Plate Forming by Line Heating

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DEPARTMENT OF
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AND OFFSHORE ENGINEERING
Plate Forming by Line Heating

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Preface

This thesis is submitted as a partial fulfillment of the requirements for the Danish PhD degree. The study has been carried out at the Department of Naval Architecture and Offshore Engineering, the Technical University of Denmark, in the period from February 1997 to April 2000. Associate Professor Jan Baatrup and Professor Jørgen Juncher Jensen supervised the project.

The study is supported financially by the Technical University of Denmark, for which I am very grateful.

During this period of three years, I have been in contact with many people who have influenced my thoughts and work in many ways. Thanks to everyone! Especially Jan Baatrup, Jørgen Juncher Jensen, and Lars Fuglsang Andersen for educational and invaluable discussions.

Two periods of 1 and 6 months, respectively, in 1998 and 1999 were spent at the College of Engineering, Seoul National University, where Professor Jong Gye Shin was so kind to invite me and my family to stay in Seoul. These trips were partly supported by the Danish Society for Naval Architecture and Marine Engineering (Skibsteknisk Selskab), which is greatly acknowledged. In Seoul, I had the opportunity to work with Professor Shin and his students in a laboratory concentrating specifically on aspects of automated ship production— including plate forming by line heating. Special thanks to Professor Shin and all his students for their heartwarming hospitality, which made the stay a delightful experience.

Thanks also to my wife and son for their support and patience and for taking the chance of travelling to South Korea with me.

Henrik Bisgaard Clausen
April, 2000
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Executive Summary

The purpose of the present work is to examine and explain the mechanisms of the forming process called ‘line heating’ and to develop numerical tools for efficient calculation and prediction of its behaviour. The forming process consists of heating at a (steel) plate in a predetermined pattern of lines by means of e.g. a gas torch so that the plate assumes a certain, curved shape. Thus, the method is an alternative or supplement to other forming methods such as pressing and rolling.

Today, few skilful shipwrights are capable of performing the art of line heating, as the amount of heating and the position of the lines are entirely based on experience. As this knowledge is difficult to categorise it takes several years to learn the method, and hence the production method is a bottleneck impeding rapid increase of capacity. A rational method for the determination of heating line patterns and heating amount would be very beneficial. It can help the shipwright to determine the most efficient heating parameters, it may allow less skilled craftsmen to carry out the heat treatment, and in the long term it may lead to automation of the process.

At the IHI shipyard in Kure (near Hiroshima in Japan) a system which automatically calculates the heating parameters and carries out the heat treatment on moderately curved plates has been developed. The only human intervention takes place when the plate is turned over by a crane. The NKK shipyard (also in Japan) has a robot too with a gas torch mounted, but as it needs off-line teaching by a shipwright it is most suitable for production of series of identical curved sections. Much research is carried out in industry and at universities in the USA and South Korea to achieve automation, as the potential economic benefit is obvious. The savings are not directly connected with the production of the curved plates themselves, but rather with the subsequent production stages where even the smallest deviation in curvature and cutting of the plates involves expensive corrections.

This thesis focuses on the relation between heating parameters and resulting deformations, and further a method for predicting the position of the heating lines is implemented.

The following items are dealt with.

- A finite element model is built to investigate how the resulting deformations depend on varying heating parameters. The model includes effects of temperature-dependent
mechanical and thermal properties, among these plasticity, conduction and heat loss by both convection and radiation. A convergence test has shown that six linear elements through the plate thickness and 18 perpendicular to the heating path are sufficient to model the process. Outside the heated zone, a much coarser mesh can be used.

- The validity of the numerical model is verified by experiments. Good agreement between the simulated and the measured deformations is found.

- A method for experimental determination of the heat flux distribution from a gas torch is developed. The temperature distribution caused by a stationary torch is measured on the bottom side of a plate, after which a finite element optimisation procedure tries to find the same temperature distribution by varying the heat flux on this numerical model.

- A parameter study consisting of 27 variations of the maximum temperature, the plate thickness and the torch velocity is made. Hence, relations between heating parameters and deformations are obtained.

- A method for linearisation of the plastic strains from the heated zone is developed so that they are described only by their contribution to shrinkage and bending, along and perpendicularly to the heating path. Thus, the strain field is described by four parameters only.

- The linearised plastic strains can be applied to an elastic analysis, which can reproduce the results of the fully elastoplastic analysis with good accuracy. The advantage is that already known strains (from e.g. a database as described above) can be applied to a plate of any shape, and that the deformations are computed very fast. While it takes eight to ten hours to analyse a plate heated over 20 cm, it takes but a few minutes to carry out the elastic analysis for a plate of any configuration.

- The effect of heating at the edge of a plate is investigated. It is demonstrated that if the plate is heated from the edge towards the centre of the plate, the strains are almost unaffected by the edge. Thus, the simplified elastic analysis can be used to model this particular way of heating as well.

- Two methods—a general but rather cumbersome and a less general but simpler—are developed to interpolate in the database of relations mentioned above. Thus, a continuous function for the relation between heating parameters and deformation measures exists.

- A sensitivity analysis is carried out to investigate which parameters other than those from the study above may influence the results. It shows that material properties and heat source modelling are important factors, which must be known before simulations are carried out.

To find the relation between heating parameters and deflections the following approach can be used:
1. Determine the material properties for the type of steel to which the relations should correspond, as they are determined for each type of material separately.

2. Find the flux distribution from the heat source by the method described in Chapter 6.

3. Select representative ranges of heating parameters. It is recommended to use the temperatures 500, 600 and 700 °C for steel. However, the plate thicknesses should cover the range needed in the production and the torch velocities should match the power of the available torches.

4. Divide the parameter ranges into 3x3x3 points (a total of 27 combinations) and carry out the numerical analyses for the 20 ‘outer’ cases as shown in Figure 4.6, p. 54. Thus, the relations between heating parameters and deformations are determined for the given combination of steel and heat source.

5. The plastic strain is linearised by the method described in Section 3.5.

6. The linearised strains can be interpolated by Eq. (4.16) to yield a continuous distribution of the strains as functions of the heating parameters.

Thus, the relations are determined. However, the heating paths, which will produce a certain shape, remain to be determined and this can be accomplished by the rational method proposed in Chapter 5:

7. By forcing the plate into shape by an elastic finite element analysis, the heating direction can be chosen perpendicularly to the most negative principal strain direction at any point. This also provides information on how to cut the plate before forming, as a good approximation to the actual deflection of the edge is computed by the analysis.

8. Finally, the heating line positions must be converted into information on how much each line should be heated. This is obtained by turning the heating lines into soft zones, which will absorb most of the deformations when the plate is forced into shape by another elastic analysis. The obtained deformations can be compared to the results of the case study in point 6 and thus the corresponding heating parameters can be found.

The method described in 7 and 8 is not fully developed, but it provides a rational means of determining the heating paths in a reproducible manner, which can be adjusted to the actual behaviour of the process.
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Synopsis


Det er i dag kun få, dygtige skibsbyggere, som kender kunsten at formgive med metoden, da varmemængden og placeringen af varmelinierne alene baseres på erfaring. Da denne viden er svær at kategorisere, opleves det på værfter, hvor metoden allerede er i brug, at det kan tage flere år at oplære folk til metoden. Dermed er den en flaskehals, der forhindrer hurtigt forøgelse produktionen. En rationel metode til at bestemme varmeliniernes position og varmemængde kan støtte skibsbyggeren i valget af varmeparametre, den kan betyde at mindre erfarne er i stand til at forestå formgivningen, og endelig kan den på længere sigt føre til automation af processen.

På IHIs værf i Kure (nær Hiroshima i Japan) findes allerede i dag et system, der automatisk beregner varmeparametrene og udfører behandlingen på moderat krumme plader ved hjælp af induktionsvarme - den eneste menneskelige indgriben foregår, når pladen skal vendes med kran. NKK (ligeledes i Japan) har også indført en robot med en gasbrænder monteret, men den skal oplæres fra gang til gang og er således bedst egnet til serieproduktion. I USA og Sydkorea forskes der i flere universitære og industrielle sammenhænge også intenst på at automatiser, da man har indset, at der er store besparelser i sigte. Besparelserne kommer ikke direkte i forbindelse med produktionen af selve de krumme plader, men i alle de efterfølgende produktionstrin, hvor selv den mindste fejl i pladernes krumhed og tilskæring kræver omkostningsfulde tilretninger.

Denne afhandling fokuserer på at finde sammenhængen mellem varme og deformationer, og derrudover er også en metode til at forudsige liniernes position implementeret.

Følgende undersøgelser er foretaget:

- En finite element model er opbygget til at undersøge hvordan de blivende deformationer afhænger af ændringer i varmeparametrene. Modellen inkluderer såvel temperatur- afhængige mekaniske egenskaber som varmeledningsegenskaber, herunder plasticitet,

• Den numeriske model er verificeret ved hjælp af eksperimenter. Det viser sig, at den numeriske model giver resultater, der stemmer godt overens med de eksperimentelt bestemte deformationer.

• En metode til eksperimentelt at bestemme fluxfordelingen mellem en varmekilde og en plade er udviklet. Temperaturfordelingen som følge af en stillestående varmekilde er målt på undersiden af en plade, hvorpå en optimeringsrutine finder den samme temperaturfordeling i en numerisk model ved at variere fordelingen af varmekildens flux.

• Der er udført et parameterstudie over 27 variationer af fremføringshastighed, maksimumtemperatur og pladetykkelse. Dette er gjort for at opbygge en database over sammenhængen mellem varmeparametre og deformationer.

• En metode til at linearisere de plastiske tøjninger, så de er beskrevet udelukkende ved deres bidrag til hhv. krympning og bojning på langs og på tværs af varmeliniens retning er fundet. Hele det plastiske tøjningsfelt hørende til den varmebehandlede zone kan således beskrives med kun fire parametre.

• De lineariserede plastiske tøjninger kan påføres en simpel elastisk model, som med god præcision kan gengive resultaterne fra den fulde analyse. Fordelen er, at hvis man på forhånd kender de lokale deformationer (fra f.eks. en database som nævnt ovenfor), er det muligt meget hurtigt at få en beregning af en vilkårlig plades deformation. Beregningerne er meget korte med denne metode - det tager få minutter at beregne en plade af vilkårlig størrelse, hvor det tager 8-10 timer at beregne en plade opvarmet over en 20 cms længde med den fulde elastoplastiske model.

