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Nash-Williams' cycle-decomposition theorem.

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Abstract

We give an elementary proof of the theorem of Nash-Williams that a graph has an edge-decomposition into cycles if and only if it does not contain an odd cut. We also prove that every bridgeless graph has a collection of cycles covering each edge at least once and at most 7 times. The two results are equivalent in the sense that each can be derived from the other.

Keywords: infinite graphs, cycle-decompositions and coverings
MSC(2010):05C38,05C40,05C63

1 Introduction.

A fundamental result of Nash-Williams [6] (see also [7], [9], and [2] page 268) says that a graph has a collection of pairwise edge-disjoint cycles containing all edges of the graph if and only if the graph has no finite odd cut. This result is trivial in the finite case, an easy exercise in the countably infinite case, and remarkably difficult in the uncountable case. This difficulty in the uncountable case suggests that Nash-Williams' theorem might be a valuable bridge between results for countable graphs and their extensions to uncountable graphs. This was indeed demonstrated by Laviolette [4], [5] who used

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Nash-Williams' theorem to prove that every graph G can be decomposed into countable graphs which he called *bond-faithful*, and we call *cut-faithful*. Laviolette's theorem is for general cardinals. In the present paper we define a cut-faithful subgraph of a graph G as a subgraph H such that every finite minimal cut in H is a cut (and hence a minimal cut) in G. It follows that every finite cut in H is a cut in G. It also follows that, if a finite minimal cut D in G intersects H, then D is contained in H.

Further extensions and a new proof of Nash-Williams' result are given by Soukup [8].

In [10] it is proved that, for any tree T and any set A of vertices in T, there is a collection of pairwise edge-disjoint paths in T such that every vertex in A, except possibly one, is the end of precisely one path in the collection. We use this to prove that every bridgeless graph has a collection of cycles covering each edge at least once and at most countably many times. From this we derive Laviolette's above-mentioned decomposition result, and that immediately implies Nash-Williams' decomposition result. We then extend our first result to the last statement in the abstract.

Although Nash-Williams' result is difficult for uncountable graphs, our methods involve no knowledge of set theory such as cardinals, ordinals, or transfinite induction. Zorn's lemma is used but only in a so straightforward way that it is not even mentioned explicitely. (Zorn's lemma is used explicitely in the proof in [10] of Theorem 1, though.) Of course, some version of the axiom of choice must be used (as it is easily seen to be a consequence of Nash-Williams' theorem).

2 Cycles covering each edge at least once and at most countably many times.

The terminology is essentially the same as in [1] and [2]. The graphs in this paper are allowed to contain multiple edges but no loops. A double edge is considered to be a cycle of length 2. If G is a graph and its vertex set is divided into sets A, B, then all edges between A and B form a *cut* in G. We call A, B the *sides* of the cut. The cut is *minimal* if it contains no other cut as a proper subset. It is easy to see that a cut in a connected graph G with sides A, B is minimal if and only if both graphs $G(A), G(B)$ are connected.

Using this observation it is also easy to see that every cut can be decomposed into pairwise disjoint minimal cuts.

If G is a graph and H is a subgraph of G, then the *boundary of H* is the set of vertices in H having a neighbor outside H. If we add a collection of edges to G such that the added edges form a matching, then we call that set of edges an *external matching*.

Our proof is based on the following result in [10].

Theorem 1 *Let T be a tree and let A be a set of vertices in T. Then T has a collection of pairwise edge-disjoint paths, each joining two vertices in A such that each vertex in A, except possibly one, is the end of precisely one of the paths. If the exceptional vertex in A exists, then it is not the end of any path in the path-collection.*

■

Lemma 1 *Let T be a tree and let v be a vertex of T. The edge set of T can be decomposed into pairwise edge-disjoint paths, and these paths can be divided into two classes $\mathcal{P}_1, \mathcal{P}_2$ such that each vertex of T is the end of at most one path in $\mathcal{P}_1$ and at most one path in $\mathcal{P}_2$. Moreover, v is not the end of any path in $\mathcal{P}_1$.*

Proof of Lemma 1. We divide all edges incident with v into paths of length 2, except possibly one which has length 1. These paths all belong to $\mathcal{P}_2$. We repeat this argument for each component of $T - v$ where the neighbor of v plays the role of v and the paths of length $2, 1$ containing that vertex belong to $\mathcal{P}_1$.

