DTU Library

Bridging Economic Theory Models and the Cointegrated Vector Autoregressive Model

Møller, Niels Framroze

Published in:
Economics: the Open Access, Open Assessment,E-journal

Publication date:
2008

Document Version
Publisher's PDF, also known as Version of record

Economics
The Open-Access, Open-Assessment E-Journal

Vol. 2, 2008-36 | December 16, 2008 | http://www.economics-ejournal.org/economics/journalarticles/2008-36

Bridging Economic Theory Models and the Cointegrated Vector Autoregressive Model

Niels Framroze Møller

University of Copenhagen

Abstract
Examples of simple economic theory models are analyzed as restrictions on the Cointegrated VAR (CVAR). This establishes a correspondence between basic economic concepts and the econometric concepts of the CVAR: The economic relations correspond to cointegrating vectors and exogeneity in the economic model is related to econometric concepts of exogeneity. The economic equilibrium corresponds to the so-called long-run value (Johansen 2005), the long-run impact matrix, *C;* captures the comparative statics and the exogenous variables are the common trends. The adjustment parameters of the CVAR are related to expectations formation, market clearing, nominal rigidities, etc. Finally, the general-partial equilibrium distinction is analyzed.

JEL: C32

Keywords: Cointegrated VAR, unit root approximation, economic theory models, expectations, Hybrid New Keynesian Phillips Curve, general equilibrium.

Special Issue "Using Econometrics for Assessing Economic Models"
Editor: Katarina Juselius
http://www.economics-ejournal.org/special-areas/special-issues

Correspondence: Niels Framroze Møller, Department of Economics, University of Copenhagen; Nørregade 10, DK-1165 Copenhagen K
E-mail: Niels.Framroze.Moller@econ.ku.dk

I would like to thank: Massimo Franchi, Søren Johansen, Katarina Juselius, Diana Framroze Møller, Bent Nielsen, Heino Bohn Nielsen, Paul Sharp, Christin Tuxen, two anonymous referees and participants at the Conference "The Cointegrated VAR model: Methods and Applications", Schæffergården, July 2006.

1 Introduction

The purpose of this paper is to facilitate the formulation of economic theory models as restrictions on the Cointegrated Vector Autoregressive (CVAR) model.

It is well-known that macroeconomic time series often exhibit persistence that can be modelled as the *integrated* type, I(1), which makes the CVAR the relevant econometric model (Granger 1981, Engle and Granger 1987, Johansen 1996). It is also well-known that, in spite of their diversity, most economic models involve the *same* basic concepts, e.g. behavioral relations, comparative statics, equilibrium conditions, the endogenous-exogenous dichotomy etc..

Given the purpose at hand, it therefore seems useful to relate such basic concepts of economic models to the statistical concepts of the CVAR, such as cointegrating relations, common trends, loadings matrix, etc. (Johansen 1996). To do this, I shall consider a few examples of simple economic theory models, and suggest how they translate into restrictions on a VAR, when the data can be approximated as being I(1).

To keep the exposition accessible I shall consider static- and simple dynamic theory models, rather than the "state-of-the-art" Dynamic Stochastic General Equilibrium (DSGE) model, since the fundamental assumptions are similar in form. Though simple, the models considered represent the basic form of a wealth of models, ranging from competitive static partial-/general equilibrium models, AS-AD models, Wage- and Price setting models, to more modern models like the Hybrid New Keynesian Phillips Curve model.

The methodological background is the Cointegrated VAR Methodology (Juselius 2006, Hoover, Juselius, and Johansen 2007). This implies, that theory models are viewed as sub-models embedded in a "larger" well-specified statistical model (Johansen 2006), here the *unrestricted* VAR, in which all variables are modelled (are endogenous) from the outset. Moreover, it means that the I(1)-, or unit root assumption is generally viewed as a *statistical approximation,* used to obtain reliable inference on relationships between persistent series. For most of the cases I am considering, the imposition of I(1) does not contradict the economic model as the order of integration is not implied by the latter.

In the next section, I summarize the basic concepts for the type of economic theory models considered. A simple supply- and demand model illustrates. The notion of persistence and the econometric concepts of the CVAR are then described briefly in Section 3. Section 4.1 collects the threads by suggesting a set of restrictions on a VAR, consistent with the simple static supply- and demand model from Section 2, *and* the persistence of the data cf. Section 3. This establishes a correspondence between the economic -, and statistical concepts, upon which I shall elaborate. A few generalizations of the empirical model (longer reaction time, lag length and simultaneous effects) under which this correspondence holds, are considered in Section 4.2. Given the well-known limitations of static theory models, two simple dynamic models are analyzed in Section 4.3: One is based on expectations of the exogenous variables, the other on expectations of the endogenous variables (the Hybrid New Keynesian Phillips Curve). As an extension of the basic framework, a general equilibrium example is analyzed in Section 4.4. Discussion and further generalizations are found in Section 5, while Section 6 concludes.

2 Basic Concepts of Economic Theory Models

The purpose of most economic models is to explain a set of variables, the *endogenous* variables, as a function of *exogenous* variables[1]. How the latter are generated is, by construction of the model, not explained. In the present paper *exogeneity* is referred to as "*economic exogeneity*", to distinguish it from the econometric concepts of weak- and strong exogeneity (Engle, Hendry, and Richard 1983): *A variable is economically exogenous if it is not influenced at any point in time by any other variable in the system under study, including other exogenous variables.*

The economic model contains *behavioral relations* for the endogenous variables. These may be plans *contingent* on either *observed outcomes*, or *expectations*, and may often be regarded as solutions to optimization problems.

An *equilibrium condition* is imposed to secure a solution with consistency between the plans of different agents, and hence with no inherent tendency for the system to change. If existent, this solution defines the *economic equilibrium.* I shall consider two types of economic equilibrium conditions: One for the static models stating that demand equals supply, the other for the dynamic models, stating that expectations are correct (see below)[2]. The usual motivation for both types is the *hypothetical* state in which there have been no shocks to any of the variables for a long time, and all previous shocks have propagated fully. In such a state it seems natural that demand and supply will be equal, since otherwise incentives to change variables (e.g. prices) would remain. Likewise, it would also seem natural that expectations will become correct (expected - and actual values coincide), since *sustained* expectational errors seem implausible in this static state.

To derive positive statements from theory models, the *comparative static analysis* is conducted: the study of the effects on the endogenous variables *in economic equilibrium* from hypothetical changes in the exogenous variables (Samuelson 1941).

A basic example, building on the above concepts is the static supply- and demand model,

$$Q^d = a_0 - a_1 P + a_2 W, \tag{1}$$

$$Q^s = b_0 + b_1 P - b_2 Z, \tag{2}$$

$$Q^s = Q^d, \tag{3}$$

where Q^d and Q^s are, respectively, demanded and supplied quantity, P, the price level, W, wage income and Z, the price of an input used in the production of Q. All parameters are positive, and all variables are in logarithms. The endogenous variables are Q^d, Q^s and P, while W and Z are economically exogenous. The equations (1) and (2), define the two behavioral relations, and (3) is the equilibrium condition.

[1]For more on the concepts of economic models and related issues see e.g Intriligator (1983).
[2]The latter may imply the former (see Section 4.3.1).

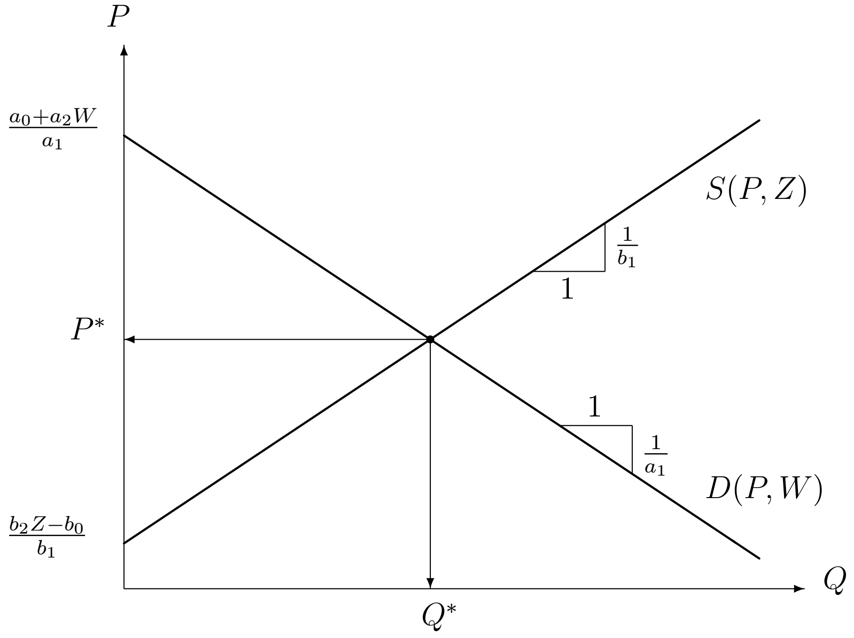

Figure 1: The Economic Cross of Supply and Demand.

The economic equilibrium is,

$$Q^* = \frac{b_1(a_0 + a_2W) + a_1(b_0 - b_2Z)}{a_1 + b_1}, \quad P^* = \frac{(a_0 + a_2W) - (b_0 - b_2Z)}{a_1 + b_1}, \tag{4}$$

with comparative static effects,

$$\frac{\partial Q^*}{\partial W} = \frac{b_1 a_2}{a_1 + b_1}, \ \frac{\partial Q^*}{\partial Z} = -\frac{a_1 b_2}{a_1 + b_1}, \ \frac{\partial P^*}{\partial W} = \frac{a_2}{a_1 + b_1}, \ \frac{\partial P^*}{\partial Z} = \frac{b_2}{a_1 + b_1}. \tag{5}$$

The model is illustrated as the famous economic cross in Figure 1, where $D(P, W)$ and $S(P, Z)$ denote the demand- and supply curves respectively. P is on the vertical axis and Q on the horizontal, following the convention in economics.

As is well-known, in order for the static model and its comparative statics to have any empirical relevance, equilibria must be attainable (stable), and the adjustment towards them cannot be too slow (see e.g. Chiang and Wainwright 2005). As these are dynamic properties, the static model can only *implicitly* assume that this is the case rather than explain it. This major limitation of static models in general, motivates the introduction of dynamic theory models, and I consider two examples in Section 4.3.

The purpose is now to suggest a set of restrictions on a VAR, consistent with simple theory models like the above, when data are persistent. However, first, a precise notion of persistence and the econometric tools of the CVAR analysis are needed.

3 The Persistence of Macroeconomic Data and the CVAR

The type of economic theory models under study can be written as sub-models of the general linear p - dimensional model,

$$Ax_t = B_1 x_{t-1} + ...B_k x_{t-k} + B_0 D_t + u_t, \tag{6}$$

where D_t is a $d \times 1$ term of d deterministic components, the initial values, $x_{1-k}, ...x_0$, are fixed, A has full rank and ones on the diagonal, $u_t \sim i.i.N(0, \Sigma)$, with $\Sigma > 0$ and diagonal, and B_i are unrestricted[3]. The corresponding *reduced form VAR(k)* model is,

$$x_t = \Pi_1 x_{t-1} + ...\Pi_k x_{t-k} + \Phi D_t + \varepsilon_t, \tag{7}$$

with $\varepsilon_t \equiv A^{-1} u_t$, $\Phi \equiv A^{-1} B_0$, and $\Pi_i = A^{-1} B_i$ for $i = 1, .., k$. This can be reparameterized in the Error- (or Equilibrium-) Correction-Mechanism form (ECM) as,

$$\Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \ \Delta x_{t-i} + \Phi D_t + \varepsilon_t, \tag{8}$$

where $\Pi \equiv \Sigma_{i=1}^{k} \Pi_i - I_p$ and $\Gamma_i \equiv -\Sigma_{j=i+1}^{k} \Pi_j$. For later, define $\Gamma \equiv I - \Sigma_{i=1}^{k-1} \Gamma_i$.

