Identification of Critical Transmission Limits in Injection Impedance Plane

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Highlights

- Derivations of critical boundaries for system stability in terms of injection impedence.
- Derivations of characteristic curves of constant $P$, $Q$, $V$ and delta in injection impedance plane.
- Example of how one of the derived mappings provides an analytical load flow solution.
Identification of critical transmission limits in injection impedance plane

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ABSTRACT

In this paper, equations are derived that describe the mapping of critical boundaries and characteristic curves from the three dimensional PQV-surface into the two-dimensional injection impedance plane (load impedance plane for both positive and negative resistance). The expressions derived for the critical and characteristic curves in the impedance plane form the basis for a new phasor measurement based situational awareness method, which uses the results in this paper to identify critical operational boundaries in real time and to visualize the system operating conditions in an informative way. The situational awareness method will be described in a later paper, where this paper focuses on the derivations of some system characteristics in the injection (or load) impedance plane. The critical curves from the PQV-surface that are mapped into the impedance plane are the ones representing the conditions where the partial derivatives of the variables P, Q and V in respect to each other become zero. In addition to the mapping of the critical curves, some characteristic curves are mapped as well. These include the mapping of the curves of constant P, Q, V and δ from the PQV-surface into the impedance plane. All of the mapped critical and characteristic curves appear as circles in the injection impedance plane.

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1. Introduction

As a consequence of world wide increased political focus on climate changes and thereby an increased focus on reduction of CO2 emission, it is anticipated that the share of electric power production that is based on renewable energy sources will continually increase in the coming decades. This trend is apparent in many countries where especially power production based on wind and solar energy have been rapidly increasing during the last decade.

In a power system where the share of power production by means of uncontrollable renewable energy sources (such as wind or solar energy) is gradually increasing, the power production becomes more decentralized where the production units are often relatively small and spread over a wide area in the power system. In systems where the amount of wind power production has been significantly increased, the existing transmission system is not always designed to cope with the new production patterns. In many cases, it would be desirable to strengthen existing transmission system, but due to an increasing political and public resistance against further expansion of high voltage transmission grids, future expansions could possibly be limited.

Another possibility, instead of strengthening the current transmission system, would be an operation of the existing power system closer to its critical boundaries. Such system operation requires a trustworthy online monitoring of these boundaries. The formation of the power system towards increasing utilization of uncontrollable sources of renewable energy combined with generally more stressed transmission system, necessitates a research within the field of methods providing situational awareness for the power system operation in real time. A situational awareness involves both a knowledge concerning the current operating point, its distance to critical operational boundaries and knowledge of how the operating point could be controlled to increase distance from the critical boundaries.

With the introduction of phasor measurement units (PMUs) [1,2], synchronized measurements of the system voltage and current phasors became possible. With a widespread usage of PMUs in electric power system, a time synchronized snapshots of the system conditions can be updated with a repetition rate equal the system frequency. Several approaches for optimal PMU placement to obtain full observability of the system conditions have been reported [3–5]. The high repetition frequency of PMUs measurements opens up for the development of new applications of wide-area monitoring, detection, protection and control [6–12]. Present and potential applications of phasor measurements have been documented in several surveys [13–15].

One potential application of phasor measurements is to utilize them for obtaining real time situational awareness, where the system stability boundaries are monitored in real time. As a part of the process of developing methods that assess system stability in real time, it can be useful to express critical system boundaries in terms of measurable system quantities, such as system injection...
impedances (the term injection impedance is used to denote a load impedance where the resistive part can be positive or negative). Expressing the system boundaries in the injection impedance plane, real time measurements of the injection impedances can be held against the boundaries, hence providing an assessment of the system stability.

This paper describes analytical derivations for the mapping of some useful characteristics of a three dimensional PQV-surface into an injection impedance plane. The presented expressions for the characteristic boundaries form the basis of a situational awareness method that exploits results from this paper for real time stability assessment and informative visualization of observed system conditions.

2. Critical curves on a PQV-surface

The two bus system and the relevant notations for system variables used in the following derivations are provided in Fig. 1. The relationship between receiving and sending end voltage (E and V), active power P and reactive power Q can be written as:

\[ V^4 + V^2(2(RP + XQ) - E^2) + (R^2 + X^2)(P^2 + Q^2) = 0 \]  

(1)

A solution for either P, Q or V has to be determined if a PQV-surface is to be plotted. Rearranging (1) and solving for P gives the two solutions below:

\[ P_{\pm} = -\frac{R^2V^2 - Q^2(R^2 + X^2)^2 - (R^2 + X^2)(E^2 - 2QX)V^2 - X^2V^4}{R^2 + X^2} \]  

(2)

\[ P_{\pm} = -\frac{R^2V^2 + Q^2(R^2 + X^2)^2 + (R^2 + X^2)(E^2 - 2QX)V^2 - X^2V^4}{R^2 + X^2} \]  

(3)

Letting \( R \geq 0 \) and \( X > 0 \), then \( P_{\pm} \) represents a solution where \( P \) is always negative while \( P_{\pm} \) represents values of \( P \) that can be positive as well as negative. All possible positive values of \( P \) are hence described by \( P_{\pm} \), given the constraints for \( R \) and \( X \) considered above.

