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# Destroying Longest Cycles in Graphs and Digraphs

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## Abstract

In 1978, C. Thomassen [Hypohamiltonian graphs and digraphs, *Proceedings of the International Conference on the Theory and Applications of Graphs*, Kalamazoo, 1976, Springer Verlag, pp. 557 – 571] proved that in any graph one can destroy all the longest cycles by deleting at most one third of the vertices. We show that for graphs with circumference  $k \leq 8$  it suffices to remove at most  $1/k$  of the vertices. The Petersen graph demonstrates that this result cannot be extended to include  $k = 9$  but we show that in every graph with circumference nine we can destroy all 9-cycles by removing  $1/5$

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of the vertices. We consider the analogous problem for digraphs and show that for digraphs with circumference  $k = 2, 3$ , it suffices to remove  $1/k$  of the vertices. However this does not hold for  $k \geq 4$ .

*Key words:* longest cycle, circumference, vertex deletion  
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## 1. Introduction

If  $G$  is a graph or digraph we denote its order (number of vertices) by  $n(G)$ . The *circumference* of  $G$ , denoted by  $c(G)$ , is defined as the length of a longest cycle in  $G$  if  $G$  has a cycle, and  $c(G) = 0$  if  $G$  is acyclic. (By a cycle in a digraph we mean a directed cycle.) If  $G$  is nonhamiltonian but removing any single vertex from  $G$  results in a hamiltonian graph, we say that  $G$  is *hypohamiltonian*.

Thomassen [9] proved the following.

**Theorem 1.1.** [9] *If  $G$  is a graph of order  $n$  with circumference  $k \geq 3$ , then there exists a set  $A \subset V(G)$  such that  $|A| \leq n/3$  and  $A$  meets all longest cycles in  $G$ .*

Thus, in any graph one can destroy all the longest cycles by deleting at most one third of the vertices. This result is best possible for graphs with circumference 3, but we shall prove that for graphs with larger circumference the desired result may be achieved with a smaller fraction of the vertices. This motivates the following definition.

**Definition 1.2.** *For each  $k \geq 3$ , let  $\alpha(k)$  denote the smallest number such that every graph  $G$  with circumference  $k$  contains a set  $A$  of  $\lfloor \alpha(k)n(G) \rfloor$  vertices such that  $G - A$  has no  $k$ -cycles.*

Theorem 1.1 shows that  $\alpha(k) \leq 1/3$  for all  $k$ . In Section 2 we extend this result by showing that  $\alpha(k) = 1/k$  for  $k \leq 8$ . However, we also show that  $\alpha(9) = 1/5$ . In fact, the existence of hypohamiltonian graphs shows that  $\alpha(k) \geq 2/(k+1)$  for all but finitely many  $k$ . Furthermore, constructions by Zamfirescu [10] and Grünbaum [7] show that  $\alpha(k) > 2/(k+1)$  for some  $k$ .

In Section 3 we consider the analogous problem for digraphs. We first show that in a digraph with circumference  $k$  we can destroy *all* the cycles by deleting a solely  $k$ -dependent fraction of the vertices. Thus we can define the directed analogue of  $\alpha(k)$  as follows.

**Definition 1.3.** For each  $k \geq 2$ , let  $\vec{\alpha}(k)$  denote the smallest number such that every digraph  $D$  with circumference  $k$  contains a set  $A$  of  $\lfloor \vec{\alpha}(k)n(D) \rfloor$  vertices such that  $D - A$  has no  $k$ -cycles.

We show that  $\vec{\alpha}(k) \leq (k-1)/k$  for all  $k \geq 2$  and that  $\vec{\alpha}(k) = 1/k$  for  $k = 2, 3$ ,  $\vec{\alpha}(4) \geq 1/3$  and  $\vec{\alpha}(k) \geq 2/(k+1)$  for every  $k \geq 5$ .

## 2. Destroying longest cycles in graphs

For undefined concepts we refer the reader to [4].

The following lemma will enable us to determine  $\alpha(k)$  for certain values of  $k$ .

**Lemma 2.1.** Suppose  $k \geq 3$  is an integer such that every 2-connected graph with circumference  $k$  has a vertex that meets every  $k$ -cycle. Then  $\alpha(k) = 1/k$ .

*Proof.* It is obvious that  $\alpha(k) \geq 1/k$ . (Consider any number of disjoint  $k$ -cycles.) To prove that  $\alpha(k) \leq 1/k$ , let  $G$  be a graph of order  $n$  and circumference  $k$ . We prove by induction on  $n$  that  $G$  contains a set  $A$  with  $|A| \leq n/k$  such that  $G - A$  has no  $k$ -cycles. If  $G$  is disconnected, the result follows immediately from our induction hypothesis. If  $G$  is 2-connected, the result follows from our assumption on  $k$ . Thus we assume that  $G$  is a connected graph with more than one block. Let  $B$  be an end-block of  $G$  and let  $v$  be the cut-vertex of  $G$  in  $B$ . Put

$$F = G - V(B - v), \text{ and } F' = G - V(B) = F - v.$$

If  $c(B) < k$ , the result follows immediately by applying our induction hypothesis to  $F$ . Thus we assume that  $c(B) = k$ . We consider two cases.

**Case 1.**  $n(B) = k$ .

In this case  $n(F') = n - k$ , so by our induction hypothesis,  $V(F')$  contains a set  $A'$  such that  $F' - A'$  has no  $k$ -cycles and  $|A'| \leq (n - k)/k$ . But  $B - v$  has no  $k$ -cycles since it has only  $k - 1$  vertices, so we put  $A = \{v\} \cup A'$ . Then  $|A| \leq n/k$  and  $G - A$  has no  $k$ -cycles.

**Case 2.**  $n(B) \geq k + 1$ .

In this case  $n(F) \leq n - k$ , so our induction hypothesis implies that  $V(F)$  contains a set  $A'$  such that  $F - A'$  has no  $k$ -cycles and  $|A'| \leq (n - k)/k$ . Furthermore, our assumption on  $k$  implies that  $B$  has a vertex  $x$  that lies on every  $k$ -cycle in  $B$ . Thus the set  $A = \{x\} \cup A'$  has the desired property.  $\square$

We shall show that every  $k \leq 8$  satisfies the condition of Lemma 2.1. Our proof uses the following result of Holton, McKay, Plummer and Thomassen, known as the “Nine Point Theorem”.

