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TETRAVALENT ONE-REGULAR GRAPHS OF ORDER 4p²

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Abstract. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper tetravalent one-regular graphs of order $4p^2$, where p is a prime, are classified.

1. Introduction

A graph is *arc-transitive* if its automorphism group acts transitively on the set of its arcs. A graph is *one-regular* if its automorphism group acts regularly on the set of its arcs. Not surprisingly arc-transitive graphs - and one-regular graphs in particular - have received considerable attention over the years, the aim being to obtain structural results and possibly a classification of such graphs of particular orders or satisfying certain additional properties. Research in one-regular graphs is interesting for two reasons, the first being their connection to regular maps, a lively area of research. Namely, the underlying graphs of chiral maps admit one-regular group actions with a cyclic vertex stabilizers (see, for example, [8, 10–12]). Second, one may argue that one-regular graphs are interesting in their own right if one's goal is a description of arc-transitive graphs. For some classes of Cayley graphs, for example, circulants, this has been achieved, whereas for others, such as Cayley graphs of dihedral groups, all 2-arc-transitive graphs have been completely classified [16], but arc-transitivity remains an open problem.

Clearly, a one-regular graph with no isolated vertices is connected, and it is of valency 2 if and only if it is a cycle. The first example of a cubic one-regular graph was constructed by Frucht [21]. Further research in cubic one-regular graphs has been part of a more general project dealing with the investigation of cubic arc-transitive graphs (see [9, 15, 17–20, 31]). Tetravalent one-regular graphs have also received considerable attention. In [4] tetravalent one-regular graphs of prime order were constructed, and in [30] an infinite family of tetravalent one-regular Cayley graphs on alternating groups is given. Tetravalent one-regular circulant graphs were classified in [41], and tetravalent one-regular Cayley graphs on abelian groups were classified in [40]. Next, one may extract a classification of tetravalent one-regular Cayley graphs on dihedral

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groups from [26, 36, 38]. Let p and q be primes. Clearly every tetravalent one-regular graph of order p is a circulant graph. Also, by [7, 32, 34, 37, 40, 41], every tetravalent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classification of tetravalent one-regular graphs of order $2pq$ is given in [43]. The aim of this paper is to classify tetravalent one-regular graphs of order $4p^2$, see Theorem 5.1. (For more results on tetravalent arc-transitive graphs, see [22, 23, 27, 33].)

In the next section we gather various concepts that are needed in the analysis of tetravalent one-regular graphs in Section 4 and in the proof of our main result in Section 5. In Section 3, we give examples of tetravalent one-regular graphs of order $4p^2$, where p is a prime.

2. Preliminaries

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, its edge set, its arc-set and its full automorphism group, respectively. For $u, v \in V(X)$, denote by uv the edge incident to u and v in X. By C_n and K_n we denote the cycle of length n and the complete graph of order n, respectively.

A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. The graph X is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive, respectively. An arc-transitive graph is also called a *symmetric* graph. An arc-transitive graph X is said to be *one-regular* if $\mathrm{Aut}(X)$ acts regularly on $A(X)$. A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *k-arc-transitive* if it acts transitively on the set of k-arcs, and it is said to be *k-regular* if it is k-arc-transitive and the stabilizer of a k-arc in G is trivial.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. The permutation group $R(G) = \{R(g) \mid g \in G\}$ on G is called the *right regular representation* of G. It is easy to see that $R(G)$ is isomorphic to G, and it is a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [42, Proposition 1.5] proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$.

Given a transitive group G acting on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding G-invariant partition as to an *imprimitive block system* of G.

2.1. Group theoretic results

Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of units of $\mathbb{Z}_n$. For two groups M and N, $N \leq M$ means that N is a subgroup of M and $N < M$ means that N is a proper subgroup of M.

For a permutation group G on a set Ω and $\alpha \in \Omega$ we let G_α denote the stabilizer of α in G, that is, the subgroup of G fixing the element $\alpha \in \Omega$. The group G is said to be *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and it is said to be *regular* if it is both transitive and semiregular on Ω.

Below we gather various group-theoretic results that are needed in the subsequent sections of this paper. The first one is about transitive abelian permutation groups.

Proposition 2.1. [35, Proposition 4.4] *Every transitive abelian group G on a set Ω is regular.*

For a subgroup H of a group G, let $C_G(H)$ be the centralizer of H in G, and let $N_G(H)$ be the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.2. [25, Chapter I, Theorem 4.5] *Let G be a group and H a subgroup of G. Then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

The following result can be extracted from [13, P.285, summary].

Proposition 2.3. [13] *Let $G = \mathrm{PSL}(2,7)$ and let $A = \mathrm{PGL}(2,7)$. Then Sylow 2-subgroups of G and A are, respectively, isomorphic to D_8 and D_{16}. Moreover, all involutions of G are conjugate, and G has no subgroup of order 14.*

The following classical result is due to Wielandt [35, Theorems 3.4].

Proposition 2.4. [35] *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $w \in \Omega$. If p^m divides the length of the G-orbit containing ω, then p^m also divides the length of the P-orbit containing w.*

2.2. Graph covers

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p\colon V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})}\colon N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The set $\mathrm{fib}_v = p^{-1}(v)$ is a *fibre* of a vertex $v \in V(X)$. The subgroup K of all those automorphisms of $\widetilde{X}$ which fix each of the fibres setwise is called the *group of covering transformations*. If the group of covering transformations is regular on the fibres of $\widetilde{X}$, we say that $\widetilde{X}$ is a *regular K-covering*. We say that $\alpha \in \mathrm{Aut}(X)$ *lifts* to an automorphism of $\widetilde{X}$ if there exists $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$, called the *lift* of α, such that $\widetilde{\alpha}p = p\alpha$.

Let X be a graph and K a finite group. A *K-voltage assignment* of X is a function $\phi\colon A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$, where a^{-1} denotes the reverse arc of the arc a. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $X \times_\phi K$ derived from a voltage assignment $\phi\colon A(X) \to K$ has vertex set $V(X) \times K$ and edges of the form $(u, g)(v, g\phi(a))$ where $a = (u, v) \in A(X)$ and $g \in K$. Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p\colon X \times_\phi K \to X$. By letting K act on $V(X \times_\phi K)$ as $(u, g')^g = (u, gg')$, $(u, g') \in V(X \times_\phi K)$, one obtains a semiregular subgroup of $\mathrm{Aut}(X \times_\phi K)$, showing that $X \times_\phi K$ can in fact be viewed as a K-covering. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Moreover, Gross and Tucker [24] showed that every regular covering $\widetilde{X}$ of a graph X can in fact be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. (Given a spanning tree T of a graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are all equal to the identity of K.) If $X \times_\phi K \to X$ is a connected K-covering derived from a T-reduced voltage assignment ϕ then the problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\overline{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by $(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha)$, where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on C and C^α, respectively. Note that if K is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X. The next proposition is a special case of [30, Theorem 4.2].

Proposition 2.5. [30] *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

For more results on graph covers we refer the reader to [1, 2, 14, 28, 29].

2.3. Tetravalent arc-transitive graphs

In this subsection we gather known results about tetravalent arc-transitive graphs that will be needed in subsequent sections. The first two propositions can be deduced from [40, Theorem 3.5].

Proposition 2.6. [40] *Let p be a prime, and $G \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then there exists a tetravalent one-regular Cayley graph on G if and only if $p-1$ is a multiple of 4. Moreover in each of these two cases exactly one such graph exists.*

Proposition 2.7. [40] *Let p be a prime and $G \cong \mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$. Then there is no tetravalent one-regular Cayley graph on G.*

Let X be a connected symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of $\mathrm{Aut}(X)$. For a normal subgroup N of G, the *quotient graph* X_N of X relative to the set of orbits of N is defined as the graph whose vertices are orbits of N on $V(X)$ with two orbits being adjacent in X_N if there is an edge between these two orbits in X. The following proposition is a 'reduction' theorem which is deduced from [22, Theorem 1.1].

Proposition 2.8. [22, Theorem 1.1] *Let X be a tetravalent connected symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of $\mathrm{Aut}(X)$. Then for each normal subgroup N of G one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N acts transitively on each of the two bipartition sets;*
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} of X_N;*
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-symmetric graph and X is a regular cover of X_N.*

To state the next result we need to introduce three families of tetravalent graphs that were first defined in [23]. First, let $C^{\pm 1}(p;4,2)$ be the graph with vertex set $\mathbb{Z}_p^2 \times \mathbb{Z}_4$, and adjacencies in $C^{\pm 1}(p;4,2)$ satisfying the following conditions: for $i, j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4$

$$(i,j,k) \sim \begin{cases} (i \pm 1, j, k+1) & \text{if } k \text{ is even} \\ (i, j \pm 1, k+1) & \text{if } k \text{ is odd} \end{cases}.$$

Second, for a prime $p \equiv \pm 1 (\mathrm{mod}\ 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv 2 \pmod{p}$ the graph $\mathcal{NC}^0_{4p^2}$ is defined to have vertex set and edge set

$$\begin{aligned} V(\mathcal{NC}^0_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x,y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^0_{4p^2}) &= \{(x,y,0)(x \pm 1, y, 1) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x, y \pm 1, 2) \mid x,y \in \mathbb{Z}_p\} \cup \\ &\quad \{(x,y,2)(x \mp 1, y \pm k, 3) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x \mp k, y \pm 1, 0) \mid x,y \in \mathbb{Z}_p\}. \end{aligned}$$

And third, for a prime p, $p \equiv 1 (\mathrm{mod}\ 8)$ or $p \equiv 3 (\mathrm{mod}\ 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv -2 \pmod{p}$ the graph $\mathcal{NC}^1_{4p^2}$ is defined to have vertex set and edge set

$$\begin{aligned} V(\mathcal{NC}^1_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x,y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^1_{4p^2}) &= \{(x,y,0)(x \pm 1, y, 1) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x, y \pm 1, 2) \mid x,y \in \mathbb{Z}_p\} \cup \\ &\quad \{(x,y,2)(x \pm 1, y \pm k, 3) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x \pm k, y \mp 1, 0) \mid x,y \in \mathbb{Z}_p\}. \end{aligned}$$

The graphs $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are extracted from [23, Lemma 8.4, Lemma 8.7]. We can now state the result of Gardiner and Praeger [23, Theorem 1.2] about connected tetravalent graphs admitting arc-transitive subgroups of automorphisms with normal elementary abelian p-groups N such that the corresponding quotient graph X_N is a cycle.

