can see that the agreement between averaging and numerical simulations decrease with the coefficient of restitution, having a 16% difference in amplitude for $R = 0.6$ at $\Omega = 2$. 

Figure A2.9: (a) Frequency response curve and minimum amplitude for $\beta = 1 \times 10^{-3}$ and $\Delta = 0.7 \times 10^{-3}$, (b) zoom of (a), and (c) gap boundary value.
A2.4 Analytical Techniques

Figure A2.10: Influence of (a) gap width $\Delta$ and (b) coefficient of restitution $R$ on the frequency response using averaging.

Figure A2.11: Comparison of numeric and averaging frequency responses for (a) $R = 0.6$, (b) $R = 0.7$, (c) $R = 0.8$ and (d) $R = 0.9$.

A2.4.2 Pointwise mapping

Another way to obtain solutions for the vibro-impact model (A2.1) and (A2.2), is to use the impact condition (A2.2) to link the linear solutions of non-impacting regimes,
given by:

\[ q(t) = e^{-\beta t} \left( A_1(X_0) \cos(\omega_D t) + A_2(X_0) \sin(\omega_D t) \right) + C \sin(\Omega t + \theta), \quad \text{if} \quad q < \Delta, \]  

(A2.28)

where \( X_0 = (t_0, q_0, \dot{q}_0) \) and \( \omega_D = \sqrt{1 - \beta^2} \) represent the initial conditions and damped natural frequency for non-impacting configuration respectively.

The post-impact state from a previous collision \((t_\pm^{(i-1)}, \dot{q}_\pm^{(i-1)})\) can be used as an initial condition for (A2.28). Then, the next collision instant \( t_\pm^{(i)} \) can be obtained by applying a root-finding algorithm to \( q(t_\pm^{(i)}) = \Delta \). From that, the collision velocity \( \dot{q}_\pm^{(i)} \) can be found by substituting \( t_\pm^{(i)} \) on the derivative of (A2.28). The next post-impact velocity \( \dot{q}_\pm^{(i+1)} \) can be found by using the impact relation (A2.2) restarting the cycle, as illustrated in Figure A2.12.

Some drawbacks of this approach are that it:

- does not provide any expression for the system’s resonances;
- is not purely analytical, requiring a root-finding algorithm to work;
cannot be used if other nonlinearities are also present;

An exception to the last item is given by (A2.7), which is linear for \( z \) strictly positive/negative, enabling the application of pointwise mapping to (A2.7) and (A2.8). In this case, the solution would be similar to (A2.28) with an additional term to account for the particular solution to \( \Delta \text{sgn}(z) \).

The procedure described above was used by Shaw and Holmes [19] to derive a two-dimensional discrete map of the pre-contact instant and velocity, \( (t^{(i+1)}, \dot{q}^{(i+1)}) \). Given the need for root-finding techniques, it is not possible to obtain an analytical formula for this map. However, it is possible to analyze the stability of its equilibrium states using the map’s Jacobian, which can be obtained analytically by deriving the linear solutions of the non-impacting configuration with respect to its initial conditions.

Considering that \( t^{(i)}_+ \approx t^{(i)}_+ \) and \( \dot{q}^{(i)}_+ = -R\dot{q}^{(i)}_- \), the Jacobian’s determinant and trace are given by:

\[
\det(J) = -R\frac{\dot{q}^{(i-1)}_+}{\dot{q}^{(i-1)}_-} \exp(-2\beta S \tau), \tag{A2.29}
\]

\[
\text{tr}(J) = -\frac{\exp(-\beta S \tau)}{\omega_D \dot{q}^{(i+1)}_-} \left( (N + R\beta \dot{q}^{(i)}_+) \sin(\omega_D \tau) + \omega_D (\dot{q}^{(i-1)}_+ - R\dot{q}^{(i)}_-) \cos(\omega_D \tau) \right), \tag{A2.30}
\]

\[
\tau = t^{(i)}_- - t^{(i-1)}_-, \tag{A2.31}
\]

\[
N = \gamma \tilde{V} \Omega^2 \sin(\Omega t^{(i-1)}_+) - \beta S \dot{q}^{(i-1)}_+ - \Delta, \tag{A2.32}
\]

where \( \tau \) represents the non-contacting period. To analyze the stability of period-\( n \) oscillations, Shaw and Holmes [19] assumed that:

\[
t^{(i)}_- = t^{(i-1)}_- + nT, \quad \dot{q}^{(i-1)}_- = -R\dot{q}^{(i-1)}_-, \tag{A2.33}
\]

\[
\dot{q}^{(i)}_- = \dot{q}^{(i-1)}_-, \quad T = \frac{2\pi}{\Omega}, \tag{A2.34}
\]

where \( T \) is the period. With those assumptions (A2.29) and (A2.30) become:

\[
\det(J) = R^2 \exp(-\beta s n T), \tag{A2.35}
\]

\[
\text{tr}(J) = \frac{\exp(-\beta s n T)}{\omega_D \dot{q}^{(i)}_-} \left( (R\beta \dot{q}^{(i)}_- + N) \sin(\omega_D n T) - 2\omega_D R\dot{q}^{(i)}_- \cos(\omega_D n T) \right), \tag{A2.36}
\]

\[
N = \gamma \tilde{V} \Omega^2 \sin(\Omega t^{(i-1)}_-) + R\beta \dot{q}^{(i)}_- - \Delta, \tag{A2.37}
\]

The frequency response for \( \Delta = 0.5 \times 10^{-3} \) obtained using numerical simulation and pointwise mapping is shown in Figure A2.13(a). Its boxed region is shown in detail in Figure A2.13(b), together with marks to indicate instability of period-1 oscillations. The stability was accessed using both full and simplified versions of the Jacobian’s determinant and trace, given by (A2.29) and (A2.30) and (A2.35) and (A2.36), respectively.
In this figure, one can see the full version of the Jacobian’s determinant and trace is able to identify more unstable orbits than its simplified version. Also, only two orbits were considered unstable by both methods and the oscillations at some frequencies were considered to be period-1 stable but presented non-periodic behavior, see the phase-portrait for $\Omega = 2.35$ on Figure A2.13(c). Despite that, both approaches identify the beginning of instability at virtually the same frequency. The averaging analysis made in Subsection A2.4.1 is valid only in a small neighborhood of $\Omega = 2$, being not able to identify the unstable region, which started for $\Omega \approx 2.3$. On the other hand, the stability equations provided by averaging analysis are more compact than those provided by pointwise mapping.

Figure A2.13: (a) FRC for $\Delta = 0.5 \times 10^{-3}$. (b) Zoom at the boxed region shown in (a) indicating how the instability was accessed. (c) Phase-portrait at $\Omega = 2.35$.

Observing Figure A2.13(a) one can see that pointwise mapping gives the same quantitative results as numerical simulations for period-1 orbits. Both approaches also produce equivalent qualitative results for more complex behavior, as shown in Figure A2.13(c).

This approach can be used to piecewise linear systems as well. In this case, the transition between $t^{(i)}$ and $t^{(i)}_+$ is made by using the state at $t^{(i)}$ as an initial condition to the other linear system in contact configuration ($q \geq \Delta$). The solution for $q(t^{(i)}_+) = \Delta$ provides the final contact instant $t^{(i)}_+$, which should be substituted on the time derivative of the contact configuration linear solution to find $\dot{q}^{(i)}_+$. Finally, the state at $t^{(i)}_+$ is used as
A2.5 Experimental setup

The experimental setup has been described previously in [66, 67, A1] and is shown in Figure A2.15. In Figure A2.15(a, b) an electrodynamic shaker (1) [B&K® 4808] is
used to apply a harmonic excitation to a platform (2), containing a cantilever beam with attached mass and a stop (5) located at the left of the mass, to restrain its lateral movement. The shaker is driven by a power amplifier [B&K® 2712] and is connected to the platform by a stinger. The displacement of both platform and attached mass are measured by inductive sensors (3) [Pulsotronic® KJ4-M12MN50-ANU] and filtered by a Brickwall® filter [Wavetek 752A] hardware with low pass cut-off frequency of 500 Hz. Data acquisition is performed at 5 kHz sampling frequency using a dSPACE® DS1104 R&D controller board.

The impacting beam can be seen in detail in Figure A2.15(c,d), where \( U \) represents the beam’s transversal displacement, \( L_S = 100 \text{ mm} \) is the displacement’s sensor axial location, \( L_m = 132 \text{ mm} \) is the mass axial location, \( L = 180 \text{ mm} \) is the beam’s length, \( \Delta_C \) is the gap width and \( b(t) \) is the base excitation. The attached mass \( m \) weights 0.2 kg and it consists of 4 steel discs with 7 mm thickness and 35 mm diameter hold together with bolts.

To avoid saturating the displacement sensors, measurement and impact locations \( L_S \) and \( L_m \) are not coincident. Thus the measured grazing width \( \Delta_G \) will differ from the gap width \( \Delta_C \) due to beam deflection, see Figure A2.16. The measured grazing amplitude will be used to define the model’s impact rule later on.