• Randeffekter som følge af opvarmning i nærheden af en kant er undersøgt. Det viser sig, at hvis man varmer fra kanten og ind imod centrum af pladen, er de plastiske tøjninger næsten, som hvis varmelinien havde ligget langt fra kanten. Dermed kan den simplificerede elastiske metode fra forrige punkt også bruges til beregninger af opvarmning i nærheden af en kant.

• To metoder - en lidt omstændelig men meget generel og en simpel men mindre generel - er udviklet til at interpolere i de lineariserede data. Dermed kan man forudsige de plastiske tøjninger for varmeparametre, der ikke allerede foreligger beregninger for.

• En følsomhedsanalyse er gennemført for at undersøge hvilke parametre, som ikke er medtaget i ovennævnte parameterstudie, er vigtige at kendte nøjagtigt. Konklusionen er, at resultaterne (tøjningerne) afhænger meget af materialetype og af den anvendte varmekildes fluxfordeling.
Hvis man vil opbygge en database over sammenhængen mellem varmeparametre og deformationer, kan man benytte sig af følgende fremgangsmåde:

1. Bestem materialedata hørende til den type stål, som datasættet skal høre til - idet parameterstudiene foretages for et bestemt sæt materialedata ad gangen.


3. Find et repræsentativt område at variere varmeparametrene indenfor. Det anbefales at benytte temperaturområdet fra 500 til 700 °C for stål, hvorimod pladetykkelserne skal tilpasses behovet og fremføringshastighederne skal være af en størrelse, så de til rådighed værende varmekilder kan levere den nødvendige effekt.

4. Inddelt området i 3x3x3 punkter (i alt 27 kombinationer) og gennemfør numeriske beregninger for de 20 'ekstreme' tilfælde som angivet i figur 4.6, side 54. Dermed er relationerne mellem parametre og deformationer bestemt for en givet type materiale og varmekilde.

5. Den plastiske tøjning lineariseres ved hjælp af metoden beskrevet i afsnit 3.5.


Dermed er grundlaget for at omsætte krævede deformationer til information om den dertil hørende varmemængde skabt. Endnu mangler en metode til at forudsige, hvor der skal varmes, men et forslag til en rationel metode er givet i kapitel 5:

7. Ved at tvinge pladen i form med en rent elastisk finite element analyse, kan man i alle punkter på pladen beregne retningen af varmelinierne som værende vinkelret på den mest negative hovedtøjningsretning. Samtidig vil denne metode angive, hvordan pladen skal skæres ud inden formgivningen, da man får et godt bud på kanternes deformation.


Metoden i punkt 7 og 8 er ikke færdigudviklet, men da den giver reproducerbare resultater er den et godt udgangspunkt for en rationel metode, der kan kalibreres med den observerede opførsel af metoden.
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Symbols

Roman Symbols

\(A\)  \hspace{1cm} \text{Area}
\(B_{x,y}\)  \hspace{1cm} \text{\(x\)- and \(y\)-components of the linearised \textit{bending} contribution of the plastic field}
\(c_p\)  \hspace{1cm} \text{Specific heat}
\(d_i\)  \hspace{1cm} \text{Dimensions in dimension analysis}
\(E, F, G\)  \hspace{1cm} \text{Coefficients of the first fundamental form}
\(E\)  \hspace{1cm} \text{Young’s modulus}
\(E_t\)  \hspace{1cm} \text{Tangent modulus}
\(e_j^T\)  \hspace{1cm} \text{Error fraction of thermocouple No. \(j\)}
\(g\)  \hspace{1cm} \text{Gravity}
\(H\)  \hspace{1cm} \text{Enthalpy}
\(h\)  \hspace{1cm} \text{Plate thickness}
\(\hat{h}\)  \hspace{1cm} \text{Natural coordinate version of \(h\). \(\hat{h}\) \(\in\) \([-1; 1]\)}
\(h_f\)  \hspace{1cm} \text{Convective film coefficient}
\(i, j\)  \hspace{1cm} \text{Indices}
\(k\)  \hspace{1cm} \text{Total number of possible interchanges between \(A\) and \(B\)}
\(L\)  \hspace{1cm} \text{Characteristic length}
\(L, M, N\)  \hspace{1cm} \text{Coefficients of the second fundamental form}
\(Nu_E\)  \hspace{1cm} \(= \frac{h_f L}{\lambda_a}\) \text{Nusselt’s number}
\(N_d\)  \hspace{1cm} \text{Number of dimensions}
\(N_p\)  \hspace{1cm} \text{Number of dimensionless parameters}
\(N_s\)  \hspace{1cm} \text{Number of independent dimensional sets}
\(N_V\)  \hspace{1cm} \text{Number of variables}
\(N_{z,i}\)  \hspace{1cm} \text{Number of zeros in row} \(i\) \text{ of \(C\)}
\(Q''(r)\)  \hspace{1cm} \text{Heat flux from heating torch as a function of} \(r\)
\(Q_c\)  \hspace{1cm} \text{Convective heat loss}
Symbols

\( Q''_{max} \)  
Peak value of the Gaussian distributed heat flux

\( Q_R \)  
Radiative heat loss

\( Q_{tot} \)  
Total heat flux from torch to plate

\( Ra_L = \frac{\rho c a (T_s - T_B)}{\nu \beta} \)  
Rayleigh’s number

\( r \)  
Distance from centre of heat source

\( r_{torch} \)  
Radius of the gas torch \((Q''(r_{torch}) = 0.01Q''_{max})\)

\( S_{x,y} \)  
\( x \) - and \( y \)-components of the linearised shrinkage contribution of the plastic field

\( s_{ij} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{3} \)  
Deviatoric stress tensor

\( T \)  
Temperature

\( \bar{T} \)  
Average temperature

\( \hat{T} \)  
Natural coordinate version of \( T \). \( \hat{T} \in [-1; 1] \)

\( T_B \)  
Bulk (air) temperature

\( T_s \)  
Plate surface temperature

\( t \)  
Time

\( U_c \)  
Number of interchanges between \( A \) and \( B \) that yields equivalent sets of dimensionless parameters

\( U_R \)  
Number of prohibited interchanges between \( A \) and \( B \) due to singularity of \( A \)

\( V_j \)  
Variables in dimension analysis

\( v \)  
Velocity

\( \hat{v} \)  
Natural coordinate version of \( v \). \( \hat{v} \in [-1; 1] \)

\( W_h = \frac{Q_{tot} v}{\rho c a} \)  
Heat per unit length

\( w^p \)  
Width of the plastic zone

\( x \)  
Coordinate. Aligned along the heating path

\( y \)  
Coordinate. Perpendicular to \( x \) and in the plane of the plate

\( z \)  
Coordinate. Thickness direction

### Greek symbols

\( \alpha \)  
Thermal expansion coefficient (of steel)

\( \alpha_a \)  
Thermal expansion coefficient of air

\( \beta \)  
Thermal diffusivity

\( \delta_{ij} \)  
Kronecker’s delta function

\( \epsilon \)  
Radiant exitance

\( \varepsilon \)  
Strain

\( \varepsilon^e \)  
Elastic strain

\( \varepsilon_{1,2} \)  
Principal elastic strains

\( \varepsilon^p \)  
Plastic strain
**Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\dot{\varepsilon}^p$</td>
<td>Plastic strain increment</td>
</tr>
<tr>
<td>$\overline{\varepsilon}^p$</td>
<td>Average plastic strain</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Width factor of Gaussian heat flux distribution</td>
</tr>
<tr>
<td>$\kappa_{1,2}$</td>
<td>Principal curvature</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Thermal conductivity (of steel)</td>
</tr>
<tr>
<td>$\lambda_a$</td>
<td>Thermal conductivity of air</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson’s ratio. Kinematic viscosity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Stress, Stefan-Boltzmann’s constant</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>Yield stress</td>
</tr>
<tr>
<td>$\theta_p$</td>
<td>Principal strain direction</td>
</tr>
<tr>
<td>$\vartheta$</td>
<td>Number of duplicate prohibited interchanges between $A$ and $B$</td>
</tr>
</tbody>
</table>

**Matrices and vectors**

Vectors are written as $\vec{a}$, matrices as $B$

- **$A$** Square sub-matrix of $H$
- **$B$** Sub-matrix of $H$
- **$\bar{b}$** Coefficient matrix for $X$
- **$C = -D(A^{-1}B)^T$**
- **$D$** Matrix of independent columns
- **$D$** Conductivity matrix
- **$\epsilon$** Powers to variables to form dimensionless parameters
- **$H$** Dimensional matrix
- **$\bar{\pi}$** Dimensionless parameters
- **$\vec{q}$** Heat flux vector $\{q_x, q_y, q_z\}$
- **$\bar{q}$** Powers to dimensions ($\bar{0}$ for dimensionless parameters)
- **$\bar{v}$** Eigenvector for principal curvature
- **$X$** Multivariate parameters
- **$\bar{y}$** Result vector for Multivariate Analysis

**Operators**

- **$|A|$** Determinant of matrix $A$
- **$\nabla \bar{a}$** Gradient of $\vec{a} = \{\frac{\partial \bar{a}}{\partial x_1}, \frac{\partial \bar{a}}{\partial x_2}, \frac{\partial \bar{a}}{\partial x_3}\}$
- **$\nabla \cdot \bar{a}$** Divergence of $\vec{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{A}^T )</td>
<td>Transpose of ( \mathbf{A} )</td>
</tr>
<tr>
<td>( \mathbf{A}^{-1} )</td>
<td>Inverse of ( \mathbf{A} )</td>
</tr>
<tr>
<td>( E_{\alpha} )</td>
<td>( E ) differentiated with respect to ( \alpha )</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

Of the many skills gathered by engineers and shipwrights in a shipyard, the ability to form compound curved plates for the outer hull is a speciality. This production process is indispensable, as only the simplest barges are constructed entirely of flat plates.

Whereas a single curved plate is simply produced by rolling, the forming of double-curved plates requires skilled labour and frequent use of heavy equipment. The double-curved plates can be shaped by applying force by means of pressing (with or without a die), peening, or forming with narrow rollers. Although these are proven techniques, there are some problems. In the case of press-forming it is difficult to predict how overbent the plate should be to obtain the correct shape after ‘spring-back’, and further peening and narrow roller forming make the plate slightly thinner in the worked areas.