■

Laviolette [4] derived Theorem 2 below from his decomposition theorem, Theorem 3 below. We now derive Theorem 2 below more directly using Theorem 1.

Theorem 2 *Let G be a 2-edge-connected graph. Then G has a collection $\mathcal{C}$ of cycles such that every edge of G is in at least one cycle in $\mathcal{C}$ and is in at most countably many cycles in $\mathcal{C}$.*

Proof of Theorem 2. Let v be any vertex of G. We prove, by induction on n, that there exists a sequence $\mathcal{C}_n$, $n \in \mathbb{N}$, of collections of cycles with the following properties.

(i) Every cycle in $\mathcal{C}_{n-1}$ is also in $\mathcal{C}_n$. Every edge of G is in at most finitely many cycles in $\mathcal{C}_n$. For each natural number n, the union of all cycles in $\mathcal{C}_n$ is an induced subgraph G_n of G.

(ii) G_n contains all vertices of distance $< n$ to v.

(iii) If M is an external matching joining vertices in the boundary of G_n, then $G_n \cup M$ has a collection of cycles such that every edge in M is in at least one cycle in the collection, and every edge of $G_n \cup M$ is in only finitely many edges in the collection.

Having proved the existence of $\mathcal{C}_n$ for each n, Theorem 2 follows with $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$. We let $\mathcal{C}_0$ be empty. Put $G_0 = v$. Let n be a natural number, and assume that $\mathcal{C}_{n-1}$ exists. Let H denote a connected component of $G - V(G_{n-1})$, and let S be the vertices in G_{n-1} joined to H. We now construct a collection of pairwise edge-disjoint paths starting and ending with an edge between G_{n-1} and H such that all intermediate edges are in H. If a vertex x in H is joined to more than one vertex in G_{n-1}, then we consider a maximal collection of pairwise edge-disjoint paths of length 2 having x as mid-vertex and having their endvertices in G_{n-1}. We call these *H-paths*. If there are still edges left between G_{n-1} and H, then we let A denote the ends of these edges in H. We let T be a spanning tree in H, and we now apply Theorem 1 to T, A. We have now constructed a collection of pairwise edge-disjoint paths (or cycles), which we also call H-paths (even though some of them may be cycles), containing all edges between G_{n-1} and H, except possibly one which we call the *exeptional H-edge*. We consider a path (or cycle), called the *exceptional H-path* (or cycle) starting with the exceptional edge, ending with an edge from H to G_{n-1} and having all intermediate edges in T. The exceptional H-path (or cycle) exists because G is 2-edge-connected. Now the vertex set of G_n will consist of $V(G_{n-1})$, the vertices of the H-paths (for all components H of $G - V(G_{n-1})$), and also the vertices of each exceptional H-path, if it exists. Thus G_n satisfies (ii).

We now describe how to extend $\mathcal{C}_{n-1}$ to $\mathcal{C}_n$. We consider all H-paths for all components H of $G - V(G_{n-1})$. We take a maximal collection of pairwise edge-disjoint cycles, each of which is the union of H-paths (but not exceptional H-paths), and we add those cycles to $\mathcal{C}_n$. The remaining H-paths can be thought of as edges in a forest defined on the vertices in the boundary of G_{n-1}. We apply Lemma 1 to that forest. Now all H-paths for all components H are combined to form cycles and paths.