The dynamic properties are summarized in the *roots*, z, of the *characteristic equation* corresponding to (8),

$$\det(A(z)) = 0, \tag{9}$$

where,

$$A(z) \equiv (1 - z)I - \Pi z - \sum_{i=1}^{k-1} \Gamma_i \, (1 - z) z^i. \tag{10}$$

In practice, we typically have a *relatively short* sample of time series which gives a set of estimated roots, $\widehat{z}$, all with $|\widehat{z}| > 1$, but *some close to 1*, and where $|\cdot|$ denotes the modulus. I refer to such time series as being *persistent*. To conduct inference, assumptions about the underlying Data Generation Process (DGP) are needed, so that asymptotic distributions can be used as approximations of the unknown finite sample distributions of estimators and statistics. In this case, the choice is between assuming that all roots have $|z| > 1$, or, that some are at 1 while the rest have $|z| > 1$. Under the first assumption *asymptotics* are standard Gaussian based. However, when some roots are close to 1, as suggested by the estimates, the asymptotic distributions will be poor approximations for typical sample lengths, implying unreliable inference (See e.g. Johansen 2006). From a statistical inferential point of view, it is then probably more useful to impose $z = 1$ for some roots, as an approxi-

[3] For technical details and applications of the CVAR see Johansen (1996) and Juselius (2006) respectively.

mation, and use the corresponding asymptotic inference theory for cointegrated I(1) processes described in Johansen (1996), cf. the second assumption.

However, though a useful statistical approximation, this unit root restriction may, or may not, contradict the economic model. We can distinguish between three cases:

First, if the economic theory predicts unit roots, we of course impose them and continue the analysis, to find out whether these are generated in the manner according to the theory.

Second, the theory model may instead involve a steady state, implying a stationary VAR model. Given the persistence and the sample at hand, it is however, not possible to conduct inference on the steady state relations and multipliers to a satisfactory extent. As a result, the price of valid inference is that we are forced to give up the stationarity assumption of the model, hopefully in order to learn about other assumptions of the model (Møller and Sharp 2008). In this case, one would not necessarily claim that the evidence is inconsistent with the stationary "theory-VAR", but simply that inference under stationarity is unreliable.

Third, it may also be the case that the assumption of stationarity or non-stationarity - here the order of integration - is not implied by the theory model. This is the case for the static model and the simple examples of dynamic models considered here (see the discussion however): The level of the economically exogenous variables cause the level of the endogenous variables, that adjust passively. As a result, persistence in the system variables must originate from the generation of the former. As mentioned in Section 2, this is outside the theory model, implying that imposing $z = 1$, i.e. estimating a CVAR, is not contradicting the theory model, and since it delivers better inference, it may be the obvious thing to do (see Section 4.1).

Whichever of the three cases, it follows from (9) and (10), that imposing a root at 1, means $\det(A(1)) = \det(-\Pi) = 0$, and therefore imposing reduced rank on Π, which can be parameterized as,

$$\Pi = \alpha\beta', \tag{11}$$

where the matrices α and β are $p \times r$, of rank r, α being the *adjustment coefficients,* and β, the r *cointegrating vectors.*

The model (8), under the restriction on Π in (11), but otherwise *unrestricted* parameters, *including* α and β, and $r < p$, is thus a sub-model of the VAR, and is called a cointegrated I(1) model, denoted by $H(r)$. The theory models below are viewed as sub-models of $H(r)$.

As alluded to above, a relevant assumption about the DGP is that,

$$\text{The roots of (9) have } |z| > 1 \text{ or } z = 1. \tag{12}$$

Under (12) and,

$$\det(\alpha'_{\perp}\Gamma\beta_{\perp}) \neq 0, \tag{13}$$

where $\alpha_{\perp}$ and $\beta_{\perp}$ are the orthogonal complements, the I(1) model can be represented in *Moving Average (MA) form,*

$$x_t = C\sum_{i=1}^{t}(\Phi D_i + \varepsilon_i) + C(L)(\Phi D_t + \varepsilon_t) + C_0, \tag{14}$$

where $C \equiv \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp$ is the *long-run impact matrix*, $C(L)$, a convergent lag polynomial, and C_0 depends on initial values, with $\beta' C_0 = 0$ (Theorem 4.2, Johansen 1996).

The long-run movement of the series is described by the $p-r$ dimensional vector of *common (stochastic) trends,* (CT_t), given by,

$$\mathrm{CT}_t \equiv \alpha'_\perp \Sigma_{i=1}^{t}\varepsilon_i. \tag{15}$$

Usually, the stochastic trend, $C\Sigma_{i=1}^{t}\varepsilon_i$, is decomposed into CT_t, and the so-called *loadings matrix*, given by,

$$L \equiv \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}, \tag{16}$$

describing how each of the $p-r$ common trends affect the individual variables.

Below, the focus is primarily on the VAR with one lag. This keeps the analysis simple while still illustrating the main points clearly. More importantly, the VAR(1) has a particular status since any VAR(k) can be rewritten as a VAR(1), using the *companion form.*

I shall also assume that the deterministic term, ΦD_t, is a constant term, which is restricted, so that it does not produce a trend in the series. Again, this is to keep it simple and generalizing deterministics (trends, indicator variables etc.), does not affect the conclusions, but merely blurs the illustrations. Hence, I assume that,

$$D_i = 1 \text{ and } \Phi = \alpha s, \tag{17}$$

in (8), where s is $r \times 1$. The resulting CVAR(1), used repeatedly below, can be written as,

$$\Delta x_t = \alpha(\beta' x_{t-1} + s) + \varepsilon_t. \tag{18}$$

In a VAR(1), $\Gamma = I$, and the condition in (13) reduces to,

$$\det(\alpha'_\perp\beta_\perp) \neq 0, \tag{19}$$

and the MA representation becomes,

$$x_t = C\sum_{i=1}^{t}\varepsilon_i + \sum_{i=0}^{\infty}C_i^*(\alpha s + \varepsilon_{t-i}) + Cx_0, \tag{20}$$

with $C = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp$, $C_i^* = \alpha(\beta'\alpha)^{-1}(I_r + \beta'\alpha)^i\beta'$.

Under the assumption that $r(\Pi) = r < p$, the assumptions, (12), and (19) *together*, are equivalent to $A(z)$ having exactly $p-r$ roots at $z = 1$, while the rest have $|z| > 1$. Either of these equivalent conditions imply that the eigenvalues of the matrix $I_r + \beta'\alpha$, all have modulus less than 1, or equivalently that,

$$\rho(I_r + \beta'\alpha) < 1, \tag{21}$$

where $\rho(\cdot)$ is the spectral radius, which in turn implies that $\beta'\alpha$ has full rank, r. From these assumptions one can then establish the identity,

$$\alpha(\beta'\alpha)^{-1}\beta' + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp = I_p, \tag{22}$$

which can be used to derive (20).

Given (20), the *Impulse Response Function* (IRF) is,

$$\frac{\partial E(x_{t+h} \mid x_t)}{\partial x_t} = \frac{\partial E(x_{t+h} \mid x_t)}{\partial \varepsilon_t} = C + C_h^* \to C, \text{ for } h \to \infty, \tag{23}$$

where $C_h^* \to 0$ follows from (21). This coincides with the structural IRF, when $A = I$ in (6), as in Section 4.1 .The structural IRF is the one of economic interest, as it captures the economically interpretable dynamic response to a given *isolated* (i.e., statically and dynamically uncorrelated) shock. For $A \neq I$, it is based on (20) with $\widetilde{C} = CA^{-1}$ instead of C, $\widetilde{C}_i^* = C_i^*A^{-1}$ instead of C_i^*, and u_t instead of ε_t, and is therefore given by $\widetilde{C} + \widetilde{C}_h^*$.

The so-called *attractor set,* for the VAR(1), is usually defined as,

$$\mathcal{A} = \{x \in R^p \mid \beta'x = 0\} = sp(\beta_\perp). \tag{24}$$

Finally, two concepts of econometric exogeneity are needed, *weak*- and *strong exogeneity*, (See Engle, Hendry, and Richard 1983). They are both defined with respect to the parameters of interest, which is β in this context. These concepts are usually discussed in connection with efficient estimation and forecasting from partial models respectively (Ericsson, Hendry, and Mizon 1998). Here, the focus is on their relation to (and distinction from) the above concept of economic exogeneity. A variable is said to be *weakly exogenous* for β, if it has a zero row in α (Johansen 1992). This in turn implies that the cumulation of shocks to this variable is a common trend. If, in addition, this variable is not *Granger Caused* by the endogenous variables, the variable is said to be *strongly exogenous for* β (Johansen 1992). Partitioning x_t as $(x'_{1t}, x'_{2t})'$, and correspondingly the matrices in (8), under the restriction, (11), as,

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \text{ and } \Gamma_i = \begin{pmatrix} \Gamma_{11,i} & \Gamma_{12,i} \\ \Gamma_{21,i} & \Gamma_{22,i} \end{pmatrix}, \tag{25}$$

weak exogeneity of x_{2t} for β, is the restriction that $\alpha_2 = 0$, while strong exogeneity requires $\Gamma_{21,i} = 0$, in addition. For the VAR(1) the two concepts coincide.

4 Analyzing Theory Models in the CVAR Model

4.1 A Static Theory Model

Suppose that a VAR(1) describes the variation in the time series (Q_t, P_t, W_t, Z_t), corresponding to the variables in Section 2. Assume that these series are *persistent,* cf. Section 3. Under this assumption, a set of restrictions on the VAR, consistent with the simple demand- and supply model (1) - (3) is now suggested.

In general, the equations defining an economic model involve *latent* constructs, such as expectations and plans. Hence, they are not directly empirically implementable. Here, the relations, (1) and (2), are the plans involving the latent variables, Q^s and Q^d. In particular, introducing a time index, I shall assume that these relations are *contingent plans,* conditional on *observed* outcomes (see e.g. Hendry 1995): For example, for the demand relation, (1), $Q_t^d = a_0 - a_1 P_t + a_2 W_t$, denotes *demand at time* t, where the plan, Q_t^d, is unobserved while P_t and W_t are *realized* values. This is a *point* on the *demand curve at time* t, which is denoted by $D(P, W_t) \equiv a_0 - a_1 P + a_2 W_t$.

In contrast, the VAR model is formulated in the observables. As a consequence, a mapping relating the latent variables and plans to the observables is needed. Usually, such mappings come in the form of an *observation equation*, for Q_t outside equilibrium, and an *adjustment equation* for P_t outside equilibrium[4].

Consider the *price adjustment mapping.* I assume that it has the general form,

$$\Delta P_t = g(Q_{t-1}, Q_{t-1}^d, Q_{t-1}^s), \tag{26}$$

where $g()$ is a continuous and locally differentiable function. It seems reasonable that the adjustment in prices from period $t-1$ to t depends on what is learned or observed in period $t-1$: Say, at the end of period $t-1$, firms realize the reduction in inventories and the increased willingness to buy. As a consequence, they probably charge higher prices the next period.