Line heating is a method of forming double-curved plates by means of local heat treatment. It is used in parts of the shipyard industry, and much attention is paid to it in the automotive industry (Ji & Wu, 1998; Magee et al., 1998) and within sheet metal forming technologies (Geiger et al., 1994). Despite its popularity the process is difficult to control, and it is generally regarded as an art performed only by the most skilled shipwrights. The main problem is to tell in a reproducible way where and how much to heat the plate in order to obtain a certain shape. However, it has some advantages over the above mentioned methods, including that line heating makes the plates locally thicker, so that designers do not have to worry about the thickness of the plate with respect to approval by classification societies. Further, production equipment for this method is very cheap, as it may be as simple as an oxyacetylene gas torch. Of the mentioned methods, line heating seems to be best suitable for automation—again because of the simplicity of the equipment. A disadvantage of using oxyacetylene torches is the difficult temperature control (i.e. mixture, clearance and velocity), and material degradation at the surface by diffusion of either excess carbon or oxygen from the combustion product.

When the plate is subjected to local heating, two things happen: The material becomes softer (lower yield limit) and at the same time it expands. The adjacent material still has its
original strength, why the hot and soft steel will yield and make the plate slightly thicker. Upon cooling the material will regain its strength and the thermal contraction bends/shrinks the plate. This sequence is illustrated in Figure 1.1.

The overall goal of the current work is to investigate and analyse the mechanics of the process and to find a method for predicting where and how much to heat a plate to obtain a certain shape. The thesis is organised as follows:

**Literature review.** An overview of past and present methods from literature is given. Chapter 2.

**Numerical analysis.** To understand the mechanisms and the relation between heating parameters and final deflections a numerical model is established, and a set of calculations with varying heating parameters is carried out. Chapter 3.

**Simplified elastic analysis.** To save computation time a simplified numerical method which can predict the response to applying already known plastic strains (from e.g. a database) is derived. Section 3.6.

**Empirical relations.** To be able to interpolate in an existing set of results (as those from the numerical analysis) empirical relations are derived from multivariate analysis of the numerical analysis. Further, a simpler but less general method for interpolation is suggested. Chapter 4.

**Heating paths.** Determination of heating lines and heating parameters can be found according to the method outlined in Chapter 5.

**Experiments.** Experiments for validation of the numerical modelling and for assessment of the heat flux from a gas torch are carried out. Chapter 6.
Chapter 2

Literature Review

Until today, quite a few researchers have addressed the topic of line heating in the search for better control of the process. A range of different methods which help to understand the mechanics has been developed, among others beam analysis approximations, equivalent force calculations and three-dimensional finite element analyses.

That the line heating method is efficient and popular is proved by the fact that many shipyards use it in the production. Those include (to the author’s knowledge) Odense-Lindø (Denmark), Astilleros Españoles (Spain) (de la Bellacasa, 1992; Sarabia & de la Bellacasa, 1993), Fincantieri (Italy), Daewoo (South Korea), Mitsubishi, IHI (Ishiyama et al., 1999), and NKK (Kitamura et al., 1996) (all Japan), Todd Pacific Shipyards (Chirillo, 1982), Atlantic Marine Shipyards, NASSCO, and Norfolk Naval Shipyard (all USA).

In the following, a brief review is given—sorted by topic—of the methods described in the literature.

2.1 Production of Compound Curved Plates

According to Chirillo (1982) there are a number of benefits from using line heating over powerful plate pressing facilities, including that the process is faster, safer, and allegedly more accurate. However, the described methods are based on experienced, manual labour and are as such trial-and-error procedures.

Table 2.1 shows the results of a questionnaire (Kitamura et al., 1996) describing the amount of time spent in seven Japanese shipyards shaping double-curved shells. On average it takes 6.8 manhours (MH) in the line heating shop (hot bending) per plate to work it into the desired shape. Prior to that the plate is typically shaped by means of rolling (cold bending), which takes 2.7 hours on average. From Table 2.1 it is also seen that the average manhour
Table 2.1: Number of plates and manhours per plate according to Kitamura et al. (1996).

consumption is 4200. However, it is not clear whether this is per ship or per year. Any savings of manhours in these workshops will only constitute a small fraction of the total manhour consumption in shipbuilding—considered alone it would not influence the overall cost of a ship. However, all the plates from this workshop must be used at the later stages of the production, and as all errors are added up from step to step it is crucial to have a good degree of accuracy from the plate-forming shop.

Not only the ‘downstream’ assembly stages will benefit from improved accuracy of the curved shells. Also in the ‘upstream’ production—in the plate cutting facility—proper knowledge of the forming process is important. Various mapping techniques emulate the actual forming process (Lamb, 1995; Letcher, 1993) so that the cut plates fit one another perfectly at the erection stage. Unfortunately, this does not work very well, as the mappings are typically based on differential geometry (which assumes evenly distributed deformation) in contrast to line heating and other forming methods which deform the plates locally. Therefore, better control of the method improves cutting.

2.2 Types of Heat Sources

The line heating process can be divided into a few categories according to the heat sources:

**Gas torch** is far the cheapest to buy and maintain. It is somewhat difficult to control as regards repeatability in gas amount (unless flowmeters as described in de la Bellacasa (1992) are used) and as regards keeping a constant distance to the plate.

**High-frequency induction heating** allows for control of the heat penetration, as it depends on the frequency of the induced electrical field. It is unsuitable for heating at the edges of a plate as overheating is almost inevitable. Further the equipment is rather heavy, so it cannot replace the gas torch in manual line heating.

**Laser beam** is the most well defined heat source although it is very expensive. Laser is very well suited for automation and in combination with a protection gas it reduces the risk of oxydation of the surface.

**Welding arc** is a possibility but due to the low penetration of the heat, the surface is prone to melt with material degradation as a result (Kitamura et al., 1996).
2.3 Finding the Heating Paths

In principle, all the heat sources can be automated by mounting them on a gantry crane, which makes it possible to treat large plates.

2.3 Finding the Heating Paths

An imminent problem is to determine the locations of the heating lines. At least four different methods are described in the literature:

**Connection of points of extreme curvature on the deflection difference surface.** One method (Jang & Moon, 1998) compares the target surface with the current (initially flat) surface. The objective is to make the difference surface (target minus current) as flat as possible. This is achieved by finding the points of maximum curvature on the difference surface and then grouping them into simulated heating lines. By iteration, the difference surface is gradually becoming flat.

**Use of elastic analysis and check with ‘similarity’ measure.** An algorithm is developed by Lee (1996), which fits heating line candidates through points of similar curvature on the surface. These are subjected to bending moments which emulate the heating, and the obtained shape is compared to the target shape by a square root norm of the differences between the points on the two surfaces. Then an optimisation procedure varies the bending moments to achieve a better fit. The final bending moments translate into heating parameters by comparison to a database.

**Use of principal strain directions from numerical elastic analyses.** In Ueda et al. (1994a), the following procedure is formulated:

- Compute the strain caused by deformation from the initial configuration to the final one by using elastic FE analysis.
- Decompose the computed strain into in-plane and bending components and display the distribution of their principal values on a graphic display.
- From the distribution of the in-plane strain, the region where the magnitude of compressive principal strain is large is chosen to be the heating zone, and the heating direction is assumed to be normal to the direction of the principal strain with the maximum absolute value.
- From the distribution of the bending strain, the region where the absolute value of the bending strain is large is selected to be an additional heating zone, and the heating direction is assumed to be normal to the direction of the principal strain with the maximum absolute value.

In other words, the principal bending or principal compressive strain directions in every point determine the path. In addition, it is described how to find the amount of
deformation necessary in each heating line. By making the elements at the lines found by the above approach very soft (Young’s modulus 1/1000 of normal) the strain will be concentrated here, as it would be in the real case of line heating.

**Normal strain directions determined by differential geometry.** Shin & Kim (1997) use a method where differential geometry takes account of the mapping between the initially flat plate and the curved plate. Hence, bending and in-plane principal strains are found. The most crucial part is to select a mapping method which resembles the line heating process as much as possible. Present work at Seoul National University combines the methods of shell expansion and tracing of Nutbourne et al. (1972), Manning (1980), and Hinds et al. (1991). More specifically, Manning (1980) describes a method of isometric trees to develop the surface: A tree with a long trunk with attached branches represents the surface, and each of the branches and the trunk are mapped onto the plane by involutes (unrolling). Hinds et al. (1991) refine this method by replacing the involutes by traces of geodesic curvature\(^1\). The geodesic curvature can be traced onto the plane by the method developed by Nutbourne et al. (1972), where any line can be traced given a curvature and an arc length.

An initial shape may also be chosen as close to the target shape as possible. Such an initial shape is available by a method developed by Randrup (1997), where the cylinder shape which—in some sense—is closest to the double-curved shape is found. This information can be used to roll the plate first and thus minimise the forming effort required by line heating.

### 2.4 Temperature Field Analyses

An early description of the total heat flux from a torch is made by Fay (1967). With a calorimeter he measured the power from a range of different burners and fuels.

Concerning the flux distribution, most references to line heating assume a model where the gas torch is Gaussian distributed as \(Q''(r) = Q''_{\text{max}} e^{-\gamma r^2}\) (see also Section 3.1.2). This idea comes from Rykalin (1960) and is adopted by among others Moshaiov & Latorre (1985), Ueda et al. (1994c) and Shin et al. (1996). Yu et al. (1999) adopt a modified Gaussian distribution, in which the heat flux outside a core radius is slightly increased to model a laser beam. Others use the solution of Rosenthal (1946) for a point source on an infinite plate of finite thickness, for example Jang et al. (1997) or Moshaiov & Latorre (1985) in an analytical solution. Yet others (Tomita et al. (1998)) calculate the heat flux by measuring the velocity profile of the torch flame. Section 6.1 of the present thesis shows how the heat flux is evaluated from temperature measurements during heating. A simulation of heating by high frequency induction is treated by Ogawa et al. (1994).

\(^1\)Geodesic curvature is the curvature projection of any curve, \(C\), onto the tangent plane at points of \(C\).
2.5 Structural Analysis

Various analytical and numerical analyses have been carried out to predict the response to heat treatment:

- A solution to a two-dimensional problem is found by Iwamura & Rybicki (1973). It consists of a finite difference implementation of a beam perpendicular to the heating direction.

- Jang et al. (1997) use a simplification where a circular, axisymmetric disc with springs represents the heated region. The spring constants are determined from the theory of an infinite plate with a hole. The disc is then distributed along the heating path to simulate the actual heating and the resulting strains are converted into equivalent forces by integration.