Lemma 1 gives two collections of paths (or, more precisely, walks, since vertex repetitions may occur). Each such path is now thought of as an edge in a matching M. We apply (iii) (with $n-1$ instead of n) to G_{n-1}, M. This results in a collection of cycles in $G_{n-1} \cup M$. We replace each edge in M by a union of H-paths and obtain thereby a collection of closed walks. There may be repetition of vertices but not edges in each such closed walk, and so it can be further decomposed into pairwise edge-disjoint cycles. All these cycles will be in $\mathcal{C}_n$. Now all edges in H-paths (for all H) are contained in cycles in $\mathcal{C}_n$. We treat the exceptional H-path (for all H) in the same way. (Another way of saying this is that the argument above is not applied to two collections of paths but three collections.) This enlarges $\mathcal{C}_n$, but we still keep the condition that no edge is in infinitely many cycles in $\mathcal{C}_n$.

Now (i) is satisfied except that G_n is not yet an induced graph. All edges which join vertices of G_n which are not contained in cycles of $\mathcal{C}_n$ are called *residual edges*. Every residual edge joins two vertices of $V(G_n) \setminus V(G_{n-1})$. We shall now dispose of the residual edges so that G_n is induced and satisfies (i). (We note that the same argument will be used to prove that G_n satisfies (iii).) First we consider (and add to $\mathcal{C}_n$) a maximal collection of pairwise edge-disjoint cycles consisting of residual edges. The remaining residual edges form a forest. We decompose that forest into paths, by Lemma 1. Consider one of these paths, Q say, with ends x, y, say. Then x is on an H-path (or on the exceptional H-path). And y is on an H-path (or on the exceptional H-path). We use those two H-paths (or exceptional H-paths) to extend Q to a walk Q' which joins two vertices in the boundary of G_{n-1} and such that the edges of Q' are outside G_{n-1}. The walk Q' has no repetition of edges and can therefore be edge-decomposed into cycles and a path with the same ends as Q'. With a slight abuse of notation we also call that path Q'. The paths Q' need not be edge-disjoint. They may share edges in H-paths. However, each edge in an H-path is contained in only finitely many paths Q' (because paths Q have distinct ends, and H-paths have only finitely many vertices). Thus we may make the paths Q' pairwise edge-disjoint by replacing each edge of an H-path by a multiple edge of finite multiplicity. We form a graph F whose vertex set is the boundary of G_{n-1} such that two vertices u, v are neighbors in F if there is a path Q' joining u, v. After deleting pairwise edge-disjoint cycles from F (and adding the corresponding cycles in G to $\mathcal{C}_n$) we may assume that F is a forest. (As we have created multiple edges, it is possible that some of the afore-mentioned cycles are double edges. We do not add these to $\mathcal{C}_n$ as they are not cycles in G. Note that the underlying edges of these double edges are already covered by $\mathcal{C}_n$.) We apply Lemma 1 to F and obtain two collections of paths such that, in either collection, no two paths have a common end. Thus we may think of these paths (in either collection) as a matching M consisting of external edges added to the boundary of G_{n-1}. An edge in M corresponds to a path in F and that path corresponds to a walk in G. That walk in G can be edge-decomposed into cycles and a path Q'' in G with the same ends as an edge in M. By applying (iii) to G_{n-1}, M we find a collection of cycles which contain all paths Q'' and hence all paths Q. We add this collection to $\mathcal{C}_n$, and now G_n is an induced graph. (Note again that this argument is used for each of the two collections of paths in F.) This proves (i).

In order to prove (iii), consider a matching M consisting of external edges whose ends are in the boundary of G_n. Consider such an edge e with ends x, y, say. In the previous case we considered a path Q with ends x, y. We now treat the edge e in exactly the same way as we treated Q. The only difference is that now x may be in an H-path, and y may be in an H'-path where possibly $H' \neq H$. But this does not affect the argument. In this way we complete the proof of (iii).

This completes the proof of Theorem 2.

