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Abstract

We prove a general result on graph factors modulo k. A special case says that, for each natural number k, every $(12k-7)$-edge-connected graph with an even number of vertices contains a spanning subgraph in which each vertex has degree congruent to k modulo $2k$.

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1 Introduction

Jaeger [7], [8] generalized Tutte's 3-flow conjecture to the following conjecture which he called the *the circular flow conjecture*:

If k is an odd natural number, and G is a $(2k-2)$-edge-connected graph, then G has an orientation such that each vertex has the same indegree and outdegree modulo k.

This conjecture does not extend to the case where k is an even number (because a vertex of odd degree cannot be balanced modulo an even number) also not in the weak version where we replace the edge-connectivity $2k-2$ by a larger function of k. However, the weak version becomes true also when

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k is odd in the following version proved in [12]: Let k be a natural number, and let G be a $(2k^2 + k)$-edge-connected graph with n vertices $v_1, v_2, \ldots, v_n$. Let d_i be an integer for each $i = 1, 2, \ldots, n$ such that the sum of all d_i is congruent $(mod\ \ k)$ to the number of edges of G. Then G has an orientation such that each v_i has outdegrees d_i modulo k. In [9] the quadratic bound $(2k^2 + k)$ was improved to the linear bound $3k - 3$.

This orientation result has applications to instances of the graph decomposition conjecture in [3] that, for every tree T, every graph of large edge-connectivity (depending on T only) has an edge-decomposition into copies of T (provided the size of T divides the size of the graph), see [12], [13].

It also implies the $(2 + \epsilon)$-flow conjecture by Goddyn and Seymour. In [12] an application of the weak 3-flow conjecture to the $(2+\epsilon)$-flow conjecture was discussed. However, as pointed out by a referee of an early version of the present paper, the general orientation result in [12] implies the $(2 + \epsilon)$-flow conjecture in its full strength by the discussion in Section 9.2 in [15]. In [14] it is shown precisely which flow values can be used in the $(2 + \epsilon)$-flow conjecture. Prior to that, the $(2 + \epsilon)$-flow conjecture had been verified first for planar graphs [6] and then for graphs on a fixed surface [16]. Apart from these results not much was known about the conjecture, as pointed out in [17]

In this paper we reformulate the orientation result in the weak circular flow conjecture as a factor result for bipartite graphs and derive the special case mentioned in the Abstract. This special case is related to results of Alon Friedland, and Kalai [1] concerning non-empty subgraphs where all degrees are divisible by k. Those results are based on edge-densities only, whereas the results in the present paper need some kind of connectivity as well. To illustrate the different nature of the results in [1] and the present note, a special result in [1] (see also [2]) says that every graph with n vertices and at least $2n + 1$ edges contains a non-empty subgraph in which each vertex has degree divisible by 3. Such a subgraph may be small as it may contain vertices of degree zero. By replacing edge-density by edge-connectivity we obtain subgraphs where all vertices have positive degrees, all divisible by 3. Specifically, every 29-edge-connected graph with an even number of vertices has a spanning subgraph in which each vertex has degree 3 modulo 6.

2 Graph factors modulo k

The terminology and notation are the same as in [12] which are essentially the same as in [4], [5], [10]. In the present paper, however, a *graph* may have multiple edges (but no loops).

Theorem 1 *Let k be a natural number, and let G be a $(3k-3)$-edge-connected bipartite graph with n vertices $v_1, v_2, \ldots, v_n$ and with partite sets A, B. Let d_i be an integer for each $i = 1, 2, \ldots, n$ such that the sum of all d_i where v_i is in A is congruent $(mod\ k)$ to the the sum of all d_i where v_i is in B. Then G has a spanning subgraph H such that*

$$d(v_i, H) \equiv d_i(mod\ k)$$

for $i = 1, 2, \ldots, n$.

Proof of Theorem 1:

For each vertex v_i in A, put $p_i = d_i$. For each vertex v_i in B, put $p_i = d(v_i, G) - d_i$. Then the sum of all p_i is congruent to the number of edges modulo k. By the strengthening in [9] of Theorems 1 and 3 in [12], the edges of G can be oriented such that each vertex v_i has outdegree p_i modulo k. The edges oriented from A to B can now play the role of H.

So, Theorem 1 is an immediate consequence of Theorems 1 and 3 in [12] and their extension in [9]. Conversely, it is easy to derive these results (except for a weaker upper bound on the edge-connectivity needed) from Theorem 1 above because every $(2k-1)$-edge-connected graph G contains a spanning bipartite k-edge-connected subgraph H, as pointed out in Proposition 1 in [11]. (The proof is easy: Consider a spanning bipartite subgraph with as many edges as possible. If that subgraph has a cut with fewer than k edges, then the corresponding partition of the vertex set can be used to find a spanning bipartite subgraph with more edges, a contradiction.) If we wish to orient all edges in G such that the vertices have prescribed outdegrees modulo k, then we orient the edges in G but not in H at random, and then we apply Theorem 1 to H resulting in a subgraph H'. All edges in H' are directed from one partite class to the other, and the edges in H but not in H' are directed in the opposite direction. By choosing the degrees in H appropriately (modulo k), we obtain the desired orientation of G.

Thus we may regard Theorem 1 as a reformulation of Theorem 3 in [12] and its extension in [9]. We apply this to a result for general graphs.

Theorem 2 *Let k be a natural number, and let G be a $(6k-7)$-edge-connected graph with n vertices. Let d_i be an integer for each $i = 1, 2, \ldots, n$ such that, for any m in $\{1, 2, \ldots, n-1\}$, there is a partition of $\{1, 2, \ldots, n\}$ into sets A, B of cardinality $m, n-m$, respectively, such that*

the sum of all d_i where i is in A is congruent $(mod\ k)$ to the the sum of all d_i where i is in B.

Then G has a spanning subgraph H such that the degrees of H are $d_1, d_2, \ldots, d_n$ modulo k.

Proof of Theorem 2:

By the above-mentioned observation in Proposition 1 in [11], G has a spanning $(3k-3)$-edge-connected bipartite subgraph M. Then apply the partition condition of $d_1, d_2, \ldots, d_n$ where $m, n-m$ are the number of vertices in the two partite sets of M. Theorem 2 now follows from Theorem 1.

The partition condition of $d_1, d_2, \ldots, d_n$ is necessary because G might be bipartite to begin with. Unfortunately, that condition puts a severe restriction on the applications to non-bipartite graphs. Another weakness is that we do not specify which vertices have which degrees, and therefore Theorem 2 is not really a factor result. However, special cases are about factors, for example the following.

Theorem 3 *Let k be a natural number, and let G be a $(12k-7)$-edge-connected graph with an even number of vertices.*

Then G has a spanning subgraph H such that each vertex in H has degree congruent to k $(mod\ 2k)$.

Proof of Theorem 3: The prescribed degrees modulo $2k$ satisfy the partition condition in Theorem 2 because the number of vertices is even. Note that Theorem 3 is not true if k is odd and the number of vertices is odd.

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