The entire PQV-surface is described by the two solutions \( P_{\pm} \) and \( P_{\pm} \). By defining a region \( S \) that contains all points on the PQV-surface and subregions \( S_1 \) and \( S_2 \) that contain the solutions of \( P_{\pm} \) and \( P_{\pm} \) respectively, then \( S = S_1 \cup S_2 \) is valid.

Similarly, by manipulating (1) expressions for \( Q \) and \( V \) can be written as:

\[ Q_{\pm} = -\frac{XV^2 - \sqrt{-P^2(R^2 + X^2)^2 + (R^2 + X^2)(E^2 - 2RP)V^2 - R^2V^4}}{R^2 + X^2} \]  

(4)

\[ Q_{\pm} = \frac{-XV^2 - \sqrt{-P^2(R^2 + X^2)^2 + (R^2 + X^2)(E^2 - 2RP)V^2 - R^2V^4}}{R^2 + X^2} \]  

(5)

There are two solutions for \( Q \) and four solutions for \( V \). The solutions \( V_{\pm} \) give negative values for \( V \) which has no physical meaning and are therefore not of interest in the following. The solutions \( V_{\pm} \) result in positive values for \( V \), and are used in the following derivations. If the regions \( S_1 \) and \( S_2 \) represent the set of points described by \( Q_{\pm} \) and \( Q_{\pm} \) respectively and the regions \( S_3 \) and \( S_4 \) represent the set of points described by \( V_{\pm} \) and \( V_{\pm} \) respectively, then the equalities \( S = S_1 \cup S_2 = S_3 \cup S_4 \) are valid.

The expressions provided in (2)–(6) can be used to visualize the relationship between P, Q, and V as a three dimensional surface. Such surfaces are depicted in Fig. 2 where the values \( E = 1 \), \( X = 0.1 \) and \( R = 0.01 \) are used. Both surfaces are plotted for positive values of \( P \).

On the surface in Fig. 2a several curves of constant load power factor \( \theta \) are shown. Projecting these curves onto the PV-plane gives the traditional nose curves (PV-curves), which are often used in voltage stability analysis. Curves of constant receiving end power \( P \) are plotted onto the surface in Fig. 2b. Projecting the curves of constant \( P \) onto the PQ-plane gives the PV-curves, which are as well commonly used for voltage stability analysis. The resulting PV-curves and PQ-curves from the surfaces in Fig. 2a and b are depicted in Fig. 3a and b respectively.

Besides the curves of constant power factor \( \theta \) and constant power \( P \), two curves of critical importance for stability studies are depicted as well. The first curve intercepts the curves of constant power factor \( \theta \) when \( P \) is at maximum and represents the conditions where the partial derivative \( \frac{\partial \theta}{\partial P} \) is zero. The second curve represents conditions where \( \frac{\partial \theta}{\partial Q} = 0 \).

Both curves represent limits for maximum deliverable power \( P \) for different system condition. The curve \( \frac{\partial \theta}{\partial P} = 0 \) represents maximum deliverable power \( P \) when the sending end voltage \( E \) is fixed and the relationship between \( P \) and \( Q \) is linear (that is changes in \( P \) are proportional to changes in \( Q \)). The curve represents as well the condition for maximum deliverable power \( P \) when the value for the reactive power \( Q \) is constant.

The curve \( \frac{\partial \theta}{\partial Q} = 0 \) represents the maximum deliverable power \( P \) when the sending and receiving end voltages (E and V) are constant. The curve could be used as boundary for the maximum deliverable power between two busses, where the voltage at each bus is kept constant. This boundary may be relevant for rotor angle stability as the boundary represents the conditions for maximum power \( P \) that a source of constant voltage magnitude can inject into the system. The curve described by \( \frac{\partial \theta}{\partial Q} = 0 \) represents the situation when the maximum deliverable power \( P \) to the load is reached for a fixed load factor \( \theta \) and is of interest in voltage stability studies.

With regard to the development of real time stability assessment methods, it might be of interest to monitor an instantaneous operating point and determine its margin from the critical boundaries where the partial derivatives in Fig. 2 become zero. Monitoring and visualizing multiple operating points on a PQV-surface is not convenient as the shape of a PQV-surface is dependent on the value of \( Z_L \) associated with each operating point. On the other hand, a mapping of the main characteristics of a PQV-surface into an injection impedance plane enables the monitoring of multiple operating conditions with real time capacity.
195 points since it enables normalization of the monitored operating points in such a way that the same critical boundaries apply to all points. Before mappings of characteristic curves from the PQV-surface can be described, it is necessary to identify some important equalities that enable an analytical derivation of the mappings.