Proposition 2.9. [23, Theorem 1.2] *For an odd prime p let X be a connected, G-symmetric, tetravalent graph of order $4p^2$, let $N = \mathbb{Z}_p^2$ be a minimal normal subgroup of G with orbits of size p^2, and let K be the kernel of the action of G on $V(X_N)$. If $X_N = C_4$ and $K_v = \mathbb{Z}_2$ then X is isomorphic to one of the following graphs: $C^{\pm 1}(p; 4, 2)$, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$.*

In [23] it is proven that the three graphs in the above proposition all admit a one-regular subgroup of automorphisms. In the following two lemmas we improve this result by showing that $C^{\pm 1}(p; 4, 2)$ is not one-regular whereas $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are.

Lemma 2.10. *Let p be a prime. Then $C^{\pm 1}(p; 4, 2)$ is not one-regular.*

Proof. First recall that the vertex set of $X = C^{\pm 1}(p, 4, 2)$ is equal to $V(X) = \{(i, j, k) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_p, k \in \mathbb{Z}_4\}$ and the edges are of the form

$$\begin{aligned} (i, j, 2l) &\sim (i \pm 1, j, 2l + 1), \text{ where } i, j \in \mathbb{Z}_p \text{ and } l \in \{0, 1\} \\ (i, j, 2l - 1) &\sim (i, j \pm 1, 2l), \text{ where } i, j \in \mathbb{Z}_p \text{ and } l \in \{0, 1\}. \end{aligned}$$

Then the reader can check that a permutation α of $V(X)$ defined by $(i, j, k)^\alpha = (-i, j, k)$ maps edges to edges, and hence α is an automorphism of X. Since α fixes the arc $(0, 0, 1)(0, 1, 2) \in A(X)$ it follows that X is not one-regular. □

Lemma 2.11. *Let p be a prime. Then $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are both one-regular graphs.*

Proof. Let $X \in \{\mathcal{NC}^0_{4p^2}, \mathcal{NC}^1_{4p^2}\}$ and let X^2 be the distance-2-graph of X, that is, $V(X^2) = V(X)$ with two vertices being adjacent in X^2 if and only if they are at distance 2 in X. Let

$$\Delta_i = \{(x, y, i) \mid x, y \in \mathbb{Z}_p\}, \ \ i \in \mathbb{Z}_4.$$

Then for every $i \in \mathbb{Z}_4$ the subgraph $X^2[\Delta_i]$ of X^2 induced by the vertices in Δ_i is a 2-dimensional grid $C_p \times C_p$, whereas any edge uv in X^2 with endvertices $u \in \Delta_i$ and $v \in \Delta_j$, where $i \neq j$, is contained in an induced subgraph of X^2 isomorphic to the complete graph K_4. Moreover this induced subgraph isomorphic to K_4 containing the edge uv is unique. Take four vertices $u_1, u_2, u_3, u_4 \in \Delta_i$ such that the subgraph Y of X^2 induced on these four vertices is isomorphic to a 4-cycle C_4. Then Y^g for any $g \in \mathrm{Aut}(X^2)$ is an induced subgraph of X^2 isomorphic to C_4. Since there is no set of four vertices containing vertices from different sets Δ_i such that the induced subgraph of X^2 is isomorphic to C_4 it follows that Y^g is a subgraph of $X^2[\Delta_j]$ for some $j \in \mathbb{Z}_4$. This shows that the sets Δ_i, $i \in \mathbb{Z}_4$, are blocks of imprimitivity for $\mathrm{Aut}(X)$. Therefore every automorphism $g \in \mathrm{Aut}(X)$ that fixes the vertices $(0, 0, 0)$ and $(1, 0, 1)$, and thus the arc $(0, 0, 0), (1, 0, 1)$, also fixes the vertices $(2, 0, 0)$ and $(-1, 0, 1)$. Now looking at the action of g on X^2 we get that g fixes both Δ_0 and Δ_1 pointwise. Since all the vertices in Δ_1 are fixed by g and the induced bipartite subgraph $X[\Delta_1, \Delta_2]$ is a disjoint union of p $2p$-cycles it follows that also Δ_2 is fixed pointwise by g. Using the same argument for $X[\Delta_0, \Delta_3]$ one can see that g also fixes the vertices in Δ_3 and thus $g = 1$, which shows that X is one-regular. □

To state the next result we need to introduce two additional families of tetravalent graphs that were first defined in [23]. The graph $C^{\pm 1}(p; 4p, 1)$ is defined to have the vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and the edge set $\{(i, j)(i \pm 1, j + 1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The graph $C^{\pm \varepsilon}(p; 4p, 1)$ is a graph with vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies in $C^{\pm \varepsilon}(p; 4p, 1)$ satisfying the following conditions:

$$(i, j) \sim \begin{cases} (i \pm \varepsilon, j + 1) & \text{if } j \text{ is odd} \\ (i \pm 1, j + 1) & \text{if } j \text{ is even} \end{cases},$$

where $i \in \mathbb{Z}_p$, $j \in \mathbb{Z}_{4p}$ and ε is an element of order 4 in $\mathbb{Z}_p^*$.

Proposition 2.12. [23, Theorem 1.1] *Let p be an odd prime and let X be a connected, G-symmetric, tetravalent graph of order* $4p^2$*. Let* $N = \mathbb{Z}_p$ *be a minimal normal subgroup of G with orbits of size p and let K denote the kernel of the action of G on* $V(X_N)$*. If* $X_N = C_{4p}$ *and* $K_v = \mathbb{Z}_2$ *then X is isomorphic either to* $C^{\pm 1}(p; 4p, 1)$ *or to* $C^{\pm \varepsilon}(p; 4p, 1)$*.*

We end this subsection with a result on tetravalent arc-transitive graphs of order $4p$, where p is a prime. In order to state the result, first recall that the *lexicographic product* $X[Y]$ (sometimes also called the *wreath product*) of two graphs X and Y has vertex set $V(X) \times V(Y)$, and two vertices (a, u) and (b, v) are adjacent in $X[Y]$ if $ab \in E(X)$ or if $a = b$ and $uv \in E(Y)$. Second, following [44], for a prime p congruent to 1 modulo 4, an element w of order 4 in $\mathbb{Z}_p^*$ and the group $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, we use notation $C\mathcal{A}_{4p}^0 = \mathrm{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $C\mathcal{A}_{4p}^1 = \mathrm{Cay}(G, \{a, a^{-1}, a^{w}b, a^{-w}b\})$. For the definition of the graph $C(2, p, 2)$ stated in the sixth row of Table 1 see Section 4. Finally, by [44, Example 3.7], $\mathfrak{g}_{28} = \mathrm{Cos}(G, T, TaT)$ is a coset graph of the group $G = \mathrm{PGL}(2, 7)$ with respect to a subgroup T isomorphic to A_4 and an involution a from the center of the normalizer of a Sylow 3-subgroup of T in G.

Proposition 2.13. [44, Theorem 4.1] *Let s be a positive integer and let p be a prime. Then a connected tetravalent graph of order* $4p$ *is s-arc-transitive if and only if it is isomorphic to one of the graphs listed in Table 1. Furthermore, all graphs listed in Table 1 are pairwise non-isomorphic.*

X	s	$\mathrm{Aut}(X)$	comments
$K_{4,4}$	3	$\mathbb{Z}_2 \ltimes (S_4 \times S_4)$	p=2
$C_{2p}[2K_1]$	1	$D_{4p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$C\mathcal{A}_{4p}^0$	1	$\mathbb{Z}_2^2 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2)$,	$p \equiv 1 (\mathrm{mod}\ 4)$
$C\mathcal{A}_{4p}^1$	1	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2)$,	$p \equiv 1 (\mathrm{mod}\ 4)$
$C(2, p, 2)$	1	$D_{2p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$\mathfrak{g}_{28}$	3	$\mathrm{PGL}(2, 7) \times \mathbb{Z}_2$	$p = 7$

Table 1: Tetravalent s-arc-transitive graphs of order $4p$.

3. Examples

In this section, we give examples of tetravalent one-regular graphs of order $4p^2$, where p is a prime. In this paper, the abbreviations $C\mathcal{A}$ and $C\mathcal{N}$ will mean a Cayley graph on abelian group and a Cayley graph on non-abelian group, respectively.

Example 3.1. Introduced by Wilson [39] the *bicycle wheels* are defined in the following way. Given natural numbers n, a, r and s, the graph $X = \mathcal{BW}_n(a, r, s)$ is defined to be the graph of order $3n$ with vertex set $V(X) = \{A_i, B_i, C_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$E(X) = \{A_iB_i, B_iA_{i+1}, B_iC_i, C_iB_{i+a}, A_iA_{i+r}, C_iC_{i+s} \mid i \in \mathbb{Z}_n\}.$$

With the help of computer software package MAGMA [3] one can see that $\mathcal{BW}_{12}(5, 1, 5)$ is one-regular. In addition, it is a Cayley graph $\mathrm{Cay}(G_{36}, S)$ on the group $G_{36} = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1 = [a, b] = [a, c] = [b, c] = [c, d], d^{-1}ad = b, d^{-1}bd = ab \rangle$ with respect to the generating set $S = \{ad, (ad)^{-1}, bdc, (bdc)^{-1}\}$, and $\mathrm{Aut}(C\mathcal{A}_{36}^2) \cong G_{36} \rtimes \mathbb{Z}_2^2$.

Remark: The automorphism group of the graph $\mathcal{BW}_{12}(5, 1, 5)$ has a non-normal Sylow 3-subgroup. Since, by Theorem 5.1, the automorphism groups of the graphs $C\mathcal{A}_{4p^2}^i$, $i \in \{0, 1, 2\}$, given in Examples 3.3 and 3.4 and Lemma 3.6, all have normal Sylow p-subgroups, the graph $\mathcal{BW}_{12}(5, 1, 5)$ is not isomorphic to any of these graphs.

Example 3.2. Given natural numbers k and m, and a 2×2 matrix M over $\mathbb{Z}_n$ the 2-*dimensional generalized power spidergraph* $\mathcal{GPS}2(k,n,M)$ is defined to be the graph with vertex set $\mathbb{Z}_k\times\mathbb{Z}_n\times\mathbb{Z}_n$, and edge set $\{(i,x)(i+1,x+a_i),(i,x)(i+1,x+b_i)\mid i\in\mathbb{Z}_k, x\in\mathbb{Z}_n\times\mathbb{Z}_n\}$ where $a_i=(1,0)M^i$ and $b_i=(-1,0)M^i$ (see [39]). With the use of MAGMA [3] one can see that $\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$ is a one-regular graph. In addition, it is not a Cayley graph and the stabilizer of a vertex in the automorphism group is isomorphic to $\mathbb{Z}_4$.