■

3 Cut-faithful countable subgraphs.

Laviolette's proof of the decomposition theorem [4] depends on Nash-Williams' decomposition theorem. In this section we prove Laviolette's decomposition theorem [4] without using Nash-Williams' decomposition theorem. Then the latter follows as a corollary.

Theorem 3 *Every graph has an edge-decomposition into connected, countable, cut-faithful graphs.*

Proof of Theorem 3.

It suffices to prove the theorem for connected graphs. As every bridge together with its two ends is cut-faithful, it suffices to prove the theorem for 2-edge-connected graphs.

Let G be a 2-edge-connected graph. For any two natural numbers p, q we define an index set $I(p, q)$ and a decomposition of G into pairwise edge-disjoint 2-edge-connected countable graphs $G_{i,p,q}$ where $i \in I(p, q)$.

We first consider the case where $p = q = 1$. By Theorem 2, G has a collection $\mathcal{C}$ of cycles such that every edge of G is in at least one cycle in $\mathcal{C}$ and is in at most countably many cycles in $\mathcal{C}$. We form a new graph J where each vertex in J is a cycle in $\mathcal{C}$. Two vertices in J are joined by an edge if the two corresponding cycles in $\mathcal{C}$ have at least one edge in common. In J each vertex has countable degree, and hence each component of J is countable. Let $J_i, i \in I(1,1)$ be the components of J. For each $i \in I(1,1)$, let $G_{i,1,1}$ be the union of the cycles in $\mathcal{C}$ corresponding to the vertices in J_i. Then the graphs $G_{i,1,1}$, $i \in I(1,1)$ are pairwise edge-disjoint 2-edge-connected, countable subgraphs of G.

Since the graph $G_{i,1,1}$ is countable, it has only countably many finite edge-sets and hence only countably many finite, minimal cuts, say $D_{1,i,1}, D_{2,i,1}, \ldots$. Consider the cut $D_{1,i,1}$. Assume that $D_{1,i,1}$ is not a cut in G. As $D_{1,i,1}$ is a minimal cut in $G_{i,1,1}$, $G_{i,1,1} - D_{1,i,1}$ has precisely two components with vertex sets A, B, say, and $D_{1,i,1}$ consists of edges $a_1b_1, a_2b_2, \ldots, a_nb_n$, where $a_1, a_2, \ldots, a_n$ are in A and $b_1, b_2, \ldots, b_n$ are in B. We now insert two new vertices x_j, y_j on the edge a_jb_j such that $a_jx_jy_jb_j$ becomes a path. Then we identify all the vertices $x_1, x_2, \ldots, x_n$ into a vertex x, and we identify all the vertices $y_1, y_2, \ldots, y_n$ into a vertex y. We replace the n edges between x, y by a single edge. In this way $G_{i,1,1}$ is modified into a connected graph $G'_{i,1,1}$. Note that in each $G_{i,1,1}$ we modify only one cut. As the graphs $G_{i,1,1}$, $i \in I(1,1)$, are pairwise edge-disjoint, we can perform these modifications simultaneously. The union of all $G'_{i,1,1}$, $i \in I(1,1)$ is called G'. Clearly, G' is 2-edge-connected. By Theorem 2, G' has a collection $\mathcal{C}'$ of cycles such that every edge of G' is in at least one cycle in $\mathcal{C}'$ and is in at most countably many cycles in $\mathcal{C}'$. We form a new graph J'. Each cycle in $\mathcal{C}'$ is a vertex in J'. Each graph $G'_{i,1,1}$, $i \in I(1,1)$, is also a vertex of J'. Two vertices in J' are joined by an edge if the two corresponding subgraphs in G' have at least one edge in common. In J' each vertex has countable degree. Let $J'_i, i \in I(2,1)$ be the components of J'. For each $i \in I(2,1)$, let $G_{i,2,1}$ be the union of the subgraphs in G (not G') corresponding to the vertices in J'_i. Then the graphs $G_{i,2,1}$, $i \in I(2,1)$ are pairwise edge-disjoint, 2-edge-connected, countable subgraphs of G. The cut $D_{1,i,1}$ is contained in $G_{i,1,1}$ and hence it is contained in some graph $G_{j,2,1}$, $j \in I(2,1)$. Since the edge xy defined above is contained in a cycle C in $\mathcal{C}'$, the graphs of the form $G_{s,1,1}$, $s \in I(1,1)$, which C has edges in common with are also in $G_{j,2,1}$. Hence C can be modified into a cycle in $G_{j,2,1}$ which contains precisely one edge in $D_{1,i,1}$. This implies that $D_{1,i,1}$ is not a cut in $G_{j,2,1}$.