**Theorem 2.2.** [8] *Every 3-connected cubic graph contains a cycle through any nine prescribed vertices.*

**Corollary 2.3.**

- (a) *Every 3-connected graph of order at least 10 has circumference at least 9.*
- (b) *The Petersen graph is the only 3-connected cubic graph of order at least 10 that has circumference less than 10.*

*Proof.* Theorem 2.2 immediately implies (a). To prove (b), suppose  $G$  is a 3-connected cubic graph with  $n(G) \geq 10$  and  $c(G) \leq 9$ . Let  $H$  be an induced subgraph of  $G$  with  $n(H) = 10$ . Then Theorem 2.2 implies that  $H$  is hypohamiltonian. But the Petersen graph is the only hypohamiltonian graph of order 10, so  $H$  is the Petersen graph and since  $G$  is cubic,  $G$  itself is the Petersen graph.  $\square$

It is well-known that the Petersen graph is the smallest nonhamiltonian 3-connected cubic graph. By considering each of the four 3-connected cubic graphs of order 8, we observe the following.

**Observation 2.4.** *Let  $H$  be a 3-connected cubic graph of order 8 and let  $I$  be a set of three independent edges in  $H$ . Then  $H$  has a Hamilton cycle that contains at least two of the edges in  $I$ .*

Next we state three observations concerning longest cycles in graphs.

**Observation 2.5.** *Any two longest cycles in a 2-connected graph have at least two vertices in common.*

**Observation 2.6.** *Let  $C$  be a longest cycle in a graph  $G$  and let  $u$  and  $w$  be two distinct vertices on  $C$ . If  $P$  is a  $u - w$  path whose internal vertices are in  $G - V(C)$ , then each of the two  $u - w$  paths on  $C$  is at least as long as  $P$ .*

**Observation 2.7.** *Let  $C = u_1 \dots u_k u_1$  be a longest cycle in a graph  $G$ . Let  $v \in V(G) - V(C)$  and let  $u_i$  and  $u_j$  ( $i \neq j$ ) be two neighbours of  $v$  on  $C$ . Then any  $u_{i+1} - u_{j+1}$  path (similarly, any  $u_{i-1} - u_{j-1}$  path) has at least one internal vertex on  $C$ .*

The following form of Menger's Theorem is well-known (see [4], Lemma 9.4).

**Theorem 2.8.** *Let  $G$  be a  $k$ -connected graph and let  $X$  and  $Y$  be subsets of  $V(G)$  of cardinality at least  $k$ . Then there exists in  $G$  a family of  $k$  pairwise disjoint  $(X, Y)$ -paths.*

**Corollary 2.9.** *Let  $v$  be a vertex in a 3-connected graph  $G$  and  $C$  a cycle in  $G - v$ . Then, for any two neighbours  $x_1$  and  $x_2$  of  $v$ , there are three paths  $P, Q_1, Q_2$  from  $v$  to  $C$  that are pairwise disjoint except for  $v$ , such that  $Q_i$  contains the edge  $vx_i$ ,  $i = 1, 2$ .*

*Proof.* It follows from Theorem 2.8 that there are three vertex disjoint paths  $P, Q'_1, Q'_2$  from the set  $\{v, x_1, x_2\}$  to  $C$ . (If  $x_i$  lies on  $C$  for  $i = 1$  or  $2$ , the corresponding path is a single vertex.) Let  $Q_i$  be the concatenation of the edge  $vx_i$  with the path  $Q'_i$ ,  $i = 1, 2$ . Then  $P, Q_1, Q_2$  are the required paths.  $\square$

**Theorem 2.10.** *Let  $G$  be a 2-connected graph with circumference  $c(G) \leq 9$ . Then  $G$  has a vertex meeting all longest cycles unless  $G$  is the Petersen graph.*

*Proof.* Let  $3 \leq k \leq 9$  and suppose that  $G$  is a smallest counterexample to the theorem for the case  $c(G) = k$ .

We claim that  $G$  is a subdivision of a 3-connected graph. For this, let us consider a separating set of two vertices  $x, y$ . We consider a  $k$ -cycle  $C_1$  in  $G - x$  and a  $k$ -cycle  $C_2$  in  $G - y$ . By Observation 2.5,  $C_1 - y$  and  $C_2 - x$  are in the same component of  $G - x - y$ . Call this component  $Q$ .

Let  $P$  be a longest  $x - y$  path in  $G - V(Q)$ . We claim that  $G - V(Q) = P$ . If not, then the union  $G'$  of  $Q, x, y$  and  $P$  is smaller than  $G$ . By the minimality of  $G$ , the subgraph  $G'$  has a vertex  $v$  such that  $G' - v$  has no  $k$ -cycles. However,  $G - v$  has a  $k$ -cycle, say  $C$ . If  $C$  intersects  $Q$ , we can modify it to a cycle of length  $k$  in  $G' - v$  (by the maximality of  $P$ ). Also, if  $C$  is disjoint from  $Q$ , then it has only one vertex in common with  $C_1$  or  $C_2$ , contradicting Observation 2.5.

This proves that  $G$  is a subdivision of a 3-connected graph  $H$ .

For the remainder of the proof we only consider the case  $k = 9$ . (The case  $k < 9$  is similar and easier.)

**Claim 1.** *The edges in  $H$  which are subdivided form a matching.*

*Proof of (1):* Suppose to the contrary that there are two distinct subdivided edges  $vx_1$  and  $vx_2$  in  $H$  which share a common vertex  $v$ . Let  $C$  be a

longest cycle in  $G - v$ . Since  $H$  is 3-connected, it follows from Corollary 2.9 that there are three paths from  $v$  to  $C$  that are mutually disjoint except for  $v$  such that two of them contain the subdivided edges. But then it follows from Observation 2.6 that  $C$  has more than 9 vertices.