Example 3.3. Let $p\equiv 1 \pmod 4$ be a prime and w an element of order 4 in $\mathbb{Z}_p^*$ with $1\le w\le p-1$. Let $G^0_{4p^2}=\langle a\rangle\times\langle b\rangle\cong\mathbb{Z}_{2p^2}\times\mathbb{Z}_2$. Then, by [40, Proposition 3.3(iv)], the Cayley graph $\mathcal{CA}^0_{4p^2}=\mathrm{Cay}(G^0_{4p^2},\{a,a^{-1},a^wb,a^{-w}b\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^0_{4p^2})\cong(\mathbb{Z}_{2p^2}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2^2$.

Example 3.4. Let p be an odd prime and $G^1_{4p^2}=\langle a,b\,|\,a^{4p}=b^p=1, ab=ba\rangle\cong\mathbb{Z}_{4p}\times\mathbb{Z}_p$. Then, by [40, Proposition 3.3], the Cayley graph $\mathcal{CA}^1_{4p^2}=\mathrm{Cay}(G^1_{4p^2},\{ab,a^{-1}b,ab^{-1},a^{-1}b^{-1}\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^1_{4p^2})\cong(\mathbb{Z}_{4p}\times\mathbb{Z}_p)\rtimes\mathbb{Z}_2^2$. The graph $\mathcal{DW}(12,3)$ of order 36 given in [39] is the smallest example of such graphs.

For an odd prime p, the tetravalent graph $C^{\pm1}(p;4p,1)$ is defined in the paragraph preceding Proposition 2.12. In the following lemma we prove that $C^{\pm1}(p;4p,1)$ is isomorphic to $\mathcal{CA}^1_{4p^2}$, and thus it is one-regular in view of Example 3.4.

Lemma 3.5. *Let p be an odd prime, let $G^1_{4p^2}=\langle a,b\mid a^{4p}=b^p=1, ab=ba\rangle\cong\mathbb{Z}_{4p}\times\mathbb{Z}_p$, and let $S=\{ab,a^{-1}b,ab^{-1},a^{-1}b^{-1}\}$. Then $C^{\pm1}(p;4p,1)\cong\mathrm{Cay}(G^1_{4p^2},S)=\mathcal{CA}^1_{4p^2}$.*

Proof. Recall that $C^{\pm1}(p;4p,1)$ has vertex set $\mathbb{Z}_p\times\mathbb{Z}_{4p}$ and edge set $\{(i,j)(i\pm1,j+1)\mid i\in\mathbb{Z}_p, j\in\mathbb{Z}_{4p}\}$. The map defined by $(i,j)\mapsto a^jb^i$ is an isomorphism from $C^{\pm1}(p;4p,1)$ to the Cayley graph $\mathcal{CA}^1_{4p^2}$. We leave the details to the reader. □

Let $p\equiv 1 \pmod 4$ be a prime and let $\varepsilon\in\mathbb{Z}_p$ be such that $\varepsilon^2\equiv -1 \pmod p$. The following lemma shows that $C^{\pm\varepsilon}(p;4p,1)$ is a Cayley graph.

Lemma 3.6. *Let $p\equiv 1 \pmod 4$ be a prime, let $\varepsilon\in\mathbb{Z}_p$ be such that $\varepsilon^2\equiv -1 \pmod p$, let $G^2_{4p^2}=\langle a,b\mid a^{4p}=b^p=1, a^{-1}ba=b^\varepsilon\rangle$, and let $S=\{ab,a^{-1}b^\varepsilon,ab^{-1},a^{-1}b^{-\varepsilon}\}$. Then $\mathcal{CN}^2_{4p^2}=\mathrm{Cay}(G^2_{4p^2},S)$ is a symmetric graph isomorphic to $C^{\pm\varepsilon}(p;4p,1)$.*

Proof. Recall that the graph $C^{\pm\varepsilon}(p;4p,1)$ has vertex set $\mathbb{Z}_p\times\mathbb{Z}_{4p}$ with adjacencies defined as follows:

$$(i,j)\sim\begin{cases}(i\pm\varepsilon,j+1) & \text{if } j \text{ is odd}\\ (i\pm1,j+1) & \text{if } j \text{ is even}\end{cases}$$

where $i\in\mathbb{Z}_p$ and $j\in\mathbb{Z}_{4p}$.

Let $G=G^2_{4p^2}$ and $X=Cay(G;S)$. Then the map defined by $(i,j)\mapsto a^jb^i$ is an isomorphism from $C^{\pm\varepsilon}(p;4p,1)$ to X. Since, by [23], the graph $C^{\pm\varepsilon}(p;4p,1)$ is symmetric, the lemma holds. □

4. Analysis of tetravalent one-regular graphs of order 4p²

Let p be an odd prime. Then define $C(2,p,2)$ to be a graph with $V(C(2,p,2))=\mathbb{Z}_4\times\mathbb{Z}_p$ and adjacencies in $C(2,p,2)$ satisfying the following conditions:

$$\begin{aligned}
(0,i)\sim(0,j) &\iff j-i=\pm1,\\
(0,i)\sim(1,j) &\iff j-i=-1,\\
(0,i)\sim(2,j) &\iff j-i=1,\\
(1,i)\sim(2,j) &\iff j-i=\pm1,\\
(1,i)\sim(3,j) &\iff j-i=-1,\\
(2,i)\sim(3,j) &\iff j-i=1,\\
(3,i)\sim(3,j) &\iff j-i=\pm1.
\end{aligned}$$

Let $X = C(2, p, 2)$ and let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$. Observe that for each $j \in \mathbb{Z}_p$, $j \neq i$, the subgraph $X[B_i, B_j]$ induced on the union $B_i \cup B_j$ is not an independent set of vertices if and only if $j = i \pm 1$. Moreover, for each such j we have that $X[B_j, B_{j+1}] \cong 2C_4$, see also Figure 1. The following lemma shows that there is no one-regular $\mathbb{Z}_p$-cover of $C(2, p, 2)$.

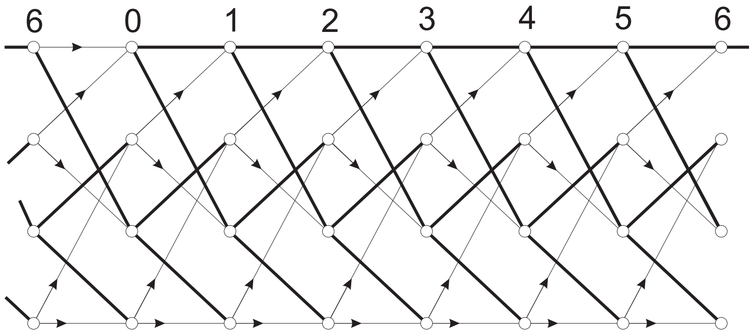

Figure 1: A spanning tree in the base graph $C(2, p, 2)$ for $p = 7$.

Lemma 4.1. *Let Y be a tetravalent one-regular graph of order $4p^2$, $p > 3$ a prime, such that there exists a normal subgroup H of* $\mathrm{Aut}(Y)$ *of order p. Then Y is not a regular $\mathbb{Z}_p$-cover of the graph $C(2, p, 2)$.*

Proof. Let $\mathcal{K} = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein 4-group acting on $\mathbb{Z}_4$ so that $\tau_1 = (0\,1)(2\,3)$, $\tau_2 = (0\,2)(1\,3)$ and $\tau_3 = (0\,3)(1\,2)$. Let $X = C(2, p, 2)$, let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$, and let K be the kernel of the action of $\mathrm{Aut}(X)$ on $\mathcal{B}$. We shall be sloppy and shall identify restrictions of elements of K to sets B_i by elements of $\mathcal{K}$. For instance, when we say that the restriction γ_i of $\gamma \in K$ to B_i is, for example, τ_1, we mean that $\gamma_i = ((0, i)(1, i))((2, i)(3, i))$. Now, the structure of X indicated in Figure 1 implies that the restrictions γ_i must satisfy the following conditions:

$$\gamma_i \in \{1, \tau_1\} \iff \gamma_{i+1} \in \{1, \tau_2\} \quad \forall i \in \mathbb{Z}_p. \tag{1}$$

Let the vertices of X be labeled in the following way: $a_i = (0, i)$, $b_i = (1, i)$, $c_i = (2, i)$ and $d_i = (3, i)$. Let $E = \langle \gamma_i \mid i \in \mathbb{Z}_p \rangle$. It is well known, see for instance [33, 44], that $\mathrm{Aut(X)} = E \rtimes \langle \rho, \tau \rangle \cong \mathbb{Z}_2^p \rtimes D_{2p}$ where

$$\rho = (a_0\, a_1\, \ldots\, a_{p-1})(b_0\, b_1\, \ldots\, b_{p-1})(c_0\, c_1\, \ldots\, c_{p-1})(d_0\, d_1\, \ldots\, d_{p-1})$$

and

$$\tau = (a_0)(b_0\, c_0)(d_0) \prod_{i=1}^{p-1} (a_i\, a_{-i})(b_i\, c_{-i})(c_i\, b_{-i})(d_i\, d_{-i}).$$

Now let Y be a tetravalent one-regular graph of order $4p^2$. Assume that $\mathrm{Aut}(Y)$ contains a normal subgroup H isomorphic to $\mathbb{Z}_p$ such that the corresponding quotient graph Y_H is isomorphic to $X = C(2, p, 2)$. Then, since the orbits of H form an $\mathrm{Aut}(Y)$-invariant partition, the whole automorphism group $\mathrm{Aut}(Y)$ of Y projects to a subgroup of $\mathrm{Aut}(X)$. On the other hand, the graph Y can be viewed as an H-covering graph (that is, a $\mathbb{Z}_p$-covering) of X, and it can therefore be derived from X through a suitable voltage assignment ζ. To find this voltage assignment fix the spanning tree T of X as indicated on Figure 1.

Let G be the largest subgroup of $\mathrm{Aut}(X)$ which lifts with respect to the natural projection $X \times_\zeta \mathbb{Z}_p \cong Y \to Y_H \cong X$, where ζ is as given in Figure 1. Clearly, since Y is arc-transitive, we may assume that $\rho, \tau \in G$. Let F denote the largest subgroup of E which lifts. Then $G = F \rtimes \langle \rho, \tau \rangle$ and thus $|G| = 2p|F|$. We will show that $|F| > 8$. This will then imply that the lift $\bar{G}$ of G is of order $|\bar{G}| = 2p^2|F| > 16p^2$, and consequently that Y is not one-regular.