Suppose, that (26) has the specific, though still general, form,

$$\Delta P_t = g(Q_{t-1} - Q_{t-1}^d, Q_{t-1} - Q_{t-1}^s), \tag{27}$$

where prices adjust as a result of the discrepancy between plans and realizations for both consumers and producers. Compared to the equations often used in the literature (see e.g. Laroque and Salanie 1995), the mapping in (27) allows for *different* adjustment processes for demand- and supply deviations respectively, which seems empirically relevant, as these processes may involve different sets of agents.

As an equilibrium represents a state with no change, it follows that,

$$g(0,0) = 0, \tag{28}$$

so that a Taylor expansion of $g()$ around the equilibrium delivers the mapping,

$$\Delta P_t \simeq g_1'(0,0)(Q_{t-1} - Q_{t-1}^d) + g_2'(0,0)(Q_{t-1} - Q_{t-1}^s), \tag{29}$$

[4] For a treatment of macroeconometric models and these related concepts see Hendry (1995), p. 781 ff.

where $g'_\cdot$ is a partial derivative. A special case of (29), in which, excess demand causes prices to rise, follows from assuming $g'_1(0,0) < 0$ and $g'_2(0,0) = -g'_1(0,0)$, since this implies $\Delta P_t \simeq g'_1(0,0)(Q^s_{t-1} - Q^d_{t-1})$, resembling equation A6.11 in Hendry (1995).

The Taylor approximation is only useful provided that $Q_t - Q^d_t$ and $Q_t - Q^s_t$ are stationary. However, as shown below, this is exactly what cointegration means in this case.

As the form of the *observation equation* is similar to (29), this is given by,

$$\Delta Q_t \simeq h'_1(0,0)(Q_{t-1} - Q^d_{t-1}) + h'_2(0,0)(Q_{t-1} - Q^s_{t-1}). \tag{30}$$

The partial derivatives, h'_1, h'_2 and g'_1, g'_2, are evaluated in the equilibrium. They are thus constants and henceforth they are denoted as, α_{11}, α_{12} and α_{21}, α_{22}, respectively.

The price- and quantity adjustment in equations (29) and (30), represent the systematic, or anticipated part of the change from one period to the next. It seems reasonable to add the error terms, ε_{Pt} and ε_{Qt}, respectively in these equations, representing *unanticipated* and *unmodelled* influences. Their stochastic properties are given below.

As argued in Section 3 the economically exogenous variables, W and Z, are the source of persistence, and since the theory model is not concerned with *how* these are generated, it seems uncontroversial to *empirically model* them as I(1) processes. Assume therefore that,

$$W_t = W_{t-1} + \varepsilon_{Wt}, \tag{31}$$

$$Z_t = Z_{t-1} + \varepsilon_{Zt}. \tag{32}$$

Hence, this is where the persistence is approximated by imposing the unit roots (see below).

Finally, it is assumed, as is usual, that,

$$\varepsilon_t \sim i.i.N(0, \Omega), \tag{33}$$

where $\varepsilon_t = (\varepsilon_{Qt}, \varepsilon_{Pt}, \varepsilon_{Wt}, \varepsilon_{Zt},)'$ and $\Omega > 0$ is diagonal. The shocks, ε_W and ε_Z, are referred to as demand- and supply shocks respectively, not to be confused with ε_Q and ε_P.

Collecting all this, the system, (29) - (32) can be written as the following CVAR(1),

$$\begin{aligned}
\Delta Q_t &= \alpha_{11}(Q_{t-1} - (a_0 - a_1 P_{t-1} + a_2 W_{t-1})) + \alpha_{12}(Q_{t-1} - (b_0 + b_1 P_{t-1} - b_2 Z_{t-1})) + \varepsilon_{Qt}, \\
\Delta P_t &= \alpha_{21}(Q_{t-1} - (a_0 - a_1 P_{t-1} + a_2 W_{t-1})) + \alpha_{22}(Q_{t-1} - (b_0 + b_1 P_{t-1} - b_2 Z_{t-1})) + \varepsilon_{Pt}, \\
\Delta W_t &= \varepsilon_{Wt}, \\
\Delta Z_t &= \varepsilon_{Zt},
\end{aligned} \tag{34}$$

or, in the compact notation from Section 3,

$$\Delta x_t = \alpha(\beta' x_{t-1} + s) + \varepsilon_t, \tag{35}$$

with $x_t' = (Q_t, P_t, W_t, Z_t)$, and matrices given by,

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 1 \\ a_1 & -b_1 \\ -a_2 & 0 \\ 0 & b_2 \end{pmatrix}, \text{ and } s = \begin{pmatrix} -a_0 \\ -b_0 \end{pmatrix}, \tag{36}$$

and the corresponding orthogonal complements,

$$\alpha_\perp = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_\perp = \begin{pmatrix} \frac{a_2}{1+\frac{a_1}{b_1}} & \frac{-b_2}{1+\frac{b_1}{a_1}} \\ \frac{a_2}{b_1+a_1} & \frac{b_2}{b_1+a_1} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

Under the assumption (12), the model generates I(1) variables only, since $\det(\alpha_\perp'\beta_\perp) = 1$. The MA representation thus implies the following components,

$$\mathrm{CT}_t = \begin{pmatrix} \Sigma_{i=1}^t \varepsilon_{Wi} \\ \Sigma_{i=1}^t \varepsilon_{Zi} \end{pmatrix}, \ L = \begin{pmatrix} \frac{a_2}{\frac{a_1}{b_1}+1} & -\frac{b_2}{\frac{b_1}{a_1}+1} \\ \frac{a_2}{a_1+b_1} & \frac{b_2}{a_1+b_1} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 0 & \frac{a_2}{\frac{a_1}{b_1}+1} & -\frac{b_2}{\frac{b_1}{a_1}+1} \\ 0 & 0 & \frac{a_2}{a_1+b_1} & \frac{b_2}{a_1+b_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

using equations (15) and (16). Note that, whereas $\det(\alpha_\perp'\beta_\perp) = 1$ is implied by the theory model, (12) is not.

The model in (35) and (36) is an identified sub model of the I(1) model, $H(2)$. It can be tested in $H(2)$ by jointly imposing the zero restrictions on α, the normalization, and the corresponding two *generically identifying* zero restrictions on β (See Johansen 1996). This can be done using the software CATS in RATS (Dennis, Hansen, and Juselius 2006).

From (36), it is seen that the consequence of the I(1) approximation is that the theoretical parameters of interest, a_i and b_i, should be modeled as cointegrating parameters, and that the assumption of economic exogeneity implies strong exogeneity *in this case.*

As alluded to above, the I(1) -, or unit root approximation corresponds to the *empirical modelling* of the exogenous variables, in that (31) and (32) can be interpreted as approximations of the processes, $W_t = \rho_w W_{t-1} + \varepsilon_{Wt}$ and $Z_t = \rho_z Z_{t-1} + \varepsilon_{Zt}$, with $\rho_j < 1$ but close to 1, respectively. In such a case, instead of the unit roots, corresponding to (35) and (36), the true underlying process has two roots $\frac{1}{\rho_w}$ and $\frac{1}{\rho_z}$, both close to, but above 1, while the rest are unaltered also with $|z| > 1$, provided that (12) applies. The true process is therefore a stationary "near unit root" process generating persistent series. Approximating ρ_j by 1, is thus equivalent to approximating the two borderline unit roots, $\frac{1}{\rho_w}$ and $\frac{1}{\rho_z}$, by 1. Note that, as opposed to the parameters a_i and b_i, ρ_w and ρ_z are *not theoretical parameters of interest*, and hence, $\rho_w = \rho_z = 1$ is not inconsistent with the theory model, as argued in Section

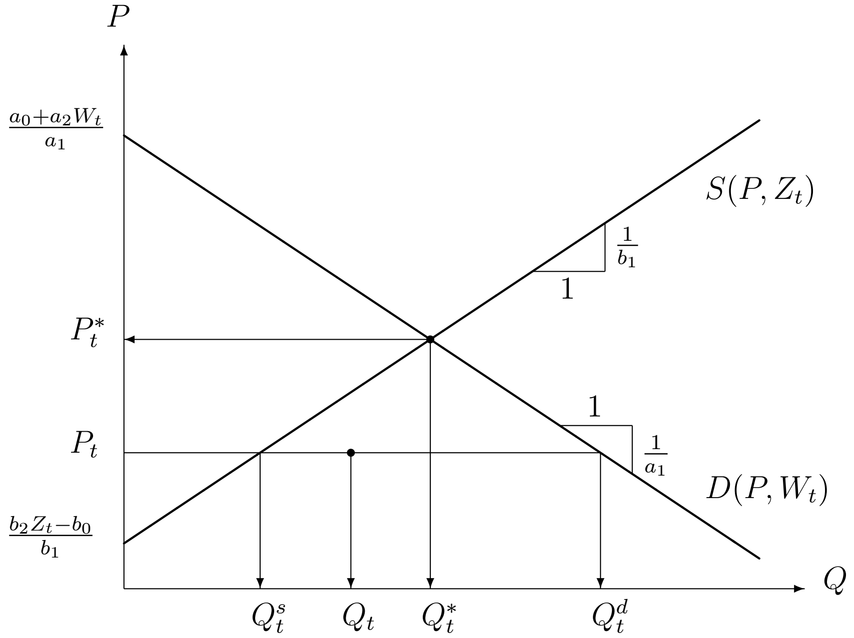

Figure 2: The Supply and Demand Curves, $S(P, Z_t)$ and $D(P, W_t)$, Demanded - and Supplied Quantities, Q_t^d and Q_t^s, and the Actual Outcome, (Q_t, P_t), - all at time t.

3.

The model is illustrated in Figure 2. The diagram looks like that in Figure 1, but there is a fundamental difference: In the equations, (1) - (3), underlying Figure 1, the time subscript is t for all variables, and is therefore suppressed. Since the system, (1) - (3), is in equilibrium, t must be interpreted as an "*end period*" in this case. That is, a period for which there have been no shocks for a long time and all previous shocks have fully propagated (see e.g. Sørensen and Whitta-Jacobsen 2005, p. 17). As a result, Figure 1 illustrates the economic equilibrium. In contrast, the time subscript in Figure 2 denotes any arbitrary period. As drawn the *actual* point, (Q_t, P_t), differs from the equilibrium at time t, (Q_t^*, P_t^*), determined by the intersection of the demand curve at time t, $D(P, W_t)$, and the supply curve at time t, $S(P, Z_t)$.

The equilibrium at time t acts as a *pulling force* on the observed point, in the sense that, in the hypothetical absence of any other shocks from period $t+1$ and onwards, the (Q, P)-point would converge towards (Q_t^*, P_t^*), starting in (Q_t, P_t). To see this, we simply set $\varepsilon_\tau = 0$ for $\tau \geq t+1$, so that the values of W and Z are fixed at W_t and Z_t. Using the identity in (22) it can be shown that,

$$x_{t+h} = \alpha(\beta'\alpha)^{-1}(I_r + \beta'\alpha)^h(\beta' x_t + s) + Cx_t - \alpha(\beta'\alpha)^{-1}s, \tag{39}$$

for $h \geq 0$. Given (21), $(I_r + \beta'\alpha)^h \to 0$, for $h \to \infty$, so that the limit of (39) is,

$$Cx_t - \alpha(\beta'\alpha)^{-1}s \equiv x^*. \tag{40}$$

This expression implies that $x^* = (Q_t^*, P_t^*, W_t, Z_t)'$, and since $W_{t+h} = W_t$ and $Z_{t+h} = Z_t$ this shows the convergence of (Q_{t+h}, P_{t+h}) towards the "*economic equilibrium at* *time t*", (Q_t^*, P_t^*). Since $x^* = x_{\infty,t} \equiv \lim_{h\to\infty} E[x_{t+h} \mid x_t]$ this economic equilibrium thus corresponds to the so-called *long-run value*, defined in Johansen (2005).