By the same argument we prove:

**Claim 2.** *No edge of  $H$  is a path of length more than 2 in  $G$ .*

**Claim 3.** *Let  $u$  and  $w$  be two distinct vertices on a 9-cycle  $C$  of  $G$ , and suppose  $P$  is a  $u - w$  path whose internal vertices are in  $G - V(C)$ . Then  $P$  has at most 2 internal vertices.*

*Proof of (3):* By Observation 2.6,  $P$  does not have more than 3 internal vertices. So suppose  $P$  is a path  $ux_1x_2x_3w$ . If for some  $i \in \{1, 2, 3\}$  there is a path  $Q$  from  $x_i$  to a vertex  $v$  on  $C - \{u, w\}$  with no internal vertex on  $V(C) \cup V(P)$ , then the concatenation of the  $u - x_i$  subpath of  $P$  or the  $w - x_i$  subpath of  $P$  with  $Q$  violates Observation 2.6. Hence, for  $i = 1, 2, 3$ , every path from  $x_i$  to  $C$  contains either  $u$  or  $w$ . But  $H$  is 3-connected, so  $x_i \notin V(H)$  for  $i = 1, 2, 3$ . This contradicts Claim 2.

**Claim 4.**  *$G$  has no vertex of degree 5 or more.*

*Proof of (4):* Suppose  $v$  is a vertex of degree at least five in  $G$  and let  $C$  be a 9-cycle in  $G - v$ . By Observation 2.6,  $v$  cannot have five neighbours on  $C$ . By Claim 3 and the fact that  $G$  is 2-connected,  $v$  cannot have two neighbours in  $G - V(C)$ . Hence  $v$  has exactly four neighbours on  $C$  and one neighbour  $x$  in  $G - V(C)$ . But then there is no path from  $x$  to  $C$  that does not contain  $v$ , for otherwise it follows from Observation 2.6 that  $G$  has a cycle of order bigger than nine.

**Claim 5.** *If  $v$  is a vertex of degree 4 in  $G$ , then  $v$  is not incident with a subdivided edge.*

*Proof of (5):* Let  $C$  be a 9-cycle in  $G - v$ . Suppose to the contrary that  $vw$  is an edge in  $H$  that is subdivided by a vertex  $x$  in  $G$ . By Claim 3, the four neighbours of  $v$  in  $H$  lie in  $C$ . But then by Observation 2.6,  $C$  has more than 9 vertices, a contradiction.

Since the Petersen graph is the only hypohamiltonian graph of order 10, we may assume that  $n(G) \geq 11$ . Since  $H$  is 3-connected, every vertex in  $H$  has degree at least 3.

Now suppose  $H$  is cubic. If  $n(H) \leq 6$ , then Claims 1 and 2 imply that  $n(G) \leq 9$ , a contradiction. If  $n(H) = 8$ , then  $n(G) \leq 10$ , since it follows from Observation 2.4 that if three independent edges of  $H$  are subdivided, then a cycle of length at least 10 is created. If  $n(H) \geq 10$ , then by Corollary 2.3,  $H$  is the Petersen graph. But then  $G$  itself is the Petersen graph, since

subdividing any edge of the Petersen graph creates a 10-cycle. Thus we may assume that  $H$  is not cubic.

Now let  $v$  be a vertex of degree 4 in  $G$  (and  $H$ ). Let  $C_0$  be a 9-cycle in  $G - v$ .

Consider first the case where  $v$  has four neighbors on  $C_0$ . Then by Observation 2.6, the notation can be chosen such that  $C_0 = v_1v_2\dots v_9v_1$  and  $v$  is joined to  $v_1, v_3, v_5, v_7$ .

Let  $U = \{v_2, v_4, v_6, v_8, v_9\}$  and  $W = \{v_1, v_3, v_5, v_7\}$ . Since  $n(G) \geq 11$  and  $G$  is 2-connected, there are two distinct vertices  $v_i, v_j$  on  $C_0$  such that each has a neighbour in  $V(G) - V(C_0) - v$ . It follows from Observations 2.6 and 2.7 that  $i, j \notin \{2, 4, 6\}$ . Suppose  $v_9$  has a neighbour  $x \in V(G) - V(C_0) - v$ . Then there is a path  $P$  from  $x$  to  $C_0 - v_9$ . It is easy to see that since  $c(G) = 9$ ,  $P = xv_7$ . By Claim 1,  $H$  contains at least one of  $v_8$  and  $x$ , say  $v_8$ . But then, since  $H$  is 3-connected, there is a path from  $v_8$  to  $C_0 - \{v_7, v_9\}$ . Since  $c(G) = 9$ , this is not possible. Hence  $v_9$  has no neighbour in  $V(G) - V(C_0)$ , and the same holds for  $v_8$ . Hence  $v_i, v_j \in W$ . Moreover, it follows from Claims 1 and 2 that at least three vertices in  $U$  are in  $H$  and hence adjacent to at least three vertices in  $W$ . But then  $W$  has a vertex of degree at least 5, contradicting Claim 4.

Hence  $v$  has less than four neighbours on  $C_0$ . It now follows from Claim 3 that  $v$  has exactly one neighbour  $u$  in  $V(G) - V(C_0)$  and three neighbours on  $C_0$ . By Claim 5,  $u$  is in  $H$  and hence  $u$  has at least two neighbours on  $C$  (by Claim 3). By Observation 2.6, we may assume that the neighbours of  $v$  on  $C$  are  $v_1, v_4, v_7$  and the neighbours of  $u$  on  $C$  are  $v_4, v_7$ . But then, since  $c(G) = 9$ , it follows that both  $v_2$  and  $v_9$  have degree 2 in  $G$ , contradicting Claim 1.  $\square$

**Theorem 2.11.**

1.  $\alpha(k) = 1/k$  for  $k = 1, \dots, 8$ .
2.  $\alpha(9) = 1/5$ .
3.  $\alpha(k) \geq 2/(k + 1)$  for  $k \in \{12, 14, 15\}$  and for every  $k \geq 17$ .

*Proof.*

1. This follows immediately from Lemma 2.1 and Theorem 2.10.

2. Let  $G$  be a graph of order  $n$  and circumference 9. We prove by induction on  $n$  that  $G$  contains a set of  $n/5$  vertices that meets every 9-cycle in  $G$ . If  $G$  is disconnected, this follows by applying induction to each component of  $G$ . If  $G$  is 2-connected, the result follows from Theorem 2.10. Thus we assume that  $G$  is connected and has more than one block. Let  $B$  be an end-block of  $G$ , let  $v$  be the cut-vertex of  $G$  in  $B$  and put

$$F = G - V(B - v), \text{ and } F' = G - V(B) = F - v.$$

If  $n(B) \leq 9$ , the proof is obvious, so we assume  $n(B) \geq 10$ .