3. Characteristics of the surface

As necessary background for the mapping of the critical curves on the PQV-surface into the injection impedance plane, the following relationship must be valid:

When \( \frac{\partial P}{\partial V} = 0 \), then \( \frac{\partial Q}{\partial V} = 0 \). \( \tag{9} \)

When \( \frac{\partial P}{\partial Q} = 0 \), then \( \frac{\partial V}{\partial Q} = 0 \). \( \tag{10} \)

When \( \frac{\partial Q}{\partial P} = 0 \), then \( \frac{\partial V}{\partial P} = 0 \). \( \tag{11} \)

In the following subsections it is shown that (9)–(11) are valid statements.

3.1. Proof of when \( \frac{\partial P}{\partial V} = 0 \) then \( \frac{\partial Q}{\partial V} = 0 \)

Differentiating the expression for \( P_{[1]} \) in (2) with respect to \( V \) yields:

\[
\frac{\partial P}{\partial V} = \frac{2RV}{R^2 + X^2} + \frac{V(E^2 - 2QX)(R^2 + X^2) - 2X^2V^3}{(R^2 + X^2)^2 - Q^2(R^2 + X^2)^2 + (R^2 + X^2)(E^2 - 2QX)V^2 - X^4V^4} \tag{12}
\]

Setting \( \frac{\partial P}{\partial V} = 0 \) and solving for \( Q \) results in:

\[
Q_{(\frac{\partial P}{\partial V} = 0)} = \frac{X(E^2 - 2V^2) \pm ER\sqrt{4V^2 - E^2}}{2(R^2 + X^2)} \tag{13}
\]

If \( P_{[2]} \) is used instead of \( P_{[1]} \) when determining the partial derivative in (12), the same expression for \( Q \) as in (13) would be obtained. The result in (13) reveals two solutions for \( Q \), that describe the curve where \( \frac{\partial P}{\partial V} = 0 \).

It can be seen that (13) is valid for \( V \) in the range \([E/2, \infty]\). An description of the entire PQV-surface can be obtained from Eqs.
By constraining expression for $P_{(4)}$ in (2) by inserting the expression with the positive sign for $Q_{(4)}$ in (13) (which lays entirely in $S_1$) gives:

$$P = -2RV^2 + \sqrt{2E^2(R^2 + E^2)(2V - E)(2V + E) + 4X^2V^2E^2 - X^4E^4}$$

$$= -2RV^2 + \sqrt{E^2(2E^2 + 4XV^2E^2 - X^4E^2)}$$

$$= E^2 + 4XV^2E^2 - X^4E^2$$

This result would be obtained as well if $P_{(2)}$ is differentiated in (17) instead of $P_{(1)}$. By inspecting the expressions for $Q_{(1)}$ and $Q_{(2)}$ in (4) and (5) it can be seen that the expression for $Q_{(4)}$ is identical to $Q_{(1)}$ and $Q_{(2)}$ when the term under the root equals zero. This means that the curve representing $Q_{(4)} = 0$ is at the boundary of regions described by $Q_{(1)}$ and $Q_{(2)}$ ($S_1$ and $S_2$ respectively). Therefore, the expression for $Q_{(4)}$ represents the intersection between $Q_{(1)}$ and $Q_{(2)}$ ($Q_{(4)}$ and $S_1$ respectively). This result would be obtained as well if $P_{(2)}$ is differentiated in (17) and (18) (4) and (5) respectively.

The aim of this subsection is to show that when $\partial P / \partial Q = 0$ then $\partial V / \partial Q = 0$. Expressions for the positive voltage magnitudes ($V_{(1)}$ and $V_{(2)}$) were derived in (6) and (7). Differentiation of the expression for $V_{(1)}$ in (4) with respect to $Q$ yields:

$$\frac{\partial P}{\partial Q} = \frac{-Q(R^2 + X^2) - XV^2}{\frac{Q^2(R^2 + X^2)^2 + (R^2 + X^2)(E^2 - 2QX)V^2 - X^4V^4}{2(R^2 + X^2)}}$$

This result would be obtained as well if $P_{(2)}$ is differentiated in (17) instead of $P_{(1)}$. By inspecting the expressions for $Q_{(1)}$ and $Q_{(2)}$ in (4) and (5) it can be seen that the expression for $Q_{(4)} = 0$ is at the boundary of regions described by $Q_{(1)}$ and $Q_{(2)}$ ($S_1$ and $S_2$ respectively). Therefore, the expression for $Q_{(4)}$ represents the intersection between $Q_{(1)}$ and $Q_{(2)}$ ($Q_{(4)}$ and $S_1$ respectively). This result would be obtained as well if $P_{(2)}$ is differentiated in (17) and (18) respectively.

