Optimal Design of Composite Structures by Advanced Mixed Integer Nonlinear Optimization

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Optimal Design of Composite Structures by Advanced Mixed Integer Nonlinear Optimization Techniques

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Composite Materials

- Composite Material
- Fibres
  - Glass Fibres
  - Carbon Fibres
  - Cellulose Fibres
  - and more …
- Matrix
  - Polymers
  - Metals
  - Ceramics

We consider discrete multi material minimum compliance problems.

We are dealing with plies that have constant and equal thicknesses and the number of layers is constant and non varying through the entire structure.

Our aim is to solve the considered problems to global optimality by modern special purpose methods and heuristics.
**Problem Formulation – Minimum Compliance**

\[
\begin{align*}
\text{minimize} & \quad \text{Compliance} \\
\text{subject to} & \quad \text{Equilibrium equations} \\
& \quad \text{Mass constraint} \\
& \quad \text{Manufacturing constraints} \\
& \quad \text{Single material selection per layer and element (DMO)}
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad \sum_{l=1}^{L} w_l f_l^T u_l \\
\text{subject to} & \quad K(x)u_l - f_l = 0, \quad l = 1, \ldots, L \\
& \quad m(x) \leq m_c, \\
& \quad Ax \leq b, \\
& \quad \sum_{i=1}^{N^i} x_{ijk} = 1, \quad \forall(j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall(i, j, k)
\end{align*}
\]

**Problem Formulation – Minimum Weight**

\[
\begin{align*}
\text{minimize} & \quad \text{Weight} \\
\text{subject to} & \quad \text{Equilibrium equations} \\
& \quad \text{Compliance Constraint} \\
& \quad \text{Manufacturing constraints} \\
& \quad \text{Single material selection per layer and element (DMO)}
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad m(x) \\
\text{subject to} & \quad K(x)u_l - f_l = 0, \quad \forall l \\
& \quad f_l^T u_l \leq c_l^\text{max}, \quad \forall l \\
& \quad Ax \leq b, \\
& \quad \sum_{i=1}^{N^i} x_{ijk} = 1, \quad \forall(j, k) \\
& \quad x_{ijk} \in \{0, 1\}, \quad \forall(i, j, k)
\end{align*}
\]
Problem Formulation – Minimum Compliance

minimize \( x, u_1, \ldots, u_L \) \( \sum_{l=1}^{L} w_l f_l^T u_l \)

subject to \( K(x)u_l - f_l = 0, \quad l = 1, \ldots, L \)

\( m(x) \leq m_c, \quad Ax \leq b, \)

\( \sum_{i=1}^{N_l} x_{ij} = 1, \quad \forall (j, k) \)

\( x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k) \)

- Our models closely follow the **Discrete Material Optimization** (DMO) parameterization scheme.

- The design variables associate candidate materials from a given set to each layer and every finite element.

- All **materials** behave **linearly elastic** and the structural behavior of the laminate is described using an **equivalent single layer theory** (ESL).

- The finite element formulations are based on the **first order shear deformation theory** (FSDT).
Reformulation – Nested Analysis and Design

\[
\text{minimize} \quad \text{Compliance} \\
\text{subject to} \quad \text{Mass constraint} \\
\text{Manufacturing constraints} \\
\quad \text{Single material selection per layer and element (DMO)}
\]

\[
\begin{align*}
\text{minimize} \quad & \sum_{l=1}^{L} w_l f_l^T K(x)^{-1} f_l \\
\text{subject to} \quad & m(x) \leq m_c, \\
& Ax \leq b, \\
& \sum_{i=1}^{n} x_{ijk} = 1, \quad \forall (j, k) \\
& x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k)
\end{align*}
\]

- (A1) The topology of the structure does not change and the stiffness matrix is symmetric and positive definite. The stiffness matrix is linear (or affine) in the design variables

\[
K(x) = \sum_{ijk} x_{ijk} K_{ijk} = \sum_{ijk} x_{ijk} B_j^T C_{ik} B_j = \sum_{j} B_j^T (\sum_{ik} x_{ijk} C_{ik}) B_j
\]

- (A2) The external loads \( f_l \in R^{n_d}\{0\} \). Furthermore, we assume that the load vectors are independent of the design variables.