Since $\rho, \tau \in G$, we have that

$$\text{if } \phi \in F \text{ then } \phi^\rho, \phi^\tau \in F. \tag{2}$$

It is convenient to view elements γ in E as vectors in $\mathbb{Z}_4^p$. Namely, we write $\gamma = (e_0, \ldots, e_{p-1})$ where $e_i = s$ if and only if $\gamma_i = \tau_s$ (where $e_i = 0$ means that $\gamma_i = \tau_0 = id$). Note that in this context (2) can be interpreted as follows: F is invariant under the "cyclic shift"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f_{p-1}, f_0, \ldots, f_{p-2}),$$

and under the "reflection around the first entry"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f_0', f_{p-1}', f_{p-2}', \ldots, f_2', f_1'),$$

where

$$f_i' = \begin{cases} 0 & , \text{ if } f_i = 0 \\ 1 & , \text{ if } f_i = 2 \\ 2 & , \text{ if } f_i = 1 \\ 3 & , \text{ if } f_i = 3 \end{cases}$$

Now choose $\phi \in F$. By (1) the first two components of ϕ can be one of the following pairs: $\phi = (0, 0, \ldots)$, $\phi = (0, 2, \ldots)$, $\phi = (1, 0, \ldots)$, $\phi = (1, 2, \ldots)$, $\phi = (2, 1, \ldots)$, $\phi = (2, 3, \ldots)$, $\phi = (3, 1, \ldots)$, or $\phi = (3, 3, \ldots)$. Since the lift of G acts arc-transitively on Y the group G must be of order $|G| = 2p|F| \geq 16p$ and thus $|F| \neq 1$.

Suppose first that there exist $\psi \in F$ such that $\psi \notin \{id, (3, 3, \ldots, 3)\}$. Since ρ is of prime order, the conjugacy class of ψ under $\langle\rho\rangle$ is of size p. But then, by (2), we have that $|F| > 8$, which implies that $\bar{G}$ is not acting one-regularly on Y.

Suppose now that $(3, 3, \ldots, 3)$ belongs to F. Then, since $\langle(3, 3, \ldots, 3)\rangle \leq F$ is of order 2 and $|G| = 2p|F| = 16p$, we have that there must also exist a non-identity automorphism $\psi \in F$ which is different from $(3, 3, \ldots, 3)$. But then, as above, the conjugacy class of ψ is of size p, and consequently $|F| > 8$. This shows that $\bar{G}$ is not acting one-regularly on Y, and the proof is completed. □

By the following lemma there are only two normal one-regular Cayley graphs on the group $G = \langle a, b, c, g \,|\, a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a\rangle$.

Lemma 4.2. *Let p be a prime and $G = \langle a, b, c, g \,|\, a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a\rangle$. Then a tetravalent normal Cayley graph X of order $4p^2$ on G is one-regular if and only if it is either isomorphic to*

$$\mathcal{CN}_{4p^2}^3 = \mathrm{Cay}(G, \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\}) \text{ or to } \mathcal{CN}_{4p^2}^4 = \mathrm{Cay}(G, \{ag,\, b^{\varepsilon}cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}).$$

Moreover, $\mathrm{Aut}(\mathcal{CN}_{4p^2}^3) \cong G \rtimes \mathbb{Z}_2^2$ *and* $\mathrm{Aut}(\mathcal{CN}_{4p^2}^4) \cong G \rtimes \mathbb{Z}_4$.

Proof. Let X be a tetravalent one-regular normal Cayley graph $\mathrm{Cay}(G, S)$ on the group G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on S, and either $\mathrm{Aut}(G, S) \cong \mathbb{Z}_2^2$ or $\mathrm{Aut}(G, S) \cong \mathbb{Z}_4$. This implies that elements in S are all of the same order.

Observe that G contains elements of order 2, p and $2p$. In particular, elements of the form c, a^ib^jg and a^ib^jcg, where $p \mid i + j$, are of order 2; elements of the form a^ib^j are of order p; and elements of the form a^ib^jc, a^mb^ng and a^mb^ncg, where $p \nmid m + n$, are of order $2p$. In the following, we will show that up to isomorphism, there are only two generating sets of size 4 such that the corresponding Cayley graphs are normal and one-regular.

First, observe that neither four involutions nor two elements of order p can generate G. Moreover, G cannot be generated by the following pairs of elements of order $2p$: $a^{i_1}b^{j_1}c$ and $a^{i_2}b^{j_2}c$, $a^{m_1}b^{n_1}g$ and $a^{m_2}b^{n_2}g$, $a^{m_1}b^{n_1}cg$ and $a^{m_2}b^{n_2}cg$, where $m_i + n_i \neq 0\, (1 \leq i \leq 2)$. Second, $Z(G) = \langle ab, c\rangle = \langle ab\rangle \times \langle c\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_2$, and thus $\langle c\rangle$ char G. Also, since $\mathrm{Aut}(G, S)$ is transitive on S, we have that $S \neq \{a^ib^jc,\, a^mb^ng,\, (a^ib^jc)^{-1},\, (a^mb^ng)^{-1}\}$ and $S \neq \{a^ib^jc,\, a^mb^ncg,\, (a^ib^jc)^{-1},\, (a^mb^ncg)^{-1}\}$, where $m + n \neq 0$. Now suppose that G is generated by

$$S_0 = \{a^ib^jg,\, a^{m'}b^{n'}cg,\, (a^ib^jg)^{-1},\, (a^{m'}b^{n'}cg)^{-1}\},$$

where $p \nmid i+j$ and $p \nmid m'+n'$.

Case 1. $\mathrm{Aut}(G,S_0) = \langle\alpha\rangle \times \langle\beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where α and β are such that $a^\alpha = a^{i_1}b^{j_1}$, $b^\alpha = a^{j_1}b^{i_1}$, $c^\alpha = c$, $g^\alpha = a^x b^{-x} cg$, $a^\beta = a^{i_2}b^{j_2}$, $b^\beta = a^{j_2}b^{i_2}$, $c^\beta = c$ and $g^\beta = a^y b^{-y} g$.

Subcase 1.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 if and only if $m' \neq n'$. Now take an automorphism σ of G such that

$$a^\sigma = a^i,\ b^\sigma = b^i,\ c^\sigma = c,\ g^\sigma = g.$$

Then $(abg)^\sigma = a^i b^i g$, and hence

$$S = {S_0}^{\sigma^{-1}} = \{abg,\ a^m b^n cg,\ (abg)^{-1},\ (a^m b^n cg)^{-1}\} = \{abg,\ a^m b^n cg,\ a^{-1}b^{-1}g, a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$. Moreover, it can be easily seen that $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = abg$, $(a^{-1}b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = a^{-1}b^{-1}g$. It follows that either $m+n = 2$ or $m+n = -2$. If $m+n = 2$ then, since $m \neq n$, we have that $m \neq 1$ and

$$a^\alpha = b,\ b^\alpha = a,\ c^\alpha = c,\ g^\alpha = a^{m-1}b^{1-m}cg.$$

If $m+n = -2$, then since $m \neq n$, we have $n \neq -1$ and

$$a^\alpha = a^{-1},\ b^\alpha = b^{-1},\ c^\alpha = c,\ g^\alpha = a^{-1-n}b^{1+n}cg.$$

Suppose now that $(abg)^\beta = a^{-1}b^{-1}g$. Then $(a^{-1}b^{-1}g)^\beta = abg$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. By a similar argument as above, one can get that

$$a^\beta = b^{-1},\ b^\beta = a^{-1},\ c^\beta = c,\ g^\beta = g.$$

Consequently, either $S_0 = S_1 = \{abg,\ a^m b^{2-m} cg,\ a^{-1}b^{-1}g,\ a^{m-2}b^{-m}cg\}$, where $m \neq 1$, or

$$S_0 = S_2 = \{abg,\ a^{-2-n}b^n cg,\ a^{-1}b^{-1}g,\ a^{-n}b^{n+2}cg\},$$

where $n \neq -1$. In addition, replacing $-n$ with m, it can be seen that $S_2 = S_1$. Moreover, it can be easily seen that G can indeed be generated by S_1. Namely, since $(abg)^p = g$ we have $g, ab \in \langle S_1\rangle$. Then, since $a^m b^{2-m} cg \in \langle S_1\rangle$, we get that $a^m b^{2-m} c \in \langle S_1\rangle$. Further, since $(a^m b^{2-m} c)^p = c$, also $c, a^m b^{2-m} \in \langle S_1\rangle$. Now, since $a^m b^{2-m} = a^m b^m b^{2-2m}$, $m \neq 1$, and $ab \in \langle S_1\rangle$, we get that $b^{2-2m} \in \langle S_1\rangle$. Finally, the fact that $b^g = a$ implies that $G = \langle S_1\rangle$.

Subcase 1.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$ and

$$S = {S_0}^{\sigma^{-1}} = \{ag,\ a^m b^n cg,\ (ag)^{-1},\ (a^m b^n cg)^{-1}\} = \{ag,\ a^m b^n cg,\ b^{-1}g,\ a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = ag$, $(b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$. In addition, either $m+n = 1$ or $m+n = -1$. If $m+n = 1$ then, since $\{ag,\ acg,\ b^{-1}g,\ b^{-1}cg\}$ cannot generate G, we have that $m \neq 1$. Thus α is mapping according to the rule: $a^\alpha = b$, $b^\alpha = a$, $c^\alpha = c$, and $g^\alpha = a^m b^{-m} cg$. If on the other hand $m+n = -1$ then, since $\{ag,\ b^{-1}cg,\ b^{-1}g,\ acg\}$ cannot generate G, we have that $n \neq -1$, and hence α is mapping according to the rule: $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$, $c^\alpha = c$, and $g^\alpha = a^{-n}b^n cg$.

Suppose now that $(ag)^\beta = b^{-1}g$. Then we have that $(b^{-1}g)^\beta = ag$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. Whenever $m+n = 1$ or $m+n = -1$, we can get that β is mapping according to the rule: $a^\beta = b^{-1}$, $b^\beta = a^{-1}$, $c^\beta = c$, and $g^\beta = g$. Thus, we can conclude that either $S_0 = S_3 = \{ag,\ a^m b^{1-m} cg,\ b^{-1}g,\ a^{m-1}b^{-m}cg\}$, where $m \neq 1$, or $S_0 = S_4 = \{ag,\ a^{-n-1}b^n cg,\ b^{-1}g, a^{-n}b^{n+1}cg\}$, where $n \neq -1$. Moreover, replacing $-n$ with m, it can be easily seen that $S_4 = S_3$. Also, since $(ag)^2 = ab$ and $aga^m b^{1-m} cg = a^{2-m} b^m c$, we get that $c, a^{2-m} b^m \in \langle S_3 \rangle$. Further, the facts that $a^{2-m} b^m = a^{2-2m} a^m b^m$, $m \neq 1$ and $ab \in \langle S_3 \rangle$ combined together imply that $a^{2-2m} \in \langle S_3 \rangle$. Since $ag \in \langle S_3 \rangle$, it follows that $g \in \langle S_3 \rangle$. Finally, since $a^g = b$, G is indeed generated by S_3.

