Enumeration of Combinatorial Classes of Single Variable Complex Polynomial Vector Fields

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A vector field \( f \) in the space \( \mathbb{C}^2 \) of degree \( d \) monic, centered complex polynomial vector fields has a combinatorial structure which can be fully described by an equivalence relation on the \( 2d - 2 \) separatrices. This equivalence relation can be equivalently represented in a combinatorial disk model, by labelling the points \( \epsilon_{i, j} = \epsilon_{2i - 1, 2j - 1} \) with \( i = 0, \ldots, 2d - 3 \) on \( \Sigma_2 \) by \( \epsilon_1 \) and joining the points in the same equivalence class by geodesics in \( \Sigma_2 \) with respect to the Poincaré metric. Here, the combinatorics can be fully described by non-crossing inversions between even and odd separatrices \( \epsilon_{i, j} \) corresponding to homoclinic separatrices and between the even and odd ends \( \epsilon_1 \). The principal points defined by the \( 2d - 2 \) accesses to infinity between the separatrices \( \epsilon_{i, j} \) and \( \epsilon_{i, 2j - 1} \) which correspond to \( \epsilon_1 \)-zones. These disk models can be completely described by bracketing problems: combinatorial problems involving pairings of parentheses placed in a string of elements \( \epsilon_{i, j}, \epsilon_{i, 2j - 1} \) in our case. This specific problem is similar to the general bracketing problem of Schröder [C], and a similar method is used to calculate a recursion equation for the number of combinatorial classes \( C_{d,n} \), for vector fields in \( \Sigma_2 \). Furthermore, an implicit expression for the generating function \( C_{d,n} \) is computed, and asymptotic growth questions are considered.

**Structurally Stable Vector Fields**

For structurally stable vector fields, this disk model is due to a non-crossing inversion on the \( 2d - 2 \) ends \( \epsilon_i \). An involution on the ends can be seen as one way that \( 2d - 2 \) people can shake hands without any handshake crossing another. You might recognize then that the number of combinatorial classes (with the labelling of the separatrices) for the structurally stable vector fields is just the Catalan number \( C_{d-1} \) [DES]. Note that this is equivalent to the number of ways to make "valid" pairings of \( d-1 \) pairs of parentheses [D]. An example is given below to demonstrate.

**Valid Bracketings**

The bracketing must satisfy certain rules so they can be in accordance with what can happen for a vector field. Pairs of parentheses placed in a string of elements is called a valid bracketing if:

1. there are an equal number of right and left parentheses
2. the number of left parentheses must be greater than or equal to the number of right, reading from left to right Example: \( 0(0) \) not valid
3. there must be at least one element between successive left (resp. right) parentheses Example: \( (0)(0) \) not valid
4. There must be an even number of elements in each pair of parentheses Example: \( (0)(0) \) not valid

**The Complete Solution**

If we allow for non-structurally stable vector fields, then an involution on the ends is not enough to describe the combinatorial structure. We also need to include pairings of separatrices, corresponding to homoclinic separatrices for the vector field. This is represented by a bracketing problem where we distinguish between pairs of separatrices and pairings of ends. This will be represented by placing round brackets \( ( \) and square brackets \( [ \) in the string of the \( 2d-1 \) elements \( \epsilon_{i, j+1}, \epsilon_{i, 2j-1} \) in the valid way. An example is given below to demonstrate.

**Generating Function and Asymptotic Growth**

The generating function for \( C_{d,n} \) can be computed (after simplification) to be:

\[
G(t) = \sum_{n=0}^{\infty} C_{d,n} t^n = \frac{1}{1 - (4d^2 - 1)t + t^2}. 
\]

Now \( G(t) \) is algebraic since equation (5) gives \( G(4d^2 - 1) + (1 + t^2)G' - 1 = 0 \). A simple calculation and application of the implicit function theorem shows that \( G(1) \) has positive radius of convergence at \( t = 0 \), where the radius can be determined by solving the system of equations

\[
\frac{d}{dt} \left[ G(t)^2 - G(4d^2 - 1) + (1 + t^2)G - 1 \right] = 12G^2 - 8G^2 + 2 + t^2 = 0,
\]

and giving \( R = \sqrt{\frac{2 - 22 + 10\sqrt{3}}{3 \pi}} \approx 0.3028350.995 \). Now

\[
\lim_{t \to +\infty} G(t)^2 = \lim_{t \to +\infty} G(t)^2 = 2\sqrt{\frac{1}{\pi} - 1} < c_2 < \frac{1}{\pi} + \frac{1}{2} + e^{1/2},
\]

since \( c_2 \) is a strictly positive and increasing sequence. By the definition of limit, one can conclude that for large enough \( n \),

\[
\left( \frac{1}{\pi} + e^{1/2} \right)^n < c_2 < \left( \frac{1}{\pi} + \frac{1}{2} + e^{1/2} \right)^{n/2},
\]

that is, \( c_2(n) \approx 2^{n/2} \) for a large enough. Numerical evidence is given below.

**Conclusions/Future work**

A recursion equation is successfully computed for the number of combinatorial classes for vector fields in \( \Sigma_2 \) when the separatrices are labeled. An estimate for the asymptotic growth of the sequence is given.

I have also computed a recursion equation for the number of combinatorial classes of a given dimension within \( C_{d,k} \), however, a two-variable generating function has not successfully computed as of date.

**References**