The setup’s main natural frequencies are identified using experimental modal analysis with an accelerometer [B&K® 4397] and an impact hammer [B&K® 8206]. Excitations were applied at 10 mm, 30 mm, 50 mm, 70 mm, 90 mm, 132 mm and 160 mm from the beam’s clamped end and measured at 90 mm from the beam’s clamped end. Using a sampling frequency of 512 Hz, the measured data is post-processed using B&K® PULSE frequency analyzer. The system’s frequency response for an excitation at 50 mm from the beam’s clamped end can be seen in Figure A2.17, where one can identify four natural frequencies: 3.25 Hz, 8 Hz, 67 Hz and 98.75 Hz obtained with a resolution of 0.25 Hz. At 3.25 Hz only the platform oscillates. The beam’s first and second bending resonances occur at 8 Hz and 98.75 Hz respectively, with a torsional resonance at 67 Hz.

From literature [19, 56, 74], it is known that SDOF single-sided vibro-impact systems present resonant behavior around even multiples of their natural frequency. Based on that the frequency range of interest in this work is lies between 14 Hz to 17 Hz. Different gap configurations are also analyzed: pre-stressed \(-0.1 \text{ mm}\), loose \(0.1 \text{ mm}\), and neutral \(0 \text{ mm}\).

A compact view of this analysis is shown in Figure A2.18 for different gap configurations: pre-stressed with \( \Delta_G \approx -0.1 \text{ mm} \) in Figure A2.18(a); neutral with \( \Delta_G \approx 0 \text{ mm} \) in Figure A2.18(b) and loose with \( \Delta_G \approx 0.1 \text{ mm} \) in Figure A2.18(c). There one can see an alternation between period one and two oscillations with the blank regions presenting more complex behavior, such as quasi-periodicity and chaos. The concentration of period-1 orbits before and after twice the system’s fundamental resonance \( (2f_n \approx 15.5 \text{ Hz}) \) for loose and pre-stressed configurations respectively is in accordance with the predictions made by [32] using (A2.24) in Subsection A2.4.1. The pre-stressed configuration in Figure A2.18(a), demands a higher level of excitation in order to overcome the initial deformation. Period two oscillations are mainly present between 15 Hz.
Figure A2.15: Experimental setup (a,c) and its schematic representation (b,d).
A2.5 Experimental setup

Figure A2.16: Deflection of a cantilever beam.

Figure A2.17: Frequency response: (a) amplitude, (b) coherence.

to 15.5 Hz.

The experimental frequency response for period one orbits can be seen in Figure A2.19, where the different excitation amplitudes and gap configurations are organized as follows:
excitation amplitude from 0.2 V to 0.4 V can be seen in Figure A2.19(c,e), while excitation amplitude from 0.5 V to 0.7 V can be seen in Figure A2.19(a,b,d). The tight configuration with ∆G ≈ −0.1 mm can be seen in Figure A2.19(a), the neutral configuration with ∆G ≈ 0 mm can be seen in Figure A2.19(b,c), and the loose configuration with ∆G ≈ 0.1 mm can be seen in Figure A2.19(d,e). In this figure, it is not possible to identify an amplitude pattern for tightened configuration, shown in Figure A2.19(a). The frequency response for neutral configuration Figure A2.19(b,c) is similar to the linear frequency response curve of SDOF systems with resonance around 15.5 Hz to 15.6 Hz. For loose configuration Figure A2.19(d,e), the amplitude is almost constant between 14 Hz to 15 Hz, after that, it decreases, passing by a state of complex oscillations until there are no more vibro-impact oscillations.

The harmonic decomposition of period one orbits is shown in Figure A2.20(a) to (c), for tight, neutral and loose configurations, respectively. There one can see that the fundamental harmonic 1Ω is the one with higher amplitude. The static component 0Ω also has high amplitude because the oscillations are asymmetric with respect to U = 0. For neutral configuration Figure A2.20(b) one can see resonance-like curves for 0Ω and 1Ω with a peak close to 2f_n.

The different gap values should be understood as nominal measures because intermittent impacts during a frequency sweep and wear on the contacting surfaces can modify the initial gap width. Although not well studied, this is very common in engineering practice. This variation according to the forcing frequency is shown in Figure A2.21, where the mass amplitude and measured gap width are shown in Figure A2.21(a, b), (c, d) and (e, f) for tight, neutral and loose gap configurations, respectively. In this figure, it is possible to correlate some gap variations with the frequency response, such as the
A2.5 Experimental setup

Figure A2.19: Experimental frequency response for different excitation levels 0.2 V to 0.7 V and gap configurations: (a) pre-stressed $-0.1 \text{ mm}$, (b,c) neutral 0 mm and (d,e) loose 0.1 mm.

Figure A2.20: Harmonic decomposition of period one orbits for $\tilde{V} = 0.6 \text{ V}$ and different gap configurations: (a) pre-stressed $-0.1 \text{ mm}$, (b) neutral 0 mm and (c) loose 0.1 mm.
pattern similarity for $|U(L_S)|$ and $\Delta G$ around 16 Hz on Figure A2.21(a, b) and the discontinuity of the FRC and the sudden increase of $\Delta G$ around 15 Hz on Figure A2.21(c, d). The actual gap width was measured as the maximum value of vibrations for impacting orbits.

Figure A2.21: Period-1 frequency response (a, c, e) and gap variation (b, d, f). Excitation: 0.6 V.

The results from measurements made at the neutral configuration and excitation level 0.6 V can be seen in Figures A2.22 and A2.23 showing the different dynamic behaviors in the neighborhood of a period-doubling bifurcation. The stroboscopic map $S_T$ has one fixed point before bifurcation and two after that. Chaotic behavior is observed in the experimental data shown in Figure A2.24 for pre-stressed conditions and 0.6 V. In this case the stroboscopic map has no equilibrium points. The velocity was obtained from the displacement data using central finite differences.

It is not possible to ensure that the contact location will remain unchanged during the tests with different gap configurations, due to imperfections on the setup assembling, and on the surfaces of both the attached mass and constraint. Rotational vibrations of the attached rigid mass can occur due to eccentric collisions between the constraint and the mass, which has a significant moment of inertia. However, the amplitudes of the corresponding higher modes are generally much smaller than that of the beams first bending mode. The relevance of modal interactions for the present setup can be determined with an analysis of its vibrations in the frequency domain, as shown in Figures A2.20 and A2.22 to A2.24, where only harmonics and sub-harmonics of the forcing frequency can be identified. This indicates that the higher natural frequencies determined by experimental modal analysis do not seem to present a significant role for the vibro-impact response, thus justifying modeling the beam-mass setup as a SDOF system with a point mass.
Figure A2.22: (a) Time series, (b) Frequency spectrum (c) phase-portrait.

Figure A2.23: (a) Time series, (b) Frequency spectrum (c) phase-portrait.
Figure A2.24: (a) Time series, (b) Frequency spectrum (c) phase-portrait.
A2.5 Experimental setup

A2.5.1 Coefficient of restitution

The most common way of obtaining the coefficient of restitution is to pick the velocities before and after impact from an experimental time series and divide them ($R = -\dot{U}_+ / \dot{U}_-$). In general, the choice of $\dot{U}_\pm$ is made by picking up the initial and final points of intervals where velocity present an abrupt variation, changing its sign.

Figure A2.25: Displacement (a) and velocity (b) for $\Delta_G \approx 0$ mm, $\tilde{V} = 0.6$ V at $\tilde{\Omega} = 15.6$ Hz.

This approach was used on the first three impacts of period-one orbits, such as the one in Figure A2.25, resulting in Figure A2.26, which contains the value of $R$ for tight (a, b), neutral (c, d) and loose (e, f) gap configurations as a function of forcing frequency (a, c, e) and pre-impact velocity (b, d, f). In this figure, each dot represents the mean value of $R$ and/or $\dot{U}(L_S, \tilde{t}_-)$ for the three impacts at each forcing frequency.

In Figure A2.26(a) one can see that $R$ presents a piecewise linear growing pattern with the forcing frequency around 16 Hz and 16.5 Hz. Similar behavior can be seen in Figure A2.26(c) for $R$ around 15.5 Hz. Also, $R$ presents a decreasing pattern for $\dot{U}_-$ from 0 to 30 mm s$^{-1}$, see Figure A2.26(d, f), which turns into a growing pattern for higher $\dot{U}_-$, see Figure A2.26(d). The mean and standard deviation of $R$ for each gap are shown in Table A2.1.

While the forcing frequency is a pre-defined parameter, the pre-contact velocity is influenced by both forcing frequency and gap configuration. That explains the dependence of the coefficient of restitution to the gap width shown in Table A2.1, where the mean value of $R$ decrease as its standard deviation grows together with the gap width. Alternatively, the gap dependence could indicate that the contact deformations are not
Figure A2.26: Coefficient of restitution for tight $\Delta_G \approx -0.1\,\text{mm}$ (a, b), neutral $\Delta_G \approx 0\,\text{mm}$ (c, d) and loose $\Delta_G \approx 0.1\,\text{mm}$ (e, f) gap configurations as a function of forcing frequency (a, c, e) and pre-impact velocity (b, d, f). $\tilde{V} = 0.6\,\text{V}$.