- A combination of the differential equation for a Kirchhoff plate and a membrane theory coupled by the constitutive relations (plasticity) is derived by Moshaiov & Vorus (1987). Unfortunately, only a boundary element solution to the plastic Kirchhoff plate alone was found and solved.

- A theory where a strip perpendicular to the heating line supported by springs represents the heated plate is developed by Moshaiov & Shin (1991) and Shin & Moshaiov (1991). This is successfully applied to the elastic case, but when plasticity is included it is difficult to determine the spring constants.

- Three-dimensional finite element modelling with different commercial programs is carried out by Lee (1999), Yu et al. (1999), Clausen (1999), Ishiyama et al. (1999), and Ogawa et al. (1994). In the three former, the relation between input heat and output deflection is also described.

2.6 Implementation of Manual Line Heating

Interesting reports on procedures for implementation of manual line heating are available through the National Shipbuilding Research Program (NSRP). The first published report from Todd Pacific Shipyards (Chirillo, 1982) is a manual on how to perform line heating for forming purposes of plates and stiffeners and distortion removal due to welding. This work is supplemented by Scully (1987) with an investigation of the possibility of using laser as heat source.

The manual line heating process is transferred to the Spanish shipyard Astilleros Españoles in the early 1990s, and a manual is published (de la Bellacasa, 1992) on that occasion. A status report is also available (Sarabia & de la Bellacasa, 1993).
Of course, it can be avoided to use compound forming techniques, if (parts of) a ship are designed by means of developable spline surfaces as described by Chalfrant & Maekawa (1998). This is, however, not very interesting in the case of most ship types—and in the context of the current study.
Chapter 3

Numerical Modelling

Numerical modelling is a very important supplement/alternative to experiments, and in the case of line heating, computerised calculations yield information which is otherwise unavailable from experiments. For example, the in-plane deflections are barely measurable and no simple method can give information on plastic strains.

As a first approach to numerical modelling, a formulation based on the v. Kármán plate with plasticity was established. The intention was to use this in a finite difference analysis (FDA), but unfortunately the idea had to be abandoned due to problems with translating the intricate equations into a discrete version to be solved numerically. The derivation of the differential equations is, however, included in Appendix A.

Instead, a numerical model has been built by use of the finite element code ANSYS. Below a thorough description of the model and its subsequent validation is given.

3.1 Modelling

Several assumptions and choices have been made for the formulation of the numerical models.

**Kinematics.** A small strain and small deflections formulation has been applied. This is justified by the fact that the deformations induced by line heating are very small.

**Constitutive formulations.** The temperature field is unaffected by the structural response so that the thermal and structural problems can be solved in sequence—with the results of the former as input to the latter. The steel is modelled as isotropic and with plasticity taken as $J_2$ flow theory with strain hardening, see Section 3.1.1. Cooling is implemented as both conduction, convection and radiation.
Modelling. Linear brick elements are used in a fine mesh near the heated zone to capture the temperature and stress gradients. The mesh is gradually refined towards a coarser mesh representing the surrounding plate. Only cases where the heat is applied far from the edges (on a sufficiently large plate) are examined as this is more general than including edge effects. It is investigated later how heating near the edges affects the deformations. The plate is considered free of residual stresses, even though fabrication, handling, cutting etc. and will all induce stresses. How severe the residual stresses are is impossible to foresee, so this effect is not included in the present modelling.

A number of cases within these assumptions are analysed in Section 3.4.

3.1.1 Material Properties

Data on temperature-dependent material properties is generally very hard to obtain. Here, three sources are used, namely Patel (1985), Richter (1973), and Birk-Sørensen (1999), and Figures 3.1 to 3.4 show comparisons of the thermal and structural properties of the material as a function of temperature, $T$.

As it is seen all the data except that for the tangent modulus is much alike, so any of it may adequately describe the material behaviour. The Patel data set is used throughout this thesis as it seems to be the most complete.
3.1 Modelling

Figure 3.2: Comparison of Young’s modulus, $E$, and yield limit, $\sigma_y$, for mild steel.

Figure 3.3: Comparison of tangent modulus, $E_t$, and Poisson’s ratio, $\nu$, for mild steel.
3.1.2 Heat Source

Proper modelling of the heat source is important, as this is the ‘driving force’ of the forming method. As mentioned in Chapter 2 most researchers in the field employ a Gaussian distributed, axisymmetric heat flux as a means of modelling the gas torch. This Gaussian distribution is expressed as

$$Q''(r) = Q''_{\text{max}} e^{-\gamma r^2} \quad (3.1)$$

Thus, the pointwise distribution, $Q''$, is expressed by a peak value, $Q''_{\text{max}}$, a width factor, $\gamma$, and the distance from the centre, $r$. The total heat input, $Q_{\text{tot}}$, is calculated as the integral of the distribution:

$$Q_{\text{tot}} = \int_0^{2\pi} \int_0^{\infty} Q'' r \, dr \, d\theta = \frac{\pi Q''_{\text{max}}}{\gamma} \quad (3.2)$$

To make a proper choice of the torch width and to understand the term $\gamma$ better, a ‘torch radius’, $r_{\text{torch}}$, is defined as the distance where $Q''$ is 1% of $Q''_{\text{max}}$. Thus,

$$Q''_{\text{max}} e^{-\gamma r_{\text{torch}}^2} = 0.01Q''_{\text{max}} \quad \uparrow$$

$$\gamma = \frac{\ln 100}{r_{\text{torch}}^2} \quad (3.3)$$

In the following calculations, $r_{\text{torch}}$ is fixed at 4 cm unless otherwise stated, which means that $\gamma$ is 2878 m$^{-2}$.

The true distribution from a gas torch can be determined experimentally as described in Section 6.2 p. 68.

3.1.3 Conduction, Convection and Radiation

Heat conduction is governed by the first law of thermodynamics, which states that energy is conserved. Written in the form of a differential control volume formulation, neglecting heat
3.1 Modelling

transfer caused by mass transport and heat generation, it becomes (Kohnke, 1998)

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot \bar{q} = 0$$  \hspace{1cm} (3.4)$$

where $\rho$ is density, $c_p$ is specific heat, $T$ is temperature, $t$ is time, $\nabla \cdot$ is the divergence operator, and $\bar{q}$ is the heat flux vector. $x$ is along the heating path, $y$ is perpendicular to it and $z$ is in the thickness direction.

Heat flux and temperature gradients are related by Fourier’s law:

$$\bar{q} = -D \nabla T$$  \hspace{1cm} (3.5)$$

where $D = \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{bmatrix}$ is the conductivity matrix, as $\lambda$ denotes thermal conductivity, and $\nabla$ is the gradient operator. Eqs. (3.5) and (3.4) combine in the well-known form:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda_z \frac{\partial T}{\partial z} \right)$$  \hspace{1cm} (3.6)$$

which is the governing equation for the thermal problem.

The convection properties of the surrounding air are found from the description of free convection from a horizontal plate in Incropera & Dewitt (1993, pp. 460ff). Although this theory is derived for an infinite plate with uniform temperature distribution, it is assumed that it may describe locally heated regions as well. As shown below, the convection on the upper side does not depend on a characteristic length on the plate.

The basic assumption used in the ANSYS is that heat, $Q_c$, convected from a surface may be described in the general case as Newton’s postulate

$$Q_c = \int_A h_f (T_s - T_B) dA$$  \hspace{1cm} (3.7)$$

where $h_f$ is a film coefficient which may vary with temperature, $T_s$ is the surface temperature, and $T_B$ is the bulk air temperature.

The film coefficient is given by the mean Nusselt number, $\overline{Nu}_L$, relating to the Rayleigh number, $Ra_L$, as

$$\overline{Nu}_L = 0.15 Ra_L^{1/3}$$  \hspace{1cm} (3.8)$$

Here $\overline{Nu}_L = h_f L/\lambda_a$ is a non-dimensional measure of the heat convection from the surface. $L$ is a characteristic length describing the system and $\lambda_a$ is the thermal conductivity of air. $Ra_L = g \alpha_a (T_s - T_B) L^3$ is a measure of the instability of the air given the physical constants and a temperature difference. $g$ is gravity, $\alpha_a$ is the thermal expansion coefficient of air, $\nu$ is the kinematic viscosity, and $\beta$ is the thermal diffusivity.
Chapter 3. Numerical Modelling

<table>
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<td>427</td>
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<td>5.32</td>
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<td>7.76</td>
<td>7.76</td>
<td>7.63</td>
<td>7.43</td>
<td>7.27</td>
<td>7.10</td>
</tr>
</tbody>
</table>

Figure 3.5: *Film coefficient based on average of surface and bulk temperature, $T_B = 20$ °C.*

Solving the above equations for $h_f$ yields

$$h_f = \frac{\lambda_a}{L} \left( 0.15 \left( \frac{g \alpha_a (T_s - T_B) L^3}{\nu \beta} \right)^{\frac{1}{2}} \right) = \lambda_a \left( 0.15 \left( \frac{g \alpha_a (T_s - T_B)}{\nu \beta} \right)^{\frac{1}{2}} \right)$$  \hspace{1cm} (3.9)

Hence, the convection is not influenced by the characteristic length, $L$, of the plate.

By use of Table A.4 ‘Thermophysical properties of gases at atmospheric pressure’ in Incropera & Dewitt (1993) and the relation $\alpha_a = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right) \approx -\frac{1}{\rho} \left( \frac{\Delta \rho}{\Delta T} \right)$, it is possible to establish the table above of film coefficient versus temperature. Similarly, for the lower side of the plate a relation between the Nusselt number and the Rayleigh number exists:

$$\overline{Nu_L} = 0.27 R o_L^{1/4}$$  \hspace{1cm} (3.10)

It is evident that the convection on the lower side *does* depend on a characteristic length in contrast to the upper side. However, for simplicity convection on the lower side is assumed to be equal to that of the upper side.

The heat radiated from the plate is modelled by Stefan-Boltzmann’s law for radiation:

$$Q_R = \int_A \epsilon \sigma (T_s^4 - T_B^4) dA$$  \hspace{1cm} (3.11)

where $A$ is an area, $\epsilon$ is the radiant emissivity, and $\sigma$ is the Stefan-Boltzmann constant.

Here the only variable other than temperature is the emissivity, $\epsilon$. As suggested in Yagla et al. (1995) the emissivity is approximately 0.5.