The aim of this subsection is to show that when $\partial P / \partial Q = 0$ then $\partial V / \partial Q = 0$. Expressions for the positive voltage magnitudes ($V_{(1)}$ and $V_{(2)}$) were derived in (6) and (7). Differentiation of the expression for $V_{(1)}$ in (4) with respect to $Q$ yields:

$$\frac{\partial V}{\partial Q} = \frac{-2RXP - RX^2 - X^4\sqrt{2d^2d^2 + 2d^2(E^2 - 2(RP + QX))}}{2d^2d^2 + 2d^2(E^2 - 2(RP + QX))}$$

where $d = -(RX - XP)^2 + E^2(E^2 - 4(RP + QX))$. Setting $\partial V / \partial Q = 0$ and solving for $P$ results in:

$$P_{(2)} = \frac{RX + E\sqrt{2d^2 - 2d^2(E^2 - 2(RP + QX))}}{X}$$

This expression would have been obtained as well if any one of the four solutions for the voltages ($V_{(1)}$) in (6)–(8) would have been differentiated in (15) and used to determine $P_{(2)}$. Eq. (20) is valid for negative $X$ when $Q$ is positive (inductive) and vice versa.
To show that \( \partial P/\partial Q = 0 \) and \( \partial V/\partial Q = 0 \) are identical, the expression for \( Q_{i,(i\neq 0)} \) from (18) is put into the expression for \( P_{i,1} \) given by (2). Doing so gives:

\[
P_{i,1} = \left( -RV + \sqrt{R^2E^2 + X^2E^2} \right) \cdot V \over R^2 + X^2 \tag{21}
\]

This expression can be achieved by inserting \( Q_{i,(i\neq 0)} \) from (18) into the expression for \( P_{i,(i\neq 0)} \) with the positive sign given by (20):

\[
P_{i,(i\neq 0)} = \frac{RQ}{Q_{i,(i\neq 0)}} + E \sqrt{-Q_{i,(i\neq 0)}} \cdot X \over X
\]

\[
= \left( -RV + \sqrt{R^2E^2 + X^2E^2} \right) \cdot V \over R^2 + X^2 \tag{22}
\]

Since the same result is obtained by constraining the surface described by \( P_{i,1} \) with \( Q_{i,(i\neq 0)} \) as by constraining the curve described by \( P_{i,(i\neq 0)} \) with \( Q_{i,(i\neq 0)} \), it is proven that the curve on the surface where \( \partial V/\partial Q = 0 \) is the same as the curve where \( \partial P/\partial Q = 0 \).

3.3. Proof of when \( \partial Q/\partial P = 0 \) then \( \partial V/\partial P = 0 \)

Differentiation of the expression for \( Q_{i,1} \) in (4) with respect to \( P \) yields:

\[
\partial Q \over \partial P = \frac{RV^2 + P(R^2 + X^2)}{\sqrt{P^2(R^2 + X^2)^2 + (V^2E^2 - 2V^2RP)(R^2 + X^2) - R^2V^4}} \tag{23}
\]

Rearranging and solving for \( P \), when \( \partial P/\partial Q = 0 \) results in:

\[
P_{i,(i\neq 0)} = -\frac{RV^2}{X^2} \tag{24}
\]

This is the same result as would have been obtained if \( Q_{i,1} \) in (5) would have been differentiated in (23) instead of \( Q_{i,1} \). Inspection of Eqs. (2) and (3) for \( P_{i,1} \) and \( P_{i,2} \) respectively, reveals that \( P_{i,(i\neq 0)} \) is identical to \( P_{i,1} \) and \( P_{i,2} \) when the term under the root equals zero. This means that the curve representing \( \partial Q/\partial P = 0 \) is at the boundaries of \( P_{i,1} \) and \( P_{i,2} \) \( (\partial Q/\partial P = 0) \) at \( S_1 \cap S_2 \).

The aim is to show that when \( \partial Q/\partial P = 0 \) then \( \partial V/\partial P = 0 \). Differentiating the expression for \( Q_{i,1} \) in (6) in respect to \( P \) yields:

\[
\partial Q \over \partial P = \frac{2X(RQ - XP) - E^2R - RVd}{2d(E^2 - 2RP + QX)} \tag{25}
\]

where \( d = -4(RP - QX)^2 + E^4 - 2RP + QX) \).