The uncertainty level of the coefficient of restitution and pre-impact velocity is influenced by various factors, namely:

- Deviations on the gap width during operation, see Figure A2.10;
- Inherent mounting errors make it difficult to ensure identical impact conditions when the constraint location is changed;
- The velocity is not measured directly, being obtained through central finite differences from noise-contaminated position measurements;
- The pre- and post-contact velocities are chosen by visual observation of the time series, a task that can be difficult due to the presence of high-frequency components (HFC) as shown in Figure A2.27 for the velocity;
Table A2.1: Coefficient of restitution against gap width

<table>
<thead>
<tr>
<th>$\Delta G$ [mm]</th>
<th>$R$ mean</th>
<th>std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.80</td>
<td>0.08</td>
</tr>
<tr>
<td>0</td>
<td>0.72</td>
<td>0.11</td>
</tr>
<tr>
<td>+0.1</td>
<td>0.59</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Observing the velocity’s frequency spectrum, shown in Figure A2.27(b), only harmonics of the forcing frequency have noticeable amplitude, while the setup’s resonances (8 Hz, 67 Hz, and 98 Hz) obtained in Section A2.5 are not prominent.

However, it is possible to observe the velocity’s HFC in the time domain, see Figure A2.27(c). Removing the low-frequency component of the velocity between impacts and using the average period between peaks, one obtains the signal’s frequency as 103 Hz. Looking at the velocity frequency spectrum in Figure A2.27(b) it is possible to observe a small peak at this frequency. It is also possible to notice a small peak after 100 Hz in Figure A2.17, that shows the results of the modal analysis. The low-frequency velocity in Figure A2.27(a) was obtained through the optimization of the numeric response to match experimental data. The time series and frequency spectrum for the displacement associated with the velocity data shown in Figure A2.27 can be seen in Figure A2.22.

One can estimate the vibration amplitude of the HFC by considering that the velocity’s amplitude is equal to the displacement amplitude times its oscillation frequency. The mean peak velocity amplitude of the HFC is $35 \text{ mm s}^{-1}$ and its frequency is approximately 103 Hz, resulting in a displacement amplitude of 0.05 mm, which is more than 10 times smaller than the low-frequency displacement amplitude. Also, this component is not activated in the same way during the whole frequency and gap width domains, as one can see in Figure A2.25(b).
Figure A2.27: (a) Velocity time series for $\Delta G \approx 0$ mm, $\tilde{V} = 0.6$ V, $\tilde{\Omega} = 15$ Hz. (b) Frequency spectrum for velocity. (c) High-frequency component of velocity at the boxed region shown in (a).

### A2.5.2 Platform dynamics

The influence of the impacting dynamics on the platform can be identified by comparing the frequency spectrum of colliding mass and platform shown in Figure A2.28(c, d). Also, the platform acts as a low-pass filter on the excitation provided by the shaker, delivering reduced stimulation to the beam, as shown in Figure A2.29, where its frequency response between 6 Hz and 9 Hz is at a higher level than at 14 Hz and 17 Hz. Its variation between 7.5 Hz and 8 Hz is due to beam’s resonance. This happens because the electrodynamic shaker is not feedback controlled. Thus, the chosen system’s external excitation source is the single harmonic electric signal sent to the shaker’s power amplifier. From that, one can write the system’s platform displacement excitation as:

$$ b(\tilde{t}) = \tilde{\gamma} \tilde{V} \sin(\tilde{\Omega} \tilde{t}), $$

(A2.38)

where $\tilde{V}$ is the voltage amplitude in V and $\tilde{\gamma}$ is the unit conversion gain in m V$^{-1}$.

The presence of a high-frequency component in the experimental data, together with the attenuating characteristics of the platform can be seen as motivation to update the mathematical model, adding more degrees of freedom, however, that would also make the use of techniques such as averaging and pointwise mapping impractical. Since the high-

110
A2.5 Experimental setup

frequency component (103 Hz) is higher than the second and third beam’s resonances 67 Hz and 98 Hz, the inclusion of their corresponding modes on the model is not well-supported. Finally, incorporating the platform dynamics does not directly affect the impact behavior.

Figure A2.29: Platform response for $\bar{V} = 0.1$ V around 1$\times$ and 2$\times$ resonance $f_n = 7.6$ Hz.
A2.6 Model for the experimental setup

The setup’s model is equivalent to the one presented in Section A2.2, making it possible to investigate the occurrence of the predictions made in Sections A2.3 and A2.4 in experiments. Also, it is similar to the ones proposed by Elmegård et al. [47] and Rebouças et al. [A1]. In [47] the impact was modeled using a piecewise relation for the spatial discretization of the equations of motion, while [A1] modeled the impact as a force, normalizing the transverse displacement $U(X, \tilde{t})$ with the grazing amplitude $\Delta_C$. This would be a problem for the current analysis, which considers single-sided impacts whose gap can be null, leading to infinite normalized transverse vibrations. To avoid that, the beam length $L$ was used as a normalizing parameter.

In the schematic representation of the forced impacting beam in Figure A2.15, $L_m$, $L_S$ and $L$ represent the axial location of the attached mass, displacement sensor, and beam length, respectively, $\Delta_C$ represents the gap width. The stop has the same axial location of the attached mass, being placed on its right side. Using Bernoulli-Euler beam theory, one can write its equation of motion for transverse displacement $U(X, \tilde{t})$ as [47, A1]:

$$\left(\rho A + m\delta(X - L_m)\right)\ddot{U} + D E I \dot{U}''' + E I U''' = \left(\rho A + m\delta(X - L_m)\right)\Omega^2 \gamma \tilde{V} \sin(\tilde{\Omega} \tilde{t}), \quad \text{if} \quad U_m < \Delta_C, \tag{A2.39}$$

$$U(0, \tilde{t}) = U'(0, \tilde{t}) = U''(L, \tilde{t}) = U'''(L, \tilde{t}) = 0, \tag{A2.40}$$

where $\dot{} \equiv \partial(\cdot)/\partial \tilde{t}$, $(\cdot)' \equiv \partial(\cdot)/\partial X$, $\rho$, $D$, $E$, $A$ and $I$ represent the beam’s density, damping proportional to stiffness, elasticity modulus, cross-sectional area, and cross-section area moment of inertia, respectively, $m$ is the attached mass, $\delta(\cdot)$ is Dirac’s delta function and $U_m = U(L_m, \tilde{t})$ is the transverse displacement of the attached mass. Impact occurs when the mass reaches the stop, in which case:

$$U(L_m, \tilde{t}_-) = \Delta_C \Rightarrow \begin{cases} U(L_m, \tilde{t}_+) = U(L_m, \tilde{t}_-), \\ \dot{U}(L_m, \tilde{t}_+) = -R \dot{U}(L_m, \tilde{t}_-), \end{cases} \tag{A2.41}$$

where $R$ is the coefficient of restitution, $\tilde{t}_-$ and $\tilde{t}_+$ represent instants before and after impact, respectively. Following the same steps taken in [A1], namely:

- Normalize transversal displacement and axial position: $U = uL$ and $X = xL$;
- Assume that the attached mass is much larger than the beams: $m \gg \rho AL$;
- Perform spatial discretization:

$$u(x, \tilde{t}) = \sum_{i=1}^{N} \phi_i(x) q_i(\tilde{t}), \tag{A2.42}$$

where the assumed modes $\phi_i(x)$ satisfy all essential boundary conditions and $q_i(\tilde{t})$ are modal coordinates;
Neglect higher modes: \( N = 1 \);

One arrives at:

\[
\ddot{q} + 2\beta_S \dot{q} + q = \gamma \tilde{V} \Omega^2 \sin(\Omega t), \quad \text{if} \quad q < \Delta,
\]

(A2.43)

where:

\[
\beta_S = \frac{1}{2} D \omega_n, \quad \Omega = \tilde{\Omega} / \omega_n, \quad t = \omega_n \tilde{t},
\]

(A2.44)

\[
\Delta = \Delta_G / L, \quad \omega_n^2 = K_S / M_S, \quad M_S = m \phi(x_m)^2, \quad \gamma = \frac{\gamma}{\phi(x_m) L},
\]

(A2.45)

are the structural damping ratio, normalized forcing frequency, normalized time, normalized gap width, squared fundamental linear natural frequency, equivalent structural mass, equivalent structural stiffness and normalized unit conversion gain. The assumed mode \( \phi(\cdot) \) is chosen as the static deformation pattern of a cantilever beam loaded at \( L_m \), normalized such that \( \phi(L_S) = 1 \). Also, from now on \( \dot{(\cdot)} \equiv \partial(\cdot) / \partial t \). Applying the same steps to the impact condition, (A2.2), one arrives at:

\[
\dot{q}(t_+) = -R \dot{q}(t_-) \quad \text{and} \quad q(t_+) = q(t_-) \quad \text{if} \quad q(t_-) = \Delta,
\]

(A2.47)

where \( t_+ \) and \( t_- \) are the normalized instants after and before impact, respectively.