From (40) it follows that, $\frac{\partial x^*}{\partial x_t'} = C$, describing the long-run impact of unit changes in the variables. Thus, the C matrix in (38) captures the comparative static effects given in (5).

Starting from the point (Q_t, P_t) the expression for x_{t+h} in (39), tells us exactly where the (Q, P)-allocation is located in the diagram after h periods, in the absence of shocks. Pre-multiplying with β' in (39), we get an expression for the equilibrium error at time $t + h$,

$$\beta' x_{t+h} + s = (I_r + \beta'\alpha)^h(\beta' x_t + s), \tag{41}$$

and by writing $\beta' x_{t+h} + s$, as $(Q_{t+h} - Q_{t+h}^d, Q_{t+h} - Q_{t+h}^s)'$ we find that,

$$Q_{t+h}^s - Q_{t+h}^d = k'(I_r + \beta'\alpha)^h(\beta' x_t + s), \tag{42}$$

$k' = (1, -1)$. This shows that the assumption, $(I_r + \beta'\alpha)^h \to 0$, i.e. $\rho(I_r + \beta'\alpha) < 1$, has the economic interpretation of market clearing.

For a given deviation from equilibrium, $(\beta' x_t + s) = (Q_t - Q_t^d, Q_t - Q_t^s)'$, the expression (42) shows how (and how fast) the market clears. It may involve oscillations or smooth convergence, fast or slow, depending on the eigenvalues of $(I_r + \beta'\alpha)$. Thus, (42) offers a framework for formulating interesting hypotheses about the market clearing process, which could be formulated in terms of restrictions α given β, provided that that $\rho(I_r + \beta'\alpha) < 1$. It should be possible to formulate Keynesian type of hypotheses about nominal -, or real rigidities, in this manner. For example, loosely illustrated in the right context of a simple AS-AD, with the same form as (1) - (2), a small value of α_{21}, combined with a large value of α_{11}, would describe little adjustment in prices while more adjustment in quantities, in the wake of a demand shock, i.e. "nominal rigidities".

The above "long run in the hypothetical absence of shocks" essentially resembles the theory model in pure form, but clearly, in each period unanticipated shocks hit all variables in the system: The demand- and supply curves are shifted by ε_W and ε_Z respectively, and in addition to the anticipated changes in Q and P, the shocks ε_Q and ε_P occur. An unanticipated realized position and an unanticipated equilibrium position have thus resulted, and adjustment in Q and P towards *this* equilibrium will take place *in the next period*, in which new shocks occur etc.. The economic equilibrium thus moves and corresponds to an *attractor*. This is also captured by the fact that *existence* of the economic equilibrium (4), requires $a_1 + b_1 \neq 0$, which is the requirement for the attractor set, $\mathcal{A} = sp(\beta_\perp)$, to exist, as seen from (37).

This moving equilibrium will induce lagged *error correction*, or, in other words, x_{t+1} will depend on the *equilibrium error at time t*, as is seen from,

$$\beta' x_{t+1} + s = (I_r + \beta'\alpha)(\beta' x_t + s) + \beta'\varepsilon_{t+1}, \tag{43}$$

resembling (41) for $h = 1$, when shocks occur. Given (21) this is (asymptotically)

stationary, which supports the use of the Taylor expansion in (29) and (30), as

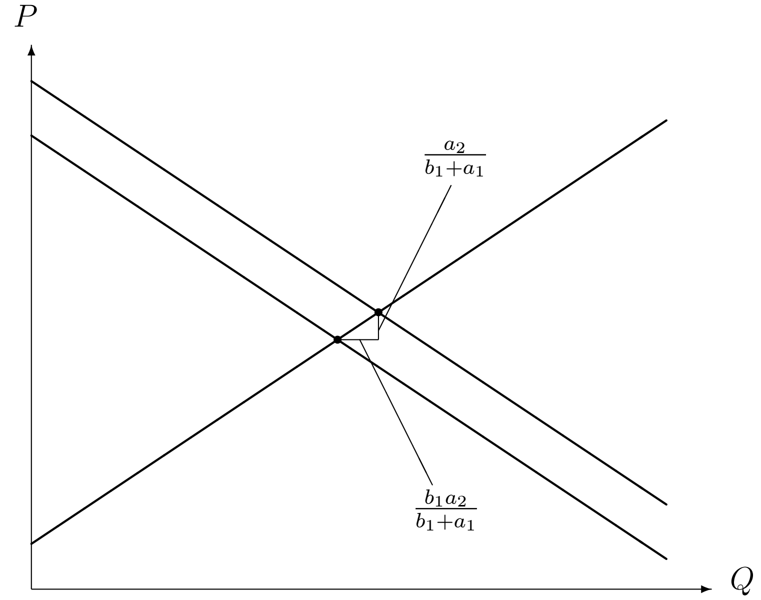

Figure 3: Illustration of the Long-run Impact of a Unit Rise in ε_W (Positive Demand Shock) on the Endogenous Variables, Q and P.

$\beta' x_t + s = (Q_t - Q_t^d, Q_t - Q_t^s)'$.

As the demand- and supply shocks, ε_W and ε_Z, change the locations of respectively the demand- and the supply curves *permanently,* it is seen from the C matrix in (38), that they have a long-run impact on the endogenous variables. In contrast, the shocks, ε_Q and ε_P, have no long-run impact. This is essentially because they do not affect the positions of the curves: Starting from an equilibrium, an unanticipated price shock say, $\varepsilon_P < 0$, will introduce excess demand inducing upward price adjustment until the initial equilibrium is restored. It is therefore the cumulation of ε_W and ε_Z, and not ε_Q and ε_P that determines the long-run position of the endogenous Q and P, which is what CT_t, in (38) shows.

The loadings matrix, L, in (38) shows how these common trends affect the endogenous variables. The interpretation of the elements in L is facilitated by use of the demand- and supply diagram: Consider a unit rise in ε_{Wt}, which according to L in (38) will have a long-run impact of $\frac{a_2}{\frac{a_1}{b_1}+1}$ units on Q, and of $\frac{a_2}{b_1+a_1}$ units on P. In Figure 3, this corresponds to a unit shock to W_t which shifts the demand curve upwards by $\frac{a_2}{a_1}$ units, eventually resulting in a rise in the equilibrium value of Q and P of the same magnitudes, $\frac{a_2}{1+\frac{a_1}{b_1}}$ and $\frac{a_2}{b_1+a_1}$ respectively.

Similarly, from L, we can see that the unit shock in W will have the full impact, $\frac{a_2}{a_1}$, on P, while no effect on Q, for $b_1 \to 0$. That is, when the supply curve is vertical, demand shocks will have no effect on quantity, while full effect on prices, e.g. as in a simple classical AS-AD model. In contrast, when $b_1 \to \infty$, the supply curve is horizontal and there is no impact on the price level from the demand shock, while full effect on Q, equal to a_2, since the horizontal shift in the demand curve is the vertical shift, $\frac{a_2}{a_1}$, times the absolute inverse slope, a_1.

Thus, the impact on the endogenous variables from demand and supply shocks is completely determined by the slopes (partial derivatives) of the curves, an this is exactly what the loadings matrix captures (or C, as $C = L\alpha'_\perp = L(0, I_{p-r}) = (0, L)$, when there are $p - r$ weakly exogenous variables).

4.2 Some Generalizations of the Empirical Model

The framework of the previous section is now generalized in three directions: First, since the mappings $g()$ and $h()$ are not part of the theory model, they are, to some extent, arbitrary. These are therefore generalized with respect to the lag of response, which was 1 above (see e.g. 27). Second, as the data often suggest more than one lag, I consider this case as well. Third, the endogenous variables may also be allowed to respond to current changes in the exogenous variables.

The theory model is still the same, and hence, α, β, $\alpha_\perp$ and $\beta_\perp$ are unaltered, and it is investigated whether the interpretations of the CVAR parameters, C and L, as describing comparative statics, can be retained.

Suppose that the response time were longer than one period, so that instead of (27), the mapping were,

$$\Delta P_t = g(Q_{t-u} - Q^d_{t-u}, Q_{t-u} - Q^s_{t-u}), \tag{44}$$

for $u > 1$ ($u = 0$ is considered below). Hence, the reaction to market disequilibrium takes place u periods later. I assume the same for quantity, i.e. for $h()$. This implies, that instead of (35) we have,

$$\Delta x_t = \alpha(\beta' x_{t-u} + s) + \varepsilon_t, \tag{45}$$

where as before α and β are given by (36). As this can always be rewritten as,

$$\Delta x_t = \alpha(\beta' x_{t-1} + s) + \Gamma_1 \Delta x_{t-1} + \Gamma_2 \Delta x_{t-2}.. + \Gamma_{u-1}\Delta x_{t-(u-1)} + \varepsilon_t, \tag{46}$$

with the restrictions that, $\Gamma_i = -\alpha\beta'$, for $i = 1, ...u-1$, we find that $\Gamma = I + (u-1)\alpha\beta'$, which implies that C and L are unaltered as, $C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp$, and $\alpha'_\perp\alpha = 0$, $\beta'\beta_\perp = 0$. Hence, the result that C and L can be interpreted as above, is invariant with respect to reaction time u. This is intuitively expected, as the *long-run* effect is the same as before. This holds also if P and Q react to equilibrium errors with *different* lags.

Now, suppose that the CVAR needs more than one lag. Since $\alpha_\perp$ and $\beta_\perp$ are the same, we see from, $C = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp = L\alpha'_\perp$, that Γ needs to fulfill certain requirements, for the interpretations to be unaltered. It suffices to have two lags i.e.,

$$\Delta x_t = \alpha(\beta' x_{t-1} + s) + \Gamma_1 \Delta x_{t-1} + \varepsilon_t. \tag{47}$$

Introduce the block notation,

$$x_t = \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}, \ \Gamma_1 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \ \alpha_\perp = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}, \ \beta_\perp = \begin{pmatrix} B_{1\perp} \\ I_2 \end{pmatrix}, \tag{48}$$

where $x_{1t} = (Q_t, P_t)'$ and $x_{2t} = (W_t, Z_t)'$, and the definition of $B_{1\perp}$ follows from $\beta_\perp$ in (37). From this, it follows that,

$$\alpha'_\perp \Gamma \beta_\perp = \alpha'_\perp (I_4 - \Gamma_1)\beta_\perp = I_2 - \Gamma_{22} - \Gamma_{21} B_{1\perp}, \tag{49}$$

which enters the expression for C.

