Comment on “Temporal Correlations of the Running Maximum of a Brownian Trajectory”

Bénichou et al. [1] use the running maximum (RM) position in a single experimental trajectory of a particle exhibiting 1D Brownian motion (BM) to estimate its diffusion coefficient. This is unreliable: While the estimator’s precision (reproducibility) increases with the suggested parameter tuning, so does its inaccuracy (bias), as increasing emphasis is put on the RM’s maximum value.

In the mathematical idealization for BM used in Ref. [1], \(B_t\) is the position of a particle diffusing with coefficient \(D\). However, \(B_t = \sqrt{2D}W_t\), where \(W_t\) is the Wiener process. In this model, BM is a scale-free process.

Experimentally, one samples positions \(x_{i=1,...,N}\) at time points \(t_i = i\Delta t\) and \(T = N\Delta t\). For BM, measured positions relate as \(x_{i+1} = x_i + \sqrt{2D}\Delta t_i\), where \(\eta_i = W_{t_{i+1}} - W_{t_i}\) is a Gaussian white noise with \(\langle \eta_i \rangle = 0\) and \(\langle \eta_i\eta_j \rangle = \Delta t \delta_{ij}\) for all \(i, j\). Each of the \(N-1\) displacements \(\Delta x_i = x_{i+1} - x_i\) contains information about \(D\); hence, variances of estimators in this discrete case are limited by \(N\), not \(T\), due to the scale invariance of BM.

A reasonable estimator \(\hat{D}\) for \(D\) should (i) be unbiased, i.e., \(\langle \hat{D} \rangle = D\), and (ii) have a variance that decreases with, respectively, large \(N\) and \(N\) large and practically relevant \(N\). The discretized version \(\hat{D}_{\text{msd}}^{(N)}\) of \(D_{\text{msd}}\) [1] with \(\tau = \Delta t\), i.e., \(\hat{D}_{\text{msd}}^{(N)} = \sum_{i=1}^{N-1} (\Delta x_i)^2/(2(N-1)\Delta t)\), complies with (i) and (ii) for \(N \geq 2\) in the present case of instantaneous recording of positions and in the absence of measurement noise. It is even optimal: It achieves the Cramér-Rao lower bound [2,3] and thus has the lowest possible variance among unbiased estimators.

With discrete sampling, the RM is \(M_t = \max_{j=1,...,N} x_j\), and thus the RM-based estimator of Ref. [1] must read \(\hat{D}_{\text{es}}^{(N,k)} = C(k)\sum_{i=1}^{N-1} M_{i+1}^{2k}/k\), with \(C(k) = \langle (\Delta t \sqrt{\pi}(k/2 + 1))/\{2\Gamma[(k+1)/2]2^{k+1}\}\rangle\) and \(k > 0\). As a function of \(N\), the information available to \(\hat{D}_{\text{es}}^{(N,k)}\) increases so slowly that its variance approaches a constant value [1]. This is in conflict with (ii). The variance can be made arbitrarily small, however, by increasing \(k\) [1]; thus it is argued that \(\hat{D}_{\text{es}}^{(N,k)}\) is superior to \(\hat{D}_{\text{msd}}^{(N)}\) for small \(T\) [1].

Application of both estimators to Monte Carlo (MC) simulated BM shows, however, that the estimates of \(\hat{D}_{\text{msd}}^{(N)}\) scatter with a normal distribution around \(D\), while the estimates of \(\hat{D}_{\text{es}}^{(N,k)}\) are skewed [Figs. 1(a) and 1(b)]. This results in a bias, \(\langle \hat{D}_{\text{es}}^{(N,k)} \rangle \neq D\), which is in conflict with (i). The bias becomes worse with increasing \(k\) [Fig. 1(c)], while the variance indeed decreases [Fig. 1(d)]. The bias of \(\hat{D}_{\text{es}}^{(N,k)}\) vanishes too slowly with \(N\) to ensure any practical relevance of \(\hat{D}_{\text{es}}^{(N,k)}\) relative to \(\hat{D}_{\text{msd}}^{(N)}\) [Figs. 1(c) and 1(d)].

In summary, the estimator suggested by Bénichou et al. [1] unfortunately yields biased values for the diffusion coefficient, while optimal, plug-and-play alternatives already exist [2,3].

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