Rearranging and solving for \( Q \), when \( \partial P/\partial Q = 0 \) results in:

\[
Q_{i,(i\neq 0)} = \frac{PX + E\sqrt{RP}}{R} \tag{26}
\]

If any one of the four solutions for the voltages \((V_{1-4}) \) in (6) \(-(8) \) would have been differentiated in (25) and used to determine \( Q_{i,(i\neq 0)} \), the same expression is obtained. Eq. (26) is valid for negative \( P \) if \( R \) is positive and vice versa.

Inserting the expression for \( P_{i,(i\neq 0)} \) form (24) into the expressions for \( Q_{i,1} \) and \( Q_{i,2} \) in (4) and (5) yields:

\[
Q_{i,1} = \frac{XV^2 + EV\sqrt{R^2 + X^2}}{R^2 + X^2} \tag{27}
\]

\[
Q_{i,2} = \frac{XV^2 - EV\sqrt{R^2 + X^2}}{R^2 + X^2} \tag{28}
\]

Same results would be obtained if the expression for \( Q_{i,(i\neq 0)} \) in (26) would have been constrained by inserting \( P_{i,(i\neq 0)} \) from (24).

This proves that the curve on the PQV-surface where \( \partial Q/\partial P = 0 \) is the same as the curve where \( \partial V/\partial P = 0 \).

4. Transformation of the critical curves into injection impedance plane

The following sections describe the mapping of the curves on the PQV-surface, that represent the set of points where the partial derivatives become zero, into the injection impedance plane.

4.1. Transformation of the curve that represents \( \partial P/\partial Q = \partial V/\partial Q = 0 \) into injection impedance plane

By writing the expression for \( Q_{i,(i\neq 0)} \) from (18) in terms of apparent injection impedance \( Z_{iD} \) results in:

\[
Q_{i,(i\neq 0)} = -\frac{XV^2}{R^2 + X^2} = \frac{V^2\sin\theta}{Z_{iD}} \tag{29}
\]

and solving for \( Z_{iD} \) results in:

\[
Z_{iD} = -\frac{Z_{iD} \sin\theta}{\sin\phi} \tag{30}
\]

From (30) it can be seen that the curve that represents \( \partial P/\partial Q = \partial V/\partial Q = 0 \) is mapped as a circle in the injection impedance plane. The circle intercepts the origin of the plane, the complex conjugate of the line impedance and the negative of the line impedance. That is, the curve where \( \partial P/\partial Q = \partial V/\partial Q = 0 \) appears as a circle with center in the impedance plane at \( (R_{iD} = 0, X_{iD} = -r) \) where \( r \) is the radius of the circle and can be expressed as:

\[
r = \frac{Z_{iD} \sin\theta}{2 \sin\phi} \tag{31}
\]

4.2. Transformation of the curve that represents \( \partial Q/\partial P = \partial V/\partial P = 0 \) into injection impedance plane

Revisiting the expression for \( P_{i,(i\neq 0)} \) in (24), which satisfies the conditions \( \partial Q/\partial P = \partial V/\partial P = 0 \), and equating it to an expression for \( P \) in terms of apparent injection impedance \( Z_{iD} \) results in:

\[
P = -\frac{RV^2}{R^2 + X^2} = \frac{V^2\cos\theta}{Z_{iD}} \tag{32}
\]

and solving for \( Z_{iD} \) yields:

\[
Z_{iD} = -\frac{Z_{iD} \cos\theta}{\cos\phi} \tag{33}
\]

From (33) it can be seen that the curve that represents \( \partial Q/\partial P = \partial V/\partial P = 0 \) is mapped as a circle in the injection impedance plane. The circle intercepts the origin of the plane, the real conjugate of the line impedance and the negative of the line impedance. That is, the curve where \( \partial Q/\partial P = \partial V/\partial P = 0 \) appears as a circle with center in the impedance plane at \( (R_{iD} = -r, X_{iD} = 0) \) where \( r \) is the radius of the circle and is expressed as:

\[
r = \frac{Z_{iD}}{2 \cos\phi} \tag{34}
\]

4.3. Transformation of the curve that represents \( \partial P/\partial V = \partial Q/\partial V = 0 \) into injection impedance plane

Previously, it was shown that the expression for \( P_{i,(i\neq 0)} \) in (15) and the expression for \( Q_{i,(i\neq 0)} \) in (13) describe the same curve on the PQV-surface where \( \partial P/\partial V = \partial Q/\partial V = 0 \).

Furthermore, it was stated in Section 3.1 that the solution for \( P_{i,(i\neq 0)} \) with the negative sign lies entirely on the same surface as the solution of \( Q_{i,(i\neq 0)} \) with the positive sign. By rewriting and manipulating the solutions for \( P_{i,(i\neq 0)} \) and \( Q_{i,(i\neq 0)} \) that lie on the same surface gives:
The previously derived mappings of the critical curves where the partial derivatives on the PQV-surface are zero, are useful in stability studies [10]. For the purpose of establishing meaningful visualization of a given operating condition in injection impedance plane, the mapping of curves of constant $P$, $Q$, $V$ and $\delta$ become of interest. In the following section, the mappings of those curves are derived.