Vibro-impact oscillations measured from the experimental setup can be related to (A2.43) by multiplying the model solution by the measured grazing amplitude and by the first mode shape at the measurement location, \( U(L_S, \tilde{t}) = q(\omega_n \tilde{t}) L \phi(L_S / L) \). This relation can be simplified by normalizing the static deformation pattern \( \phi(\cdot) \) with the measurement location, \( L_S \). By doing so, the measured vibrations become proportional to the beam’s length, i.e., \( U(L_S, \tilde{t}) = q(\omega_n \tilde{t}) L \).

**A2.6.1 Linear parameter estimation**

The system’s structural damping ratio obtained through the decay envelope of the unforced SDOF response, such as the one shown in Figure A2.30(a) together with its model prediction, indicating good agreement between them. So, from now on \( \beta_S = 17 \times 10^{-3} \). The frequency spectrum of the unforced response, shown in Figure A2.30(b) gives \( f_n = 7.75 \) Hz, while the frequency sweep around this value, as shown in Figure A2.30(c) indicates that \( f_n = 7.6 \) Hz, which is the value of the system’s fundamental natural frequency from now on.

The normalized unit conversion gain \( \gamma = 1.5 \times 10^{-3} V^{-1} \) was obtained by fitting the experimental data shown in Figure A2.30(c) around \( f_n \) to its theoretical estimative obtained with (A2.43). Given the attenuating characteristic of the platform, c.f. Subsection A2.5.2, it may be necessary to modify this parameter during vibro-impact analysis (around \( 2f_n \)).
A2.7 Comparing Experimental and Numerical results

Remembering the attenuating characteristic of the platform, one can find a new value for $\gamma$ by fitting the frequency-amplitude obtained by numerical simulation to the experimental data shown previously. Using the mean values of $R$ for each gap width, given in Table A2.1, one obtains the results shown in Figure A2.31 with $\gamma = 3.78 \times 10^{-4}$, $5.16 \times 10^{-4}$ and $7.66 \times 10^{-4}$ respectively, which corresponds to 25, 34 and 51% of the initial value, obtained in Subsection A2.6.1 for measurements around resonance.

On tightened configuration Figure A2.31(a), the experimental amplitude does not have a clear pattern, making it difficult to associate with numerical results. At neutral configuration Figure A2.31(b), the numerical results present a reasonable agreement with the experimental ones. For loose configuration Figure A2.31(c), the numerical frequency response is close to the experimental one for $\Omega \geq 2$, being able to predict the end of period-1 orbits before $\Omega = 2.2$, and the beginning of non-periodic behavior.
A2.8 Conclusions

The experimental vibro-impact behavior of a cantilever beam with unilateral impacts and base-excitation was presented considering different excitation levels and gap configurations (tight, neutral and loose). Analysis of the experimental data showed different qualitative behaviors near vibro-impact resonance and the effect of gap variation during a frequency sweep on the setup’s amplitude response. Since the experiment’s parameter identification and analysis were made in different frequency ranges, it was possible to show the challenges involved in tuning its vibro-impact model. The use of non-smooth transformations on numerical simulations and analytical schemes was discussed, showing the differences between Zhuravlev and Ivanov’s substitutions and among averaging and pointwise mapping.

It was possible to obtain an agreement between numerical and experimental results for some gap configurations and frequency ranges. In the time domain, numerical and analytical predictions were able to match experimental data despite the occasional pres-
ence of high-frequency oscillations. Investigations on the coefficient of restitution $R$ showed its dependence on the pre-impact velocity and forcing frequency, presenting a linear behavior at some ranges.

Motivated by the work of Grace et al. [41], Zhuravlev and Ivanov transformations were used as pre- and post-processing steps on the numerical simulation of a model for a SDOF oscillator with unilateral impacts and coefficient of restitution. Comparison with direct numerical simulation showed that those transformations can decrease computation time without compromising the results. Ivanov’s approach provided the smallest simulation time because it eliminated the impact condition. Results obtained with Zhuravlev’s coordinates were different from the direct simulation for the velocity variable during impacts. When compared to Ivanov, Zhuravlev’s transformation has a clear physical interpretation, with a straightforward derivation.

Stability boundaries in terms of the amplitude of oscillations and gap width were obtained through averaging analysis of the Zhuravlev transformed SDOF model. Interpretation of parametric relations from perturbation methods can be useful in the analysis of empirical systems that do not satisfy some of the theoretical assumptions. Despite the mismatch between numerical and experimental results showed in this work, the concentration of period-1 orbits according to the gap sign (before/after impact resonance if the gap is positive/negative) agreed with predictions made by Zhuravlev [32] using averaging analysis for a SDOF model.

The frequency responses obtained with pointwise mapping and numerical simulation showed an unstable region for period-1 orbits which was not identified by the averaging analysis. With pointwise mapping, it was possible to obtain two stability equations, one for generic orbits and a simplified one for period-$n$ behavior. Both formulas were able to identify the instability threshold, with the general formula being able to identify more unstable orbits than its simplified version. However, both methods could not identify instability at some frequencies.

The experimental analysis carried out in this manuscript could be improved by finer tuning of the gap width, followed by a further empirical exploration of the vibro-impact behavior in tightened configuration. A feedback controlled shaker would diminish the concerns created by the attenuating characteristics of the platform. Alternatively, one could propose another model for the excitation source and/or an appropriate way to insert the high-frequency component on the model. Further contributions can be made through the application of the methods presented here for systems with bilateral impacts and rigid stops. In this case, the response magnitude is limited by the clearance width, and the response’s frequency becomes the output of interest.

This work presents a structured discussion of common techniques for analysis of unilateral vibro-impact problems, discussing the common analytical and numerical schemes. Despite experimental challenges, using the coefficient of restitution it was possible to reproduce empirical observations to a certain extent, validating analytical predictions.
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On the influence of contact stiffness in unilateral vibro-impact

As of the writing of this Ph.D. thesis, the following article is under preparation and its content can suffer modifications before its submission to a peer-reviewed journal. The article’s highlights are:

• Discussion of the most common model formulations to analyze soft and hard impacts;

• Derivation of different equivalence relations between the coefficient of restitution and the parameters from the piecewise linear contact model;

• Experimental investigation of the contact stiffness with different springs;
Abstract

The influence of the contact stiffness on the steady-state response of unilateral vibro-impact systems is analyzed experimentally using four different compression helical springs. Equivalence relations between the coefficient of restitution and parameters of the piecewise linear contact force model are established and analyzed. Experimental observations around one and twice the setup’s fundamental linear natural frequency are used to investigate the validity of the results provided by both model formulations and their equivalence relations.

Keywords: vibro-impact; contact stiffness; piecewise linear contact force; coefficient of restitution; model comparison;

A3.1 Introduction

To obtain an appropriate model for a physical system one should identify its relevant characteristics and formulate model assumptions and mathematical expressions. For impacting systems, the contact duration and deformation play a significant role on the choice of the appropriate model. When both contact deformation and duration are negligible with respect to the amplitude and period of oscillations the contacting parts are treated as rigid bodies suffering hard impacts where the collisions can be adequately modeled using the coefficient of restitution, CoR. For softer impacts, one needs to model collisions with finite-time duration and non-zero deformation using contact forces.

The objective and main originality of the present work is to investigate the influence of contact stiffness in unilateral vibro-impact using experimental observations and theoretical predictions. The CoR and the piecewise linear contact force models are used to model hard and soft impacts, respectively. These models were chosen because of their simplicity and popularity on the vibro-impact literature despite their inaccuracies and limitations. Equivalence relations for both impact models are obtained using a base excited single-degree of freedom system with unilateral constraint.