We repeat this argument for each cut $D_{2,i,1}$, $i \in I(1,1)$, whenever $D_{2,i,1}$ is a cut (and hence a minimal cut) in the graph $G_{j,2,1}$ containing $D_{2,i,1}$, but $D_{2,i,1}$ is not a cut in G. In this way we obtain an decomposition of G into pairwise edge-disjoint 2-edge-connected, countable subgraphs $G_{j,3,1}$, $j \in I(3,1)$. $D_{2,i,1}$ is contained in one of these graphs but it is not a cut in that graph.

We repeat this argument defining decompositions of G into $G_{j,4,1}$, $j \in I(4,1)$, $G_{j,5,1}$, $j \in I(5,1)$, ... We now define the decomposition of G into graphs $G_{j,1,2}$, $j \in I(1,2)$ as follows. Consider an edge e in G. Then e is in one of the graphs $G_{i,1,1}$, and in one of the graphs $G_{j,2,1}$, and in one of the graphs $G_{k,3,1}$ etc. These graphs form an increasing sequence of graphs and we define their union to be one of the graphs $G_{j,1,2}$, $j \in I(1,2)$. Note that each of $D_{1,i,1}, D_{2,i,1}, \ldots$ is contained in some $G_{j,1,2}$, $j \in I(1,2)$, but none of $D_{1,i,1}, D_{2,i,1}, \ldots$ is a cut in $G_{j,1,2}$ unless it is a cut in G. But, $G_{j,1,2}$ may have new cuts which are not cuts in G. As before, we enumerate them $D_{1,j,2}, D_{2,j,2}, D_{3,j,2} \ldots$, and we define decompositions of G into $G_{j,2,2}$, $j \in I(2,2)$, $G_{j,3,2}$, $j \in I(3,2)$, ..., and after this sequence of decompositions we define the decomposition of G into $G_{j,1,3}$, $j \in I(1,3)$. We repeat the argument.

Having defined all decompositions of G into $G_{i,p,q}$, $i \in I(p,q)$, where p, q are natural numbers we define the final decomposition as follows. Consider an edge e. Then e is in one of the graphs $G_{i,1,1}$, and in one of the graphs $G_{j,1,2}$, and in one of the graphs $G_{k,1,3}$ etc. These graphs form an increasing sequence of graphs, and we define their union to be one of the graphs H in the final decomposition.

We claim that every finite minimal cut in H is also a cut in G. To prove this claim, we first observe that H is the union of an increasing sequence of graphs of the form $G_{i,1,q}$. This implies that any finite set of edges in H is contained in a graph of the form $G_{i,1,q}$. Consider now a minimal cut D in H. In each side of $H - D$ there is a finite connected subgraph of H containing all ends of D in that side. Let D' be the edges in those two connected subgraphs. The observation above implies that $D \cup D'$ is contained in a subgraph of H of the form $G_{i,1,q}$. Clearly D is a minimal cut in $G_{i,1,q}$. Now, if $G - D$ is connected, then there is a subgraph $G_{j,1,q+1}$ of H containing D such that also $G_{j,1,q+1} - D$ is connected. But this contradicts the assumption that $H - D$ is disconnected. This proves the claim that every finite minimal cut in H is also a cut in G.