Now considering the automorphism γ of G defined by $a^\gamma = a^{\frac{1}{2}}$, $b^\gamma = b^{\frac{1}{2}}$, $c^\gamma = c$, and $g^\gamma = a^{\frac{1}{2}} b^{-\frac{1}{2}} g$ we get that ${S_1}^\gamma = \{ag,\ a^{\frac{m+1}{2}} b^{1-\frac{m+1}{2}} cg,\ b^{-1}g,\ a^{\frac{m+1}{2}-1} b^{-\frac{m+1}{2}} cg\}$, where $m \neq 1$. Thus we only need to consider the generating set $S_3 = \{ag,\ a^m b^{1-m} cg,\ b^{-1}g,\ a^{m-1} b^{-m} cg\}$, where $m \neq 1$.

Case 2. $\mathrm{Aut}(G, S_0) = \langle \alpha \rangle \cong \mathbb{Z}_4$, where α is such that $a^\alpha = a^{i_1} b^{j_1}$, $b^\alpha = a^{j_1} b^{i_1}$, $c^\alpha = c$, and $g^\alpha = a^x b^{-x} cg$.

Subcase 2.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 (where $p \nmid i$ and $p \nmid m' + n'$) if and only if $m' \neq n'$. Now take an automorphism σ of G such that $a^\sigma = a^i$, $b^\sigma = b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(abg)^\sigma = a^i b^i g$, and consequently

$$S = {S_0}^{\sigma^{-1}} = \{abg,\ a^m b^n cg,\ (abg)^{-1},\ (a^m b^n cg)^{-1}\} = \{abg,\ a^m b^n cg,\ a^{-1} b^{-1} g, a^{-n} b^{-m} cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$, and $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = a^{-1} b^{-1} g$, $(a^{-1} b^{-1} g)^\alpha = a^{-n} b^{-m} cg$, $(a^{-n} b^{-m} cg)^\alpha = abg$. Hence either $m + n = \omega$ or $m + n = -\omega$, where $\omega^2 = -4$. If $m + n = \omega$ then since $m \neq n$, we have that $m \neq \frac{\omega}{2}$. It follows that $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}} b^{\frac{\omega}{2}-m} cg$, where $i = \frac{(m+1)\omega+2-2m}{2(2m-\omega)}$. If on the other hand $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n} b^{\frac{\omega}{2}+n} cg$, where $i = \frac{2-2n-(n+1)\omega}{2(2n+\omega)}$.

Suppose now that $(abg)^\alpha = a^{-n} b^{-m} cg$. Then $(a^{-n} b^{-m} cg)^\alpha = a^{-1} b^{-1} g$, $(a^{-1} b^{-1} g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = abg$. Hence, either $m + n = \omega$ or $m + n = -\omega$, where $\omega^2 = -4$. If $m + n = \omega$ then, since $m \neq n$, we have that $m \neq \frac{\omega}{2}$, and thus $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}} b^{\frac{\omega}{2}-m} cg$, where $i = \frac{(1-m)\omega-2m-2}{2(2m-\omega)}$. If however $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n} b^{\frac{\omega}{2}+n} cg$, where $i = \frac{(n-1)\omega-2n-2}{2(2n+\omega)}$.

We can conclude that either $S_0 = S_5 = \{abg,\ a^m b^{\omega-m} cg,\ a^{-1} b^{-1} g,\ a^{m-\omega} b^{-m} cg\}$, where $m \neq \frac{\omega}{2}$, or $S_0 = S_6 = \{abg,\ a^{-\omega-n} b^n cg,\ a^{-1} b^{-1} g,\ a^{-n} b^{n+\omega} cg\}$, where $n \neq -\frac{\omega}{2}$. Moreover, replacing $-n$ with m, it can be easily seen that $S_5 = S_6$. Also, the group G is indeed generated by S_5. Namely, since $(abg)^p = g$ we have that $g, ab \in \langle S_5 \rangle$. Further, since $a^m b^{\omega-m} cg \in \langle S_5 \rangle$, also $a^m b^{\omega-m} c \in \langle S_5 \rangle$, and the fact that $(a^m b^{\omega-m} c)^p = c$ implies that $c, a^m b^{\omega-m} \in \langle S_5 \rangle$. Finally, since $a^m b^{\omega-m} = a^m b^m b^{\omega-2m}$, $m \neq \frac{\omega}{2}$, and $ab \in \langle S_5 \rangle$, it follows that $b^{\omega-2m} \in \langle S_5 \rangle$. Now this fact and $b^g = a$ combined together imply that $G = \langle S_5 \rangle$.

Subcase 2.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$, and consequently

$$S = {S_0}^{\sigma^{-1}} = \{ag,\ a^m b^n cg,\ (ag)^{-1},\ (a^m b^n cg)^{-1}\} = \{ag,\ a^m b^n cg,\ b^{-1} g,\ a^{-n} b^{-m} cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = b^{-1} g$, $(b^{-1} g)^\alpha = a^{-n} b^{-m} cg$, and $(a^{-n} b^{-m} cg)^\alpha = ag$. Also, either $m + n = \varepsilon$ or $m + n = -\varepsilon$, where $\varepsilon^2 = -1$. If $m + n = \varepsilon$ then, since $\{ag,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg,\ b^{-1} g,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg\}$ cannot generate G (namely, for $\varphi \in \mathrm{Aut}(G)$ such that $a^\varphi = a^2$, $b^\varphi = b^2$, $c^\varphi = c$, and $g^\varphi = a^{-1} bg$ we have $\{ag,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg,\ b^{-1} g,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg\}^\varphi = \{abg,\ a^\varepsilon b^\varepsilon cg,\ a^{-1} b^{-1} g,\ a^{-\varepsilon} b^{-\varepsilon} cg\}$), we have that $m \neq \frac{\varepsilon+1}{2}$. It follows that

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i} b^i,\ c^\alpha = c,\ \text{and}\ g^\alpha = a^{m-i} b^{i-m} cg,$$

where $i = \frac{m\varepsilon-m+1}{2m-\varepsilon-1}$. If on the other hand $m + n = -\varepsilon$ then, since G cannot be generated by

$$\{ag,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg, b^{-1} g,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg\},$$

we have that $n \neq -\frac{\varepsilon+1}{2}$, and so

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i} b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{-\varepsilon-i-n} b^{\varepsilon+i+n} cg,$$

where $i = -\frac{(n+1)\varepsilon+n}{2n+\varepsilon+1}$.

Suppose now that $(ag)^\alpha = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$, $(b^{-1}g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = ag$. Also, either $m+n=\varepsilon$ or $m+n=-\varepsilon$, where $\varepsilon^2=-1$. If $m+n=\varepsilon$ then, since $\{ag,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg,\ b^{-1}g,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg\}$ cannot generate G, we have that $m \neq \frac{\varepsilon+1}{2}$, and thus

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i} b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{m-\varepsilon-i} b^{\varepsilon+i-m} cg,$$

where $i = \frac{\varepsilon(1-m)-m}{2m-\varepsilon-1}$. If however $m+n=-\varepsilon$ then, since $\{ag,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg,\ b^{-1}g,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg\}$ cannot generate G, we have that $n \neq -\frac{\varepsilon+1}{2}$, and consequently

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i} b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{-i-n} b^{i+n} cg,$$

where $i = \frac{n(\varepsilon-1)-1}{2n+\varepsilon+1}$.

We can conclude that either $S_0 = S_7 = \{ag,\ a^m b^{\varepsilon-m} cg,\ b^{-1}g,\ a^{m-\varepsilon} b^{-m} cg\}$, where $m \neq \frac{\varepsilon+1}{2}$, or $S_0 = S_8 = \{ag,\ a^{-n-\varepsilon} b^n cg,\ b^{-1}g, a^{-n} b^{n+\varepsilon} cg\}$, where $n \neq -\frac{\varepsilon+1}{2}$. Further, replacing $-n$ with m, one can see that $S_8 = S_7$. That G is indeed generated by S_7 can be seen in the following way. Since $(ag)^2 = ab$ and $aga^m b^{\varepsilon-m} cg = a^{\varepsilon+1-m} b^m c$, we have that $c, a^{\varepsilon+1-m} b^m \in \langle S_7 \rangle$. Then, since $a^{\varepsilon+1-m} b^m = a^{\varepsilon+1-2m} a^m b^m$, $m \neq \frac{\varepsilon+1}{2}$, and $ab \in \langle S_7 \rangle$, we get that $a^{\varepsilon+1-2m} \in \langle S_7 \rangle$. Finally, since $ag \in \langle S_7 \rangle$, it follows that also $g \in \langle S_7 \rangle$. Now the fact that $a^g = b$ implies that $G = \langle S_7 \rangle$.

Now considering the automorphism γ of G defined by

$$a^\gamma = a^{\frac{1}{2}},\ b^\gamma = b^{\frac{1}{2}},\ c^\gamma = c, \text{ and } g^\gamma = a^{\frac{1}{2}} b^{-\frac{1}{2}} g,$$

gives that ${S_5}^\gamma = \{ag,\ a^{\frac{m+1}{2}} b^{\frac{\omega}{2}-\frac{m+1}{2}} cg,\ b^{-1}g,\ a^{\frac{m+1}{2}-\frac{\omega}{2}} b^{-\frac{m+1}{2}} cg\}$, where $m \neq \frac{\omega}{2}$. So we only need to consider the generating set $S_7 = \{ag,\ a^m b^{\varepsilon-m} cg,\ b^{-1}g,\ a^{m-\varepsilon} b^{-m} cg\}$, where $m \neq \frac{\varepsilon+1}{2}$ and $\varepsilon^2 = -1$. Observe also, that this implies that $p \equiv 1 \pmod 4$.

We have proved that when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there always exists an automorphism σ of G such that ${S_0}^\sigma = S = \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}$. Moreover, $\mathrm{Aut}(G, S) = \langle \alpha, \beta \rangle$, where

$$a^\alpha = b,\ b^\alpha = a,\ c^\alpha = c,\ g^\alpha = cg\, a^\beta = b^{-1},\ b^\beta = a^{-1},\ c^\beta = c, \text{ and } g^\beta = g.$$

One the other hand when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_4$ there always exists an automorphism δ of G such that ${S_0}^\delta = S = \{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon} cg\}$. Moreover, in this case $\mathrm{Aut}(G, S) = \langle \rho \rangle$, where

$$a^\rho = a^{\frac{\varepsilon-1}{2}} b^{\frac{\varepsilon+1}{2}},\ b^\rho = a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}},\ c^\rho = c, \text{ and } g^\rho = a^{\frac{1-\varepsilon}{2}} b^{\frac{\varepsilon-1}{2}} cg.$$

Observe also that the following hold:

(1) If $\varepsilon^2 = -1$ then $\{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon} cg\}^\tau = \{ag,\ b^{-\varepsilon} cg,\ b^{-1}g,\ a^\varepsilon cg\}$, where τ is an automorphism of G mapping according to the rule $a^\tau = b^{-\varepsilon}$, $b^\tau = a^{-\varepsilon}$, $c^\tau = c$, and $g^\tau = cg$.
(2) Since $agbcg = a^2 c$, $(a^2c)^2 = a^4$, $(a^2c)^p = c$, $a^g = b$ and p is an odd prime, we can conclude that $\langle \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\} \rangle = \langle ag,\ bcg \rangle = \langle a,\ b,\ c,\ g \rangle = G$.
(3) Let $\varepsilon^2 = -1$. Then $agb^\varepsilon cg = a^{1+\varepsilon} c$, $(a^{1+\varepsilon}c)^2 = a^{2(1+\varepsilon)}$, and $(a^{1+\varepsilon}c)^p = c$. Since p is an odd prime and $a^g = b$, we can conclude that $\langle \{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon} cg\} \rangle = \langle ag,\ b^\varepsilon cg \rangle = \langle a,\ b,\ c,\ g \rangle = G$.