Precise modelling of the surface heat loss is not very crucial to the modelling, as the heat loss is quantitatively much smaller than the conduction within the steel. This is verified
3.2 Meshing

A mesh with as few nodes and elements as possible is built to save computation time. This is done by using small elements in a region around the heated zone with a transition into a coarser mesh by using only elements with six sides (not tetrahedra or wedges). The transition can be done as shown in Figure 3.7 where four or six elements in the thickness direction are transformed into only one element. A sample mesh is given in Figure 3.8.

Figure 3.6: Ratio of z- and x-direction heat flux along the heating path. The ‘distance’ is measured from the centre of the torch.

Figure 3.7: Transition in the corner of the inner mesh. (A) Four elements through the thickness. (B) Six elements through the thickness.

by Figure 3.6, showing the ratio of conduction (x-direction flux) and surface heat loss (z-direction heat flux) on the heated side, along a line in the centre of the heating path ($y = 0$). The distance on the graph is measured from the centre of the torch. Over a large part of this line of evaluation, the ratio is negative, which means that the flux is directed away from the surface. Not until far away from the torch (as the temperature becomes more evenly distributed), the convection becomes larger than the z-direction conduction, as can be seen from the small positive values of the ratio. Thus, the heat loss from the surface of the plate will not induce significant shrinkage of the outer layers.

3.2 Meshing
Generally, it should be avoided to use a single brick element without rotational degrees of freedom to model plate bending, as shear locking—the inability of the elements to represent the true response to bending moments—is inevitable. However, the linear eight-noded element is implemented in ANSYS with ‘extra shapes’ from Taylor et al. (1976) which reduces this problem. With additional internal degrees of freedom which represent bending, “it models pure bending exactly regardless of element aspect ratio” (Cook et al., 1989) provided that the element is rectangular in the bending plane. Using internal degrees of freedom makes the element incompatible, but when the mesh is refined this will not be a problem. Further, the validity of the results calculated with this element is verified by the agreement between numerical results and experiments, see Chapter 6.

3.3 Convergence Analysis

It is investigated how many elements through the thickness are required to achieve a converged solution and whether linear or parabolic elements are most time-efficient. This is done by comparing different configurations to a ‘reference’ analysis with six parabolic elements in the thickness direction. To ensure this is actually a converged solution, a comparison is also made with four parabolic elements in the thickness direction. All analyses are carried out on a 10 mm plate with a target maximum temperature on the heated side of 550 °C, and a velocity of 5 mm/s. Figures 3.9 to 3.12 show plastic strain distributions along and perpendicular to the heating path, on the upper and lower side of the plate. The legend is: $xyz_n$/type, $xyz$ = component of plastic strain, $n$ = number of elements, type = linear or parabolic elements. The results from the ‘reference’ case are drawn on the graph by a thicker line. The resemblance of the results indicates that the solutions in fact converge.

It is seen that four linear elements are not enough to obtain convergence (see Figures 3.9 and 3.11). If six linear elements are used instead, the results match those from the (more
3.3 Convergence Analysis

Figure 3.9: Comparison of plastic strains perpendicular to the heating path on the bottom side.

Figure 3.10: Comparison of plastic strains perpendicular to the heating path on the heated side.
Figure 3.11: *Comparison of plastic strains along the heating path on the bottom side.*

Figure 3.12: *Comparison of plastic strains along the heating path on the heated side.*
3.4 Numerical Test Cases

The factors controlling the final shape of the plate include plate size, heating line position, torch speed, power of the heat source, plate thickness, and material properties. To formulate relations from a reasonable number of simulations, the number of unknowns must be reduced. Therefore, focus is on the local strains, as they are more general than the description of the precise) parabolic elements. By comparison of the computer time consumption for the cases, it is obvious that linear elements should be used in favour of parabolic ones: $6/\text{lin} = 11.8 \text{ CPU hours}$, $4/\text{par} = 68.9 \text{ CPU hours}$ and $6/\text{par} = 128.7 \text{ CPU hours}$.

The influence of the width of the fine part of the mesh is examined with linear elements, and the result is shown in Figure 3.13. The legend gives the zone width and whether it is a reference solution (the solution from 6 parabolic elements shown as $\text{ref}$ and in a thick line). The graph shows that the finely meshed area should just envelope the plastic zone, i.e. be neither too wide nor too narrow. A width of 8 cm is chosen for all later analyses. Apparently, the chosen 18 elements perpendicular to the heating direction are adequate.

Figure 3.13: Influence of changing the width of the mesh around the heated zone. Plastic strains on the heated side, constant number of elements.

1 All computations are made on an HP Visualize C200 w/ 1GB RAM, SPECfp 21.4
total deflection of the plate and this more or less removes the dependence on plate size. By considering only plates of identical material and neglecting edge effects, the problem is reduced to depending only on physical measures as thickness, \( h \), torch speed, \( v \), and amount of heat, \( Q \). During the forming process, the quantities \( h \) and \( v \) are easily measured, but how much heat is actually absorbed by the plate is difficult to say. Meanwhile, the maximum temperature is very important as there must be some proportionality between the temperature and the local deflection. The surface temperature can be measured and is a function of the other parameters, \( Q \), \( v \), and \( h \).

For these reasons, only the maximum temperature measured on the heated side, \( T_{\text{max}} \), the plate thickness, \( h \), and the torch velocity, \( v \), are considered in the simulation programme. For the simulations the heat input that produces a certain maximum temperature is found by manual iteration. Once these three parameters have been set, the temperature field will be fully governed, as thermal gradients, heat penetration depth, and heat input depend directly upon those parameters. Thus, to obtain a certain temperature, the heat input must be varied depending on the thickness and the torch velocity. An error of the heat modelling is that the torch width, \( r_{\text{torch}} \), is always the same, no matter the heat input. For a gas torch this is physically impossible, whereas induction heating or a laser beam are more likely to have this characteristic.

The simulation programme has to cover a representative range of the possible parameter values, so three thicknesses (10, 15 and 20 mm), three velocities (5, 10 and 15 mm/s) and three maximum temperatures (500, 600 and 700 °C, reference temperature at 20 °C) are chosen.

Table 3.1: Numbering of test programme.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>( h ) [mm]</th>
<th>( v ) [mm/s]</th>
<th>( T_{\text{max}} ) [°C]</th>
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</table>

For the simulations the heat input that produces a certain maximum temperature is found by manual iteration. Once these three parameters have been set, the temperature field will be fully governed, as thermal gradients, heat penetration depth, and heat input depend directly upon those parameters. Thus, to obtain a certain temperature, the heat input must be varied depending on the thickness and the torch velocity. An error of the heat modelling is that the torch width, \( r_{\text{torch}} \), is always the same, no matter the heat input. For a gas torch this is physically impossible, whereas induction heating or a laser beam are more likely to have this characteristic.
3.5 Linearisation of Plastic Strains

Thus 27 \((3^3)\) simulations are to be carried out. For details and numbering, see Table 3.1. All simulations are made on a 1 by 1 m plate with symmetry conditions at the centre line and heat is applied along a 20 cm path in the \(x\)-direction in the centre of the plate (from point A to B in Figure 3.14). As mentioned earlier, a torch radius, \(r_{\text{torch}}\), of 4 cm is used. A model as shown in Figure 3.8 consists of 5100 elements and 6400 nodes and solves in about 500 iterations or eight hours. The results of the 27 test cases and a thorough discussion is given in Section 3.8.

Although the line heating process is used and approved by the American Bureau of Shipping up to 900 °C, lower temperatures may be sufficient “... because efficient bending was already being performed with lower temperatures” (quoted from Chirillo (1982, p. 6)). Embrittlement is also more likely to occur at higher temperatures due to the formation of martensite. Further, the constitutive model (bilinear plasticity law) is unable to model temperatures above the phase transformation temperature of low carbon steel (723 °C). Among others, the thermal expansion coefficient, the yield limit and Young’s modulus differ in heating and cooling near the phase transformation temperature (Ueda et al., 1985). Using temperatures higher than 700 °C in the numerical test programme is therefore not recommendable, even though it may work fine in reality.

3.5 Linearisation of Plastic Strains

To facilitate the analysis of the vast amounts of data from the three-dimensional simulations, some simplifications must be made. The goal is to describe the local deformation field with as few parameters as possible, leading to the following assumptions:

- The only area of interest is the vicinity of the heat affected zone (HAZ) where the plastic strain field is analysed, as shown in Figure 3.15.

- The plastic strain does not vary along the heating path, so effects of starting and stopping are not considered. Hence, only a section, labelled \(D\) in Figure 3.15, perpendicular to the centrepoint of the heating path \((x = 0)\) is analysed.

![Figure 3.15: Definition of zone of interest.](image)
It is further assumed that the plastic strain field can be described by four parameters only, namely shrinkage, $S$, and bending, $B$, with contributions to the longitudinal ($x$) and the transverse ($y$) directions, respectively. Thus, the strain distribution is assumed to be linear in the thickness ($z$) direction.

The first step towards simplification of the strains is to find an average strain distribution, $\hat{\varepsilon}_{x,y}^p(z)$, which applied uniformly to $D$, will yield the same result as the original strain field. The average $\hat{\varepsilon}_{x,y}^p$ is found by integration over the width of the plastic zone in every layer of nodes in the $z$-direction. This yields what may be called a ‘plastic displacement’ or the amount of deformation caused by all the elements in the zone, due to the plastic strains alone. Therefore, the average strain contribution, $\hat{\varepsilon}_{x,y}^p$, can be found by dividing the ‘plastic displacement’ by the width of the HAZ, $w^p$:

$$\hat{\varepsilon}_{x,y}^p(z) = \frac{1}{w^p} \int_0^{w^p} \varepsilon_{x,y}^p(y, z) dy$$ \hspace{1cm} (3.12)

The resulting strain distribution, $\hat{\varepsilon}_{x,y}^p(z)$, is then linearised by means of the least squares method as shown in Figure 3.16 to yield the acquired bending and shrinkage contribution, $B_{x,y}$ and $S_{x,y}$, to the plastic strain. To make a simplified analysis equivalent to the fully elasto-plastic analysis, this strain must—of course—be applied to an area of the width, $w^p$. 

\[ \begin{array}{cc}
A & B \\
C & D \\
E & F \\
G & H \\
I & J
\end{array} \]

Figure 3.17: Result representation.
Figure 3.18: Test case 4. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}^p$. 