Blazejczyk-Okolewska et. al. [11] used the model of a ball bouncing on a fender to derive parametric relations between the CoR and the contact’s stiffness and damping, performing numerical comparisons of the outcomes of both impact types. This analysis stimulated the development of the present work, which contains the derivation of different parametric relations between the CoR and contact’s damping and stiffness. Helical springs with different rates are used to investigate experimentally the influence of the contact stiffness on the steady-state response of an unilateral vibro-impact setup.
A3.2 One phenomenon and two modeling approaches

Contact forces and the coefficient of restitution are two of the most common approaches to model VI problems. The application of both formulations to a classical linear oscillator time-normalized using its linear natural frequency is represented in Figure A3.1. For contact force models [A1], the equations of motion are:

$$\ddot{q} + 2\beta \dot{q} + q + f_C(q, \dot{q}) = B\Omega^2 \sin(\Omega t), \tag{A3.1}$$

and for the coefficient of restitution [A2]:

$$\ddot{q} + 2\beta \dot{q} + q = B\Omega^2 \sin(\Omega t), \quad \text{if} \quad q < \Delta, \tag{A3.2}$$

$$q(t_+) = q(t_-), \quad \dot{q}(t_+) = -R\dot{q}(t_-), \quad \text{if} \quad q = \Delta, \tag{A3.3}$$

where $q(t) = y(t) - b(t)$ is the relative displacement between the inertial coordinates $y(t)$ and base excitation $b(t) = B \sin(\Omega t)$. Model parameters for both cases are: $\beta$ is the damping ratio, $f_C(q, \dot{q})$ is the impact force, $B$ and $\Omega$ are the forcing amplitude and frequency, respectively, $R$ is the coefficient of restitution, $\Delta$ is the gap width, and $t_+$ and $t_-$ are the time instants after and before impact, respectively. From now on terms such as $q(t_\pm)$ will be represented by $q_\pm$.

While the CoR ignores the contact dynamics, having a single straightforward description, there are various ways of defining contact forces, with some of them using the CoR as a parameter. One of the most common models for impact forces is given by a piecewise linear function, such as:

$$f_C(q, \dot{q}) = \begin{cases} 
0, & \text{if} \quad q < \Delta \\
\omega_R^2(q - \Delta) + 2\beta_C\omega_R\dot{q}, & \text{if} \quad q \geq \Delta 
\end{cases} \tag{A3.4}$$

where $\beta_C$ is the contact damping ratio and $\omega_R$ is the ratio between contact and linear natural frequencies. To study the contact phase, one can substitute (A3.4) in (A3.1), obtaining:

$$\ddot{q} + 2\beta \dot{q} + q + \omega_R^2(q - \Delta) + 2\beta_C\omega_R\dot{q} = B\Omega^2 \sin(\Omega t),$$

$$\ddot{q} + 2\xi w\dot{q} + w^2 q = P + B\Omega^2 \sin(\Omega t), \quad \text{if} \quad q \geq \Delta, \tag{A3.5}$$

$$w^2 = \omega_R^2 + 1, \quad \xi = \frac{\beta + \omega_R\beta_C}{w}, \quad P = \omega_R^2 \Delta, \tag{A3.6}$$
where $w$, $\xi$, and $P$ are the equivalent natural frequency, damping ratio and step forcing for contact configuration, respectively.

The next subsection shows some ways of manipulating (A3.5) to obtain relations between contact force parameters and the coefficient of restitution.

### A3.2.1 Equivalence relations

Shaw and Holmes [19] obtains an expression for the CoR by solving (A3.5) supposing a very high contact natural frequency ($w \to +\infty$), obtaining:

$$q(\tau) = \Delta + \dot{q}_-(1 - \xi(\tau - \tau_0)) \sin(\tau - \tau_0) + O(w^{-2}), \quad (A3.7)$$

where $\tau = wt$ is the rescaled time coordinate and $\dot{q}_- = \dot{q}(\tau_0)$ is the initial velocity before impact. Making (A3.7) equal to the gap width $\Delta$, one finds the contact duration to be $\tau_C = \pi$. Substituting this expression on the derivative of (A3.7):

$$\dot{q}(\tau_C + \tau_0) = \dot{q}_+ = -\dot{q}_-(1 - \xi \pi),$$

$$R_1 = 1 - \pi \xi. \quad (A3.8)$$

Another expression for $R$ was obtained by Blazejczyk-Okolewska et al. [11], through energy analysis of a modified version of the bouncing ball problem where the ground is replaced by a spring-damper massless support. The differences between the model analyzed in this article and [11] are the presence of restoring and dissipative forces in the non-contacting phase and external excitation on both phases. Following their approach, the energy dissipated during an impact is given by:

$$E_D = -2\xi w \int_0^t [\dot{q}(\tau)]^2 \, d\tau \quad (A3.9)$$

where $q(\cdot)$ is the solution of (A3.5) for $B = 0$ and $q(0) = \Delta$, $\dot{q}(0) = \dot{q}_-$. Considering the contact duration, $\tau_C$, to be equal to half of the damped natural period of (A3.5), one arrives at:

$$E_D = -\frac{1}{2} \left( \dot{q}_-^2 + \Delta^2 \right) \left( 1 - \exp \left( \frac{-2\pi \xi}{\sqrt{1-\xi^2}} \right) \right) \quad (A3.10)$$

Considering the energy after impact to be equal to the energy before contact $E_0$ plus the impact dissipation, one obtains:

$$R_2 = \sqrt{1 + \frac{E_D}{E_0}}, \quad (A3.11)$$

$$E_0 = \frac{1}{2} \Delta^2 + \frac{1}{2} \dot{q}_-^2 (1 - 2\beta), \quad (A3.12)$$

For steady-state period-one oscillations, the pre-contact velocity should be constant, and so $R_2$. Variations in $R_2$ could occur during transient or chaotic oscillations. If $\Delta = 0, R$
becomes velocity independent and given by:

\[
R_3 = \sqrt{\exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right) - 2\beta} \quad (A3.13)
\]

This expression can be further simplified by considering a small equivalent damping ratio \(\xi\):

\[
R_4 = 1 - \frac{\pi\xi}{1-2\beta} + O(\xi^2), \quad (A3.14)
\]

which becomes equivalent to \((A3.8)\) for \(\beta \ll 1\).

Another relation between the coefficient of restitution and the piecewise linear force is given by evaluating the solution derivative of \((A3.5)\) for \(B = 0\) and \(q(0) = \Delta, \dot{q}(0) = \dot{q}_-\) at half of its damped natural period, which is the assumed contact duration in \([11]\). The result is:

\[
\dot{q}_+ = -\dot{q}_- \exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right),
\]

\[
R_5 = \exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right), \quad (A3.15)
\]

considering the equivalent damping ratio to be small one obtains the formula derived by Shaw and Holmes \([19]\), given by \((A3.8)\). It is possible to obtain expressions for the contact deformation by finding out the maximum of \(q\) during contact configuration, but the expressions are cumbersome.

All of the expressions for \(R_i\) depend indirectly on contact damping ratio and stiffness through the term \(\xi\). The energy-based analysis shows the influence of gap width and pre-contact velocity, which are supported by empirical observations, such as in \([18, A2]\). The assumptions used to obtain the expression for \(R_i\) are:

- \(R_1\): high contact stiffness \((w \to +\infty)\);
- \(R_2\) and \(R_5\): contact duration is half of the damped natural period \((\tau_C = \pi/w)\);
- \(R_3\): gap width negligible \((\Delta = 0)\);
- \(R_4\): small equivalent damping ratio \((\xi \ll 1)\);

**A3.2.2 Parameter Analysis**

The influence of contact parameters on the different expressions for the CoR can be seen in Figure A3.2(a-e), with the contact damping ratio \(\beta_C\) being the most influential on \(R\), while the contact duration \(\tau_C\) is more sensitive to natural frequency ratio \(\omega_R\), see Figure A3.2(f). The similarity of curves for \(R_2\) and \(R_5\) indicates that gap width and
pre-contact velocity have a small influence on $R_i$. The curves for $R_1$ and $R_4$ are also similar because $\beta = 15 \times 10^{-3} \ll 1$. The values of gap width and pre-contact velocity are: $\Delta = 0.5 \times 10^{-3}$ and $\dot{q}_- = 0.01$.

Comparing CoR and contact duration, one can see that while it is possible to obtain near-elastic CoR for low values of $\beta_C$ and $\omega_R$, small impact duration, which is the main assumption for CoR modeling, demands high natural frequency ratio, $\omega_R$. With the exception of (A3.8), due to Shaw and Holmes [19], none of the expressions derived above explicitly demanded that.

The sensitivity of $R_2$ (A3.11) due to variations on linear damping ratio $\beta$ can be seen in Figure A3.3, where lower $R_2$ are more sensitive. Influence of natural frequency ratio $\omega_R$ on $R_2$ grows when the gap width $\Delta$ increases as shown in Figure A3.4, or for small contact velocities as can be seen in Figure A3.5.

Time domain comparison of the different expressions for the coefficient of restitution (blue dashed lines) against piecewise-linear formulation (red lines) using the models (A3.2) and (A3.3) and (A3.1) can be seen in Figure A3.6(a, e) for $\beta_C = 0.03$ and $\omega_R = 150$, resulting in collisions with $R_{1-5} = 0.9$ and small contact duration $\tau_C = 0.02$. Other parameters are $B = 0.5 \times 10^{-3}$, $\Omega = 2$, $\Delta = 0.5 \times 10^{-3}$ and $\beta = 15 \times 10^{-3}$. In this case, all expressions for $R_i$ give the same result, being able to replicate piecewise linear response.