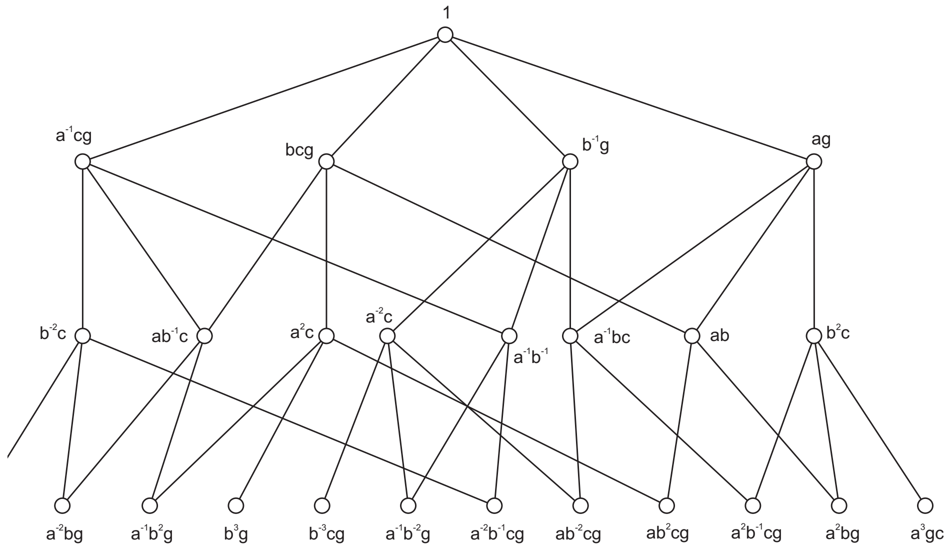

Figure 2: A local structure of the graph $CN^3_{4p^2}$.

To finish the proof, it is sufficient to prove that the graphs

$$Cay(G, \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}) \text{ and } Cay(G, \{ag,\ b^{\varepsilon}cg,\ b^{-1}g,\ a^{-\varepsilon}cg\})$$

are normal Cayley graphs.

First, let $X = Cay(G, \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing the set $S = \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}$ pointwise. Then, since the 2-arc $(1, ag, a^{-1}bc)$ lies on a 6-cycle but the 2-arc $(1, ag, ab)$ does not, one can see that A_1^* fixes every vertex at distance 2 from 1 in X (see also Figure 2). By connectivity of X and transitivity of A on $V(X)$, A_1^* fixes every vertex in X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $\mathrm{Aut}(G, S) = \mathbb{Z}_2^2 \leq A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_2^2, D_8, A_4, S_4\}$. If $A_1 \in \{A_4, S_4\}$ then there exists a permutation δ in A_1 of order 3. We can, without loss of generality, assume that δ fixes ag, and cyclically permutates the other three neighbors of 1. But, however, considering the images of the vertices at distance 2 from 1, one can see that this is impossible (see Figure 2). If $A_1 = D_8$ then we may, without loss of generality, assume that there exists an involution $\gamma \in A_1$ such that $\gamma \notin \mathrm{Aut}(G, S)$, $(ag)^{\gamma} = ag$, $(b^{-1}g)^{\gamma} = b^{-1}g$, $(bcg)^{\gamma} = a^{-1}cg$ and $(a^{-1}cg)^{\gamma} = bcg$. However, ab is a common neighbor of ag and bcg in X, but there is no common neighbor of ag and $a^{-1}cg$, and thus this case cannot occur. It follows that $A_1 = \mathrm{Aut}(G, S) = \mathbb{Z}_2^2$, and so X is a normal one-regular Cayley graph as claimed.

Now let $X = Cay(G, \{ag,\ b^{\varepsilon}cg,\ b^{-1}g,\ a^{-\varepsilon}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing S pointwise. Then considering 6-cycles passing through the vertex 1 one can see that A_1^* fixes all the vertices at distance 2 from 1 in X (see also Figure 3). Then, connectivity and vertex-transitivity of X combined together imply that A_1^* fixes every vertex of X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $\mathrm{Aut}(G, S) \cong \mathbb{Z}_4 \lesssim A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_4, D_8, S_4\}$. If $A_1 \in \{D_8, S_4\}$ then, without loss of generality, we may assume that there exists an involution $\zeta \in A_1$ such that $\zeta \notin \mathrm{Aut}(G, S)$, $(ag)^{\zeta} = ag$, $(b^{-1}g)^{\zeta} = b^{-1}g$, $(b^{\varepsilon}cg)^{\zeta} = a^{-\varepsilon}cg$, and $(a^{-\varepsilon}cg)^{\zeta} = (b^{\varepsilon}cg)$. Since there is no 6-cycle passing through $b^{-1}g$, 1, ag and ab, it follows that ζ fixes ab. On the other hand, since ζ normalizes a Sylow p-subgroup P of G ($P \trianglelefteq A$, see Theorem 5.1), we have that $(xy)^{\zeta} = 1^{R(xy)\zeta} = 1^{\zeta^{-1}(R(x)R(y))\zeta} = 1^{R(x)^{\zeta}R(y)^{\zeta}} = R(x)^{\zeta}R(y)^{\zeta} = 1^{R(x)^{\zeta}}1^{R(y)^{\zeta}} = x^{\zeta}y^{\zeta}$, for every $x, y \in \langle a, b\rangle$. In other words, ζ induces an automorphism on $\langle a, b\rangle$. Thus, ζ fixes $\langle ab\rangle$ pointwise, and, in particular, ζ fixes both $a^{\varepsilon}b^{\varepsilon}$ and $a^{-\varepsilon}b^{-\varepsilon}$, a contradiction. This means that $A_1 = \mathrm{Aut}(G, S) = \mathbb{Z}_4$, and thus X is a normal one-regular Cayley graph as claimed.

□

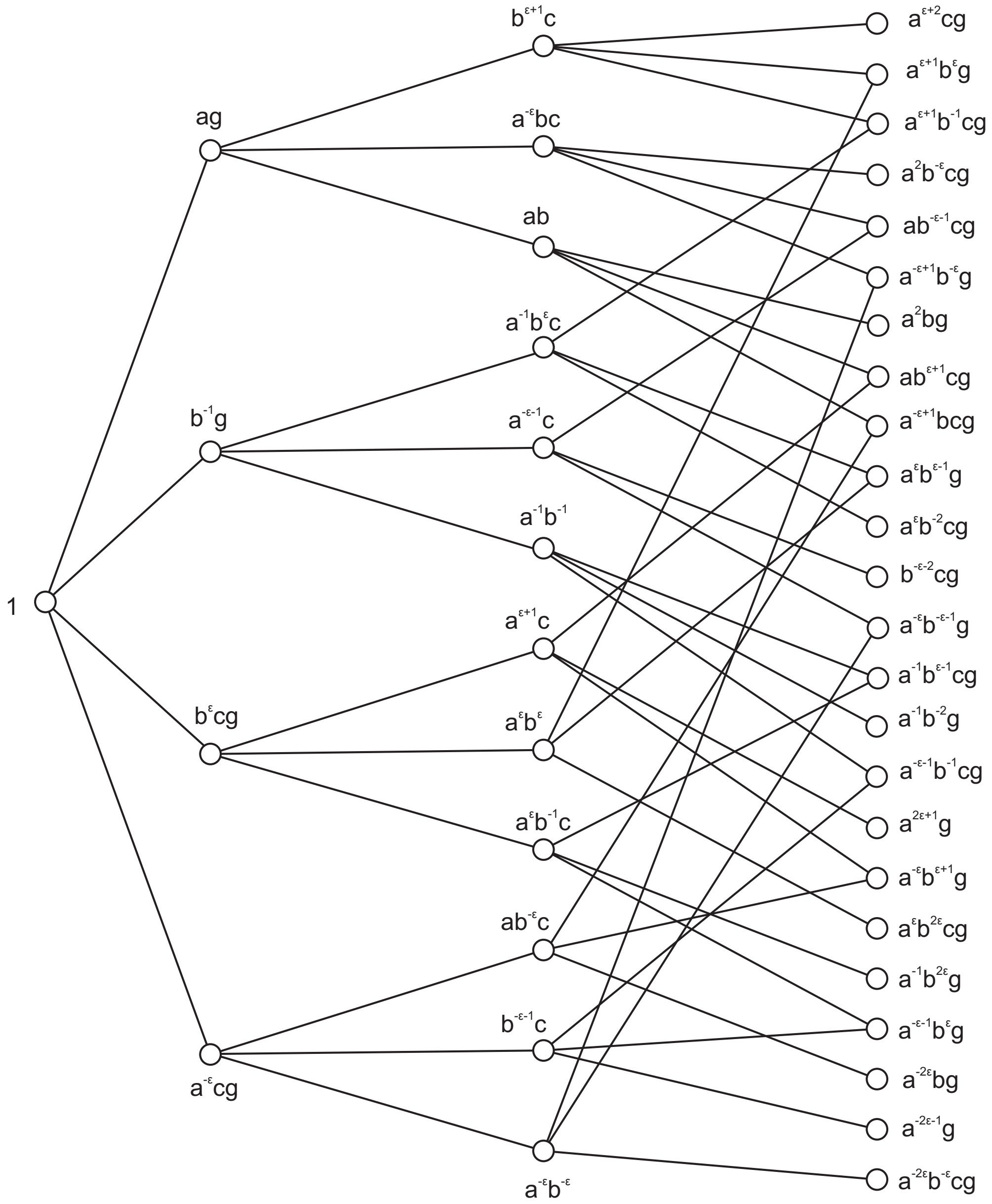

Figure 3: A local structure of the graph $\mathcal{CN}^4_{4p^2}$.