If  $B$  is not the Petersen graph, then by Theorem 2.10 it has a vertex that meets every 9-cycle in  $B$ . In this case we apply our induction hypothesis to  $F$  and deduce that  $V(G)$  has a set  $A$  whose removal destroys all the 9-cycles in  $G$  such that  $|A| = 1 + (n - n(B) + 1)/5 < n/5$ .

Now suppose  $B$  is the Petersen graph. Then  $n(F') = n - 10$  and, by our induction hypothesis,  $V(F')$  has a subset  $A$  with  $|A| = (n - 10)/5$  such that  $F' - A$  has no 9-cycles. Thus, if  $w$  is any vertex in  $V(B) - \{v\}$ , then  $G - (A \cup \{v, w\})$  has no 9-cycles and  $|A \cup \{v, w\}| = n/5$ .

3. This follows immediately from the fact that there exists a hypohamiltonian graph of order  $n$  for every  $n \geq 10$  except for  $n \in \{11, 12, 14, 17\}$  – see [2].

□

We also know that  $\alpha(k) > 2/(k + 1)$  for some  $k$ . For example, Grünbaum [7] constructed a 3-connected graph of order 90 and circumference 72 with the property that at least 3 vertices need to be removed in order to destroy all the longest cycles. Zamfirescu [10] found that Grünbaum's graph can be contracted to a graph of order 75 and circumference 63 with the same property. Hence  $\alpha(72) \geq 3/90 > 2/73$  and  $\alpha(63) \geq 3/75 > 2/64$ .

The following questions remain open.

### Questions

1. Does there exist for any  $k \in \{10, 11, 13, 16\}$  a 2-connected graph with circumference  $k$  that has no vertex meeting every  $k$ -cycle? (We know that if such a graph exists, its order will be at least  $k + 2$ .)
2. Does there exist a 2-connected graph with the property that more than three vertices need to be removed in order to destroy all the longest cycles?

### 3. Destroying longest cycles in digraphs

A directed path (directed walk) in a digraph is simply called a path (walk). An  $x \rightsquigarrow y$  path ( $x \rightsquigarrow y$  walk) refers to a path (walk) with initial vertex  $x$  and terminal vertex  $y$ . An *oriented graph* is a digraph without 2-cycles. For undefined digraph concepts we refer the reader to [3].

First we prove a general result concerning the destruction of *all* cycles in a digraph.

**Theorem 3.1.** *If  $D$  is a digraph of order  $n$  with  $2 \leq c(D) \leq k$ , then  $V(D)$  has a subset  $A$  such that  $D - A$  is acyclic and  $|A| \leq (k - 1)n/k$ .*

*Proof.* Our proof is by induction on  $n$ . The case  $n = 2$  is trivial. Since  $c(D) \leq k$ , it follows that  $D$  has a vertex  $v$  such that  $d^+(v) \leq k - 1$ . Let  $A_1$  be a subset of  $V(D) - \{v\}$  such that  $|A_1| = k - 1$  and  $N^+(v) \subseteq A_1$ . Let  $F = D - (A_1 \cup \{v\})$ . Then  $n(F) = n - k$  and hence, by our induction hypothesis, there is a set  $A_2 \subseteq V(F)$  with  $|A_2| \leq (k - 1)(n - k)/k$  such that  $F - A_2$  is acyclic. Then the graph induced by  $V(F - A_2) \cup \{v\}$  is also acyclic, since  $v$  has no out-neighbours in  $F - A_2$ . Thus, if we put  $A = A_1 \cup A_2$ , then  $D - A$  is acyclic and  $|A| \leq k - 1 + (k - 1)(n - k)/k = (k - 1)n/k$ .  $\square$

**Corollary 3.2.**  $\vec{\alpha}(k) \leq (k - 1)/k$  for every  $k \geq 2$ .

It follows from Corollary 3.2 that  $\vec{\alpha}(2) = 1/2$ . We shall characterize strong digraphs with circumference 3 and then use our characterization to prove that  $\vec{\alpha}(3) = 1/3$ .

First, we define an *oriented basis family*  $\mathcal{B}$ .

**Definition 3.3.**  $B \in \mathcal{B}$  if and only if each of the following holds.

- (a)  $B$  is a strong oriented graph with circumference 3.
- (b)  $B$  has a vertex  $x$  (called a central vertex) such that  $x$  is contained in every 3-cycle of  $B$ .

Next, we define two operations by which two digraphs can be combined to form a new digraph. Let  $D_1$  and  $D_2$  be two disjoint digraphs. If a vertex  $v_1$  of  $D_1$  is identified with a vertex  $v_2$  of  $D_2$ , we say that the resulting digraph is obtained by *attaching*  $D_1$  to  $D_2$  (at the vertex  $v_1$ ). If we identify an arc  $a_1$  of  $D_1$  with an arc  $a_2$  of  $D_2$ , we say that the resulting digraph is obtained by *glueing*  $D_1$  to  $D_2$  (at the arc  $a_1$ ).

Finally, we define a family  $\mathcal{F}$  of digraphs as follows.

**Definition 3.4.**  $F \in \mathcal{F}$  if and only if one of the following holds.

- (a)  $F \in \mathcal{B} \cup \{\vec{C}_2\} \cup K_1$ .
- (b)  $F$  is obtained from two digraphs  $F_1, F_2 \in \mathcal{F}$  by attaching or glueing  $F_1$  to  $F_2$  in such a way that no cycles of length greater than 3 are created.

The following results concerning the structure of oriented graphs in  $\mathcal{B}$  follow immediately from the definition of  $\mathcal{B}$  and the observation that in a strong oriented graph with circumference 3, every vertex and every arc lie on a 3-cycle.

**Proposition 3.5.** *Let  $x$  be a central vertex of a strong oriented graph  $B \in \mathcal{B}$ . Then each of the following hold.*

- (a) *Every vertex in  $B - x$  is a neighbour of  $x$ .*
- (b)  *$B$  is a tripartite oriented graph with partite sets  $\{x\}, N^+(x), N^-(x)$ .*
- (c) *Every vertex in  $N^+(x)$  has at least one out-neighbour in  $N^-(x)$  and every vertex in  $N^-(x)$  has at least one in-neighbour in  $N^+(x)$ .*
- (d) *There are no arcs from  $N^-(x)$  to  $N^+(x)$ .*

Following [3], we denote the underlying graph of a digraph  $D$  by  $UG(D)$ . We call a 3-cycle in an undirected graph a *triangle*. From Proposition 3.5 we observe the following.