3.5 Linearisation of Plastic Strains
The plastic strain distribution and the linearised strains from the 27 analyses are presented in Appendix B, where the plastic strain is shown as in Figure 3.17 and as an example in Figure 3.18. Above the line (A through F) the graphs show how the plastic strain \( \varepsilon_p \) along and perpendicularly to the heating path on the top and bottom sides of the plate. Thus, pane A holds \( \varepsilon_{x}^p \), evaluated perpendicularly to the centre point of the heating path, while B holds the same property, evaluated along the heating path. Panes C, D, E, and F are the same except for \( \varepsilon_y^p \) and \( \varepsilon_z^p \), respectively. The G pane shows the \( \varepsilon_{x,y,z}^p \) distribution through the thickness in the centre of the heating line. Pane H shows the maximum temperature at each time step. Finally, the I and J panes show how the averaged plastic strains are linearised. The dash lines are the original plastic data, while the thick straight lines represent the linearisation in either \( \{S_x, B_x\} \) or \( \{S_y, B_y\} \).

### 3.6 Simplified Elastic Analyses

The assumption as regards the test cases is that the deformation from the line heating process is local. Therefore, an equivalent analysis can be carried out, if the linearised plastic strains are applied to a plate in an elastic analysis. Thus, a plate of any shape can be analysed using the plastic strains according to the heating parameters as input.

Unfortunately, ANSYS is unable to use the plastic strains directly as input in the initial state. Instead an ‘artificial temperature field analysis’ (ATFA) can be used, which combines artificial thermal expansion coefficients and temperatures to simulate the linearised plastic strains. The artificial material properties are applied to a region of the same size as the plastic zone shown in Figure 3.15 and the region outside is assigned a zero thermal expansion coefficient. The ‘plastic’ region is divided into two elements in the thickness direction each with separate orthotropic thermal expansion coefficients, as shown in Figure 3.19.

In the following it is assumed that the elements are equally thick and subject to a constant temperature, \( T \). Then the strain at the bottom, \( \varepsilon_x^1 \), and at the top, \( \varepsilon_x^4 \), must be equal to the linearised strain at those \( z \)-coordinates.

\[
\varepsilon_x^1 = \alpha_x^1 T = S_x - B_x \quad \Leftrightarrow \quad \alpha_x^1 = \frac{S_x - B_x}{T} \tag{3.13}
\]

and

\[
\varepsilon_x^4 = \alpha_x^2 T = S_x + B_x \quad \Leftrightarrow \quad \alpha_x^2 = \frac{S_x + B_x}{T} \tag{3.14}
\]

At the interface between the two elements, the strains will be averaged at load vector assembly, resulting in a linear distribution from top to bottom. The same derivation is valid for the \( y \)-direction.

ATFA can be applied to a FE model with relatively large elements and still yield precise results. One thing is, however, crucial to correct modelling: Consider the boundary region
3.7 Sensitivity Analysis

A number of simulations are made to examine the sensitivity of the model to variations in input parameters related to previous assumptions. Information is provided on how variations in material properties, torch width and plate size influence the linearised strains with case No. 14 as a reference. The following simulations are made:

- Two simulations investigate changes in the yield limit (sens1 and sens2). In the former, all data for the yield limit is raised by 10%. In the latter, the room temperature yield limit is changed from 250 MPa to 400 MPa leaving all other untouched. Linear interpolation is used for temperatures between 20 °C and 400 °C, where the yield limit is still 210 MPa.

between the zones with artificial expansion coefficients and the surrounding area with zero expansion coefficient. The nodes at this interface are assigned the average thermal expansion from the element in the two zones, and the elements connected to those nodes receive an incorrect expansion. If the elements have independent, coinciding nodes at the interface, and the nodes are connected by stiff ends (constraints), the elements at the interface will be thermally independent but still provide the true kinematics.

To validate the results from this elastic method, comparison to the 27 elasto-plastic analyses is made. Figures C.1 to C.4 in Appendix C present a comparison of the $z$-direction deflection at the symmetry (heated) line and at the far edge of the plate parallel to the heating path. It is seen that good agreement is found between the two types of analysis, and considering the calculation time saved (6 hours versus 30 seconds) this is very encouraging.

Figure 3.19: Strains and artificial properties for two elements in the thickness direction.
Table 3.2: Influence of change in parameters on residual plastic strains.

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<th>S_x</th>
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<td>16%</td>
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<td>8%</td>
</tr>
<tr>
<td></td>
<td>B_y</td>
<td>5%</td>
<td>-12%</td>
<td>11%</td>
<td>-24%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Figure 3.20: Comparison of averaged plastic strains, $\hat{\varepsilon}_{x,y}$, for sens1 and sens2 versus case No. 14.

Figure 3.21: Comparison of averaged plastic strains, $\hat{\varepsilon}_{x,y}$, for sens4 versus case No. 14.
3.7 Sensitivity Analysis

- Test with a narrower torch (‘sens3’). A torch radius, \( r_{\text{torch}} \), of 3 cm is used instead of the usual 4 cm. The maximum temperature is kept constant.

- Test of the influence of using kinematic hardening instead of isotropic hardening (‘sens4’).

- One test investigates the influence of the plate size on the local plastic deformation (‘sens5’). A plate size of 1.5 m by 0.75 m is used with symmetric conditions.

It is seen from Table 3.2 that the bending in the longitudinal \( x \)-direction, \( B_x \), is highly influenced by the yield limit. However, \( S_x \), \( S_y \), and \( B_y \) are much less sensitive to those changes. Figure 3.20 shows how the averaged (not linearised) plastic strains are changed with the increase in yield stress. The change is mainly on the heated side, where the \( x \)-components of the plastic strains are increased compared to the reference case, so that bending becomes more predominant.

The results have proved insensitive to the use of either isotropic or kinematic hardening.

A narrower torch will produce less deflection in general as the required heat input is smaller to attain the temperature of 600 °C. However, the \( x \)-direction bending, \( B_x \), is larger. See Figure 3.21, where the heated zone is narrower and the heat penetration is smaller for a narrower torch, which again implies that the ratio of shrinkage to bending can be controlled by the choice of heat source. A broader torch will give more shrinkage at the expense of bending—most notably in the \( x \)-direction, provided that the maximum temperature is the same. As the results are sensitive to changes in torch radius, direct measurements of the temperature distribution on a heated plate are used to find the heat flux distribution in Chapter 6.

Finally, sens 5 shows that the results are for practical matters independent of the plate size.

Figure 3.22: Temperature profiles. The maximum temperature is 500 °C and 50 °C between each contour.
Figure 3.23: Stress $x$-components, $\sigma_x$. (A) Intermediate and (B) final step.

Figure 3.24: Stress $y$-components, $\sigma_y$. (A) Intermediate and (B) final step.
3.7 Sensitivity Analysis

Figure 3.25: Plastic strain, $x$-components, $\varepsilon_p^x$. (A) Intermediate and (B) final step.

Figure 3.26: Plastic strain, $y$-components, $\varepsilon_p^y$. (A) Intermediate and (B) final step.
3.8 Analyses of Test Case Results

General Mechanics

A general description of the stress and strain development for test case No. 9 is given in the following. Thus, \( T_{\text{max}} = 500 \, ^{\circ}\text{C}, h = 20 \, \text{mm}, \) and \( v = 15 \, \text{mm/s}. \)

Figure 3.22 shows the temperature profiles from the test case. It is seen that the temperatures do not propagate very far into the surrounding material—the contour for 50 \( ^{\circ}\text{C} \) is well inside the 4 cm boundary of the finely meshed area. Of course, the heat will eventually disperse, but at much lower temperatures, which suggests that the heating paths do not interact through the temperature fields but rather because of the stresses associated with the local deflections of neighbouring paths.

When heated, the area immediately under the torch will be subjected to large compressive stresses, \( \sigma \). Figure 3.23 shows \( \sigma_x \) at an intermediate load step and after cooling and likewise Figure 3.24 shows \( \sigma_y \). The torch position in the intermediate state is denoted by an orange patch and the torch is moving from right to left. As regards the \( \sigma_x \), a large area in front of the torch is in compression, but it gradually decreases to zero further away from the torch. The \( \sigma_y \) also shows compression in an area in front of and below the torch, but due to \( y \)-direction symmetry there is additionally an area with tensile stresses, which becomes permanent on the bottom side of the plate. After cooling, \( \sigma_x \) is mostly tensile, whereas \( \sigma_y \) is tensile on the surfaces and compressive in the centre. It should also be noted that the tensile stresses are larger in the \( x \)-direction than in the \( y \)-direction, as this information is used later to explain the development of plastic strains.

The plastic strains, \( \varepsilon^p \), are shown in Figures 3.25 and 3.26 in the intermediate and final states. Here, the final state is purely compressive strains—except just in front of the torch, where the \( y \)-component plastic strain is slightly positive (shown in grey). To analyse how the strains are developed, a plot of plastic strains versus temperature is shown in Figures 3.27 and 3.28 for nodes in the thickness direction in the centre of the heating line. The time between all the points except for the last 13 (during cooling) is a quarter of a second. As the penetration of the plastic zone is low, only strains from the top four nodes are shown.

From Figure 3.27 it is seen that \( \varepsilon_x^p \) in the top layer is initiated at about 100 \( ^{\circ}\text{C} \) and that it rapidly increases to its maximum value at 500 \( ^{\circ}\text{C} \). When the torch passes by, the material cools down and the plastic strains are constant for some time, while the contraction of the steel changes the stress state from compressive to tensile. This corresponds to the green area behind the torch in Figures 3.23(A) and 3.24(A). When the top layer reaches tensile yielding, the material is still very much expanded and there is a large temperature difference between the top and bottom sides of the plate. As the temperature is still 350 \( ^{\circ}\text{C} \), there is excessive plastic straining, which is reduced during cooling. For the second layer the behaviour is the same, but with a higher temperature for the onset of plastic yielding because of smaller
Figure 3.27: Plastic strains as functions of temperature in nodes 1 to 4 in the thickness direction. $x$-components.

Figure 3.28: Plastic strains as functions of temperature in nodes 1 to 4 in the thickness direction. $y$-components.
The linearised, averaged data for the 27 test cases is given in Table 3.3, but even with these case No. 9 to limit the storage space needed. However, another eight test cases have been included to ensure the robustness of the analysis. The linearised plastic strain increments are determined from the product of the stress deviators and the strain increments. Even though the plastic strain increments are so much larger than the elastic strain increments, the plastic strain increases at the expense of the elastic strain. This means that the plastic strain becomes numerically larger than the elastic strain.