Now, economic exogeneity of W and Z (cf. Section 2) implies that $\Gamma_{21} = 0$, that is, that x_{1t} does not Granger cause x_{2t}. As α is unaltered, x_{2t} is thus strongly exogenous for β (See Section 3). Moreover, economic exogeneity requires that the exogenous variables are mutually unrelated, which amounts to Γ_{22} being diagonal. The requirement of economic exogeneity is thus stronger than strong exogeneity. Denoting the diagonal elements of Γ_{22} by γ_{ii},

$$(\alpha'_\perp \Gamma \beta_\perp)^{-1} = \begin{pmatrix} \frac{1}{1-\gamma_{11}} & 0 \\ 0 & \frac{1}{1-\gamma_{22}} \end{pmatrix} \equiv D, \tag{50}$$

provided that $\gamma_{ii} \neq 1$ (the condition in (13)). This implies that,

$$C = \begin{pmatrix} 0 & B_{1\perp} D \\ 0 & D \end{pmatrix}. \tag{51}$$

So, the C matrix has in fact changed, but this is simply because the W and Z are now modelled as AR(2) I(1) variables, implying that the long-run impact of a unit rise in ε_W on W, say, which is what C shows, is no longer 1, but $(1-\gamma_{11})^{-1}$. As comparative statics concern the effect of a unit change in W, we simply need to normalize the shock, ε_W, to obtain a long-run effect of 1 on W. So, if we change ε_{Wt} by $1 - \gamma_{11}$ we obtain a unit change in W in the long run, resembling the comparative static experiment. As described in Johansen (2005) we can add δ to the variables at time t, i.e. to ε_t, which gives the long-run impact $C\delta$. So, if we add $\delta = (0, 0, (1-\gamma_{11}), 0)'$ to the variables we essentially normalize the column in C showing the impact of ε_W shocks and get the same as before.

Hence, under economic exogeneity nothing substantial has changed, and provided that we change the current values of the exogenous variables in order to produce a *long-run unit change* in them, C still captures the comparative statics.

Now, consider current/simultaneous effects. The econometric model in Section 4.1 is (6), with $k = 1$, and $A = I$, i.e. no current/static correlations between variables. Consider a general normalization $A \neq I$. It is sufficient to consider the CVAR(1). The corresponding so-called "structural" CVAR is,

$$A\Delta x_t = a(\beta' x_{t-1} + s) + u_t, \tag{52}$$

where $a \equiv A\alpha$, $u_t \equiv A\varepsilon_t$ distributed as $i.i.N(0, \Sigma)$, Σ diagonal. From the MA representation the modified long-run impact matrix is (see Section 3),

$$\widetilde{C} = CA^{-1}. \tag{53}$$

Hence, the C matrix will in general change. However, under economic exogeneity, A

has a certain structure, which implies that we can still maintain the interpretations from before. To see this, partition matrices again, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & B_{1\perp} \\ 0 & I_2 \end{pmatrix}. \tag{54}$$

Under economic exogeneity we have that,

$$A_{21} = 0 \text{ and } A_{22} = I_2, \tag{55}$$

which in turn implies that,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I_2 \end{pmatrix}. \tag{56}$$

Inserting this into (53) along with C from (54), we get,

$$\widetilde{C} = C. \tag{57}$$

Hence, under economic exogeneity we can also generalize with respect to (identified) A_{11} and A_{12} parameters, without affecting the conclusion about C and L.

So far ΔQ_t and ΔP_t depend only on the deviation from equilibrium in the *previous* period cf. equations (30) and (29). When discussing simultaneous effects in the present context it would seem natural to replace $Q_{t-1} - Q_{t-1}^d$ and $Q_{t-1} - Q_{t-1}^s$, by the current deviations from equilibrium, $Q_t - Q_t^d$ and $Q_t - Q_t^s$. As a result quantity and price would react to disequilibrium already within the period. Instead of (35), the model would be,

$$\Delta x_t = \widetilde{\alpha}(\widetilde{\beta}' x_t + \widetilde{s}) + \widetilde{\varepsilon}_t, \tag{58}$$

defined in terms of the theoretical parameters of interest $(\widetilde{\beta}, \widetilde{s})$, the corresponding adjustment parameters, $\widetilde{\alpha}$, and a diagonal $\widetilde{\Omega} > 0$. This corresponds to the structural VAR, (6), with $A = (I_p - \widetilde{\alpha}\widetilde{\beta}')$ and $B_k \equiv B_1 = I_p$, $B_0 = \widetilde{\alpha}\widetilde{s}$, $D_t = 1$ and $u_t = \widetilde{\varepsilon}_t$

The model in (58) can be viewed as a reparameterization of the usual model with lagged responses, i.e. $\Delta x_t = \alpha(\beta' x_{t-1} + s) + \varepsilon_t$. Since the maximum likelihood estimator (MLE) is invariant, the MLEs of the parameters in (58) can be found from the MLEs of α, β, s and Ω, to be computed as described in Johansen (1996). Inference on the "new" parameters can then be conducted following the usual approach (see e.g. Hendry 1995, Section 10.11).

So, given the model, $\Delta x_t = \alpha(\beta' x_{t-1} + s) + \varepsilon_t$, consider the mapping, $\widetilde{\theta} = \tau(\theta)$, where $\theta \equiv (\alpha, \beta, s, \Omega)$ and $\widetilde{\theta} \equiv (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{s}, \widetilde{\Omega})$, given by,

$$\widetilde{\alpha} = \alpha(I_r + \beta'\alpha)^{-1}, \; \widetilde{\beta} = \beta, \; \widetilde{s} = s \text{ and } \widetilde{\Omega} = V\Omega V', \tag{59}$$

where $V \equiv (I_p - \alpha(I_r + \beta'\alpha)^{-1}\beta')$, and where $\det(I_r + \beta'\alpha) \neq 0$ has been assumed.

A reparameterization implies that τ is a *one-to-one* mapping: By construction, for a given choice of θ, (59) implies a unique value of $\widetilde{\theta}$, but a one-to-one mapping requires the opposite as well - i.e. for a given value of $\widetilde{\theta}$ there is a unique solution for θ. To see that this is the case, we first note from (59) that $\beta = \widetilde{\beta}$, $s = \widetilde{s}$. From the definition of $\widetilde{\alpha}$ it follows that $\widetilde{\alpha} = (I_p - \widetilde{\alpha}\widetilde{\beta}')\alpha$, and hence, a unique value of α exists for given $\widetilde{\alpha}$ and $\widetilde{\beta}$ if $(I_p - \widetilde{\alpha}\widetilde{\beta}')$ is invertible. As shown in Appendix A, under the usual (testable) assumptions about α, β given in Section 3, this is the case. The unique value of α is $\alpha = (I_p - \widetilde{\alpha}\widetilde{\beta}')^{-1}\widetilde{\alpha}$. To find Ω, we note that $V = (I_p - \widetilde{\alpha}\widetilde{\beta}')$ which implies that $\Omega = (I_p - \widetilde{\alpha}\widetilde{\beta}')^{-1}\widetilde{\Omega}(I_p - \widetilde{\beta}\widetilde{\alpha}')^{-1}$.

Thus, τ is a reparameterization and the MLE of $\widetilde{\theta}$ can be obtained as $\tau(\widehat{\theta})$ - i.e. from the usual estimation. The question remains whether C in the estimated model still captures the comparative statics (given by (5) with "~" notation). To see that this is the case, use that $\beta = \widetilde{\beta}$, so that $\beta_\perp = \widetilde{\beta}_\perp$, which can be written as, $\widetilde{\beta}_\perp = (\widetilde{B}'_{1\perp}, I_2)'$, as $\beta_\perp$ in (48). Note that, the 2×2 block, $\widetilde{B}'_{1\perp}$, gives the comparative static effects. Next, the usual partitionings, $\widetilde{\alpha} = (\widetilde{\alpha}'_1, 0')'$ and $\widetilde{\beta}' = (\widetilde{B}'_1, \widetilde{B}'_2)$, imply that α, which equals $(I_p - \widetilde{\alpha}\widetilde{\beta}')^{-1}\widetilde{\alpha}$, can be written as, $\alpha = (\widetilde{\alpha}'_1 K', 0')'$, with $K \equiv (I_2 - \widetilde{\alpha}_1 \widetilde{B}'_1)^{-1}$. This implies that $\alpha'_\perp = (0, I_2)$, i.e. has the same form as before. It follows that,

$$C = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp = \begin{pmatrix} 0 & \widetilde{B}_{1\perp} \\ 0 & I_2 \end{pmatrix}, \tag{60}$$

showing that C can still be interpreted as the comparative static effects.

4.3 Two Simple Dynamic Theory Models based on Expectations

Embedding a static theory model in the dynamic VAR as suggested above clearly provides a means of empirically assessing, not only the model itself - i.e. its explicit assumptions, but also its implicit albeit crucial assumptions of stability and "reasonable" speed of equilibrium adjustment (see Section 2). Now, say, that a given data sample suggests that both the explicit and the implicit assumptions are reasonable. This certainly brings more credibility to the static theory model, and if we are primarily interested in the (long-run) equilibrium effects our analysis may be sufficient - at least as a first approximation. However, knowing that the demand and supply equilibrium is stable, and that disequilibria are not too persistent, we still lack a *theory* of the adjustment process, which may be essential in understanding, for example, efficiency- and distributional properties outside equilibrium. Such a theory is part of a dynamic theory model.

Several assumptions make theory models dynamic. For example, related to the present context, ECMs may arise from dynamic optimization (Nickell 1985)[5]. Here, the focus will be on assumptions about expectations formation. I shall consider two stylized examples: A model of investment demand with expectations of exogenous variables, and a simple version of the so-called Hybrid New Keynesian Phillips Curve model introducing forward-looking expectations of endogenous variables (see e.g. Bårdsen, Eitrheim, Jansen, and Nymoen 2005).

[5] For a survey of different interpretations of ECMs, see Alogoskoufis and Smith (1991).

4.3.1 Investment Demand - Backward-Looking Expectations of Exogenous Variables

Consider a simple model of "aggregate investment in an exporting sector for the small open economy". Planned investment, I_t^p, is assumed to depend on the (subjectively) *expected* values of the real interest rate and the level of real international output, as the actual values, r_t and Y_t respectively, are unknown when the plan is implemented. Denoting the expected values as r_t^e and Y_t^e the behavioral (demand) relation is,

$$I_t^p = c_0 - c_1 r_t^e + c_2 Y_t^e, \tag{61}$$

where variables are in logarithms except for r. I assume that r and Y are economically exogenous in the theory model. As before, given the persistence, they are thus modelled empirically as,

$$r_t = r_{t-1} + \varepsilon_{rt}, \tag{62}$$

$$Y_t = Y_{t-1} + \varepsilon_{Yt}. \tag{63}$$

The expectations formation of r is assumed to follow,

$$r_t^e = \lambda_1 r_t + \lambda_2 r_{t-1} + (1 - \lambda_1 - \lambda_2) r_{t-2}, \tag{64}$$

where $0 \leq \lambda_1 \leq 1$, and $0 \leq \lambda_1 + \lambda_2 \leq 1$. Although output expectations could be described similarly, the points can be illustrated assuming that,

$$Y_t^e = Y_{t-1}. \tag{65}$$

Finally, actual and planned investment are allowed to differ by an unsystematic unanticipated error, ε_{It}, i.e.,

$$I_t = I_t^p + \varepsilon_{It}. \tag{66}$$

Compared to static models we now have expected values in the plan as well as the expectations equations, (64) and (65). However, the models are related: Recall the motivation for the (long-run) economic equilibrium (Section 2): The hypothetical absence of shocks. As mentioned, in such a case, expectations become correct, and the relevant sub-model for this hypothetical state - the long-run, would naturally have $\lambda_1 = 1$, that is, $r_t^e = r_t$, and (65) replaced by $Y_t^e = Y_t$. This would thus become the static model "for the long run" and t would have to be interpreted as an "end period" (Sørensen and Whitta-Jacobsen 2005, p. 17).