**Lemma 3.6.** *If  $B \in \mathcal{B}$ , then every triangle in  $UG(B)$  corresponds to a (directed) 3-cycle in  $B$ , i.e., three vertices lie on a triangle in  $UG(B)$  if and only if they lie on a 3-cycle in  $B$ .*

We shall prove that every strong digraph with circumference at most 3 is a member of  $\mathcal{F}$ . The following easy observation will play a key role in the proof.

**Lemma 3.7.** *Let  $D$  be a digraph with  $c(D) = 3$  and let  $T$  be a 3-cycle in  $D$ . Then at least one of the arcs of  $T$  is not contained in any 3-cycle other than  $T$ .*

Our next two lemmas will enable us to employ induction on the number of arcs.

**Lemma 3.8.** *If  $D$  is a strong digraph with  $\delta(D) \geq 3$ , then  $D$  has an arc  $a$  such that  $D - a$  is strong.*

*Proof.* Since  $D$  contains a cycle, it has a proper subdigraph which is strong. Now let  $D'$  be a maximal proper strong subdigraph of  $D$ . If  $V(D') = V(D)$ , then clearly  $D' = D - a$  for some  $a \in A(D)$ . Thus we assume  $V(D) - V(D') \neq \emptyset$ . Then there is a vertex  $v$  in  $V(D) - V(D')$  that has an in-neighbour  $x$  in  $D'$ . Now consider the shortest path  $P$  from  $v$  to  $D'$ . The arc  $xv$  followed by the path  $P$  is a path or a cycle of length at least 2. The subdigraph of  $D$  induced by the arc set  $A(D') \cup A(P) \cup \{xv\}$  is obviously strong and it is proper since its minimum degree is 2. This contradicts the maximality of  $D'$ .  $\square$

**Lemma 3.9.** *Suppose  $D$  is a strong digraph of order at least 3 without a cut-vertex. If  $D$  contains a 2-cycle  $uvu$ , then at least one of  $D - uv$  and  $D - vu$  is strong.*

*Proof.* If  $D - uv$  is not strong, then  $V(D)$  can be partitioned into two sets  $A, B$  such that no arc in  $D$  goes from  $B$  to  $A$ . Similarly, if  $D - vu$  is not strong, then  $V(D)$  has a partition  $X, Y$  such that no arc in  $D$  goes from  $Y$  to  $X$ . Thus we have a partition of  $V(D)$  into four sets, of which only the two containing  $u$  and  $v$  are nonempty and  $uv, vu$  are the only arcs joining them. But then  $u$  or  $v$  is a cut-vertex, which contradicts our assumption.  $\square$

**Theorem 3.10.**  *$D$  is a strong digraph with  $c(D) \leq 3$  if and only if  $D \in \mathcal{F}$ .*

*Proof.* If  $D \in \mathcal{F}$  then it follows from Definition 3.4 that  $c(D) \leq 3$  and also that every arc of  $D$  lies on a 2-cycle or a 3-cycle, so  $D$  is strong.

We prove the converse by means of induction on the size  $m$  of  $D$ . The result obviously holds for  $m = 3$ .

If  $c(D) \leq 2$ , then the underlying graph of  $D$  is a tree and every block of  $D$  is a 2-cycle, so  $D \in \mathcal{F}$ . Thus we assume  $c(D) = 3$ .

If  $D$  has a cut-vertex  $y$ , let  $D - y = D_1 \cup D_2$ . Then for  $i = 1, 2$ , the digraph  $D - V(D_i)$  is obviously strong and hence, by our induction hypothesis,  $D - V(D_i) \in \mathcal{F}$ . But then it follows from Definition 3.4 that  $D \in \mathcal{F}$ . Thus we may assume that  $D$  has no cut-vertex.

Now suppose  $D$  contains a 2-cycle  $uvu$ . Then, in view of Lemma 3.9, we may assume that  $D - uv$  is strong, so, by our induction hypothesis,  $D - uv \in \mathcal{F}$ . Since  $D$  can be obtained from  $D - uv$  by glueing a 2-cycle to

the arc  $vu$ , it then follows that  $D \in \mathcal{F}$ . Thus we may assume that  $D$  is an oriented graph.

**Case 1.**  $\kappa(UG(D)) = 2$ .

Let  $\{y_1, y_2\}$  be a separating set of vertices in  $UG(D)$  and let  $D - \{y_1, y_2\} = D_1 \cup D_2$ . For each  $i \in \{1, 2\}$  we now consider the oriented graph  $H_i = D - V(D_i)$ . Since  $H_i$  is connected, there is a  $y_1 - y_2$  path  $L_i$  in  $UG(H_i)$ . Since  $D$  is a strong oriented graph with  $c(D) = 3$ , every arc of  $D$  lies on a 3-cycle. Hence, by replacing some arcs of  $L_i$  with appropriately directed 3-paths where necessary, we can find a  $y_1 \rightsquigarrow y_2$  walk and hence a  $y_1 \rightsquigarrow y_2$  path  $Q_i$  in  $H_i$ . Similarly, we can find a  $y_2 \rightsquigarrow y_1$  path  $R_i$  in  $H_i$ . Therefore, if  $\{i, j\} = \{1, 2\}$  then  $Q_i$  followed by  $R_j$  is a cycle, so one of  $Q_i$  and  $R_j$  has only one arc and the other has two arcs. Thus we may assume that  $y_1y_2 \in A(D)$ . Since  $c(D) = 3$  and  $D$  is strong, it follows that each of  $H_1$  and  $H_2$  has a 3-cycle containing the arc  $y_1y_2$ . The fact that  $D$  is strong also implies that every arc in  $H_i - y_1y_2$  lies on a 3-cycle in  $H_i$  for  $i = 1, 2$ . Hence, each of  $H_1$  and  $H_2$  is strong and therefore belongs to  $\mathcal{F}$  by our induction hypothesis. Since  $H_1$  and  $H_2$  are glued together at the arc  $y_1y_2$ , it follows that  $D \in \mathcal{F}$ .