- (A3) The mass limit satisfies \( \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} t_k a_j \min \{\rho_i\} < m_c < \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} t_k a_j \max \{\rho_i\} \)

- (A4) Throughout the weighting factors \( w_l \geq 0 \) of each load case and satisfy \( \sum_{l=1}^{L} w_l = 1 \)
Continuous Relaxation – Interior Point Method

\[ \text{minimize} \quad \text{Compliance} \]
\[ \text{subject to} \quad \text{Mass constraint} \]
\[ \quad \text{Manufacturing constraints} \]
\[ \quad \text{Single material} \]
\[ \quad \text{selection per layer} \]
\[ \quad \text{and element} \]
\[ \quad \text{(DMO)} \]

\[ \begin{align*}
    \text{minimize} & \quad \sum_{i=1}^{L} w_i f_i^T K(x)^{-1} f_i \\
    \text{subject to} & \quad m(x) \leq m_c, \\
    & \quad Ax \leq b, \\
    & \quad \sum_{i=1}^{N_i} x_{ijk} = 1, \\
    & \quad x_{ijk} \geq 0, \\
    & \quad \forall (j, k) \\
    & \quad \forall (i, j, k)
\end{align*} \]

- By \textbf{relaxing} the \textbf{integer constraints} on the design variables we get the \textbf{continuous relaxation} of the minimum compliance problem.

- The continuous relaxation is solved by a \textbf{primal-dual interior point method}.

- The continuous problem has a \textbf{non empty feasible set}.

- The objective function is \textbf{bounded} from \textbf{below} by zero.

- There is at least one optimal solution of the continuous problem (Weierstrass theorem).

- Finding a KKT-point to the continuous problem assures \textbf{global optimality}. 
Heuristic for Minimum Compliance Problems

\[
\text{minimize} \quad \text{Distance to 0-1 design} \\
\text{subject to} \quad \text{Mass constraint} \\
\quad \text{Manufacturing constraints} \\
\quad \text{Single material selection per layer and element (DMO)}
\]

\[
\begin{align*}
\text{minimize} \quad & \|x^0 - x\|_1 \\
\text{subject to} \quad & m(x) \leq m_c, \\
& Ax \leq b, \\
& \sum_{i=1}^{N^i} x_{ijk} = 1, \quad \forall (j, k) \\
& x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k)
\end{align*}
\]

- Smart rounding of the optimal solution of the continuous relaxation.
- The method is guaranteed to find a feasible 0-1 design.
- There are no guarantees on the quality of the obtained design.
- The 0-1 problem has a non-empty feasible set.
- There exists at least one optimal solution of the 0-1 problem.
Heuristic for the Minimum Weight Problem

minimize \( x \) \( \| x - \bar{x}^{m-1} \|_1 \)
subject to Compliance constraint
\( c_l(x^n) + \nabla c_l(x^n)(x - x^n) \leq c_l^{max}, \forall n = 0, \ldots, m - 1, \forall l \)
\( (\bar{x}^n - x^n)^T(x - x^n) \geq 0, \forall n = 1, \ldots, m - 1 \)
\( Ax \leq b, \forall l \)
\( \sum_{i=1}^{N_i} x_{ijk} = 1, \forall (j, k) \)
\( x_{ijk} \in \{0, 1\}, \forall (i, j, k) \)

Manufacturing constraints
Single material selection per layer and element (DMO)

minimize \( x \) \( \| x - \hat{x}^m \|_2 \)
subject to Compliance constraint
\( c_l(x) \leq c_l^{max}, \forall l \)
\( Ax \leq b, \forall l \)
\( \sum_{i=1}^{N_i} x_{ijk} = 1, \forall (j, k) \)
\( x_{ijk} \geq 0, \forall (i, j, k) \)

Manufacturing constraints
Single material selection per layer and element (DMO)
Approximate the non-linear objective and constraint functions with linear functions.

Solve a sequence of linear mixed 0-1 problems. Guaranteed to converge to global minimizer.

Since the function \( c(x) \) is convex the linearization constraints represent supporting hyperplanes.