Clearly, in the short run $r_t^e \neq r_t$ is natural (ruling out perfect foresight). Consider the case when $\lambda_1 = 0$ and $0 < \lambda_2 < 1$, referred to as simple backward-looking expectations. Since, $r_t^e = \lambda_2 r_{t-1} + (1 - \lambda_2) r_{t-2}$, the model for the observed $x_t' = (I_t, r_t, Y_t)$ is,

$$\Delta x_t = \alpha(\beta' x_{t-1} + s) + \Gamma_1 \Delta x_{t-1} + \varepsilon_t, \tag{67}$$

with matrices,

$$\alpha = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 \\ c_1 \\ -c_2 \end{pmatrix}, \ s = -c_0, \ \Gamma_1 = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{68}$$

where $\gamma = c_1(1 - \lambda_2)$, and the MA components,

$$\mathrm{CT}_t = \begin{pmatrix} \Sigma_{i=1}^t \varepsilon_{ri} \\ \Sigma_{i=1}^t \varepsilon_{Yi} \end{pmatrix}, \ L = \begin{pmatrix} -c_1 & c_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{69}$$

There is one characteristic root at 1, and $\det(\alpha'_\perp \Gamma \beta_\perp) = 1$. Hence, in this particular case, (12) is implied by the theory model.

The interpretations of C (and L) as capturing the comparative statics still apply: The impact from the economically exogenous variables on the endogenous variables computed in the hypothetical absence of shocks, so that expectations are realized (i.e. in economic equilibrium), is still given by C.

The equilibrium error, or the expectational error is,

$$\beta' x_t + s \equiv I_t - (c_0 - c_1 r_t + c_2 Y_t) = c_1(1 - \lambda_2)\Delta r_{t-1} + \beta' \varepsilon_t, \tag{70}$$

as $I_r + \beta'\alpha = 0$. As seen from (70), this has an *anticipated* component, $c_1(1 - \lambda_2)\Delta r_{t-1}$. Hence, as long as $\lambda_2 < 1$, systematic expectational errors will take place. In the special case when $\lambda_2 = 1$, r_t^e coincides with the conditional mathematical expectation, $E_{t-1}(r_t)$. This can be referred to as the case of "Rational Expectations", and it implies that the expectational error is, $\beta'\varepsilon_t$, i.e. unsystematic.

The model is illustrated in Figure 4. Each period, the downward sloping investment demand curve is shifted up or down by the random walk Y, while the interest rate shifts the vertical line (also as a random walk), which can be thought of as the supply curve. The adjustment in the wake of an interest shock is illustrated: Suppose that the economy has been in the equilibrium, E_0, up to and including period $t-1$. In period t there is a unit shock to the real interest rate, and no other shocks occur. Under simple backward-looking expectations, the movement is from E_0 to O_1, in period t, then to O_2, in period $t+1$, and from $t+2$ and onwards to the new equilibrium E_1. Once the shock has occurred it is known and the "optimal" level of investment in period $t+1$, is $I_t - c_1$. When $\lambda_2 < 1$, the actual investment is higher. When expectations are rational, the movement is from E_0 to O_1, in period t, and then directly to E_1 in period $t+1$. All information is used once it is available.

4.3.2 The Hybrid New Keynesian Phillips Curve - Forward-Looking Expectations of Endogenous Variables

The model is,

$$\begin{aligned} \Delta p_t &= \theta_1(\Delta p_{t+1})^e + \theta_2 \Delta p_{t-1} + \theta_3 z_t + \varepsilon_{pt} \\ z_t &= \theta_4 z_{t-1} + \varepsilon_{zt}, \end{aligned} \tag{71}$$

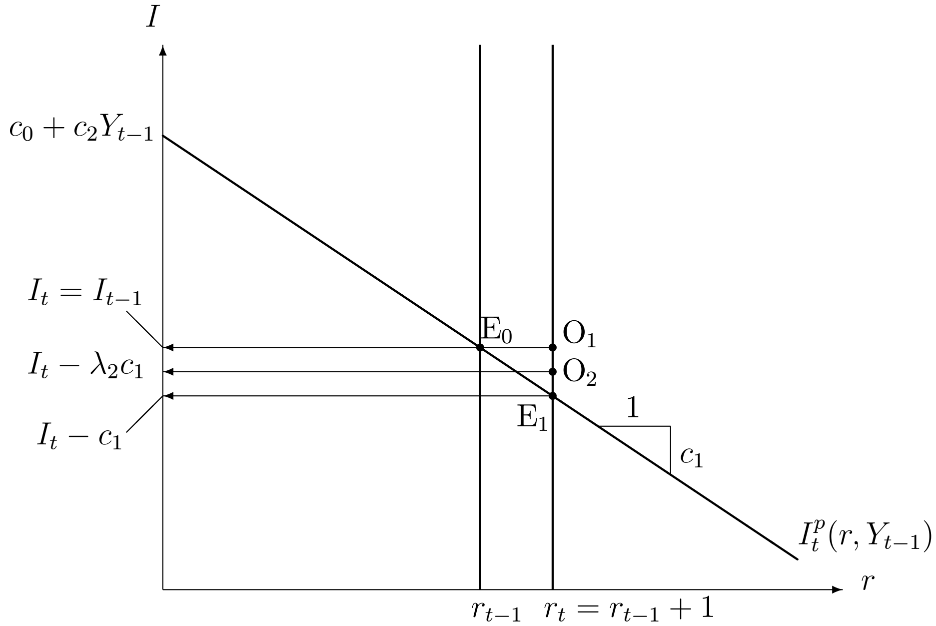

Figure 4: The Investment Demand Schedule at time t, together with a shift in the interest rate. In the case of Simple Backward-Looking Expectations, the movement is from E_0 to O_1, in period t, to O_2, in period $t+1$, and to E_1 in period $t+2$. When expectations are Rational, the movement is from E_0 to O_1, in period t, and then directly to E_1, in period $t+1$.

where Δp_t is inflation, $(\cdot)^e$ denotes the subjective expectation, z is the exogenous forcing variable stated as a steady state deviation, e.g. output gap, unemployment, or wage share (see Gali and Gertler (1999) and Bårdsen et al. 2005, Chapter 7). The shocks are $i.i.N(0, \Sigma)$, $\Sigma > 0$ and diagonal.

Compared to the previous example, there are at least three important differences: First, as seen from (71), the expectation now concerns an endogenous variable. Second, Expectations are forward-looking (they concern Δp_{t+1}), and thirdly, different types of New Keynesian models underlying (71) typically have rigorous microeconomic foundations (Roberts 1995). The presence of the forward-looking term together with the backward-looking term Δp_{t-1} motivates the adjective "Hybrid". The case, $\theta_2 = 0$, is the pure New Keynesian Phillips curve, whereas , $\theta_1 = 0$, resembles a more traditional Expectation Augmented Phillips curve (see e.g. Sørensen and Whitta-Jacobsen 2005, Chapter 18).

In Gali and Gertler (1999) - and commonly elsewhere - the assumption of Rational Expectations is invoked, i.e.,

$$(\Delta p_{t+1})^e = E_t(\Delta p_{t+1}), \tag{72}$$

where $E_t(\cdot)$ is the conditional mathematical expectation. The coefficient θ_4 is usually restricted as $|\theta_4| < 1$ (see Bårdsen et al. 2005).

For illustration, suppose that all θ_i obey, $0 < \theta_i < 1$, and insert (72) in (71). Imposing the transversality condition that $1 < |\lambda_2|$, where $\lambda_2 = \frac{1+\sqrt{1-4\theta_1\theta_2}}{2\theta_1}$, the model can be solved by e.g. repeated substitution (see Appendix A.2 in Bårdsen et al. 2005) to give,

$$\begin{aligned} \Delta p_t &= \lambda_1 \Delta p_{t-1} + \delta z_t + \breve{\varepsilon}_{pt} \\ z_t &= \theta_4 z_{t-1} + \varepsilon_{zt}, \end{aligned} \tag{73}$$

where $\lambda_1 = \frac{1-\sqrt{1-4\theta_1\theta_2}}{2\theta_1}$, $\delta \equiv \frac{\theta_3}{\theta_1(\lambda_2-\theta_4)}$, $\breve{\varepsilon}_{pt} \equiv \frac{1}{\theta_1\lambda_2}\varepsilon_{pt}$ and $|\lambda_1| < 1$ is assumed.

Under these assumptions, (73) implies the following VAR in ECM-form,

$$\Delta x_t = \Pi x_{t-1} + e_t, \tag{74}$$

with,

$$\Pi = \begin{pmatrix} -(1-\lambda_1) & \delta\theta_4 \\ 0 & \theta_4 - 1 \end{pmatrix}, \tag{75}$$

where $x_t = (\Delta p_t, z_t)'$ and $e_t = (\breve{\varepsilon}_{pt} + \delta\varepsilon_{zt}, \varepsilon_{zt})'$. This is (asymptotically) stationary as $\det(\Pi) = (\lambda_1 - 1)(\theta_4 - 1) \neq 0$ and the characteristic roots are $z_1 = \frac{1}{\lambda_1}$ and $z_2 = \frac{1}{\theta_4}$ which both have modulus greater than 1.

Now, as argued previously, given the persistence it may be better to use the approximation, $\theta_4 = 1$, to obtain more reliable inference. As a result we get (in the usual notation),

$$\beta = \begin{pmatrix} 1 \\ \frac{-\delta}{(1-\lambda_1)} \end{pmatrix}, \alpha = \begin{pmatrix} -(1-\lambda_1) \\ 0 \end{pmatrix} \text{ and } s = 0, \tag{76}$$

and as $\det(\alpha'_\perp \beta_\perp) = 1$, we have a cointegrated I(1) model.

The long-run impact matrix is,

$$C = \begin{pmatrix} 0 & \frac{\delta}{(1-\lambda_1)} \\ 0 & 1 \end{pmatrix}. \tag{77}$$

Analogous to the previous example, this gives the comparative static effect, $\frac{\delta}{(1-\lambda_1)}$, from z on Δp. To see this, first note that in the hypothetical long-run equilibrium with no shocks, z_t is fixed, expectations are realized, $(\Delta p_{t+1})^e = \Delta p_{t+1}$ and $\Delta p_{t+1} = \Delta p_t = \Delta p_{t-1}$, so that (71) implies that $\Delta p_t = \frac{\theta_3}{1-\theta_1-\theta_2} z_t$. Hence, the comparative static effect from z_t on Δp_t is $\frac{\theta_3}{1-\theta_1-\theta_2}$. It can be shown that this is equal to the element in C, $\frac{\delta}{(1-\lambda_1)}$, using that $\lambda_1 + \lambda_2 = \frac{1}{\theta_1}$ and $\lambda_1\lambda_2 = \frac{\theta_2}{\theta_1}$.

Compared to static models, these two examples show how dynamic theory models completely specify, not only β, but also the adjustment parameters, α and Γ_i. As a result, in the absence of shocks, the theory model determines the exact position of, say, (Q_t, P_t), at any t, in a diagram corresponding to Figure 2. In other words, the matrix $(I_r + \beta'\alpha)$, and hence, the process of market clearing, is fully specified by the economic model.

4.4 General Equilibrium

The framework established in Section 4.1 can readily be generalized to consider the important distinction in economics, between *partial*- and *general* equilibrium. It is well-known that general equilibrium comparative static effects may be radically different from the corresponding effects based on partial equilibrium - quantitatively but also *qualitatively*. As a result, even though we are only interested in the supply- and demand elasticities in one market, we might have to model the markets for related goods as well.