5.1. Curves of constant $P$

In order to derive expressions for curves of constant $P$ in the injection impedance plane, the expression (6) is used. In order to relate the values of $P$ to $Z_{LD}$, it is necessary to express $V$ and $Q$ as a function of those variables:

$$V^2 = Z_{LD}P \cos \theta$$

(40)

and solving for $Q_{3}$ results in:

$$Q = P \tan \theta$$

(41)

An expression for values of constant $P$ described as a function of $Z_{LD}$ is obtained by raising (6) in powers of two and inserting the expressions for $V^2$ and $Q$ from (40) and (41). After manipulation, an expression of $Z_{LD}$ as circle in polar coordinates can be obtained:

$$Z_{LD} = r_0 \cos (\theta - \phi) \pm \sqrt{r^2 + r_0^2 \sin^2 (\theta - \phi)}$$

(42)

where

$$r = \sqrt{-\frac{E^2}{4P^2} - \frac{E^4}{4P^2} + \frac{2R E P}{4P^2} - \frac{R^2}{P}}$$

and

$$r_0 = \sqrt{\frac{X^2 + R^2 + \frac{E^4}{4P^2} - \frac{R^2}{P}}{\frac{R^2}{P}}}$$

$$\tan \phi = \frac{-X}{- \left( R - \frac{2P}{E}\right)}$$

The expression in (42) represents the set of points in the injection impedance plane where the receiving end power $P$ is constant. Curves of constant power appear as circles in the injection impedance plane. In the above, $r_0$ is the distance from the origin to the center of the circle, $r$ is the radius of the circle and $\phi$ is the angle between the positive real axis and the line connecting the origin and the center of the circle.

5.2. Curves of constant $Q$

In order to derive expressions for curves of constant $Q$ in the injection impedance plane, the expression in (6) is used again along with the following expressions for $V^2$ and $Q$:

$$V^2 = Z_{LN}Q \sin \theta$$

(43)

and

$$Q = P \tan \theta$$

(44)

Inserting (43) and (44) into the square of (6) and solving for $Z_{LD}$ gives the following expression in polar coordinates:

$$Z_{LD} = r_0 \cos (\theta - \phi) \pm \sqrt{r^2 + r_0^2 \sin^2 (\theta - \phi)}$$

(45)

where

$$r = \sqrt{-\frac{E^2}{4Q^2} - \frac{E^4}{4Q^2} + \frac{2R E Q}{4Q^2} - \frac{R^2}{Q}}$$

and

$$r_0 = \sqrt{\frac{X^2 + R^2 + \frac{E^4}{4Q^2} - \frac{R^2}{Q}}{\frac{R^2}{Q}}}$$

$$\tan \phi = \frac{-X}{- \left( R - \frac{2P}{E}\right)}$$

Eq. (45) represents a circle in the injection impedance or a fixed value of $Q$. In the above, $r_0$ is the distance from the origin to the center of the circle, $r$ is the radius of the circle and $\phi$ is the angle between

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1 For interpretation of color in Fig. 4, the reader is referred to the web version of this article.

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5.3. Curves of constant $V$

In order to derive expressions for curves of constant $V$ in the injection impedance plane, (3) is used along with the following expressions for $P$ and $Q$:

$$P = \frac{V^2}{Z_{LD}} \cos \theta$$

$$Q = \frac{V^2}{Z_{LD}} \sin \theta$$

By inserting (46) and (47) into (3) and solving for $Z_{LD}$ results in the following expression in polar coordinates.

$$Z_{LD} = r_0 \cdot \cos(\theta - \varphi) + \sqrt{r^2 + r_0^2 \cdot \sin^2(\theta - \varphi)}$$

Finally, setting (53) equal (54) and solving for $Z_{LD}$ yields:

$$Z_{LD} = -\frac{Z_{IN}}{\sin \delta} \sin(\delta + \phi - \theta)$$

This equation represents a circle in the injection impedance plane for constant values of $\delta$. The circle represented in (55) has a diameter equal $Z_{IN}/\sin \delta$ and intercepts the origin of the injection impedance plane and the point $Z_m = -Z_{IN}$ for all values of constant $\delta$.