For the sake of completeness we show that every finite minimal cut in G intersecting H is contained in H. Consider a finite minimal cut D in G. Let

H be a graph in the final decomposition containing at least one edge of D. Suppose (reductio ad absurdum) that H does not contain all edges of D. Let D' be those edges in D which are in H. Then D' is a cut in H. Let D'' be a minimal cut in H contained in D'. Then D'' is a proper subset of D and hence not a cut in G, a contradiction.

This completes the proof of Theorem 3.

∎

As an immediate consequences of Theorem 3 we obtain the following result of Laviolette [4]:

Theorem 4 *If k is a natural number, and G is a k-edge-connected graph, then G has an edge-decomposition into countable, k-edge connected graphs.*

Using Theorem 4 for $k = 2$ we get Nash-Williams' decomposition theorem (because it is an easy exercise in the countable case):

Theorem 5 *Every graph with no odd cut has an edge-decomposition into cycles.*

4 Cycles covering each edge at least once and at most finitely many times.

In this paper we have derived Laviolette's decomposition theorem (Theorem 3) and Nash-Williams' decomposition theorem (Theorem 5) from the cycle covering theorem (Theorem 2). Conversely, Theorem 2 is an immediate consequence of Theorem 3 which Laviolette derived from Nash-Williams' theorem. So, in some sense these results are equivalent. In the same sense they are equivalent to Theorem 7 below. First we mention a related version, namely Theorem 6 below whose proof follows from the proof of Theorem 11 in [10].

Theorem 6 *Let G be a 2-edge-connected graph. Then the edges of G can be oriented so that the resulting directed graph has a collection of directed cycles such that each edge is in at least one and only finitely many directed cycles in the collection.*

■

We now strengthen Theorem 2 further. In the proof we use the result of Jaeger [3] that every finite 2-edge-connected graph is the union of three even graphs, that is, graphs where all vertices have even degree. Jaeger used this for his breakthrough on Tutte's 5-flow conjecture since the three even graphs easily give a nowhere-zero 8-flow. Here we note that it also gives a collection of cycles covering each edge at least once and at most 3 times.

Theorem 7 *Let G be a 2-edge-connected graph. Then G has a collection $\mathcal{C}$ of cycles such that every edge of G is in at least one cycle in $\mathcal{C}$ and is in at most 7 cycles in $\mathcal{C}$.*

Proof of Theorem 7. We refine the proof of Theorem 2. By Theorem 4 it suffices to prove Theorem 7 for countable graphs. So assume that G is countable. It is proved in [10] that every vertex of G can be split up into vertices so that each block of the resulting graph is locally finite, that is, every vertex has finite degree, and, furthermore, the edge-connectivity is preserved. (Splitting up a vertex v means that we delete v and replace it by a vertex set V_v. Each edge incident with v will be incident with precisely one vertex in V_v.)

So, we may assume that G is locally finite.

Let v be any vertex of G. We prove, by induction on n, that there exists a finite collection $\mathcal{C}_n$ of cycles with the following properties.

(i) Every cycle in $\mathcal{C}_{n-1}$ is also in $\mathcal{C}_n$. For each natural number n, the union of all cycles in $\mathcal{C}_n$ is an induced subgraph G_n of G. Every edge of G is in at most 7 cycles in $\mathcal{C}_n$. Every edge of G joining two vertices of $V(G_n) \setminus V(G_{n-1})$ is in at most 5 cycles in $\mathcal{C}_n$.

(ii) G_n contains all vertices of distance $\leq n$ to v.

(iii) For any two vertices x, y in the boundary of G_n such that $G - V(G_n)$ has a component joined to each of x, y, G_n has a path joining x, y disjoint from G_{n-1}. In other words, for each component H of $G - V(G_n)$, there is a component Q of $G_n - V(G_{n-1})$ such that all edges from H to G_n go to Q.