Simulations results with large contact duration, $\tau_C = 0.22$, are shown in Figure A3.6(f,
A3.2 One phenomenon and two modeling approaches

Figure A3.3: Variation of $R_2$ (A3.11) due to different levels of linear damping ratio $\beta$ equal to (a) $1 \times 10^{-3}$, (b) $10 \times 10^{-3}$, (c) $15 \times 10^{-3}$ and (d) $40 \times 10^{-3}$.

Figure A3.4: Variation of $R_2$ (A3.11) due to different levels of gap width $\Delta$ equal to: (a) $1 \times 10^{-3}$, (b) $2 \times 10^{-3}$, (c) $4 \times 10^{-3}$ and (d) $8 \times 10^{-3}$.
Figure A3.5: Variation of $R_2$ (A3.11) due to different levels of pre-contact velocity $\dot{q}_-$ equal to: (a) 0, (b) $2 \times 10^{-3}$, (c) $4 \times 10^{-3}$ and (d) $10 \times 10^{-3}$.

j) for $10\beta_C$, $\omega_R/10$ and $R_{1-5} = 0.03$ to 0.38. The agreement between CoR and piecewise linear contact force, in this case, is explained by the small deformations during impact. Despite the variations in $R_i$, the responses are similar between to the ones obtained with piecewise linear contact force, especially those for $R_1$ and $R_4$, see Figure A3.6(f, i).
A3.3 Analytical treatment

The analytical tools described here have been presented in [A1, A2], being briefly discussed ahead to facilitate reading.

A3.3.1 Harmonic Linearization

In this method, the solution of nonlinear models such as (A3.1) is considered to be single-periodic, i.e. \( q(t) = Q \sin \varphi(t) \) with \( \varphi(t) = \Omega t + \theta \), and its nonlinear terms are replaced by an equivalent spring-dashpot subsystem, with amplitude-dependent coefficients. The model’s frequency response is obtained by substituting the equivalent spring-dashpot subsystem into the equations of motion. For (A3.1), one has:

\[
Q^2 \left( \left( \Omega^2 - 1 - \kappa \right)^2 + 4\Omega^2 (\beta + \sigma)^2 \right) = (B\Omega^2)^2, \tag{A3.16}
\]
where:

\[ f_C(q, \dot{q}) \approx \kappa q + 2\sigma \dot{q}, \quad (A3.17) \]

\[ \kappa = \frac{1}{\pi Q} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \sin \varphi \, d\varphi, \quad (A3.18) \]

\[ \sigma = \frac{1}{2\pi Q \Omega} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \cos \varphi \, d\varphi. \quad (A3.19) \]

The harmonic linearization coefficients \( \kappa \) and \( \sigma \) are obtained from the first harmonic of the Fourier series of the nonlinear term \( f_C(q, \dot{q}) \). The integration interval on \((A3.18)\) and \((A3.19)\) is modified because the piecewise linear contact force is only active when \( q \geq \Delta \). Defining the grazing phase as \( \varphi_0 = \arcsin(\Delta/Q) \), the new interval should be \([\varphi_0; \pi - \varphi_0]\). This leads to:

\[ \kappa = \frac{1}{2} \omega_R^2 W(Q), \quad (A3.20) \]

\[ \sigma = \frac{1}{2} \beta_C \omega_R W(Q), \quad (A3.21) \]

\[ W(Q) = 1 - \frac{1}{\pi} \left( 2 \varphi_0 + \sin(2\varphi_0) \right). \quad (A3.22) \]

In \([A1]\), it was shown that this approach produces the same solution as the standard Averaging method \([27, 28]\), eliminating its intermediate steps, but without providing any information about the stability of solutions.

### A3.3.2 Averaging with Zhuravlev transformation

This approach is used only to VI systems modeled using CoR formulation. With the following coordinate transformation \([32]\):

\[ q(t) = \Delta - |z(t)|, \quad (A3.23) \]

it is possible to decrease the velocity jump due to impacts. Substituting it on \((A3.2)\) and \((A3.3)\), leads to:

\[ \ddot{z} + 2\beta \dot{z} + z = \text{sgn}(z) \left( \Delta - B\Omega^2 \sin(\Omega t) \right), \quad \text{if} \quad z \neq 0, \quad (A3.24) \]

\[ \dot{z}_+ - \dot{z}_- = -(1 - R)\dot{z}_+, \quad \text{if} \quad z = 0. \quad (A3.25) \]

Assuming that damping ratio \( \beta \), gap width \( \Delta \) and forcing amplitude \( B \) are small and impacts are near elastic \((1 - R \approx 0)\), and applying Van der Pol’s transformation \((z = Q \sin \varphi(t), \dot{z} = Q \cos \varphi(t), Q = Q(t) > 0)\) followed by the averaging theorem, it is possible to show the occurrence of impact resonances when the system is forced around even multiples of its fundamental natural frequency, \( \Omega = 2k \), for \( k = 1, 2, 3, \ldots \) and to obtain the following frequency-amplitude relationship:

\[ P(Q) = g_2 Q^2 + g_1 Q + g_0 = 0, \quad (A3.26) \]
A3.4 Experimental Setup and its Mathematical Model

where:

\[ g_2 = \pi^2 (\sigma_k^2 + \beta_k^2), \quad g_1 = 4\pi \Delta \sigma_k, \quad (A3.27) \]
\[ g_0 = 4 \left( \Delta^2 - (B_k \Omega^2)^2 \right), \quad B_k = \frac{B}{4k^2 - 1}, \quad (A3.28) \]
\[ \sigma_k = \frac{\Omega}{2k} - 1, \quad \beta_k = \beta_{\text{eff}}/2k, \quad \beta_{\text{eff}} = \beta + \frac{1 - R}{\pi}. \quad (A3.29) \]

Analysis of the lower stability boundary for the amplitude shows that resonance should occur before (after) \( \Omega = 2k \) for loose (tight) systems [32]. The advantages of this approach are: to predict the existence of infinity impact resonances, providing an insight on the role of gap configuration on those resonances together with an estimative of vibration amplitude around these regions. The disadvantage is that this method is only valid for \( R \approx 1 \) in small regions around \( \Omega = 2k \), providing no information about oscillations near linear resonance, \( \Omega \approx 1 \).

A3.4 Experimental Setup and its Mathematical Model

The experimental setup in Figure A3.7(a) is used to test the influence of contact stiffness on VI oscillations and is similar to the one used in [A2]. Given the importance of this parameter to define the model impact type, four helical springs, see Figure A3.7(c), are used to obtain unilateral impacts with different duration and stiffness.

Helical springs were used to have better estimative of contact stiffness and its variations, using (A3.30) and (A3.31), instead of placing the spots somewhere alongside the beam, as done in [A1]. This also facilitated the analysis in tightened configuration around \( 2f_n \), avoiding non-impacting vibrations with permanent contact. The stiffness and geometrical properties of all springs can be found in Table A3.1. Spring 2 has a rectangular cross-section and the formula for its rate, \( k_2 \) (A3.31) was taken from [97].

\[ k_i = \frac{Gd^4}{8ND^3}, \quad i \neq 2, \quad (A3.30) \]
\[ k_2 = \frac{Gb^3h^3}{2.62ND^3(b^2 + h^2)} \quad (A3.31) \]

One can estimate the value of the beam’s structural stiffness using the lumped mass \( m = 0.21 \text{ kg} \) and the measured linear natural frequency, \( f_n = 7.6 \text{ Hz} \), obtaining \( K_S = 0.61 \text{kN m}^{-1} \). The contact natural frequency has been predicted using the rate of each spring \( k_i \) and the beam’s lumped mass, \( m \). With that, one can obtain the contact duration as half of the contact natural period according to [11, 19]. The coefficient of restitution is obtained according to (A3.8), proposed by [19].

In Table A3.1, one can see that the estimated contact duration for spring \( k_1 \) is less than 5\% of the system’s fundamental natural period, \( T_n = 132 \text{ ms} \), justifying the use of the coefficient of restitution to represent the collisions against this spring. The values for
the CoR in Table A3.1 are close to unity because they were calculated without taking the contact damping ratio into account, $\beta_C \equiv 0$.

The mathematical model of the experimental setup discussed in [A2] will be used here for both contact formulations. Using Bernoulli-Euler theory, the governing equation for the beam shown in Figure A3.7(a) is obtained. The beam length $L$ is used to normalize transverse displacement $U(X, \tilde{t})$ and axial position $X$. Then, assuming that the lumped mass is much larger than the beam’s mass ($m \gg \rho AL$), the continuous
Table A3.1: Properties of helical springs shown in Figure A3.7(c).