The basic ideas can be illustrated with a model with two markets, and as the example we could extend the partial equilibrium model (1) - (3), by including the labour market thereby endogenizing the wage, W. Instead, another equally simple theory model is considered, which illustrates exactly the same point.

Consider the markets for two related goods, chicken and beef, say, with quantities and prices denoted Q_1, Q_2, and P_1, P_2, respectively. The demand for Q_1, is related negatively to P_1, and positively to P_2 (Q_1 and Q_2 are substitutes). Supply depends positively on P_1, and negatively on some input price, denoted P_I.

The partial equilibrium model for market 1 assumes that P_2 and P_I are exogenous (resembling the model in Section 4.1), and is given by,

$$Q_1^d = \frac{d_0}{d_1} - \frac{1}{d_1}P_1 + \frac{d_2}{d_1}P_2, \tag{78}$$

$$Q_1^s = -\frac{e_0}{e_1} + \frac{1}{e_1}P_1 - \frac{e_2}{e_1}P_I, \tag{79}$$

$$Q_1^s = Q_1^d, \tag{80}$$

where, as before, all coefficients are positive. The chosen normalization on prices (divisions with d_1 and e_1) is purely notational, implying that the inverse demand expressions, which are the ones we draw, enter the cointegrating relations.

The assumption that the price, P_2, is exogenous, in the partial equilibrium model, implies that when P_2 changes there is no feed back on it from P_1, which seems unrealistic: An increase in P_2 shifts demand, Q_1^d, which will ignite an increase in P_1, which, in turn, will raise demand for good 2, causing a higher price P_2. This will further feed back positively on demand for good 1, so that the increase in P_1 would be reinforced, and so on. Hence, the partial equilibrium results are invalidated, and we need to include the market for good 2, i.e. impose general equilibrium.

To keep the exposition as simple as possible, I shall assume that the supply of good 2, Q_2^s, and P_I are economically exogenous. Hence, I retain the endogenous-exogenous dichotomy, but P_2 has become endogenous. The demand for good two is,

$$Q_2^d = \frac{f_0}{f_1} - \frac{1}{f_1}P_2 + \frac{f_2}{f_1}P_1, \tag{81}$$

and the equilibrium condition is,

$$Q_2^s = Q_2^d. \tag{82}$$

The general equilibrium model is described by (78) - (82), with P_I and Q_2^s as exogenous. Solving the model yields the general equilibrium,

$$\begin{aligned} Q_1^* &= \frac{d_0 + d_2(f_0 - f_1 Q_2) + (d_2 f_2 - 1)(e_0 - e_2 P_I)}{D}, \\ P_1^* &= \frac{(e_0 - e_2 P_I)d_1 + e_1(d_0 + d_2(f_0 - f_1 Q_2))}{D}, \\ P_2^* &= \frac{(e_1 + d_1)(f_0 - f_1 Q_2) + f_2((e_0 - e_2 P_I)d_1 + e_1 d_0)}{D}, \end{aligned} \tag{83}$$

where $D \equiv d_1 - e_1(d_2 f_2 - 1)$ is the determinant of the coefficient matrix to the system. Thus, the equilibrium exists, if and only if, $D \neq 0$, which is assumed. Again, the comparative static effects are readily computed as the partial derivatives with respect to P_I and Q_2 in (83).

The embedding of this theory model in the VAR can be done exactly as in Section 4.1, introducing the mappings from latent plans to the observable variables. For simplicity, the observation mapping for Q_{2t} is,

$$Q_{2t} = Q_{2t}^s, \tag{84}$$

and only one new adjustment coefficient, α_{33}, is introduced. Price- and quantity adjustment for the good 1 market are as before, so that, altogether, the CVAR(1) is,

$$\begin{aligned} \Delta Q_{1t} &= \alpha_{11}(P_1 - (d_0 - d_1 Q_1 + d_2 P_2))_{t-1} + \alpha_{12}(P_1 - (e_0 + e_1 Q_1 + e_2 P_I))_{t-1} + \varepsilon_{Q_1 t}, \\ \Delta P_{1t} &= \alpha_{21}(P_1 - (d_0 - d_1 Q_1 + d_2 P_2))_{t-1} + \alpha_{22}(P_1 - (e_0 + e_1 Q_1 + e_2 P_I))_{t-1} + \varepsilon_{P_1 t}, \\ \Delta P_{2t} &= \alpha_{33}(P_2 - (f_0 - f_1 Q_2 + f_2 P_1))_{t-1} + \varepsilon_{P_2 t}, \\ \Delta P_{It} &= \varepsilon_{P_I t}, \\ \Delta Q_{2t} &= \varepsilon_{Q_2 t}. \end{aligned} \tag{85}$$

This corresponds to the matrices,

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} d_1 & -e_1 & 0 \\ 1 & 1 & -f_2 \\ -d_2 & 0 & 1 \\ 0 & -e_2 & 0 \\ 0 & 0 & f_1 \end{pmatrix} \text{ and } s = \begin{pmatrix} -d_0 \\ -e_0 \\ -f_0 \end{pmatrix}, \tag{86}$$

in equation (18), with orthogonal complements,

$$\alpha_\perp = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \beta_\perp = \begin{pmatrix} \frac{(d_2 f_2 - 1)e_2}{D} & \frac{-d_2 f_1}{D} \\ \frac{d_1 e_2}{D} & \frac{-e_1 d_2 f_1}{D} \\ \frac{d_1 f_2 e_2}{D} & \frac{-(d_1 + e_1) f_1}{D} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{87}$$

and common trends and loadings (as $\det(\alpha'_\perp \beta_\perp) = 1$),

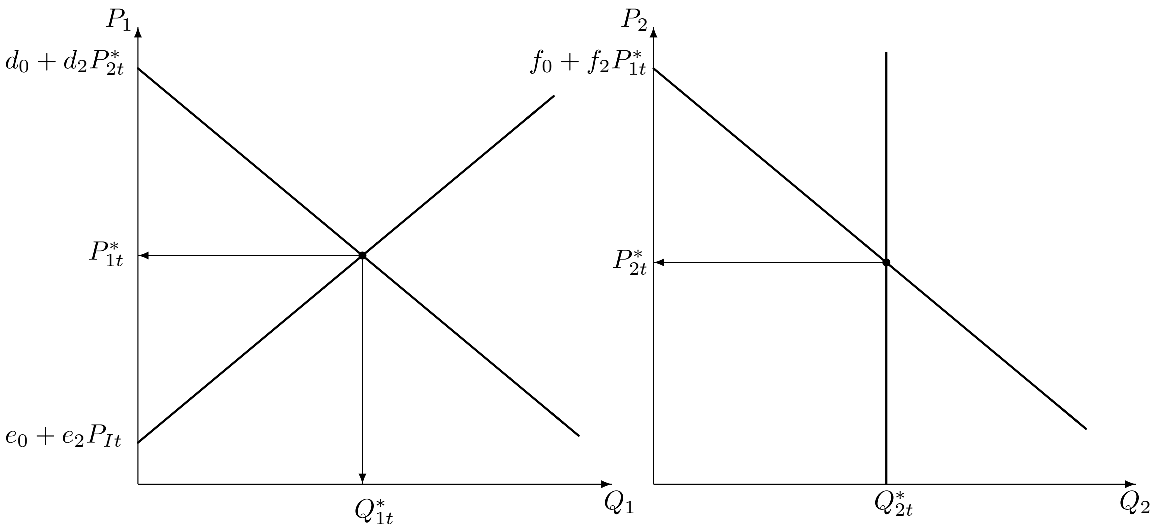

Figure 5: The Economy in General Equilibrium. Note how the demand curves are drawn for the equilibrium values of the price on the related market.

$$\mathrm{CT}_t = \begin{pmatrix} \Sigma_{i=1}^t \varepsilon_{P_I i} \\ \Sigma_{i=1}^t \varepsilon_{Q_2 i} \end{pmatrix}, \quad L = \beta_\perp, \tag{88}$$

resulting in the long-run matrix,

$$C = \begin{pmatrix} 0 & 0 & 0 & \frac{(d_2f_2-1)e_2}{D} & \frac{-d_2f_1}{D} \\ 0 & 0 & 0 & \frac{d_1e_2}{D} & \frac{-e_1d_2f_1}{D} \\ 0 & 0 & 0 & \frac{d_1f_2e_2}{D} & \frac{-(d_1+e_1)f_1}{D} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{89}$$

The general equilibrium at time t, for given values of the exogenous variables, P_I and Q_2 is illustrated in Figure 5.

Equations (86) - (89) show that the interpretations from Section 4.1 generalize straightforwardly: The economically exogenous variables become the common trends, and L and C capture the comparative static effects.

In terms of Figure 5, the transition from one equilibrium to the next, in the comparative static experiment, now involves a *sequence* of shifts in the curves, since the markets interact, as opposed to the model in Section 2. As before, this (static) theory model abstracts from this interaction altogether. It can be shown, that for the theory model to have a stable equilibrium, $D > 0$ is necessary, and gives reasonable comparative static effects, cf. Samuelson's Correspondence Principle (Samuelson 1941). This was also assumed in Section 2, as $a_1 + b_1 > 0$. This principle is discussed further in Section 5.

As opposed to the simple model in Section 2, this model involves two notions of equilibrium: The *short-run* equilibrium, i.e. the *equilibrium at time t*, which is involved in the sequential interaction between the markets, and the *long-run* equilibrium, or "steady state". In Section 2 these equilibrium concepts coincide. Furthermore, as before, this "steady state" is a moving (stochastic) equilibrium, when the shocks, $\varepsilon_{P_I t}$ and $\varepsilon_{Q_2 t}$, are introduced.

The C matrix in (89) demonstrates the central point in economics, that general equilibrium comparative statics might be qualitatively different from those in partial equilibrium: First, note that the partial equilibrium model corresponds to the restriction, $f_2 = 0$, in which case there is an influence from P_2 on P_1 but not vice versa. Now, consider the effect of, say, a supply shock, ε_{P_I}, on Q_1. The partial equilibrium effect, is $-\frac{e_2}{d_1+e_1}$, setting $f_2 = 0$ in the first row, fourth column in (89). This is unambiguously negative. In contrast, in general equilibrium, the effect is $\frac{(d_2 f_2 - 1)e_2}{D}$, which is negative only if $d_2 f_2 < 1$. In terms of the graphs, the partial equilibrium model shows the initial upward shift in the supply curve for good 1, and then the story ends. In general equilibrium, the resulting rise in P_1, spills over to the market for good 2, and shifts the demand curve on this market upwards, which feeds back and shifts demand for good 1 upwards etc.. Hence, in the wake of the shift in the supply curve in market 1, there is a sequence of upward shifts in the demand curve as well. If $d_2 f_2 > 1$, the sum of these demand shifts is greater than the initial supply shift, and Q_1^* will therefore rise.

5 Discussion and further Generalizations

So far, many practical econometric and theoretical issues have been disregarded, in order to provide an accessible and explicit exposition. A thorough discussion of all these issues is beyond the present scope, and instead a few remarks about the presented framework are given.

First, the approximation of persistence by I(1) can be generalized to I(d), $d > 1$. As an example, consider the simple static model in Section 4.2, in which more lags were added. For example, it could happen that W_t and Z_t were even more persistent, and hence, better approximated as I(2) than I(1). As a result, $\gamma_{ii} = 1$, and the MA representation in (50) would be invalid. Instead, the MA representation for I(2) processes would apply (Theorem 4.6, Johansen 1996). As before, the I(2) property would merely be an assumption about the statistical-, and not the theoretical parameters of interest.

