Polynomial Vector Fields in One Complex Variable

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POLYNOMIAL VECTOR FIELDS of ONE COMPLEX VARIABLE

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The talk is based on
Adrien's joint work with Pierrette Sentenac,
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my own joint work with Kealey Dias.

IN THE MEMORY of ADRIEN DOUADY
CONTENT

Introduction

Basic properties

Combinatorial invariants

Analytic invariants

qc-equivalence

The structure theorem
Consider the set $\mathcal{P}_d$ of monic centered polynomials of degree $d \geq 2$

$$P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \quad \text{parametrized by } a = (a_0, \ldots, a_{d-2}) \in \mathbb{C}^{d-1}$$

The associated vector field in $\mathbb{C}$ is

$$\xi_P(z) = P(z)\frac{d}{dz}$$

We study the maximal trajectories $t \mapsto \gamma(t, z_0)$ of $\xi_P$ with $t \in \mathbb{R}$ i.e. maximal solutions to

$$\gamma'(t, z_0) = P(\gamma(t, z_0)) \quad \text{and} \quad \gamma(0, z_0) = z_0$$

The space of polynomial vector fields of fixed degree $d$ is

$$\Xi_d = \{\xi_P\}_{P \in \mathcal{P}_d} \simeq \mathbb{C}^{d-1}$$
EQUILIBRIUM POINTS of $\xi_P$ – ROOTS of $P$

PROPOSITION  If $\zeta$ is a simple root of $P$ with multiplier $\lambda = P'(\zeta)$ then $\xi_P$ is holomorphically conjugate in a neighborhood of $\zeta$ to the linear vector field $\lambda z \frac{d}{dz}$.

THREE CASES:

- $\Re \lambda < 0$
  - $\Im \lambda > 0$
  - $\Im \lambda < 0$
  - sink

- $\Re \lambda = 0$
  - center

- $\Re \lambda > 0$
  - $\Im \lambda < 0$
  - source

PROPOSITION  If $\zeta$ is a multiple root of $P$ of multiplicity $m > 1$ then $\xi_P$ has $m - 1$ attracting directions and $m - 1$ repelling directions.
THREE QUADRATIC EXAMPLES

\[ P(z) = z^2 - 1 \]

\[ P(z) = z^2 + 1 \]

\[ P(z) = z^2 \]

Two simple equilibrium points
\[ \zeta = -1 \]
a sink

Two simple equilibrium points
\[ \zeta = 1 \]
a source

Two simple equilibrium points
\[ \zeta = \pm i \]
centers

One double equilibrium point
\[ \zeta = 0 \]
THREE QUADRATIC EXAMPLES

typical:
sink and source with non-real multipliers

\[ P(z) = z^2 + 1 \]
\[ P(\zeta) = \zeta^2 \]

Two simple equilibrium points

\[ \zeta = \pm i \]

a sink  a source  centers

Two simple equilibrium points

One double equilibrium point

\[ \zeta = 0 \]
THE SPACE of QUADRATIC VECTOR FIELDS $\Xi_2 \simeq \mathbb{C}$

decomposed into disjoint classes:

$$P(z) = z^2 + a$$

complex submanifold

$a \in \mathbb{C} \setminus [0, +\infty[$

real-analytic one-dim submanifold

$a > 0$

a singleton

$a = 0$
GOAL

The goal is to decompose $\Xi_d$ into disjoint classes $C$ such that

- each $C$ is connected
- all $\xi_P \in C$ have the same qualitative dynamics
- $C$ is maximal

A class is either **structurally stable** or part of the **bifurcation locus**

A class $C$ is characterized by a **combinatorial invariant** $I(C)$.