The feasible set of the 0-1 problem is non-empty.
Global Optimization – Outer Approximation

**Algorithm:** Outer Approximation for solving the minimum compliance problem.

1. Solve the continuous relaxation. Denote the optimal solution $\bar{x}^0$.
2. Generate a compliance inequality on $\bar{x}^0$, i.e.
   
   $$c(\bar{x}^0) + (\nabla c(\bar{x}^0))^T(x - \bar{x}^0) - \eta \leq 0$$

3. Solve the rounding heuristic problem. Denote the optimal solution $\hat{x}$.
4. Set the lower bound $lb = z_R$.
5. Compute the displacement vectors $u_l = K(\hat{x})^{-1}f_l$. Set the upper bound $ub = \sum w_l f_l^T u_l$.

   **while** $(ub - lb)/ub > \epsilon_0$ **do**

   1. Solve the linear mixed 0-1 problem. Denote the optimal design $\hat{x}^p$.
   2. Generate compliance inequalities for all solutions of the master problem.
   3. Update the lower bound $lb \leftarrow \max\{lb, z_{OA}\}$.
   4. For all solutions of the master problem, compute the corresponding displacement vector.
   5. Update the upper bound $ub \leftarrow \min\{ub, \sum w_l f_l^T u_l\}$.

**end**
Gap improvement Heuristic/Method – Feasibility Pump

\textbf{minimize} \quad \text{Distance to 0-1 design} \quad \text{minimize} \quad \| x - \bar{x}^{m-1} \|_1

\textbf{subject to} \quad \text{Compliance constraint} \quad \text{subject to} \quad c(\bar{x}^n) + (\nabla c(\bar{x}^n))^T(x - \bar{x}^n) \leq c^{\max}, \quad \forall n = 0, \ldots, m - 1

\quad (\bar{x}^n - \bar{x}^n)^T(x - \bar{x}^n) \geq 0, \quad \forall n = 1, \ldots, m - 1

\quad m(x) \leq m_c,

\quad A x \leq b,

\quad \sum_{i=1}^{N_i} x_{ijk} = 1, \quad \forall (j, k)

\quad x_{ijk} \in \{0, 1\}, \quad \forall (i, j, k)

\textbf{minimize} \quad \text{Distance to 0-1 design} \quad \text{minimize} \quad \| x - \hat{x}^m \|_2

\textbf{subject to} \quad \text{Compliance constraint} \quad \text{subject to} \quad c(x) \leq c^{\max},

\quad m(x) \leq m_c,

\quad A x \leq b,

\quad \sum_{i=1}^{N_i} x_{ijk} = 1, \quad \forall (j, k)

\quad x_{ijk} \geq 0, \quad \forall (i, j, k)
Algorithm: Gap improvement Heuristic/Method for the minimum compliance problem.

Solve the continuous relaxation. Denote the optimal solution $\overline{x}^0$.
Generate a compliance inequality on $\overline{x}$, i.e. $c(\overline{x}^0) + (\nabla c(\overline{x}^0))^T(x - \overline{x}^0) - c_{\text{max}} \leq 0$.

Solve the rounding heuristic problem. Denote the optimal solution $\hat{x}$.
Set the lower bound $lb = z_R$.
Compute the displacement vectors $u_l = K(\hat{x})^{-1}f_l$. Set the upper bound $ub = \sum_l w_l f_l^T u_l$.
Set the target value $c_{\text{max}} = (ub + lb)/2$.

while $(ub - lb)/ub > \epsilon_0$ do
  Set $p = 1$
  while $c(\hat{x}^p) > c_{\text{max}}$ do
    Attempt to solve the 0-1 problem.
    If it is infeasible then
      Update the lower bound $lb \leftarrow c_{\text{max}}$
      Update the target value $c_{\text{max}} \leftarrow (ub + lb)/2$
    else
      Denote the optimal solution $\hat{x}^p$
      Generate compliance inequalities on the solutions from the 0-1 problem.
      Solve the nonlinear relaxation. Denote the optimal solution $\overline{x}^p$.
      Generate a compliance inequality on $\overline{x}^p$, i.e. $c(\overline{x}^p) + (\nabla c(\overline{x}^p))^T(x - \overline{x}^p) - c_{\text{max}} \leq 0$
    Set $p \leftarrow p + 1$
  end
end