<table>
<thead>
<tr>
<th>Spring n.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire cross-sec. dimens., [mm]</td>
<td>d = 6</td>
<td>b = 5; h = 3</td>
<td>d = 3</td>
<td>d = 2</td>
</tr>
<tr>
<td>Coil diameter, D [mm]</td>
<td>25</td>
<td>24</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>Number of turns, N [-]</td>
<td>6</td>
<td>10</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>Shear modulus, G [GPa]</td>
<td>80</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density, ρ [kg m⁻³]</td>
<td>7800</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spring rate, kᵢ [kN m⁻¹]</td>
<td>138.2</td>
<td>21.9</td>
<td>8.6</td>
<td>1.5</td>
</tr>
<tr>
<td>Contact natural frequency, [Hz]</td>
<td>129</td>
<td>51</td>
<td>32</td>
<td>14</td>
</tr>
<tr>
<td>Natural frequency ratio, ωᵣ [-]</td>
<td>17</td>
<td>6.8</td>
<td>4.2</td>
<td>1.8</td>
</tr>
<tr>
<td>Contact duration, τᵣ [ms]</td>
<td>3.9</td>
<td>9.7</td>
<td>15.5</td>
<td>36.7</td>
</tr>
<tr>
<td>τᵣ/Tₙ, [%]</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>28</td>
</tr>
<tr>
<td>Rᵢ, [-]</td>
<td>0.998</td>
<td>0.996</td>
<td>0.993</td>
<td>0.985</td>
</tr>
</tbody>
</table>

The model is discretized and time-normalized. Neglecting the influence of higher modes, one obtains the following SDOF model:

\[
\ddot{q} + 2\beta_S \dot{q} + q = \gamma \tilde{V} \Omega^2 \sin(\Omega t), \quad \text{if} \quad q < \Delta \tag{A3.32}
\]

\[
q(t_+) = q(t_-), \quad \dot{q}(t_+) = -R \dot{q}(t_-), \quad \text{if} \quad q = \Delta, \tag{A3.33}
\]

for CoR formulation and:

\[
\ddot{q} + 2\beta_S \dot{q} + q + f_C(q, \dot{q}) = \gamma \tilde{V} \Omega^2 \sin(\Omega t) \tag{A3.34}
\]

for piecewise linear impact formulation, where:

\[
f_C(q, \dot{q}) = \begin{cases} 
0, & \text{if} \quad q < \Delta \\
\phi_C^2 (\omega_R^2 (q - \Delta) + 2\beta_C \omega_R \dot{q}), & \text{if} \quad q \geq \Delta 
\end{cases} \tag{A3.35}
\]

and

\[
f_C(q, \dot{q}) = \frac{1}{m \omega_n^2 \Omega \phi_C} \tilde{f}_C(\phi_C L q, \phi_C \omega_n \dot{q}), \tag{A3.36}
\]

\[
\beta_C = \frac{D_C}{2 M_S \omega_C}, \quad \omega_C^2 = K_C/M_S, \quad \omega_R = \omega_C/\omega_n, \tag{A3.37}
\]

\[
\beta_S = \frac{1}{2} D \omega_n, \quad \Omega = \bar{\Omega}/\omega_n, \quad t = \omega_n \bar{t}, \tag{A3.38}
\]

\[
\Delta = \Delta_G/L, \quad \omega_n^2 = K_S/M_S, \quad M_S = m \phi_C^2, \tag{A3.39}
\]

\[
K_S = \frac{EI}{L^3} \int_0^1 (\phi''(x))^2 \, dx, \quad \gamma = \frac{\gamma}{\phi_C L}, \quad \phi_C = \phi(x_C) = \frac{\Delta_C}{\Delta_G}. \tag{A3.40}
\]

Those parameters have been discussed in [A1] and [A2]. The geometric factor \( \phi_C \) accounts for the different axial locations of impact and measurement. It can be easily
included on the expressions derived in Sections A3.2 and A3.3 for piecewise linear impact force by replacing $\omega_R^2$ and $\beta C \omega_R$ by $\phi_C^2 \omega_R^2$ and $\phi_C \beta_C \omega_R$, respectively.

The relation between experimental transverse oscillations $U(X, \tilde{t})$ measured at axial location $X = L_S$ and the solutions for models described in this section is given by $U(L_S, \tilde{t}) = q(\omega_n t)L$.

A3.5 Experimental Observations

Two frequency ranges are examined, one around the fundamental linear resonance $f_n = 7.6$ Hz and the other around twice this value. While the VI oscillations around $f_n$ occur with a loose gap, $\Delta G = 0.4$ mm, the ones around $2f_n = 15.2$ Hz occur with a tightened one, $\Delta G = -0.1$ mm.

Information about periodicity of solutions can be obtained by observing the states of a system at every period $T$, revealing the existence of complex behavior if the periodic states do not converge. This can be seen in Figure A3.8 for the bifurcation diagrams of each spring at different frequency ranges. In this figure, one can see that complex behavior ceases to exist as the springs get softer, from $k_1$ to $k_4$. Complex behavior was not observed in [A1] due to the use of control-based continuation [66, 67].

![Bifurcation diagram for: (a, e) $k_1$, (b, f) $k_2$, (c, g) $k_3$, (d, h) $k_4$. $\Delta G = 0.4$ mm for (a-d) and $\Delta G = -0.1$ mm for (e-h). $T = 1/f$.](image)

In face of such complex data, one could use relations between maximum and minimum displacements as a condensed measure of vibration amplitude. Remembering that in [A2], vibrations around $2f_n$ were represented with a folded single-harmonic sinusoidal
A3.5 Experimental Observations

function, see (A3.23), varying between $\Delta - Q$ and $\Delta$, one can obtain the vibration amplitude $Q$ as the difference between maximum and minimum displacements. Similarly, as the frequency response around $f_n$ in [A1] was reasonably described using the assumption of a single-harmonic sinusoidal solution, varying between $\pm Q$, one could define the response amplitude around this frequency range as half of the difference between maximum and minimum displacements. After doing that, one obtains the frequency responses for different helical springs shown in Figure A3.9.

![Figure A3.9: Experimental frequency response around: (a) linear resonance $\Delta_G = 0.4 \text{ mm}$ and (b) twice linear resonance $\Delta_G = -0.1 \text{ mm}$.](image)

In general, the response amplitude for twice resonance is considerably larger than around the natural frequency. Decreasing the stiffness, from $k_1$ to $k_4$, makes the bent peak’s curvature around $f_n$ to grow. It also moves the peak location from after $2f_n$ to before that. This agrees with the discussion made in [A2] because softer springs allow higher contact deformation, $\delta$, which decrease the effective gap width. Besides $k_4$, all of the springs are able to produce the impact resonance with a peak around $2f_n$.

Figure A3.10 shows the contact deformation, duration, and CoR at some frequencies. There, one can see that contact deformations tend to increase with frequency around resonance, but it is similar for all springs around $2f_n$, being of the same order of magnitude as the gap width, $\Delta_G = -0.1 \text{ mm}$, see Figure A3.10(a, d). Contact duration does not present noticeable differences with respect to forcing frequency. The coefficient of restitution is defined as the negative ratio between the velocities at end and beginning of contact. This parameter is close to unity for most of the springs around $2f_n$, but presents considerable uncertainty level around resonance, see Figure A3.10(c, d).
Comparison between the measured and estimated contact duration, whose values are presented in Table A3.2, shows reasonable numerical agreement for $k_1$ to $k_3$. With respect to the CoR, springs $k_3$ and $k_4$ present the closest numerical agreement. The inclusion of contact damping ratio $\beta_C$ on the estimation of $R_1$ should decrease its value, making it closer to empirical data.

### Table A3.2: Comparison of contact duration and CoR

<table>
<thead>
<tr>
<th>Spring n.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_C$, [ms] (estimated)</td>
<td>3.9</td>
<td>9.7</td>
<td>15.5</td>
<td>36.7</td>
</tr>
<tr>
<td>$\tau_C$, [ms] (meas. $\approx f_n$)</td>
<td>3.3</td>
<td>10.2</td>
<td>7.4</td>
<td>26.2</td>
</tr>
<tr>
<td>$\tau_C$, [ms] (meas. $\approx 2f_n$)</td>
<td>2</td>
<td>11.4</td>
<td>14.7</td>
<td>27</td>
</tr>
<tr>
<td>$R_1$ [-] (estimated)</td>
<td>0.998</td>
<td>0.996</td>
<td>0.993</td>
<td>0.985</td>
</tr>
<tr>
<td>$R$ [-] (meas. $\approx f_n$)</td>
<td>0.77</td>
<td>0.72</td>
<td>0.91</td>
<td>0.97</td>
</tr>
<tr>
<td>$R$ [-] (meas. $\approx 2f_n$)</td>
<td>0.71</td>
<td>0.88</td>
<td>0.94</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Observation of experimental, numerical and analytical results around $f_n$ in Figure A3.11 shows that numerical simulations using piecewise linear force are able to replicate the bent peak for all springs, while simulations using the CoR fail to replicate this behavior for the soft spring $k_4$, see Figure A3.11(d). While numerical results are not able to reproduce the full impact range, results from harmonic linearization overesti-
mate this range. The values used on the simulations are $\beta_s = 17 \times 10^{-3}$, $\gamma \tilde{V} = 3 \times 10^{-4}$, $\beta_c = 0$. The natural frequency ratio and CoR for each spring can be found in Table A3.1.