Lemma 4.3. $\mathcal{CA}^1_{4p^2} \cong \mathcal{CN}^3_{4p^2}$.

Proof. Let $G^1_{4p^2} = \langle a, b \mid a^{4p} = b^p = 1, ab = ba\rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$ and let $G^3_{4p^2} = \langle a, b, c, g \mid a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a\rangle$. Then the automorphism group of $\mathcal{CN}^3_{4p^2} = \mathrm{Cay}(G^3_{4p^2}, \{ag, bcg, b^{-1}g, a^{-1}cg\})$, is equal to $\mathrm{Aut}(\mathcal{CN}^3_{4p^2}) = R(G^3_{4p^2}) \rtimes A_1 = R(G^3_{4p^2}) \rtimes \langle \alpha, \beta\rangle \cong G^3_{4p^2} \rtimes \mathbb{Z}_2^2$, where $a^\alpha = b$, $b^\alpha = a$, $c^\alpha = c$, $g^\alpha = cg$, $a^\beta = b^{-1}$, $b^\beta = a^{-1}$, $c^\beta = c$, $g^\beta = g$.

Let $H = \langle R(ag)\alpha, R(b)\rangle$. Then it is easy to see that $H = \langle R(ag)\alpha\rangle \times \langle R(b)\rangle \cong G^1_{4p^2}$. Since $H_1 \leq A_1 = \langle \alpha, \beta\rangle \cong \mathbb{Z}_2^2$ and subgroups of order 4 in H are cyclic, we have that $H_1 < A_1$. Moreover, since $(R(ag)\alpha)^{2p}$ is a unique element of order 2 in H and $1^{(R(ag)\alpha)^{2p}} \neq 1$, we have that $H_1 \notin \{\langle\alpha\rangle, \langle\beta\rangle, \langle\alpha\beta\rangle\}$. Thus $H_1 = 1$, that is, H is a regular subgroup of $\mathrm{Aut}(\mathcal{CN}^3_{4p^2})$. Now Proposition 2.6 and Example 3.4 combined together imply that $\mathcal{CA}^1_{4p^2} \cong \mathcal{CN}^3_{4p^2}$. □

Lemma 4.4. $\mathcal{CN}^2_{4p^2} \cong \mathcal{CN}^4_{4p^2}$.

Proof. Let $G^2_{4p^2} = \langle a, b \,|\, a^{4p} = b^p = 1, a^{-1}ba = b^{\varepsilon}, \varepsilon^2 \equiv -1 (\mathrm{mod}\ p)\rangle$, and let $G^3_{4p^2} = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1, a^g = b, b^g = a\rangle$. Let 4^{-1} be the inverse of 4 in $\mathbb{Z}_p$ and let $r = 4^{-1}(\varepsilon - 1)$. Observe that $8r(\varepsilon + 1) + 4 \equiv 0 \pmod{4p}$ and that $4r \neq (\varepsilon - 1)$ in $\mathbb{Z}_{4p}$.

Now define a map α from the vertex set of $\mathcal{CN}^4_{4p^2} = \mathrm{Cay}(G^3_{4p^2}, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\})$ to the vertex set of $\mathcal{CN}^2_{4p^2} = \mathrm{Cay}(G^2_{4p^2}, \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\})$ in the following way:

$$\begin{aligned} a^ib^j &\mapsto a^{4r(i-j)}b^{i+j} \\ a^ib^jc &\mapsto a^{4r(i-j+\varepsilon+1)+2}b^{i+j} \\ a^ib^jg &\mapsto a^{4r(j-i+1)+1}b^{i+j} \\ a^ib^jgc &\mapsto a^{4r(j-i-\varepsilon)-1}b^{i+j} \end{aligned}$$

where c and g are involutions in $G^3_{4p^2}$. Then

$$\begin{aligned} (a^ib^j, ag\cdot a^ib^j)^{\alpha} &= (a^ib^j, a^{j+1}b^ig)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j-1+1)+1}b^{i+j+1}) \\ &= (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i+j+1}) = (a^{4r(i-j)}b^{i+j}, ab\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{\varepsilon}cg\cdot a^ib^j)^{\alpha} &= (a^ib^j, a^jb^{i+\varepsilon}gc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i+\varepsilon-j-\varepsilon)-1}b^{i+j+\varepsilon}) \\ &= (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j+\varepsilon}) = (a^{4r(i-j)}b^{i+j}, a^{-1}b^{\varepsilon}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{-1}g\cdot a^ib^j)^{\alpha} &= (a^ib^j, a^jb^{i-1}g)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-1-j+1)+1}b^{i-1+j}) \\ &= (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i-1+j}) = (a^{4r(i-j)}b^{i+j}, ab^{-1}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, a^{-\varepsilon}cg\cdot a^ib^j)^{\alpha} &= (a^ib^j, a^{j-\varepsilon}b^igc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j+\varepsilon-\varepsilon)-1}b^{i+j-\varepsilon}) \\ &= (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j-\varepsilon}) = (a^{4r(i-j)}b^{i+j}, a^{-1}b^{-\varepsilon}\cdot a^{4r(i-j)}b^{i+j}). \end{aligned}$$

Similarly, it can be checked that for any edge $(u, s\cdot u)$, we have that $(u, s\cdot u)^{\alpha} = (v, \bar{s}\cdot v)$, where

$$\begin{aligned} u &\in \{a^ib^jc, a^ib^jg, a^ib^jgc\}, \\ v &\in \{a^{4r(i-j+\varepsilon+1)+2}b^{i+j}, a^{4r(j-i+1)+1}b^{i+j}, a^{4r(j-i-\varepsilon)-1}b^{i+j}\}, \\ s &\in \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}, \text{ and} \\ \bar{s} &\in \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}. \end{aligned}$$

From this it follows that α is an isomorphism from $\mathcal{CN}^2_{4p^2}$ to $\mathcal{CN}^4_{4p^2}$. The details are omitted. □

Lemma 4.5. *The graphs $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$, $\mathcal{CA}^i_{4p^2}$, $i \in \{0,1\}$, $\mathcal{CN}^2_{4p^2}$, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$, are pairwise non-isomorphic.*

Proof. First, by the remark subsequent to Example 3.1, the graph $\mathcal{BW}_{12}(5,1,5)$ is not isomorphic to any of the other graphs listed in the lemma. Next, Example 3.2 shows that $\mathcal{GPS}2(4,3,(0\,1):(1\,2))$ is not isomorphic to any of the other graphs listed in the lemma. Then, since the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, $\mathcal{CA}^0_{4p^2}$ is not isomorphic to $\mathcal{CA}^1_{4p^2}$ and $\mathcal{CN}^2_{4p^2}$. Also, Example 3.4 and Lemmas 4.3 and 4.4 combined together show that $\mathcal{CA}^1_{4p^2}$ and $\mathcal{CN}^2_{4p^2}$ are not isomorphic. Namely, the stabilizer of a vertex in $\mathcal{CA}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{CN}^2_{4p^2}$ is isomorphic to $\mathbb{Z}_4$. Finally, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have a minimal normal Sylow p-subgroup and the automorphism groups of $\mathcal{CA}^1_{4p^2}$, $\mathcal{CN}^2_{4p^2}$, do not have a minimal normal Sylow p-subgroups, we have that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^1_{4p^2}, \mathcal{CN}^2_{4p^2}$. Moreover, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have an elementary abelian Sylow p-subgroup and the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, which follows that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^0_{4p^2}$. The result now follows from the fact that the stabilizer of a vertex in $\mathcal{NC}^0_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_4$ (see [23, Lemmas 8.4 and 8.7] and Lemma 2.11). □

5. The classification

X	$\lvert V(X)\rvert$	$\mathrm{Aut}(X)$	*References*
$\mathcal{BW}_{12}(5,1,5)$	36	$G_{36} \rtimes \mathbb{Z}_2^2$	Example 3.1
$\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$	36	$\lvert\mathrm{Aut}(X)\rvert = 144$	Example 3.2
$\mathcal{NC}^0_{4p^2}$	$4p^2, p > 7$, $p \equiv \pm 1 \pmod 8$	given in [23, Lemma 8.4]	Lemma 2.11
$\mathcal{NC}^1_{4p^2}$	$4p^2, p > 7$, or $p \equiv 1$ or $3 \pmod 8$	given in [23, Lemma 8.7]	Lemma 2.11
$\mathcal{CA}^0_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	Example 3.3
$\mathcal{CA}^1_{4p^2}$	$4p^2, p > 2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$	Example 3.4
$\mathcal{CN}^2_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$G^3_{4p^2} \rtimes \mathbb{Z}_4$	Lemmas 4.2 and 3.6

Table 2: Tetravalent one-regular graphs of order $4p^2$.

We are now ready to state the main theorem of this paper.

Theorem 5.1. *Let p be a prime. Then a tetravalent graph X of order $4p^2$ is one-regular if and only if it is isomorphic to one of the graphs listed in Table 2. Furthermore, all the graphs listed in Table 2 are pairwise non-isomorphic.*

Proof. Let X be a tetravalent one-regular graph of order $4p^2$. Let $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer of $v \in V(X)$ in A. By [39], there is no tetravalent one-regular graph of order 16, and $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2[4,3,(0\ 1):(1\ 2)]$ and $\mathcal{CA}^1_{36}$ are the only tetravalent one-regular graphs of order 36 (see also Examples 3.1, 3.2 and 3.4). Thus, we may assume that $p > 3$. Since X is one-regular we have that $|A| = 16p^2$, and thus A is a solvable group. Let P be a Sylow p-subgroup of A.

Claim: P is normal in A.

Since $|A| = 16p^2$ Sylow's theorems imply that the number of Sylow p-subgroups of A is equal to $|A : N_A(P)| = kp+1$. In addition, this number divides 16. Hence, if $p > 7$ then we clearly have that P is normal in A as claimed. Now we will prove that P is normal in A also when $p \in \{5,7\}$.

Let $N = O_2(A)$ be the largest normal 2-subgroup of A. Suppose first that $|N| = 16$ and consider the quotient graph X_N. Then $N \leq K$, where K is the kernel of A acting on $V(X_N)$, X_N is a symmetric graph of valency 2 or 4, and, by Proposition 2.8, A/K acts arc-transitively on X_N. But then $2 \mid |A/K|$, which is clearly impossible since $|A| = 16p^2$. Therefore $|N| \mid 8$. Now we distinguish three different cases depending on the order of N. Let T be a minimal normal subgroup of A.

Case 1. $|N| = 1$.