Update the upper bound $ub \leftarrow c(\hat{x}^p)$. Update the target value $c_{\text{max}} \leftarrow (ub + lb)/2$.
Numerical Example – Layered Clamped Plate

### Problem Elements DOF Variables

<table>
<thead>
<tr>
<th>Problem</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1G1</td>
<td>4x4</td>
<td>405</td>
<td>640</td>
</tr>
<tr>
<td>P1G2</td>
<td>8x8</td>
<td>1445</td>
<td>2560</td>
</tr>
<tr>
<td>P1H1</td>
<td>16x16</td>
<td>5445</td>
<td>10240</td>
</tr>
<tr>
<td>P1H2</td>
<td>32x32</td>
<td>21125</td>
<td>40960</td>
</tr>
</tbody>
</table>

### Foam Properties

- **Ex [GPa]**
  - P1G1: 0.065
  - P1G2: -
  - P1H1: 0.47
  - P1H2: 0.29

- **Gxy [GPa]**
  - P1G1: 34.0
  - P1G2: 9.0

- **Major Poisson’s ratio**
  - P1G1: 0.29
  - P1G2: 0.29

- **Density [kg/m³]**
  - P1G1: 200.0
  - P1G2: 1910.0

- **Orthotropic Foam**
  - Ex [GPa]: 0.065
  - Gxy [GPa]: -
  - Density [kg/m³]: 200.0

- **Isotropic Polymeric Foam**
  - Allowable mass: 49.4[kg]

- **Q9 Plate Elements**
Continuous Relaxation

Layer 1
Layer 2
Layer 3
Layer 4

Rounding Heuristic – Relative Optimality Gap of 2.8%
Numerical Example – Layered Clamped Plate

Numerical Results with the Heuristics

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Itns I.P.</th>
<th>Itns Heuristic</th>
<th>Time [h:m:s]</th>
<th>Bounds</th>
<th>Gap (%)</th>
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<tbody>
<tr>
<td>P1H1</td>
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<td>10240</td>
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<td>10240</td>
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<td>17</td>
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<td>10</td>
<td>00:24:07</td>
<td>37.803 – 38.920</td>
<td>2.9</td>
</tr>
</tbody>
</table>

- The obtained results showcase the **excellent convergence properties** and the ability of the primal-dual interior point method to react swiftly to changes of scale of our problems.
- The method managed to converge in 20 to 22 iterations in all the examined cases.
### Numerical Example – Layered Clamped Plate

**Numerical Results with the gap improvement method**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Objective</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Time [h:m:s]</th>
<th>Itns</th>
<th>Bounds</th>
<th>Gap (%)</th>
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<tbody>
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<td></td>
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<td></td>
<td>Lower</td>
<td>Upper</td>
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<td>40960</td>
<td>15:32:12</td>
<td>3</td>
<td>1.787</td>
<td>1.804</td>
</tr>
<tr>
<td>P1H2</td>
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<td>21125</td>
<td>40960</td>
<td>12:25:45</td>
<td>2</td>
<td>37.803</td>
<td>38.082</td>
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</tbody>
</table>

**Numerical Results with Outer Approximation**

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<th>Objective</th>
<th>Elements</th>
<th>DOF</th>
<th>Variables</th>
<th>Time [h:m:s]</th>
<th>O.A.</th>
<th>Bounds</th>
<th>Gap (%)</th>
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<td>Itns</td>
<td>Cuts</td>
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<tr>
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</tbody>
</table>

- The gap improvement algorithm was used as a **global optimization method** and was able to solve all the considered problem instances to **global optimality**.
- **Outer approximation** is able to solve the considered problem only for **small scale** instances.
Ongoing Work – Manufacturing Constraints

- Articles covering optimal design of composite structures under manufacturing constraints are scarce in the literature and generally only (small) part of a structure and small-scale problems are considered.

- It is common practice to divide the structure into panels that may be designed independently, and consider manufacturing constraints in plane and through the thickness of the composite.
References


Acknowledgements

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THANK YOU FOR YOUR ATTENTION