To each class $C$ is associated two integers

$$s = s(C) \geq 0 \quad \text{and} \quad h = h(C) \geq 0 \quad \text{satisfying} \quad s + \frac{1}{2}h \leq d - 1$$

Within $C$ a vector field $\xi_P$ is uniquely determined by an **analytic invariant**

$$A_P = (A_P^1, \ldots, A_P^s, T_P^1, \ldots, T_P^h) \in \mathbb{H}^s \times \mathbb{R}_+^h.$$
THE STRUCTURE THEOREM

Given \( d \geq 2 \), a combinatorial data set \( \mathcal{I} \) with associated integers \( s = s(\mathcal{I}) \) and \( h = h(\mathcal{I}) \), and a tuple

\[
\mathcal{A} = (A^1, \ldots, A^s, T^1, \ldots, T^h) \in \mathbb{H}^s \times \mathbb{R}^h_+.
\]

Then there exists a unique \( P \in \mathcal{P}_d \) such that the vector field \( \xi_P \) has combinatorial invariant \( \mathcal{I}(C) = \mathcal{I} \) and analytic invariant \( \mathcal{A}_P = \mathcal{A} \).

MOREOVER

Each \( C \) is a real-analytic submanifold of \( \mathbb{C}^{d-1} \) isomorphic to \( \mathbb{H}^s \times \mathbb{R}^h_+ \), hence of real-dimension \( 2s + h \).

Any pair of vector fields \( \xi_1, \xi_2 \) in \( C \) are dynamically equivalent:

there exists a quasi-conformal mapping \( \Psi : \hat{C} \to \hat{C} \) mapping trajectories of \( \xi_1 \) onto trajectories of \( \xi_2 \), preserving orientation but not necessarily the parametrization by time.
DUAL DESCRIPTION

Meromorphic vector fields \( \xi_f(z) = f(z) \frac{d}{dz} \)

Meromorphic abelian differentials \( \omega_f = \frac{1}{f(z)} dz \)

obey similar transformation laws:

If \( \varphi : U \to V \) is a holomorphic coordinate change and \( w = \varphi(z) \) then

\[
\varphi^* (\xi_f) = \xi_g \quad \text{and} \quad \varphi^* (\omega_g) = \omega_f
\]

where

\[
\xi_g(w) = g(w) \frac{d}{dw} \quad g(\varphi(z)) = \varphi'(z) f(z) \quad \omega_g = \frac{1}{g(w)} dw
\]

The singularities of \( \xi_f \) and \( \omega_f \) are the zeros and the poles of \( f \).

The two descriptions complement each other. The advantage of the differentials are that they can be integrated.

RECTIFYING COORDINATES

In any simply connected domain avoiding zeros of \( f \) the differential \( \omega_f \) has an antiderivative, unique up to addition by a constant,

\[
\phi(z) = \int \frac{1}{f(z)} dz
\]

Note that \( \phi^* (\xi_f) = \xi_g \) where \( g(\phi(z)) = \phi'(z) f(z) = \frac{1}{f(z)} f(z) = 1 \).

The coordinates \( w = \phi(z) \) are called rectifying coordinates of \( \xi_f \).
RECTIFYING COORDINATES for THE QUADRATIC EXAMPLES

One sink and one source

Two centers

One double equilibrium point

a horizontal strip

a vertical strip

a plane
THE SINGULARITY at $\infty$

$$(\mathbb{C}^*, \xi_P)$$ is holomorphically conjugate to $$(\mathbb{C}^*, f(z)\frac{d}{dz})$$ by $z \mapsto \frac{1}{z}$

where

$$f(z) = - \frac{1}{z^{d-2}} (1 + a_{d-2}z^2 + \cdots + a_0 z^d).$$

Hence, $\xi_P$ has a pole of order $d - 2$ at $\infty$.

IN PARTICULAR

$$\xi_0(z) = z^d \frac{d}{dz}$$ is holomorphically conjugate to

$$f_0(z) \frac{d}{dz} = - \frac{1}{z^{d-2}} \frac{d}{dz}$$

$$\phi_0 : \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}, \quad \phi_0(z) = - \frac{1}{d-1} \frac{1}{z^{d-1}}$$ is a branched covering, mapping $\infty$ to 0.
RECTIFYING COORDINATES for $\xi_0(z) = z^d \frac{d}{dz}$

EXAMPLE $d = 3$
THE SPECIAL ROLE of $\infty$ – SEPARATRICES

PROPOSITION Every $\xi_P$ is holomorphically conjugate to $\xi_0$ in neighborhoods of infinity, by a conjugating map tangent to the identity at $\infty$.