Each circle obtained from (55) represents the conditions for two different values of constant $\delta$; that is when $\delta = \delta_0$ and when $\delta = -180^\circ + \delta_0$ where $\delta_0$ is arbitrary. Therefore, it has to be investigated which part of the circle represents constant $\delta = \delta_0$ and which part represents constant $\delta = -180^\circ + \delta_0$. For simplicity, it is assumed that $0^\circ \leq \delta_0 < 180^\circ$. If $\delta_0 = 0^\circ$, then (55) reveals that the circle representing $\delta = \delta_0$ and $\delta = -180^\circ + \delta_0$ has an radius going towards infinity and these conditions are represented by the straight line that intercepts the origin and the point where $Z_{LD} = -Z_{IN}$. An inspection shows that the part of the line between the two singular points (the origin and the point where $Z_{LD} = -Z_{IN}$) represents the condition where $\delta = \pm 180^\circ$ while other parts of the line represent conditions where $\delta = 0^\circ$. 

![Graphical representation of the critical lines in the injection impedance plane.](image-url)
If the line intercepting the origin and the point where \(Z_{LD} = -Z_{LN}\) is used to split the impedance plane into two halves, then by inspection it can be shown that all of the circles representing a constant \(\delta\) lie in both halves. The curves of constant \(\delta\), when \(\delta = \delta_0\) are represented by the part of the circle laying in the left half plane while the condition where \(\delta = -180 + \delta_0\) are represented by the part of the circle laying in the right half plane. Fig. 5 illustrates how the same circle in the impedance plane represents two different conditions of constant phase angle \(\delta\), depending on in which half plane the circle parts lie in.

5.5. Graphical representation of the characteristic curves in the impedance plane

Figs. 6a and 7a and b illustrate how curves of constant \(P, Q, V\) and \(\delta\) appear in the injection impedance plane. The figures represent the situation when \(Z_{IN} = (1 \angle \phi)\) and \(\phi = 75^\circ\).

Fig. 6a shows how the curves of constant receiving end power \(P\) appear in the complex injection impedance plane. Inspection of the curves of constant \(P\), reveals certain symmetry in how they appear in the impedance plane. Values of constant positive \(P\) appear as circles in the right half plane and the radius of the circle gradually decreases as the power is increased toward the point of maximum deliverable power. The point of maximum deliverable power is where the injection impedance is equal the complex conjugate of the line impedance. Values of constant negative \(P\) are represented in the left half pane of the injection impedance plane. The radius of the circles representing constant negative power (positive bus injection) gradually decreases as the absolute value of the power increases. The point of minimum deliverable power (maximum bus injection) occurs at the singular point where the injection impedance is equal to the negative of the line impedance. At this point the receiving end power \(P\) goes towards \(-\infty\). The set of points representing the situation where \(P\) is zero, appears as a straight line that satisfies \(R = 0\).

Fig. 6b shows how the curves of constant \(Q\) appear in the complex injection impedance plane. The curves for constant \(Q\) have a similar appearance as curves of constant \(P\). The circles for constant positive values of \(Q\) appear in the upper half plane, where the radius of the circles gradually decreases until the point of maximum positive \(Q\) is obtained. The point of maximum \(Q\) occurs when the value of the injection impedance is equal to the (real) conjugate of the line impedance \((X_{LD} = X_{LN}\) and \(R_{LD} = -R_{LN}\)). The circles representing negative constant values of \(Q\) appear in the lower half plane and the minimum occurs at the singular point where the injection impedance is equal the negative of the line impedance.

Fig. 7a shows how curves of constant \(V\) appear in the injection impedance plane. It can be seen that the circles representing constant values of \(V\) have center points on an imaginary line intercepting the origin and the singular point where \(Z_{LD} = -Z_{LN}\). The set of points where \(V = E\) appears as straight line that is perpendicular to the same imaginary line and intercepts that line in the point where \(Z_{LD} = -Z_{LN}/2\). The line representing \(V = E\) divides the injection impedance plane into two halves, where values of \(V\) lower than \(E\) appear as circles in the half plane containing the origin, while values of \(V\) greater than \(E\) appear as circles in the half containing the singular point \(Z_{LD} = -Z_{LN}\).

Fig. 7b shows that the curves of constant voltage phase angle \(\delta\) appear as circles in the injection impedance plane. All circles intercept the origin and the singular point where \(Z_{LD} = -Z_{LN}\). The part of the circles going in counter clockwise direction from the origin to \(Z_{LD} = -Z_{LN}\) represents positive angles of \(\delta\) (if \(\delta\) is defined to lie in the range \(-180^\circ < \delta \leq 180^\circ\)) while the part of the circle going in clockwise direction from the origin to \(Z_{LD} = -Z_{LN}\) represents the points where \(\delta\) is negative.

Furthermore, it can be noted that the circle where the partial derivative \(\partial P/\partial \delta = 0\) is the same as the circle representing fixed \(\delta\) when \(\delta = -\phi\) and \(\delta = 180^\circ - \phi\). This means that the point of maximum deliverable power \(P\) to the receiving end in Fig. 1 when \(V\) and \(E\) are fixed occurs at \(\delta = -\phi\) and the point of minimum deliverable power (maximum injectable power) occurs at \(\delta = 180^\circ - \phi\).