**Case 2.**  $\kappa(UG(D)) \geq 3$ .

In this case  $\delta(G) \geq 3$ , so Lemma 3.8 implies that  $D$  has an arc  $uv$  such that  $D - uv$  is strong. By our induction hypothesis,  $D - uv \in \mathcal{F}$ .

**Subcase 2.1.**  $D - uv \in \mathcal{B}$ .

Let  $B = D - uv$  and let  $x$  be a central vertex of  $B$ . If  $D \notin \mathcal{B}$ , there is a 3-cycle  $uvw$  in  $D$  that does not contain the vertex  $x$ . By Proposition 3.5(a),  $u, v, w \in N_B(x)$ . Suppose  $u, v, w \in N_B^+(x)$ . Then, by Proposition 3.5(c),  $w$  has an out-neighbour  $z$  in  $N_B^-(x)$  and hence  $xuvwzx$  is a 5-cycle in  $D$ , contradicting  $c(D) = 3$ . We get a similar contradiction if  $u, v, w \in N_B^-(x)$  for  $i = 1, 2, 3$ . Thus at least one of the vertices  $u, v, w$  is in  $N_B^-(x)$  and at least one in  $N_B^+(x)$ . But then the set  $\{x, u, v, w\}$  induces a strong tournament in  $D$  and consequently  $D$  contains a 4-cycle. This proves that  $D \in \mathcal{B}$  and hence  $D \in \mathcal{F}$ .

**Subcase 2.2.**  $D - uv \notin \mathcal{B}$ .

Since  $D$  is strong, the arc  $uv$  lies on a 3-cycle  $uvsu$ . Our assumption that  $\kappa(UG(D)) \geq 3$  implies that  $\kappa(UG(D - uv)) \geq 2$ . Hence, since  $D - uv \notin \mathcal{B}$ , it follows from Definition 3.4 that  $D - uv$  consists of  $k \geq 2$  oriented graphs  $B_1, \dots, B_k$ , each a member of  $\mathcal{B}$ , that are glued together in a treelike fashion. Thus  $\kappa(UG(D - uv)) = 2$ . Since  $\kappa(UG(D)) \geq 3$ , it follows that  $u$  and  $v$  lie in different  $B_i$ 's and there is a  $u - v$  path in  $UG(D)$  that visits every  $B_i$  but contains neither the vertex  $s$  nor the arc  $uv$ . Thus the labelling of the  $B_i$

may be chosen such that  $u \in B_1$ ,  $v \in B_k$  and  $B_{i+1}$  is glued to  $B_i$  at an arc  $a_i$ ,  $i = 1, \dots, k-1$ , as depicted in Figure 1. Note that  $s \in V(B_1) \cap V(B_k)$  since  $uvsu$  is a 3-cycle, and hence  $s$  is incident with  $a_i$  for  $i = 1, \dots, k-1$ , due to the acyclic glueing of the  $B_i$ . Let  $a_0 = su$ ,  $a_k = vs$ . The directions of the arcs  $a_1, \dots, a_{k-1}$  are not important for our proof. For  $i = 0, 1, \dots, k$ , let  $u_i$  be the vertex other than  $s$  that is incident with the arc  $a_i$ .

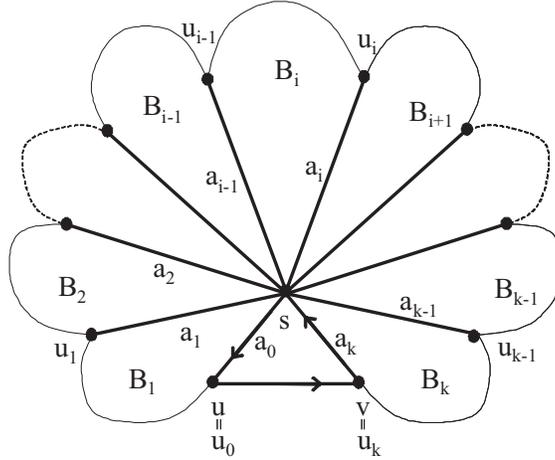


Figure 1: Case 2.2

If  $s$  is a central vertex of  $B_i$  for  $i = 1, \dots, k$ , then  $s$  lies on every 3-cycle in  $D$  and then  $D \in \mathcal{B} \subset \mathcal{F}$ . Thus we may assume that for some  $r \leq k$ , the vertex  $s$  is not a central vertex of  $B_r$ . So let  $x_r$  be a central vertex of  $B_r$ . If  $r > 1$ , let  $T_1$  be a 3-cycle in  $B_{r-1}$  that contains the arc  $a_{r-1}$ , and if  $r = 1$ , let  $T_1$  be the 3-cycle  $suv$ . Also, if  $r < k$ , let  $T_2$  be a 3-cycle in  $B_{r+1}$  that contains the arc  $a_r$ , and if  $r = k$ , let  $T_2$  denote the 3-cycle  $vsuv$ . Now we consider the subdigraph  $H$  of  $D$  induced by the arc set  $A(T_1) \cup A(B_r) \cup A(T_2)$ . We shall frequently use the following claim, which follows from Lemmas 3.6 and 3.7.

**Claim 1.** Each triangle in  $UG(H)$  has an edge that is not contained in any other triangle in  $UG(H)$ .

First suppose  $x_r = u_r$ . Then, by Proposition 3.5(a),  $u_r$  is adjacent to every vertex in  $B_r$ . Hence there is a triangle  $S$  with vertex set  $\{u_{r-1}, u_r, s\}$  in  $UG(B_r)$  (because  $u_r$  is central). But our assumption that there is a 3-cycle in  $B_r$  that does not contain  $s$  implies that there is a vertex  $w$  in  $B_r$  other than  $u_{r-1}$ ,  $s$  and  $u_r$ . Since  $UG(D)$  is 3-connected, there is a  $w - u_{r-1}$  path  $P$

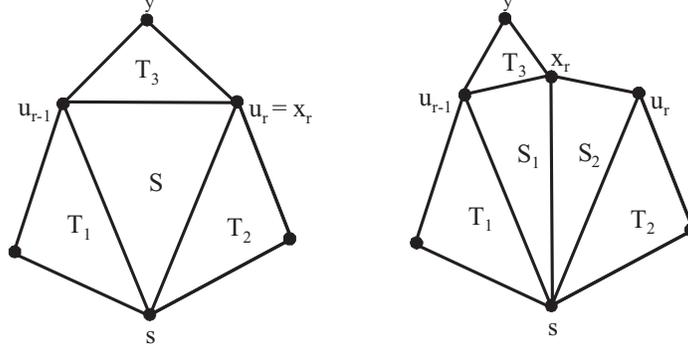


Figure 2: The cases  $x_r = u_r$  and  $x_r \notin \{u_{r-1}, u_r\}$ .

in  $UG(B_r)$  that contains neither  $u_r$  nor  $s$ . Let  $y$  be the neighbour of  $u_{r-1}$  on this path. Then there is a triangle  $T_3$  with vertex set  $\{u_{r-1}, y, u_r\}$  in  $UG(B_r)$ . But then  $S$  has an edge in common with each of the triangles  $T_1$ ,  $T_2$  and  $T_3$ , contradicting Claim 1. If  $x_r = u_{r-1}$ , we find a similar contradiction to Claim 1.