Second, one should note that in the case when the endogenous-exogenous dichotomy is rejected the common trends no longer have the simple form, i.e. $\alpha'_\perp = (0, I_{p-r})$, as in (36) say. The common trends in CT_t, will now involve linear *combinations* of different cumulated shocks.

Third, as discussed in Section 4.1, the Taylor approximation (29) and (30), rests on stationarity of the equilibrium error, (43). However, given stationarity, the approximations may work better in some case than others. In general, it depends on the degree of non-linearity of the mappings, which probably depends on whether variables are in logarithms or not, whether transaction costs are negligible or not, etc. Moreover, the continuity assumption of $g()$ and $h()$, should also be viewed as a rough approximation, as transaction costs are likely to introduce *dis*continuous adjustment to disequilibria.

Fourth, Samuelson's Correspondence Principle may be related to the CVAR analysis above (Samuelson 1941). This simple but useful principle states that *stability* of the equilibrium implies comparative statics with "reasonable" signs. For example, above the equilibrium price, it is often assumed that supply exceeds demand, so that the price level falls, implying stability. This assumption, thus involves a restriction on the slopes of the demand-, and supply curves, which make comparative statics have reasonable signs (See Samuelson 1941). This correspondence is not needed for the CVAR models in sections 4.1 and 4.4, as stability depends on the α in addition to β, of which it is the latter that the principle concerns. As a result we can have stable equilibria with "strange" comparative statics.

Finally, in Section 3 it was argued that since the economically exogenous variables cause the endogenous variables and not vice versa, persistence in the system variables must originate exclusively from the former. There is however a special case of particular interest for which this is not the case. Consider the Hybrid New Keynesian Phillips Curve again (Section 4.3.2). Now, suppose instead that $0 < \theta_4 < 1$, so that the exogenous variable is I(0). As seen from (75), it is still possible to have $\det(\Pi) = 0$, i.e. a unit root, namely when $\lambda_1 = 1$. This condition, often invoked in the literature, requires that $\theta_2 = 1 - \theta_1$, and is referred to as dynamic homogeneity (Bårdsen et al. 2005). Substituting $\lambda_1 = 1$ into (73) we see that the hybrid Phillips curve relation becomes, $\Delta^2 p_t = \delta z_t + \breve{\varepsilon}_{pt}$. This means that the rate of *change* in inflation, $\Delta^2 p_t$, rather than the level of inflation, now depends on the level of the I(0) variable, z_t, and thus becomes I(0), implying that the level, Δp_t, itself is I(1), i.e. persistent.

6 Summary and Conclusion

In an attempt to bridge the gap between economic theory models and the CVAR, I have focused on facilitating the formulation and understanding of economic theory models as restrictions on the CVAR. As most economic models build on the same fundamental concepts, simple static- and dynamic theory models were considered to keep the exposition clear.

The hypothetical point of departure was a well-specified VAR as the statistical model, with some of the estimated roots close to unity, corresponding to persistence of the series. Under the endogenous-exogenous dichotomy of the theory model, this persistence originates from the generation of the exogenous variables which is outside the theory model. Hence, roots at unity, do not contradict the theory, and should be imposed, as an approximative assumption about the DGP to obtain reliable inference from short samples of persistent series.

Approximating the exogenous variables as I(1) unit root processes, static - and simple dynamic models were thus analyzed as restrictions on a CVAR. This established an explicit correspondence between the basic concepts of theory models and the econometric concepts of the CVAR.

This correspondence shows that: The theoretical relations, i.e. demand-, and supply relations, correspond to the cointegrating vectors. The concept of exogeneity in economics is stronger than the econometric concept of strong (and thus weak-) exogeneity for β. The existence of the economic equilibrium implies the existence of the attractor set, and the economic equilibrium correspond to the so-called long-run value. The comparative statics are captured by the long-run impact matrix, C. The common trends, which determine the long-run movement of the system variables, correspond to the exogenous variables in the economic model. The loadings matrix can be interpreted to describe how the slopes of the demand-, and supply curves determine the impact on the endogenous variables, from shifts in the curves (i.e. in the exogenous variables). The matrix, $I_r + \beta'\alpha$, and, in particular, its largest eigenvalue relates to the concept of market clearing, and interesting adjustment hypotheses (e.g. nominal rigidities etc.) can be related to this matrix. The examples of the dynamic theory models, also demonstrate how hypotheses about expectations are related to the adjustment parameters of the CVAR, α and Γ_i.

In a generalization of the basic framework the distinction between general-, and partial equilibrium was also related to the CVAR: It was shown how to investigate whether comparative statics in general equilibrium differ from those in partial equilibrium, and how the empirical validity of the partial equilibrium model can be tested in the general equilibrium model.

As alluded to, given explicit hypotheses derived from detailed microeconomic assumptions about optimization, information, expectations etc., this paper should, to some extent, facilitate the formulation of such hypotheses as restrictions on the CVAR.

Appendix A

Proof of $(I_p - \widetilde{\alpha}\widetilde{\beta}')$ being Invertible

Under the assumption that all characteristic roots, z, have either $|z| > 1$ or $z = 1$, and the "I(1)-assumption", $\det(\alpha_\perp' \beta_\perp) \neq 0$, it follows that $\rho(I_r + \beta'\alpha) < 1$, ($\rho(\cdot)$ is the spectral radius), which implies that $\det(\beta'\alpha) \neq 0$ (Johansen and Hansen 1998). This then implies that the $p \times p$ matrices $(\alpha, \beta_\perp)$ and $(\beta, \alpha_\perp)$ have full rank. One can then establish the identity,

$$I_p = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha(\beta'\alpha)^{-1}\beta', \tag{90}$$

(exercise 3.7, Johansen 1996).

Using (90), the assumption that $\det(I_r + \beta'\alpha) \neq 0$, that $\widetilde{\alpha} = \alpha(I_r + \beta'\alpha)^{-1}$ and $\widetilde{\beta} = \beta$, we can write,

$$\begin{aligned} I_p - \widetilde{\alpha}\widetilde{\beta}' &= \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha(\beta'\alpha)^{-1}\beta' - \alpha(I_r + \beta'\alpha)^{-1}\beta' \qquad (91) \\ &= \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha\left[(\beta'\alpha)^{-1} - (I_r + \beta'\alpha)^{-1}\right]\beta' \\ &= \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha\left[(\beta'\alpha)^{-1}((I_r + \beta'\alpha) - \beta'\alpha)(I_r + \beta'\alpha)^{-1}\right]\beta' \\ &= \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' + \alpha((I_r + \beta'\alpha)(\beta'\alpha))^{-1}\beta' \\ &= \begin{pmatrix} \alpha((I_r + \beta'\alpha)(\beta'\alpha))^{-1}, & \beta_\perp(\alpha_\perp'\beta_\perp)^{-1} \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha_\perp' \end{pmatrix}. \end{aligned}$$

Since $\alpha((I_r + \beta'\alpha)(\beta'\alpha))^{-1}$ is in $sp(\alpha)$, $\beta_\perp(\alpha_\perp'\beta_\perp)^{-1}$ is in $sp(\beta_\perp)$, and $(\alpha, \beta_\perp)$ and $(\beta, \alpha_\perp)$ each span the whole of R^p, it follows that both matrices in the last line of (91) have full rank which establishes the result.□

References

Alogoskoufis, G. and R. Smith (1991). On error correction models: Specification, interpretation, estimation. *Journal of Economic Surveys 5*(1), 97–128.

Bårdsen, G., Ø. Eitrheim, E. S. Jansen, and R. Nymoen (2005). *The Econometrics of Macroeconomic Modelling*. Oxford University Press.

Chiang, A. C. and K. Wainwright (2005). *Fundamental methods of mathematical economics* (Fourth ed.). McGraw-Hill.

Dennis, J. G., H. Hansen, and K. Juselius (2006). *CATS in RATS. Cointegration analysis of time series, Version 2*. Evanston, Illinois, USA: Estima.

Engle, R. F. and C. W. J. Granger (1987, March). Co-integration and error correction: Representation, estimation, and testing. *Econometrica 55*(2), 251–76.

Engle, R. F., D. F. Hendry, and J.-F. Richard (1983, March). Exogeneity. *Econometrica 51*(2), 277–304.

Ericsson, N. R., D. F. Hendry, and G. E. Mizon (1998, October). Exogeneity, cointegration, and economic policy analysis. *Journal of Business and Economic Statistics 16*(4), 370–87.

Gali, J. and M. Gertler (1999, October). Inflation dynamics: A structural econometric analysis. *Journal of Monetary Economics 44*(2), 195–222.

Granger, C. W. J. (1981, May). Some properties of time series data and their use in econometric model specification. *Journal of Econometrics 16*(1), 121–130.

Hendry, D. F. (1995). *Dynamic Econometrics*. Oxford University Press.

Hoover, K., K. Juselius, and S. Johansen (2007). Allowing the data to speak freely: The macroeconometrics of the cointegrated vector autoregression. Discussion paper, University of Copenhagen.

Intriligator, M. D. (1983, January). Economic and econometric models. In Z. GrilichesâĂ and M. D. Intriligator (Eds.), *Handbook of Econometrics*, Volume 1, Chapter 3, pp. 181–221. Elsevier.

Johansen, S. (1992, June). Testing weak exogeneity and the order of cointegration in uk money demand data. *Journal of Policy Modeling 14*(3), 313–334.

Johansen, S. (1996). *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*. Oxford: Advanced Texts in Econometrics, Oxford University Press.

Johansen, S. (2005). Interpretation of Cointegrating Coefficients in the Cointegrated Vector Autoregressive Model. *Oxford Bulletin of Economics and Statistics 67*(1), 93–104.

Johansen, S. (2006). Confronting the economic model with the data. In D. Colander (Ed.), *Post Walrasian Macroeconomics: Beyond the Dynamic Stochastic General Equilibrium Model*, Volume 1, Chapter 15, pp. 287–300. Cambridge University Press.

Johansen, S. and P. R. Hansen (1998). *Workbook on Cointegration*. Oxford University Press.

Juselius, K. (2006). *The cointegrated VAR model: Econometric methodology and macroeconomics applications*. Oxford University Press.

Laroque, G. and B. Salanie (1995). Macroeconometric disequilibrium models. In H. Pesaran and M. Wickens (Eds.), *Handbook of Applied Econometrics*, Chapter 8, pp. 391–414. Basil Blackwell.

Møller, N. F. and P. Sharp (2008). Malthus in cointegration space: A new look at living standards and population in pre-industrial england. Discussion Papers 08-16, University of Copenhagen.

Nickell, S. (1985, May). Error correction, partial adjustment and all that: An expository note. *Oxford Bulletin of Economics and Statistics 47*(2), 119–29.

Roberts, J. M. (1995, November). New keynesian economics and the phillips curve. *Journal of Money, Credit and Banking 27*(4), 975–84.

Samuelson, P. A. (1941). The stability of equilibrium: Comparative statics and dynamics. *E 9*(2), 97–120.

Sørensen, P. B. and H. J. Whitta-Jacobsen (2005). *Introducing advanced macroeconomics : Growth and business cycles*. McGraw-Hill.

Economics
The Open-Access, Open-Assessment E-Journal

Please note:

You are most sincerely encouraged to participate in the open assessment of this article. You can do so by either rating the article on a scale from 1 (bad) to 5 (excellent) or by posting your comments.

Please go to:

www.economics-ejournal.org/economics/journalarticles/2008-36

The Editor