![Comparison between experimental, numerical and analytical frequency responses for different springs around resonance. (a) $k_1$, (b) $k_2$, (c) $k_3$, (d) $k_4$.](image)

Similar results for the setup around twice resonance can be seen in Figure A3.12, where one can see that numerical simulations using piecewise linear contact force are closer to the experimental frequency response. Numerical simulation using the CoR show reasonable agreement with the empirical observations for $k_1$, but it fails to do so for the other springs, which are softer. As the gap width has influence over the peak location on the impact resonance, around $2f_n$, this parameter was modified, taking into account the measured contact deformation shown in Figure A3.10(a, d) to have the same order of magnitude as the nominal gap. So the gap width used for each spring is $\Delta_G = -88 \times 10^{-3}$ mm, $-29 \times 10^{-3}$ mm, $21 \times 10^{-3}$ mm and $20 \times 10^{-3}$ mm, respectively. The other parameters remain the same.

An evidence for the influence of the frequency range on the accordance between theoretical and experimental results is the fact that analytical/numerical predictions are closer to the experimental data around $f_n$ than for the data around $2f_n$, see Figures A3.11 and A3.12.
A3.6 Conclusions

This work discussed the different connections that can be established between the coefficient of restitution and piecewise linear contact force. Parametric analysis showed that although the contact damping ratio has more influence on the CoR, the natural frequency ratio governs the contact duration, which should be small in order to have an agreement between formulations, as discussed by [11]. From five different expressions derived for $R$, only the one due to Shaw and Holmes [19] explicitly demanded that. Alternatively, an example with considerable contact duration but small contact deformation also presented a reasonable agreement between formulations.

From the experimental point of view, four helical springs were used to modify contact characteristics. Bifurcation diagrams for each spring around once and twice the setup’s natural frequency reveal the existence of complex behavior, whose incidence decreased together with springs stiffness. Using the rate of each spring and the beam’s lumped mass it was possible to estimate the contact duration and the CoR using Shaw and Holmes [19] formula. Confrontation with measurements showed reasonable agreement on the contact duration for all springs, while the estimated CoR was always higher than empirical results, agreeing only with the data for the two most soft springs.

The frequency response around resonance showed the influence of contact stiffness

Figure A3.12: Comparison between experimental, numerical and analytical frequency responses for different springs around twice resonance. (a) $k_1$, (b) $k_2$, (c) $k_3$, (d) $k_4$.  

A3.6 Conclusions
on the peak bending inclination. Numerical simulations using piecewise linear contact forces were able to reproduce this behavior, while simulations using the CoR failed to reproduce the response for the softest spring. Numerical-experimental comparisons around twice resonance showed that results from CoR formulation can replicate the response of the stiffest spring, but not for the others. On the other hand, numerical results using piecewise linear force models show reasonable agreement with empirical observations for all springs.

The results presented here showed the influence of the contact stiffness on the steady-state response of unilateral vibro-impact systems in different frequency ranges. The validity of a simple expression for the contact duration and the versatility of piecewise linear formulation in comparison to the CoR approach were analyzed experimentally, providing valuable insight on the applicability conditions of those models.

Acknowledgments

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The following extended abstract was accepted for oral presentation at the 9th European Nonlinear Dynamics Conference, ENOC 2017, held in Budapest, Hungary, from 25 to 30 of June 2017.
Experimental Validation of Vibro-Impact Force Models using Numeric Simulation and Perturbation Methods

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Department of Mechanical Engineering, Technical University of Denmark – DTU, Denmark

Summary. The frequency response of a single-degree of freedom vibro-impact oscillator is analysed using Harmonic Linearization, Averaging and Numeric Simulations considering two different impact force models, one given by a piecewise-linear function and other by a high-order polynomial. Experimental validation is carried out using control-based continuation to obtain the experimental frequency response, including its unstable branch.

Introduction

The objective and main originality of this work is to compare different impact force models using analytical, numeric and experimental techniques. The frequency response of a single-degree of freedom vibro-impact oscillator is analysed using Harmonic Linearization and Averaging considering two different impact force models, one given by a piecewise-linear function and other by a polynomial one, see [4]. Experimentally, control-based continuation [2, 3] is used to obtain the frequency response of an impacting beam, including its unstable branch. Numeric simulations are used to validate simple analytic approximations obtained by perturbation methods.

Experimental Setup

The experimental setup has been described previously by [2, 3], where a model-free controller was tuned for performing control-based continuation, obtaining the system’s frequency response and analysing the stability of the orbits found. It consists of a cantilever beam with lumped mass attached to a platform, which is connected to an electrodynamic shaker, see Fig. 1a. A pair of symmetrically situated stops restrain the lateral movement of the beam, and two electromagnetic actuators are placed on each side of the mass to execute control-based continuation.

The lumped mass dynamics dominates the oscillations in comparison to the flexible beam alone. The lumped mass is located with a reasonable distance from the stops with clearance, which can be considered as rigid supports, causing near-elastic impacts.

Mathematical Modeling

The Galerkin method is used to discretize a Bernoulli-Euler model for the test beam, giving a single DOF nondimensional model: $$\ddot{q} + 2\beta_s \dot{q} + q + F_I(q, \dot{q}) = \Omega^2 b \sin(\Omega t)$$, where $q$ is a modal displacement coordinate, $\beta_s$ is the structural damping coefficient, $b$ and $\Omega$ are the forcing amplitude and frequency. The amplitudes of modal oscillations and forcing excitations are normalized by the gap width, $\Delta = 1.6 \text{mm}$, and the forcing frequency is normalized by the system’s fundamental linear natural frequency, $f_n \approx 7.6 \text{Hz}$. In order to apply harmonic linearization and averaging, one can use the following assumptions: 1) damping, forcing and nonlinear terms are weak; and 2) a mono-frequency first order solution, i.e., $q(t) = Q \sin(\Omega t + \phi)$. The impact force $F_I(q, \dot{q})$ can be approximated by a polynomial, according to [4] or as a piecewise linear function, [1]. The impact force is a combination of restoring and dissipative forces. It can be approximated by a polynomial in the modal displacement, $F_I(q, \dot{q}) = f_s q^{2n-1} + 2\beta_I q^{n} \dot{q}$, where $f_s$ is the impacting restoring force coefficient and $\beta_I$ is the impacting damping. The piecewise linear impact force appears only when the displacement exceeds the gap, i.e. $|q| \geq 1$. If $q \geq 1$ the force is equal to $(\omega_I^2 - 1)(q - 1) + 2\beta_I q \dot{q}$ and $(\omega_I^2 - 1)(q + 1) + 2\beta_I q \dot{q}$ if $q \leq -1$, otherwise it is zero. Here, $\omega_I$ is the non-dimensional impacting linear natural frequency. In terms of model
tuning the piecewise-linear model contains fewer parameters than the polynomial one. Numerically, the discontinuity of the piecewise linear model makes its simulation more demanding. The same can be said for polynomial force with high values of $n$ and $p$.

**Results**

The aforementioned perturbation methods can be used to obtain the approximate frequency response of the impact oscillator as an implicit frequency-amplitude relationship. To obtain the numeric values of the model parameters, this relationships are fitted to the experimental data obtained by control-based continuation. The result is $\beta_I = 60 \times 10^{-3}$, $f_e = 0.12$, $n = 4$ and $p = 1$ for polynomial force model; $\omega_I = 4$ and $\beta_I = 0.16$ for piecewise linear force; and $\beta_S = 18 \times 10^{-3}$, $\delta = -0.14$ for structural damping and forcing amplitude. To check the validity of the assumptions made, the fitted values are used to obtain the frequency response of the system by numeric simulation. The results are shown in Fig. 2, where one can see that both force models give results reasonably close to experimental observations. Also, one can see that the perturbation methods are equivalent, giving the same frequency response. Thus, the assumptions made appears adequate. Comparing the force models, it is easy to see that the piecewise-linear model is able to predict the experimental behaviour more precisely, capturing both fold points, while the polynomial model fails to predict the upper folding point and has its lower fold point further away from experimental data. The precision of the piecewise linear model agrees with results found on literature, such as [5], where averaging was used to obtain the frequency response of a piecewise-linear isolator around resonance.

![Figure 2: Comparison of frequency responses.](image)

**Conclusions**

Two different impact force models (polynomial and piecewise-linear) were analysed and validated against experimental observations. Averaging and Harmonic Linearization were used to obtain the system’s frequency response and to tune the force models using experimental data. The results show that both perturbation methods provide equivalent results for the system under analysis and that the frequency-amplitude relationship obtained can be used to obtain numeric values for model parameters. Regarding the impact force models, the piecewise-linear force seems to describe the frequency response more accurately, predicting both folding points while the polynomial force predicts only one and not so accurately as the piecewise linear force.

Further experimental validation of other impact models, like kinematic impact with coefficient of restitution and analysis by nonsmooth transformations [6], can be relevant to compare different vibro-impact models in terms of ease of use, applicability and reliability.

**References**