Thus we assume that  $x_r \notin \{u_{r-1}, u_r\}$ . Then  $UG(B_r)$  has two triangles  $S_1$  and  $S_2$ , with vertex sets  $\{u_{r-1}, x_r, s\}$  and  $\{u_r, x_r, s\}$  respectively. Since  $s$  is in both these triangles, we may assume that  $B_r$  has a vertex  $w \notin \{u_{r-1}, u_r, x_r, s\}$ . Now we note a  $w - u_{r-1}$  path  $P$ , a  $w - s$  path  $Q$  and a  $w - u_r$  path  $R$  in  $UG(B_r)$  such that no vertex other than  $w$  lies on more than one of these paths. In particular,  $x_r$  cannot lie on both  $P$  and  $R$ . By symmetry, we may assume that  $x_r \notin V(P)$ . Let  $y$  be the neighbour of  $u_{r-1}$  on  $P$ . Then there is a triangle  $T_3$  with vertex set  $\{x_r, y, u_{r-1}\}$ . But then the triangle  $S_1$  has an edge in common with each of the triangles  $T_1$ ,  $S_2$  and  $T_3$ , again contradicting Claim 1.

We conclude that  $s$  lies in every 3-cycle of  $B_i$  for  $i = 1, \dots, k$  and hence in every 3-cycle in  $D$ . This proves that  $D \in \mathcal{B}$  and hence  $D \in \mathcal{F}$ . □

Suppose  $D \in \mathcal{F}$  such that  $n(D) \geq 3$  and  $D$  has no cut-vertex. Then we call any induced subdigraph of  $D$  belonging to  $\mathcal{B}$  a *blob* of  $D$  and we note that  $D$  is either in  $\mathcal{B}$  or consists of a number of blobs, glued together in a treelike fashion. The blob decomposition of  $D$  is not necessarily unique. The oriented graph depicted in Figure 3 may be viewed, for example, as consisting of two blobs of order 4 glued together, or as a blob of order 3 glued to a blob of order 5. If, in a given blob decomposition,  $B$  is a blob such that either

$B = D$ , or  $B$  has exactly one arc that it shares with some other blob(s), then we call  $B$  an *end blob* of that specific decomposition. If  $D \notin \mathcal{B}$ , then  $D$  has at least two end-blobs.

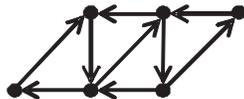


Figure 3: An orientation of the square of a path of length 5.

**Lemma 3.11.** *If  $D \in \mathcal{F}$  with  $n(D) \geq 4$  and  $D$  has no cut-vertex, then the blob decomposition of  $D$  can be chosen such that some end-blob has order at least 4.*

*Proof.* The proof is by induction on the order of  $D$ . If  $n(D) = 4$ , then it is easily seen that  $D \in \mathcal{B}$  and hence  $D$  itself is the desired end-blob. Now let  $n(D) \geq 5$  and suppose that  $B$  is an end-blob in some blob decomposition of  $D$  such that  $B$  has only three vertices  $u, v, w$ . Let  $uv$  be the glue-arc. Then  $D - w$  is in  $\mathcal{F}$  and has no cut-vertex. Hence, by our induction hypothesis, we can choose a blob decomposition of  $D - w$  that has an end-blob  $B'$  with at least four vertices. If  $B'$  is also an end-blob in some blob decomposition of  $D$ , the result is proved, so we assume this is not the case. Then  $B'$  contains the arc  $uv$  but  $uv$  is not the glue-arc of  $B'$  in  $D - w$ . Let  $x$  be a central vertex of  $B'$ .

If  $x \in \{u, v\}$ , then the subdigraph induced by  $B' \cup \{w\}$  is in  $\mathcal{B}$  (with  $x$  as central vertex), so it is an end-blob with more than 4 vertices in some blob decomposition of  $D$ .

Now suppose  $x \notin \{u, v\}$ . Then, by Proposition 3.5,  $xuvx$  is a 3-cycle in  $D$ . Since  $n(B') \geq 4$ , there is a vertex  $y \in B' - \{x, u, v\}$ . Since  $UG(D)$  is 2-connected, there is a path  $P$  from  $y$  to  $w$  that does not contain  $x$ . Then  $P$  contains a vertex  $z \notin \{u, v\}$  such that  $z$  is a neighbour of  $u$  or  $v$ , say  $u$ . Then  $xuzx$  is a 3-cycle in  $D$ . Thus, by Lemma 3.7, the arc  $vx$  is not contained in any 3-cycle in  $D$  other than  $xuvx$  and hence, by Proposition 3.5,  $u$  and  $x$  are the only neighbours of  $v$  in  $D - w$ . This implies that  $D - \{v, w\}$  is strong, so  $D - \{v, w\} \in \mathcal{F}$ . Thus,  $D$  has a blob decomposition such that the digraph induced by  $\{x, u, v, w\}$  is an end-blob with glue-arc  $xu$ .  $\square$

Theorem 3.10 now enables us to prove that in any digraph  $D$  with circumference 3 we can destroy all the 3-cycles by removing  $\lfloor n(D)/3 \rfloor$  vertices.