CONSEQUENCES

There are $d - 1$ incoming trajectories to $\infty$ and $d - 1$ outgoing trajectories from $\infty$. Their asymptotes are the half lines in directions $\delta_\ell = \exp\left(2\pi i \frac{\ell}{2(d - 1)}\right)$ where $\ell \in L = \{0, 1, \ldots, 2d - 3\}$ or $\ell \in \mathbb{Z}/2(d - 1)$.

$\infty$ is a saddle point for $\xi_P$.

Note, that for any point $z_0$ on such a trajectory it takes a finite amount of time to get to $\infty$ if incoming and to come from $\infty$ if outgoing.

SEPARATRICES

The separatrices are the maximal trajectories of $\xi_P$ incoming to and outgoing from $\infty$. A separatrix is homoclinic if both outgoing from and incoming to $\infty$. 
LIMITING BEHAVIOUR of trajectories $\gamma(t, z_0)$, $t \in ] t_-, t_+ [$

\[
\lim_{t \to t_+} \gamma(t, z_0) = \left\{ \begin{array}{ll}
\infty & \text{sink or multiple equilibrium point if } t_+ = +\infty \\
\infty & t_+ \in \mathbb{R}_+ \end{array} \right.
\]

\[
\lim_{t \to t_-} \gamma(t, z_0) = \left\{ \begin{array}{ll}
\infty & \text{source or multiple equilibrium point if } t_- = -\infty \\
\infty & t_- \in \mathbb{R}_- \end{array} \right.
\]

PROPOSITION  Each sink or source is the landing point of at least one separatrix. Each multiple equilibrium point is the landing point of at least one separatrix tangent to any of the attracting or repelling directions.
LABELING the SEPARATRICES and THE SEPARATRIX GRAPH

A separatrix $S_\ell$ is labeled according to its asymptote, $\ell \in \{0, 1, \ldots, 2d - 3\}$.

- $\ell$ even corresponds to a separatrix incoming to $\infty$.
- $\ell$ odd corresponds to a separatrix outgoing from $\infty$.

THE SEPARATRIX GRAPH $\Gamma_P$ is the closure in $\hat{\mathbb{C}}$ of the separatrices. Hence

$$\Gamma_P = \bigcup_{\ell=1,\ldots,2d-3} S_\ell \cup \bigcup_{\text{sink, source, mult.}} \zeta \cup \{\infty\}$$

SYMBOLICALLY Mark $\delta_\ell \in S^1$. 

![Diagrams for $d = 2$, $d = 3$, and $d = 4$.]
THE COMBINATORIAL INVARIANT of $\xi_P \in \Xi_d$

DEFINITION of the COMBINATORIAL INVARIANT $\mathcal{I}_P$ of $\xi_P$:

$\mathcal{I}_P$ is an equivalence relation $\sim_P$ on $L = \{0, 1, \ldots, 2d - 3\}$ with a specified subset $H_P \subset L$ satisfying:

- $H_P$ consists of the labels $\ell$ for which $S_{\ell}$ is a homoclinic separatrix. For $\ell_1, \ell_2 \in H_P$:

  $$\ell_1 \sim_P \ell_2 \iff S_{\ell_1} = S_{\ell_2}.$$

- $H_P$ is saturated by $\sim_P$.

- $L \setminus H_P$ consists of the labels $\ell$ for which $S_{\ell}$ lands at $\zeta$, a sink, a source, or a multiple equilibrium point. For $\ell_1, \ell_2 \in L \setminus H_P$:

  $$\ell_1 \sim_P \ell_2 \iff S_{\ell_1} \text{ and } S_{\ell_2} \text{ lands in } \mathbb{C} \text{ at the same } \zeta.$$

NOTE that there are three possible kinds of equivalence classes in $L \setminus H_P$:

- $[\ell]$ odd
- $[\ell]$ even
- $[\ell]$ mixed

  corresponding to $\zeta$ being a sink, source, multiple equilibrium point

For mixed $[\ell]$ the multiplicity of $\zeta$ is $m$ iff the ordered cyclic sequence of labels in $[\ell]$ changes parity $2(m - 1)$ times.
COMBINATORIAL INVARIANTS for the QUADRATIC EXAMPLES

EQUIVALENCE CLASSES of $\sim_P$

$[0]$ even
$[1]$ odd

$H_P = \{0, 1\}$
A CUBIC EXAMPLE with $H_P = \emptyset$ and its COMBINATORIAL INVARIANT

\[ P(z) = (z + 1)^2(z - 2) \]

$\zeta = -1$ a double equilibrium point

$\zeta = 2$ a source

Attracting petal of $\zeta = -1 := \mathbb{C} \setminus \{S_0 \cup S_2\}$

Repelling petal of $\zeta = -1 := \overline{S_1 \cup S_3}$

Half plane left of $\overline{S_1 \cup S_3}$

Two sepals := the intersection of the attracting and the repelling petal.

EQUIVALENCE CLASSES of $\sim_P$

[0] odd

[1, 2, 3] mixed
CUBIC EXAMPLES with $H_P \neq \emptyset$ and their COMBINATORIAL INVARIANT

one center, one sink, one source

three centers

one center, one double equilibrium

$H_P = \{1, 2\} = [1, 2]$

[0] odd

[3] even

$H_P = \{0, 1, 2, 3\}$

[1, 2], [3, 0]

$H_P = \{1, 2\} = [1, 2]$

[3, 0] mixed
SUBDIVISION of LABELS if $H_P \neq \emptyset$:

- polynomial-like of degree $d_1$
- polynomial-like of degree $d_2$

$d_1 + d_2 = d$

$d_1 = 1$
$d_2 = 2$
TYPES of ZONES

A zone $Z$ is a connected component of $\hat{\mathbb{C}} \setminus \Gamma_P$. There are three types, classified by the type of the holomorphic conjugacy

$$\phi : (Z, \xi_P) \to \left( \bullet, \frac{d}{dz} \right)$$

- An $\alpha\omega$-zone is isomorphic to a HORIZONTAL STRIP. \exists two distinct equilibrium points: $\zeta_\alpha$ a source or a multiple equilibrium, $\zeta_\omega$ a sink or a multiple equilibrium, such that $\forall z_0 \in Z$

  the $\alpha$-limit of $\gamma(t, z_0)$ is $\zeta_\alpha$
  
  the $\omega$-limit of $\gamma(t, z_0)$ is $\zeta_\omega$

  $\partial Z$ consists of one or two incoming separatrices and one or two outgoing separatrices, and possibly some homoclinics.

- A sepal-zone is isomorphic to an UPPER or LOWER HALF PLANE. \exists a multiple equilibrium $\zeta$ such that $\forall z_0 \in Z$ the $\alpha$-limit and the $\omega$-limit is $\zeta$.

  $\partial Z$ consists of one incoming separatrix and one outgoing separatrix, and possibly some homoclinics.

- A center-zone contains one center $\zeta$ and $Z \setminus \zeta$ is isomorphic to a HALF UPPER or LOWER CYLINDER. $\forall z_0 \in Z \setminus \zeta$ $\gamma(t, z_0)$ is periodic of period $T = \frac{2\pi}{|P'(\zeta)|}$.

  $\partial Z$ consists of one or several homoclinics.
A combinatorial class $\mathcal{C}$ consists of all $\xi_P$ with $\mathcal{I}_P = \mathcal{I}(\mathcal{C})$. The integers

$$s = s(\mathcal{C}) \quad \text{and} \quad h = h(\mathcal{C})$$

are the numbers of $\alpha\omega$-zones (numbers of strips) and the number of homoclinics (half the number of labels in $H_P$) respectively.
A COMBINATORIAL DATA SET

DEFINITION

Given \( d \geq 2 \), an equivalence relation \( \sim \) on \( L = \{0, 1, \ldots, 2d - 3\} \), and a specified subset \( H \subset L \) consisting of \( 2h \geq 0 \) labels, \( h \) odd and \( h \) even.