The data necessary for making plots as in Figures 3.27 and 3.28 is only available for test case No. 9 to limit the storage space needed. However, another eight test cases have been made for use in Chapter 4, and the temperature/strain plots are given in Figures 3.29 and 3.30 for comparison of results related to different input parameters (shown in Table 3.4).

The linearised, averaged data for the 27 test cases is given in Table 3.3, but even with these...
Figure 3.29: Plastic strains as functions of temperature in nodes 1 to 7 in the thickness direction. Node 1 is on the heated side.
Figure 3.30: Plastic strains as functions of temperature in nodes 1 to 7 in the thickness direction. Node 1 is on the heated side.
simplifications, the amount of data is slightly overwhelming so to assess the tendency of the data, the variation with $T_{\text{max}}$, $h$, and $v$ is shown in Figures 3.31, 3.32, and 3.33, respectively. Figure 3.31 presents the variation in temperature, and almost without exception the plastic strains become numerically larger with increasing temperature—which is not very surprising as higher plastic strains induced by higher temperatures will increase the strain hardening of the material. Thus upon cooling, the material is harder the higher the temperature, on the assumption used in the modelling that the steel is not annealed by the heat treatment.

Figure 3.32 shows the variation with velocity. With increasing velocity, the shrinkage contribution, $S_{x,y}$, decreases and the bending contribution is rather constant or even increasing,

<table>
<thead>
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<th>Case No.</th>
<th>$v$ [mm/s]</th>
<th>$h$ [mm]</th>
<th>$T_{\text{max}}$ [°C]</th>
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<td>1 5</td>
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<td>2 6</td>
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<tr>
<td>12.5</td>
<td>3 7</td>
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</tr>
<tr>
<td>12.5</td>
<td>4 8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: Heating parameters for additional test cases.
which means that bending becomes dominant at higher velocities. At high velocities, the heat is not conducted so far into the plate so the heated side mostly contracts with bending of the plate as a result. At extremely slow torch movement with full heat penetration, the strains will—ideally—only consist of shrinkage. At medium penetration depth the material on the bottom side will act as a hinge, resulting in bending.

It is seen from Figure 3.33 that the shrinkage, $S_{x,y}$, is decreasing with increasing thickness. It is also observed that the bending $x$-component is increasing with the thickness. Again, this relates to the heat penetration as explained above. However, the bending $y$-component is mostly decreasing with increasing thickness, which at first may seem very strange. To explain this, the data behind the lower, right graph is analysed further. It corresponds to test cases 25, 26, and to 27, from which the averaged and linearised data is presented in Figures B.25 to B.27 (Appendix B). For a given temperature and velocity, $\hat{\epsilon}_p^x$ is almost constant on the heated surface, but with thicker plates the value in the centre of the plate decreases. This corresponds to larger bending, $B_x$, and smaller shrinkage, $S_x$. In the $y$-direction, all the strains are generally decreasing so that bending, $B_y$, and shrinkage, $S_y$, decrease.
3.8 Analyses of Test Case Results

Comparison of $x$- and $y$-components of Shrinkage and Bending.

With the numerical data at hand, it is interesting to know how it can be compared quantitatively. For this purpose, it is relevant to know whether bending or shrinkage is predominant and how the magnitudes of the deflections can be compared in the tested temperature range.

To facilitate the interpretation, the data is compared as ratios of $y$- and $x$-components and as ratios of bending and shrinkage magnitudes, respectively. Firstly, Table 3.5 shows the relation between the longitudinal and the transversal strains, and one can see that the bending perpendicular to the heating path is larger than that along the heating path.

Further, the ratio of $y$- and $x$-components in shrinkage, $S_y/S_x$, is not far from unity, so that the instances of shrinkage along and perpendicular to the heating path are comparable in magnitude. However, the tendency is that the ratio increases with rising temperature, $T_{\text{max}}$, and decreases with rising velocity, $v$. Thus, in cases where shrinkage perpendicular to the path is mostly wanted (which is likely to be the general case), higher temperatures should be preferred. The dependence of the thickness, $h$, is rather ambiguous.

The ratio of the $y$- and $x$-components in the bending, $B_y/B_x$, has the same tendencies, apart from the dependence of $h$ where the ratio is clearly inversely proportional. In general,
transversal bending is much larger than bending along the heating line. Again, high temperatures have to be preferred to achieve a high ratio between the y- and x-components and, further, low velocities to improve the ratio can be chosen. However, the latter will reduce the absolute amount of bending and increase shrinkage.

Comparison of Bending and Shrinkage

Next, comparison between bending and shrinkage (Table 3.6) shows that their ratio, $B_{x,y} / S_{x,y}$, is highest for low temperatures and high velocities. If the required behaviour is to have a high ratio between bending and shrinkage, a low temperature should be chosen. But if—at the same time—a high ratio of y- and x-components is required, this implies high tempera-

### Table 3.5: Ratio between x- and y-components.

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<tr>
<th>$v$ [mm/s]</th>
<th>$h$ [mm]</th>
<th>Case</th>
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**$T_{max} = 500 \degree C$**

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### 3.8 Analyses of Test Case Results

#### Analyses of Test Case Results

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$T_{\text{max}} = 500 \, ^\circ C$

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$T_{\text{max}} = 600 \, ^\circ C$

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<tr>
<th>$v$ [mm/s]</th>
<th>$h$ [mm]</th>
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<th>$B_x/S_x$</th>
<th>Average $B_x/S_x$</th>
<th>Average $B_y/S_y$</th>
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<td></td>
</tr>
</tbody>
</table>

$T_{\text{max}} = 700 \, ^\circ C$

### Table 3.6: Ratio between bending and shrinkage. Test cases and averages.

Tures. Therefore, either of the two must be chosen. The $B/S$ ratios are less sensitive than the $y/x$ ratios, so the latter are the governing factor.

#### Calculations with Very Slow Movement

A sufficiently low heat source velocity will produce a ‘pure’ shrinkage state in the heated region. Table 3.6 shows that only thin plates subject to low velocities (cases 1, 10, and 19) have really low ratios of bending to shrinkage, and this is only really true for the $x$-components. In the $y$-direction it seems that even lower velocities are needed, especially for the thicker plates. Therefore, more simulations are carried out to see if ‘pure shrinkage’ is attainable for thicker plates.
Chapter 3. Numerical Modelling

<table>
<thead>
<tr>
<th>$v$ [mm/s]</th>
<th>$h$ [mm]</th>
<th>Case</th>
<th>$S_x$</th>
<th>$B_x$</th>
<th>$S_y$</th>
<th>$B_y$</th>
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<td>-1.73·10^{-3}</td>
<td>7.20·10^{-4}</td>
</tr>
</tbody>
</table>

Table 3.7: Test cases with low torch velocity.

The additional simulations along with linearised results are given in Table 3.7. See also Figures B.28 to B.32, p. 125ff. Not surprisingly, the 10 mm plate is still very close to 'pure shrinkage' at $v = 2$ mm/s. However, thicker plates still show a good deal of bending, especially in the $y$-component, and even at the unrealistically low speed of 0.5 mm/s bending is still rather conspicuous. Either the fact must be accepted that bending will always be present or methods like double sided simultaneous heating or high-frequency induction heating must be used. The latter is able to generate heat in the material as well as on the surface. Further, changing the frequency of the induced current will control the heat penetration to some degree.

**Power Consumption**

As mentioned earlier, the maximum temperature, $T_{\text{max}}$, is chosen as a parameter instead of the power transferred from the torch to the plate, $Q_{\text{tot}}$, mainly to ease the control during the forming process. Another reason is that the results from the ANSYS simulations based on power rather than temperature are not reproducible. Depending on time stepping, meshing and solution options, one specified power can lead to various temperature distributions, whereas a specified maximum temperature leads to consistent results.

However, it is interesting to know approximately how powerful the torch must be to perform shaping, especially at high velocities. Table 3.8 shows the approximate power input for each of the simulated test cases along with energy per unit length of heating, $W_{h} = \frac{Q_{\text{tot}}}{v}$. The power output from the heat source must of course be higher, as the figures in the table indicate what must be put into the plate (efficiency coefficient being ignored).

Using fast torch movement is least energy consuming, on the assumption that the total heating length is the same no matter what the chosen torch velocity is. However, there is not really a choice, as the temperature and the velocity are governed by the necessary deformation at the heating line. There may be a limitation in the heat output of the available torches which hinders the combination of high velocities and high temperatures.
3.9 Edge Effects

Until now, all the strains have been derived from cases where heating is applied far from the edges. Although this represents a large part of the potentially heated area, heating is very commonly applied near the edges when plates of overall positive Gaussian curvature (concave) are formed. Therefore, it is analysed how heating to and from an edge influences the strain distribution. Simulations based on the same parameters as case No. 14 with the torch leaving and entering the plate at right angles and 45°, respectively, are carried out.

Figure 3.35 shows the $x$-components of the plastic strain near the edge of a plate. Only half the plate (divided at $y = 0$) is shown to give a view of the strains in the interior of the plate. In the top of the figure, the result of heating towards the edge is presented, and there is clearly a large influence from the edge, which is partly connected with the increase in temperature when the edge is crossed, see Figure 3.34. Further, the area which supports the stresses in front of the heat source is gradually reduced as the edge is approached. Therefore,

<table>
<thead>
<tr>
<th>$T_{\text{max}}$ [°C]</th>
<th>$v$ [mm/s]</th>
<th>$h$ [mm]</th>
<th>Case</th>
<th>$Q_{\text{tot}}$ [W]</th>
<th>$W_p$ [MJ/m]</th>
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Table 3.8: Power consumption.
the final state at the edge will consist of large compressive $x$-direction plastic strains, $\varepsilon_p^x$, and large tensile $y$-direction plastic strains, $\varepsilon_p^y$. The development of $\varepsilon_p^x$ and $\varepsilon_p^y$ at the edge is depicted in Figures 3.40 and 3.41. Even before the heat reaches the edge the steel yields, resulting in the mentioned compressive and tensile strains.

When heating takes place from the edge towards the centre the strains are almost unaffected by the edge effect.

When heating takes place at an angle of $45^\circ$ the results are much as above: The strain field is disturbed compared to that of heating in the centre of a plate, and when heating takes place from the edge inwards, the strain field is almost normal. See figures 3.37 and 3.38. Figure 3.39 shows the plate divided in the centre of the heating line ($y = 0$).

The conclusion is that to minimise the effect of heating at an edge, the heating should be carried out from the edge towards the centre of the plate. Thus, the artificial temperature field analysis (ATFA) can be used with the plastic strains derived from the centre of the plate and reasonable results can still be obtained. Neglecting the edge effect corresponds to the assumption of neglecting the effect of starting and stopping in Section 3.5.