Then either $|T| = p^2$ or $|T| = p$. In the former case we have that $T = P$ and thus $P \trianglelefteq A$ as claimed. We may therefore assume that $|T| = p$. Let X_T be the quotient graph of X relative to the orbits of T, and let K be the kernel of A acting on $V(X_T)$. Then $T \leq K$ and A/K acts arc-transitively on X_T. If A/T is abelian then, since A/K is a quotient group of the group A/T, also A/K is abelian. But since A/K is vertex-transitive on X_T, Proposition 2.1 implies that it is regular on X_T, contradicting arc-transitivity of A/K on X_T. Thus A/T is a non-abelian group. Let $C = C_A(T)$. Then $T \leq C$ and, by Proposition 2.2, A/C is isomorphic to a subgroup of $\mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and consequently $T < C$. Let L/T be a minimal normal subgroup of A/T contained in C/T. Then $L/T \cong \mathbb{Z}_p$, and therefore $P = L \trianglelefteq A$.

Case 2. $|N| = 2$.

Then $|T| \in \{p^2, p, 2\}$. If $|T| = p^2$ then $P \trianglelefteq A$ as claimed. Suppose now that $|T| = 2$, and let $C = C_A(T)$. Then $T \leq C$ and, moreover, by Proposition 2.2, $|A/C| = 1$ which implies that $T < C$. Let L/T be a minimal normal subgroup of C/T. Then either $|L/T| = p^2$ or $|L/T| = p$. In the former case it follows that $|L| = 2p^2$, and consequently P char $L \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed. In the later case we have $L = \mathbb{Z}_2 \times \mathbb{Z}_p$. Suppose first that A/L is abelian and consider the quotient graph X_L of X relative to the orbits of L. Let K be the kernel of A acting on $V(X_L)$. Then $L \leq K$, A/K is a quotient group of A/L, and as such also abelian. But since A/K is vertex-transitive on X_L, Proposition 2.1 implies that A/K is regular on X_L, which is impossible since A/K acts arc-transitively on X_L. Thus, A/L is a non-abelian group. Let $C = C_A(L)$. Then $L \leq C$ and, by Proposition 2.2, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and so $L < C$. Let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$ and thus $M \trianglelefteq A$ and $|M| = 2p^2$. In addition, since P char $M \trianglelefteq A$, we have that $P \trianglelefteq A$ as claimed.

Assume now that $|T| = p$. Then an argument similar to the one used above shows that A/T is a non-abelian group. Let $C = C_A(T)$. Then, by Proposition 2.2, we have that $A/C \lesssim \mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. Thus A/C is abelian, which implies that $T < C$. Let L/T be a minimal normal subgroup A/T contained in C/T. Then either $L/T \cong \mathbb{Z}_p$ or $L/T \cong \mathbb{Z}_2$. If $L/T \cong \mathbb{Z}_p$, then clearly $L = P \trianglelefteq A$. If however $L/T \cong \mathbb{Z}_2$, then $L \cong \mathbb{Z}_{2p}$ and, by Proposition 2.2, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$ where $C = C_A(L)$. Hence A/C is abelian, and consequently $L < C$. Now let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$, and so $|M| = 2p^2$. But then P char $M \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed.

Case 3. $|N| \in \{4, 8\}$.

Then either $|A/N| = 2p^2$ or $|A/N| = 4p^2$. Clearly PN/N is a Sylow p-subgroup of A/N and by Sylow's theorems, $PN/N \trianglelefteq A/N$. Moreover, $PN \trianglelefteq A$. If $|N| = 4$ then for $p \in \{5, 7\}$ we have that P is characteristic in PN, and hence normal in A. Also, if $|N| = 8$ and $p = 5$ then one can easily see that P is characteristic in PN and hence normal in A. Therefore we can now assume that $|N| = 8$ and $p = 7$. Then N is isomorphic to one of the following groups: D_8, Q_8 (the quaternion group), $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$. Let $C = C_A(N)$. By Proposition 2.2, we have that $A/C \lesssim \mathrm{Aut}(N)$. If $N \not\cong \mathbb{Z}_2^3$ then $7 \nmid |\mathrm{Aut}(N)|$ and hence $7^2 \mid |C|$, which implies that $P \leq C$. It follows that P is characteristic in PN and hence normal in A. If however $N \cong \mathbb{Z}_2^3$ then $N \leq C$ and $\mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$. Observe that $|A/N| = 98$ and $A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$. But $\mathrm{Aut}(N) = \mathrm{PSL}(2, 7)$ has no subgroup of order 98 since $|\mathrm{PSL}(2, 7)| = 168$, implying that $A/N \neq A/C$, and therefore $N < C$. Note also that $|C| > 8$, but $16 \nmid |C|$. Namely, if $16 \mid |C|$, the fact that A/K acts arc-transitively on X_C, where K is the kernel of A acting on $V(X_C)$, implies that $2 \mid |A/K|$. But this is impossible since $C \leq K$. Therefore $7 \mid |C|$. If $7^2 \nmid |C|$ then $|C| = 8 \cdot 7 = 56$. But then A/C is a group of order $|A/C| = 2 \cdot 7 = 14$ isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$, which by Proposition 2.3 is impossible. Therefore $7^2 \mid |C|$, and consequently $P \leq C_A(N)$. It follows that P is characteristic in PN, and thus normal in A. This proves that A always has a normal Sylow p-subgroup as claimed.

Assume first that P is cyclic. Let X_P be the quotient graph of X relative to the orbits of P and let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, the orbits of P are of length p^2. Thus $|V(X_P)| = 4$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 2.8, we have that $X_P \cong C_4$ and hence $A/K \cong D_8$, forcing $|K| = 2p^2$. Since A/K is a quotient group of A/P, it follows that A/P is a non-abelian group. Moreover, $|K| = 2p^2$ and thus K is not semiregular on $V(X)$. Then $K_v \cong \mathbb{Z}_2$ where $v \in V(X)$. By Proposition 2.2, $A/C \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is not abelian, we have that P is a proper subgroup of C. If $C \cap K \neq P$ then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus, $C \cap K = P$ and $1 \neq C/P = C/(C \cap K) \cong CK/K \trianglelefteq A/K \cong D_8$. If $C/P \cong \mathbb{Z}_2$ then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is abelian, a contradiction. It follows that $|C/P| \in \{4, 8\}$, and hence C/P has a characteristic subgroup of order 4, say H/P. Thus, $|H| = 4p^2$ and $H/P \trianglelefteq A/P$, implying that $H \trianglelefteq A$. In addition, since $H \leq C = C_A(P)$, we have that H is abelian. Clearly, $|H_v| \in \{1, 2, 4\}$. First, suppose that $|H_v| = 4$. Then H_v is a Sylow 2-subgroup of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing $H_v = 1$, a contradiction. Second, suppose that $|H_v| = 2$, and let Q be a Sylow 2-subgroup of H. Then $Q \trianglelefteq A$ and $Q_v = H_v$. Consider the quotient graph X_Q of X relative to the orbits of Q. Since $|Q| = 4$ and $Q_v \cong \mathbb{Z}_2$, Proposition 2.8 implies that $X_Q \cong C_{2p^2}$ and hence $X \cong C_{2p^2}[2K_1]$, contradicting one-regularity of X. Thus, we have that $H_v = 1$, and since $|H| = 4p^2$, H is regular on $V(X)$. It follows that X is a Cayley graph on an abelian group with a cyclic Sylow p-subgroup P. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{4p^2}$ and $\mathbb{Z}_{2p^2} \times \mathbb{Z}_2$, where $p > 3$, are the only abelian groups with a cyclic Sylow p-subgroup. However, by Xu [41, Theorems 3], there is no tetravalent one-regular Cayley graph on $\mathbb{Z}_{4p^2}$, and so $H \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Proposition 2.6 and Example 3.3 combined together now imply that $X \cong \mathcal{CA}^0_{4p^2}$.

Now assume that P is elementary-abelian. Suppose first that P is a minimal normal subgroup of A, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, we have that the orbits of P are of length p^2, and thus $|V(X_P)| = 4$. By Proposition 2.8, $X_P \cong C_4$, and hence $A/K \cong D_8$, forcing $|K| = 2p^2$ and thus $K_v = \mathbb{Z}_2$. Proposition 2.9 now implies that X is isomorphic to $C^{\pm 1}(p, 4, 2)$, $\mathcal{NC}^0_{4p^2}$ or $\mathcal{NC}^1_{4p^2}$. However, by Lemma 2.10, $C^{\pm 1}(p, 4, 2)$ is not one-regular whereas, by Lemma 2.11, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ both are one-regular. Conditions on the prime p written in Table 2 follow from the definition of these graphs (see page 288).

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is transitive on $V(X_N)$. Moreover, we have that $|V(X_N)| = 4p$. By Proposition 2.8, X_N is a cycle of length $4p$, or N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N. If $X_N \cong C_{4p}$, and hence $A/K \cong D_{8p}$, then $|K| = 2p$ and thus $K_v = \mathbb{Z}_2$. Applying Proposition 2.12 we get that X is either isomorphic to $C^{\pm 1}(p; 4p, 1)$ or to $C^{\pm \varepsilon}(p; 4p, 1)$. By Lemmas 3.5 and 3.6 and Example 3.4, these two graphs are both one-regular and they are, respectively, isomorphic to $\mathcal{CA}^0_{4p^2}$ and $\mathcal{CA}^1_{4p^2}$. If, however, X_N is a tetravalent connected G/N-symmetric graph, then, by Proposition 2.8, X is a covering graph of a symmetric graph of order $4p$. By Proposition 2.13, there are six tetravalent symmetric graphs of order $4p$: $K_{4,4}$, $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$, $\mathcal{CA}^1_{4p}$, $C(2, p, 2)$ and $\mathfrak{g}_{28}$. But, since there is no tetravalent one-regular graph of order 16, the automorphism group of $\mathfrak{g}_{28}$ does not admit a one-regular subgroup, and since, by Lemma 4.1, there is no one-regular $\mathbb{Z}_p$-cover of $C(2, p, 2)$, we only need to consider the covering graphs of $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$ and $\mathcal{CA}^1_{4p}$. Observe that in each of these three graphs a one-regular subgroup of automorphisms contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$. Let H be a one-regular subgroup of automorphisms of X_N. Since X is one-regular graph, A is the lift of H. Since H contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$ also A contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $4p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_{2p} \times \mathbb{Z}_2 \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. The classification of groups of order $4p^2$, given in [5, 6], and a detail analysis of all these groups give that G is either isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$ or to $G = \langle a, b, c, g \,|\, a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a\rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_{2p}) \rtimes \mathbb{Z}_2$. However, by Proposition 2.7, there is no tetravalent one-regular graph on $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$, whereas for the latter group, Lemmas 4.2, 4.3 and 4.4, combined together imply that X is either isomorphic to $\mathcal{CA}^1_{4p^2}$ or to $\mathcal{CN}^2_{4p^2}$. Since, by Lemma 4.3, graphs listed in Table 2 are pairwise non-isomorphic the proof is completed. □

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