\( (\sim, H) \) is a combinatorial data set if it satisfies:

- \( \sim \) is non-crossing.
- \( H \) is saturated by \( \sim \), and each equivalence class in \( H \) consists of an odd and an even label.
- Zones in the disc model are of the three types: \( \alpha \omega \), sepal, center.
ANALYTIC INVARIANTS for a given $\xi_P$

Each homoclinic separatrix for $\xi_P$ is assigned the positive real time $T_P$ it takes to travel along the oriented trajectory from $\infty$ to $\infty$.

Each $\alpha\omega$-zone of $\xi_P$ is assigned the complex "time" $A_P$ it takes to travel along the transversal in the zone closest to the $\alpha$-limit point $\zeta\alpha$ from $\infty$ to $\infty$.

We choose $A_P \in \mathbb{H}$.

In each case the invariant can be expressed as

$$\int_{\text{loop}} \frac{1}{P} = 2\pi i \sum_{\zeta \text{ left of loop}} \text{Res} \left( \frac{1}{P}, \zeta \right) = -2\pi i \sum_{\zeta \text{ right of loop}} \text{Res} \left( \frac{1}{P}, \zeta \right)$$

where the loop is either the homoclinic or the transversal and the summation is over all equilibrium points $\zeta$ left of the loop (or all right of the loop).
QC DYNAMICAL EQUIVALENCE

Suppose $\xi_{P_1}, \xi_{P_2}$ belong to the same combinatorial class. Then they have the same qualitative dynamics. In rectifying coordinates the equivalence is given through piecewise affine mappings.

AMONG $\alpha\omega$-zones, represented in rectifying coordinates:

affine map, mapping the base $\{1, A_1\}$ to the base $\{1, A_2\}$.

piecewise affine map; on the triangle, mapping the base $\{T_1, A_1\}$ to the base $\{T_2, A_2\}$. 

QC DYNAMICAL EQUIVALENCE

Suppose $\xi_{P_1}, \xi_{P_2}$ belong to the same combinatorial class.

AMONG center-zones: affine or piecewise affine mappings,

mapping the base $\{T_1, i\}$ to the base $\{T_2, i\}$, or the base $\{T^i_j, -i\}$ to the base $\{T^j_i, -i\}$.

AMONG sepal-zones with homoclinics: piecewise affine mappings.
THE STRUCTURE THEOREM

Given $d \geq 2$, a combinatorial data set $(\sim, H)$ and

$$A = (A^1, \ldots, A^s, T^1, \ldots, T^h) \in \mathbb{H}^s \times \mathbb{R}^h$$

where $s$ is the number of $\omega$-zones for $(\sim, H)$ and $h$ is the number of equivalence classes in $H$. There exists a unique $\xi_P \in \Xi_d$ realizing the above, i.e. $(\sim_P, H_P) = (\sim, H)$ and $A_P = A$.

PROOF by surgery. From the rectified building blocks we construct a Riemann surface $\mathcal{M}$ with a vector field $\xi_\mathcal{M}$ and prove that $\mathcal{M}$ is isomorphic to $\hat{\mathbb{C}}$ and that there exists a unique $P \in \mathcal{P}_d$ such that $(\mathcal{M}, \xi_\mathcal{M})$ is holomorphically conjugate to $(\hat{\mathbb{C}}, \xi_P)$.

STRUCTURALLY STABLE

If $\xi_P$ has only sinks and sources, no homoclinic separatrices and no multiple equilibrium points, then the number of $\omega$-zones takes its maximal value $s = d - 1$. It follows that $\xi_P$ belongs to a class, which is isomorphic to $\mathbb{H}^{d-1}$. Hence $\xi_P$ is structurally stable.

BIFURCATION SET

If $\xi_P$ has a homoclinic separatrix or a multiple equilibrium point then $s < d - 1$ and $2s + h < 2(d - 1)$. Hence $\xi_P$ is in the bifurcation set.
ILLUSTRATING the SURGERY in a FAMILIAR case  \( d = 3 \)
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