**Theorem 3.12.** *Suppose  $D$  is a digraph of order  $n$  with  $c(D) = 3$ . Then  $V(D)$  contains a set  $A$  with  $|A| \leq n/3$  such that  $c(D - A) < 3$ .*

*Proof.* The results obviously holds for every digraph of order 3. Let  $D$  be a digraph with circumference 3, order  $n$  and size  $m > 3$ . The proof is by induction on the order and size of  $D$ . Thus, we assume that every proper subdigraph  $H$  of  $D$  contains a set of  $n(H)/3$  vertices whose removal destroys all the 3-cycles in  $H$ .

If  $D$  is not strong, the result follows immediately by applying our induction hypothesis to every nontrivial strong component of  $D$  and noting that each cycle in  $D$  is contained within a strong component of  $D$ . Thus we assume that  $D$  is strong. Then, by Theorem 3.10,  $D \in \mathcal{F}$ . We consider two cases.

**Case 1.**  $D$  contains a cut-vertex  $v$ .

In this case  $D$  consists of two digraphs  $D_1$  and  $D_2$  attached to one another at the vertex  $v$ . Suppose  $n(D_i) = n_i$ ,  $i = 1, 2$ . Then  $n = n_1 + n_2 - 1$ . We consider two subcases.

**Subcase 1.1**  $n_i \equiv 0 \pmod{3}$  for  $i = 1, 2$ .

We consider the digraphs  $H_i = D_i - v$ ,  $i = 1, 2$ . If  $c(H_i) = 3$ , our induction hypothesis implies that  $V(H_i)$  has a subset  $A_i$  such that  $H_i - A_i$  contains no 3-cycles and  $|A_i| \leq \lfloor (n_i - 1)/3 \rfloor = (n_i - 3)/3$ . If  $c(H_i) < 3$  we put  $A_i = \emptyset$ . Now let  $A = \{v\} \cup A_1 \cup A_2$ . Then  $c(D - A) < 3$  and  $|A| \leq (n_1 + n_2 - 3)/3 < n/3$ .

**Subcase 1.2.**  $n_1 \not\equiv 0 \pmod{3}$ .

We consider the digraphs  $D_i$ ,  $i = 1, 2$  and conclude that there is a set  $A$  in  $D$  such that  $D - A$  has no 3-cycles and  $|A| \leq \lfloor n_1/3 \rfloor + \lfloor n_2/3 \rfloor \leq (n_1 - 1)/3 + n_2/3 = n/3$ .

**Case 2.**  $D$  has no cut-vertex.

Suppose  $D$  contains a 2-cycle  $uvu$ . By Lemma 3.9, we may assume that  $D - vu$  is strong. Then  $uv$  is contained in a 3-cycle  $uvwu$ . If  $vu$  is also contained in a 3-cycle, then that 3-cycle can only be  $vuuv$  (since  $c(D) = 3$ ). Hence any set of vertices whose removal destroys all the 3-cycles in  $D - vu$  also destroys all the 3-cycles in  $D$ . The result now follows by applying our induction hypothesis to  $D - vu$ .

Thus we may assume that  $D$  is an oriented graph.

If  $D \in \mathcal{B}$ , then the result follows immediately from Definition 3.3.

Now suppose  $D \notin \mathcal{B}$  and let  $B$  be an end-blob in some blob decomposition of  $D$ . By Lemma 3.11, we can choose  $B$  to be of order at least 4. Let  $uv$  be

the glue-arc of  $B$  and let  $F = D - (V(B) - \{u, v\})$ .

**Subcase 2.1.**  $n(B) \geq 5$ .

In this case  $n(F) \leq n - 3$ , so our induction hypothesis implies that  $V(F)$  has a subset  $A'$  such that  $F - A'$  contains no 3-cycles and  $|A'| \leq (n - 3)/3$ . Let  $x$  be a central vertex of  $B$  and put  $A = A' \cup \{x\}$ . Then  $c(D - A) < 3$  and  $|A| \leq n/3$ .

**Subcase 2.2.**  $n(B) = 4$ .

In this case the underlying graph of  $B$  is a  $K_4$  minus an edge (since a strong tournament of order 4 has a 4-cycle). Hence at least one of  $u$  and  $v$ , say  $u$ , lies in every 3-cycle in  $B$ . By our induction hypothesis,  $V(F - u)$  has a subset  $A'$  with  $|A'| \leq (n - 3)/3$  such that  $(F - u) - A'$  contains no 3-cycles. Now put  $A = A' \cup \{u\}$ . Then  $c(D - A) < 3$  and  $|A| \leq n/3$ .  $\square$

The constant  $1/3$  in Theorem 3.12 is best possible, as shown by a disjoint collection of 3-cycles. There are also strongly connected examples whose underlying graphs are 2-connected: Indeed, the square of a path of length greater than 4 has a strong orientation with circumference 3 (see Figure 3 for example), and for any such orientation we need  $n/3$  vertices to destroy all 3-cycles. However, it follows from Theorem 3.10 and the definition of  $\mathcal{F}$  that every strong digraph with circumference 3 whose underlying graph is 3-connected has a vertex meeting every longest cycle.

Thomassen [9] and independently Grötschel and Wakabayashi [6] and Fouquet and Jolivet [5] showed that there exists a hypohamiltonian graph with circumference  $k$  if and only if  $k \geq 5$ . Hence  $\alpha(k) \geq 2/(k + 1)$  for  $k \geq 5$ . There is no hypohamiltonian digraph with circumference 4, but we now provide an example of an infinite family of digraphs illustrating that  $\alpha(4) \geq 1/3$ .

**Example 3.13.** *For any positive integer  $r$ , take  $2r$  undirected disjoint paths  $x_i y_i z_i$ ,  $i = 1, 2, \dots, 2r$  and add arcs to form the 4-cycles  $x_i x_{i+1} z_i z_{i+1} x_i$ ,  $i = 1, 2, \dots, 2r - 1$ . Undirected edges correspond to 2-cycles. The resulting digraph is of order  $6r$  and at least  $2r$  vertices need to be removed in order to destroy all the 4-cycles. The case  $r = 1$  is depicted in Figure 4.*

In the case of oriented graphs the situation is somewhat different. We conjecture that, in every oriented graph with circumference  $k \leq 7$ , one can destroy all the longest cycles by removing  $1/k$  of the vertices. (We know that the smallest hypohamiltonian oriented graph has circumference 8 – see [1].)

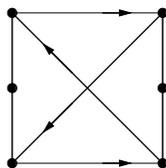


Figure 4: The case  $r = 1$ .

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