Having proved the existence of $\mathcal{C}_n$ for each n, Theorem 7 follows with $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$. We let $\mathcal{C}_0$ be empty. Put $G_0 = v$. Let n be a natural number, and assume that $\mathcal{C}_{n-1}$ exists. Let H denote a connected component of $G - V(G_{n-1})$, and let S be the vertices in G_{n-1} joined to H. Then S

belongs to a connected component Q of $G_{n-1} - V(G_{n-2})$. Hence each edge of Q is in at most 5 cycles of $\mathcal{C}_{n-1}$. Let U be the vertices in H joined to G_{n-1}. Note that there may be other components of $G - V(G_{n-1})$ which are joined to Q.

As G is 2-edge-connected and there are only finitely many edges from the finite subgraph G_{n-1} to H, it follows that the subgraph H has only finitely many bridges and hence only finitely many maximal 2-edge-connected subgraphs. In each such maximal 2-edge-connected subgraphs we select a finite 2-edge-connected subgraph which contains all ends of the bridges in H and all vertices in U which are contained in that maximal 2-edge-connected subgraphs. The union of these finite subgraphs together with the bridges of H form a finite connected subgraph H' of H. Note that H' and H have the same bridges. Also note that by adding edges to H' if necessary, we may assume that H' is induced. We now apply the afore-mentioned theorem of Jaeger to find a collection of cycles in H' covering the non-bridges in H' at least once and at most 3 times. These cycles will be part of $\mathcal{C}_n$. We shall cover the bridges of H' and the edges from H' to S as follows. Consider first the case where there is an even number, say $2s$, of edges from H' to S. Then we can find a collection of s pairwise edge-disjoint paths (and cycles) starting and ending at S such that all intermediate vertices are in H'. (We here apply the finite version of Theorem 1 (which is trivial) to a spanning tree of H'.) We call these *H-paths* (although some of them may be cycles). If there are $2s+1$ edges from H' to S, then we find an additional path (or cycle) which we call the *exceptional H-path*. We can find the s H-paths and the exceptional H-path such that the s paths are pairwise edge-disjoint. We now consider all components of $G - V(G_{n-1})$ which are joined to Q. We think of each H-path as an external edge added to Q. In Q together with the external edges we can find a collection of pairwise edge-disjoint cycles covering all external edges. (The reason is that Q is connected and has therefore a spanning tree. By deleting appropriate edges from that spanning tree we transform Q together with the external edges into a graph were all vertices have even degree.). All these cycles are part of $\mathcal{C}_n$. Now all edges of $G_n - V(G_{n-1})$ are covered at most 4 times, and all edges of $G_{n-1} - V(G_{n-2})$ are covered at most 6 times. Now we dispose of the exceptional H-paths in the same way as we disposed of the H-paths. Then all edges of $G_n - V(G_{n-1})$ are covered at most 5 times, and all edges of $G_{n-1} - V(G_{n-2})$ are covered at most 7 times.

We have earlier noted that H' above is induced, so condition (i) holds. It is trivial that (ii) holds. Condition (iii) holds because G_n is chosen such that it contains U (defined above) and all edges from U to G_{n-1}. This implies that each component of $G - V(G_n)$ is joined to a subgraph of G_n of the form H', and H' is connected.

This completes the proof of Theorem 7.

■

The *cycle double cover conjecture* by Szekeres and Seymour says that in Theorem 7 we may replace "at most 7" by "precisely 2". Although the cycle double cover conjecture is formulated for finite graphs, it is also open in the infinite case and is discussed in [4], [8].

If G satisfies the conclusion of Theorem 6, then every infinite cut is *balanced*, that is, the cardinalities of the two edge sets directed from one side of the cut to the other are the same. This suggests a possible counterpart of Nash-Williams' theorem for directed graphs.

Conjecture 1 *A directed graph has an edge-decomposition into directed cycles if and only if each cut is balanced.*

As cardinality is essential in the formulation of Conjecture 1 it seems plausible that a proof needs more set theoretic considerations than the present proof of Nash-Williams' decomposition theorem.

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