Similarly, it can be seen that the circle where the partial derivative \(\partial Q/\partial \delta = 0\) is the same as the circle representing fixed \(\delta\) when \(\delta = -90^\circ - \phi\) and \(\delta = 90^\circ - \phi\). This means that the part of the circle where \(\delta = -90^\circ - \phi\) represents the points of maximum deliverable power \(Q\) when \(V\) and \(E\) are fixed, while \(\delta = 90^\circ - \phi\) represents the minimum deliverable power \(Q\) for the same conditions.

6. Example – analytical load flow for a two bus system

The derived expressions for the PQV-characteristics can be used to provide an analytical solution of the load flow problem for a simple two bus system. Choosing the sending end as a reference bus, the power flow solution can be analytically determined by

![Fig. 5. Curves of constant \(\delta\) in the injection impedance plane. To the left, a plot of the curve of constant \(\delta\) for the special case when \(\delta\) is either 0° or 180°. These values of \(\delta\) appear as straight line, where the part between the origin and \(-Z_{LD}\) represents \(\delta = 180^\circ\) while other parts represent \(\delta = 0^\circ\). To the right, two parts of the same circle represent two different values of constant \(\delta\) depending on which side of the line \(\delta = 0^\circ\) the parts are located. Defining \(0 \leq \delta_0 < 180\), then the condition \(\delta = \delta_0\) is the part of the circle lying in the left half plane while \(\delta = \delta_0 - 180^\circ\) is represented by the same circle, just by the part lying in the right half plane.](http://dx.doi.org/10.1016/j.ijepes.2012.05.050)
655 considering the where the circles of constant $P$ and $Q$ intercept in 656 the injection impedance plane. Eq. (42) describes the curves of constant $P$ as function of $E$, $R$, $X$ and $h$. With values of $P$ and $Q$ specified, $h$ is known and hence the load impedance values corresponding to the load flow solutions can directly be determined from (42). With the load impedance known, the complex receiving end voltage $V$ can be determined which provides the load flow solution for the system of concern.

7. Conclusion

Analytical derivation for the mapping of critical and characteristic curves on a three dimensional PQV-surface into the injection plane were introduced in this paper. The critical curves of interest were those where the partial derivative of the variables $P$, $Q$ and $V$ in respect to each other becomes zero.

The curve satisfying $\partial P/\partial Q = 0$ represents a maximum receivable or injectable power at a bus when the voltage magnitude in both ends is constant. This limit appears as the circle in the impedance plane that intercepts the origin and the points where the injection impedance was equal the negative of the line impedance and equal the complex conjugate of the line impedance. This boundary can become useful for determining the maximum power that a given generator can inject a node of constant steady state voltage magnitude. If an observation of the injection impedance for a given generator crosses these limits, the generator is prone to lose its synchronism with the rest of the system. In fact, the

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**Fig. 6.** Curves of constant active and reactive power in the complex injection impedance plane, with $Z_{LD} = (1 - 75)$. Above, curves for $P$ are shown along with the previously derived curves where the partial derivatives became zero. The values shown are in per unit where $S_{base} = E^2/Z_{LN}$. Curves for positive $P$ appear as circles in the right half plane, where the circle radius decreases as the value of constant $P$ increases until the point of maximum deliverable power is reached (when $Z_{LD} = -Z_{LN}^{-1}$). Circles representing negative $P$ (power injection) lie all in the left half plane where the circle radius decreases as the absolute value of $P$ increases until the singular point where $Z_{LD} = -Z_{LN}$ is reached (where $P \to -\infty$). Below, curves of constant $Q$ (per unit values) are shown where the curves of constant positive $Q$ are represented as circles in the upper half plane and curves of constant negative $Q$ are represented in the lower half plane. The point of maximum positive $Q$ occurs when $Z_{LD} = -Z_{LN}^{-1}$ while at the singular point $Z_{LD} = -Z_{LN}$ the value of $Q$ goes towards $-\infty$. 

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boundary satisfying \( \frac{\partial P}{\partial Q} = 0 \) plays a central role in a real time stability assessment application to be described in a later publication.

The mapping of characteristic curves into the injection impedance plane are useful when an information about a given system condition is to be visualized in a real time application. The knowledge of how critical operational boundaries and other characteristic curves appear in the impedance plane is useful for obtaining a situational awareness for a given operating condition. The derived curves can illustrate where stability and operational boundaries are located in respect to the observed operating point. In that way an information concerning distance to boundaries of critical operation can be visualized in real time.

Furthermore, the analytical derivation of critical stability boundaries in the impedance plane enables a normalization of multiple operating points in such a way that the same critical boundary applies to all of the normalized points. This results in that multiple operating points can be visualized on the same screen and held against the same normalized stability boundary.

References