![Figure 3.34: Temperature as function of load steps.](image)
3.9 Edge Effects

Figure 3.35: Plastic strain $x$-components when (A) leaving and (B) entering the plate at right angles.

Figure 3.36: Plastic strain $y$-components when (A) leaving and (B) entering the plate at right angles.
Figure 3.37: Plastic strain $x$-components when leaving and entering the plate at $45^\circ$. 
Figure 3.38: Plastic strain \( y \)-components when leaving and entering the plate at 45°.

Figure 3.39: Plastic strain \( y \)-components when leaving and entering the plate at 45°. Plate divided at \( y = 0 \).
Figure 3.40: Temperature versus plastic strain, $\varepsilon_p^x$, at the edge.

Figure 3.41: Temperature versus plastic strain, $\varepsilon_p^y$, at the edge.
Chapter 4

Empirical Relations

The subject of this chapter is how functions for relations between heating parameters and linearised plastic strains, $S_x$, $B_x$, $S_y$, and $B_y$, can be found. Firstly, independent variables are established from dimensional analysis, and secondly the coefficients to the independent variables are determined by means of multivariate analysis.

4.1 Establishing Dimensionless Parameters

A rational way of finding all independent sets of dimensionless parameters which may be important to a physical problem can be found in Szirtes (1997). With manipulation of a matrix (the so-called dimensional set) the dimensionless parameters can be determined. Below derivation of the set and description of its application to the problem are given.

Firstly, the physical properties for expression of the dimensionless parameters must be chosen. A natural choice is the parameters mentioned in Section 3.4, namely $T_{\text{max}}$, $h$, and $v$. However, those are not alone sufficient to form dimensionless parameters, so additional ones must be included: density, $\rho$ [kg/m$^3$], conductivity, $\lambda$ [W/(Km)], yield limit, $\sigma_y$ [Pa], and specific heat, $c_p$ [J/(Kkg)]. Secondly, a system of independent dimensions is needed as a descriptive basis for all the variables. Here, length [m], time [s], temperature [K] and mass [kg] aptly describe all variables. The relation between the eight variables and the four dimensions in a monomial power form is

$$V_1^\epsilon_1 V_2^\epsilon_2 V_3^\epsilon_3 V_4^\epsilon_4 V_5^\epsilon_5 V_6^\epsilon_6 V_7^\epsilon_7 V_8^\epsilon_8 = d_1^{q_1} d_2^{q_2} d_3^{q_3} d_4^{q_4}$$

(4.1)

with $\epsilon_j$ being the sought powers to the variables, $V_j$, and $q_i$ the powers to the dimensions, $d_i$. If dimensionless parameters are sought, $q_i$ equals 0. Each variable, $V_j$, can be expressed in terms of the dimensions as

$$V_j = d_1^{h_{1j}} d_2^{h_{2j}} d_3^{h_{3j}} d_4^{h_{4j}}$$

(4.2)
Insertion of this in (4.1) and taking note of the fact that the powers to the dimensions on both sides of (4.1) must be the same yield

\[ h_{ij} \epsilon_j = q_i \]  

(4.3)

or

\[ H \vec{\epsilon} = \vec{q} = 0 \]  

(4.4)

for dimensionless parameters

The relation between variables and dimensions is expressed in \( H \), which is called the dimensional matrix. For example, the dimension of density, \( \rho \), is \([\text{kg/m}^3]\), which is represented by a -3 and a 1 in the \( \rho \)-column. For use in (4.7), \( H \) is divided into submatrix \( B \) and the rightmost square submatrix \( A \):

\[
\begin{array}{cccc|cccc|c}
S_x & c_p & \rho & \lambda & \sigma_y & T_{max} & h & v \\
K & -1 & -1 & & 1 & & & \\
m & 2 & -3 & 1 & -1 & 1 & 1 & \\
s & -2 & -3 & & -2 & & -1 & \\
kg & 1 & 1 & & 1 & & & \\
\end{array}
\]

Likewise, the vector \( \vec{\epsilon} \) is divided into two parts, the last as long as the number of the dimensions and the first (here called \( \vec{d} \)) holding what is left. Since the system of equations is underdetermined, \( \vec{d} \) can be chosen arbitrarily. Obviously,

\[ \vec{d} = \begin{bmatrix} I & 0 \\ B & A \end{bmatrix} \begin{bmatrix} \vec{d} \\ \vec{0} \end{bmatrix} \]

(4.6)

and therefore (4.3) can be rewritten as

\[
\begin{bmatrix} I & 0 \\ B & A \end{bmatrix} \begin{bmatrix} \vec{d} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{d} \\ \vec{0} \end{bmatrix}
\]

(4.7)

By premultiplying with the inverse of the term in the square brackets the following is obtained

\[
\begin{bmatrix} I & 0 \\ B & A \end{bmatrix}^{-1} \begin{bmatrix} \vec{d} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{d} \\ \vec{0} \end{bmatrix}
\]

(4.8)

This is, however, just one of the possible solutions to the underdetermined system of equations. Different choice of \( \vec{d} \) gives three more:

\[
\begin{bmatrix} I & 0 \\ -A^{-1}B & A^{-1} \end{bmatrix} \begin{bmatrix} D^T \\ 0 \end{bmatrix} = \begin{bmatrix} D^T \\ -A^{-1}BD^T \end{bmatrix}
\]

(4.9)
4.1 Establishing Dimensionless Parameters

The rows in $D^T$ can be chosen arbitrarily, but they have to be linearly independent. Now, the powers of the variables to form dimensionless parameters are found in $\epsilon$. For example, a dimensionless parameter, $\pi_1$, is given by the powers in the first column of $\epsilon$. If $\epsilon^T$ is written under the dimensional matrix and a new matrix $C = -D(A^{-1}B)^T$ is defined, a structure called the dimensional set is derived as outlined in Figure 4.1.

Selecting $D$ to be the identity matrix and evaluating $C$ yield the following dimensional set for the present problem:

$$
\begin{array}{cccccccc}
S_x & c_p & \rho & \lambda & \sigma_y & T_{\text{max}} & h & v \\
K & -1 & -1 & & 1 & & & \\
m & 2 & -3 & 1 & -1 & 1 & 1 & \\
s & -2 & -3 & & -2 & -1 & & \\
kg & 1 & 1 & & 1 & & & \\
\pi_1 & 1 & & & 0 & 0 & 0 & 0 \\
\pi_2 & 1 & & & 0 & 1 & 0 & -2 \\
\pi_3 & 1 & & & -1 & 0 & 0 & 2 \\
\pi_4 & 1 & & & -1 & 1 & -1 & -1 \\
\end{array}
$$

(4.10)

All empty spaces in the matrix denote a 0, but as they are needed for counting in $C$ they are included there. Horizontal reading in $D$ and $C$, reveals that $\pi_1 = S_x$, $\pi_2 = \frac{c_p T_{\text{max}}}{\sigma_y}$, $\pi_3 = \frac{\rho v^2}{\sigma_y}$, and $\pi_4 = \frac{\lambda T_{\text{max}}}{\sigma_y h v}$. This is just one of many possible dimensionless sets. By interchanging the order of the columns in the set or by selecting $D$ differently, an infinite number of dimensionless sets can be found—however, just a few of those sets will be dimensionally independent of each other. Matrix $A$ must be non-singular at all times and further, only interchanging columns between the matrices $A$ and $B$ will yield distinct sets—although this is merely a necessary and not sufficient requirement for obtaining independent sets of
Thus, for a fixed $D$ the number of distinct sets, $N_S$, is calculated as (Szirtes, 1997, Eq. (10.20)):

$$N_S = k - (U_R - \vartheta) - U_c$$  \hspace{1cm} (4.11)

The terms in (4.11) are

$$k = \binom{N_V}{N_d}$$

is the total number of possible interchanges between $A$ and $B$. $N_V$ is the number of variables and $N_d$ is the number of dimensions. Here, $N_V = 8$ and $N_d = 4$, so $k = 70$.

$$U_R = \sum_{j=1}^{N_Z,i} \binom{N_P - 1}{j - 1} \binom{N_{Z,i}}{j}$$

is the number of prohibited interchanges due to possible singularity of $A$. $N_{Z,i}$ is the number of zeroes in any row, $i$, in $C$ and $N_P$ is the number of dimensionless variables, $\bar{\pi}$. The value of $U_R$ must be summed over all the rows in $D$ which contain zeroes. As $N_P = 4$, $N_{Z,1} = 4$, $N_{Z,2} = 2$, $N_{Z,3} = 2$, and $N_{Z,4} = 0$, $U_R = 35 + 5 + 5 = 45$.

$\vartheta$ represents the number of duplicate prohibited interchanges. If for example two columns in $A$ are interchanged with two columns in $B$, this will count as two occurrences of $U_R$. However, $A$ is singular by just one of the interchanges, so that it must only count once—this is corrected by $\vartheta$. It is determined by counting zeroes: If two rows have two or more zeroes in the same columns this counts as one occurrence of $\vartheta$. In this case, the first and the third column in row one and three share zeroes, which implies one instance of $\vartheta$. Likewise, the second and the third column in row one and three share zeroes, which is another instance of $\vartheta$. Altogether, $\vartheta = 2$.

$U_c$ is the number of interchanges which yield equivalent sets. To reduce the size of the problem, an example with three rows is given: If $C$ is as shown below, $U_c = n_1 + n_2 + n_3 + n_1n_2 + n_1n_3 + n_2n_3 + n_1n_2n_3$, where $n_i$ counts the elements in row $i$ which are the only non-zero elements in its own column.

$$C = \begin{bmatrix} 100500 \\ 048000 \\ 900032 \end{bmatrix}$$

has $n_1 = 1$ (‘5’ is the only non-zero element in column 4), $n_2 = 2$ (‘4’ and ‘8’ in columns 2 and 3), and $n_3 = 2$ (columns 5 and 6). In the current set, $n_1 = 0$, $n_2 = 0$, $n_3 = 0$, and $n_4 = 1$, which means that $U_c = 1$.

Altogether, $N_S = 70 - (45 - 2) - 1 = 26$.

Performing the rather exhausting manipulation of the dimensional set to find all the sets results in a large number of apparently different dimensionless variables. However, any two
4.2 Fitting of Functions

4.2.1 Multivariate Analysis

Consider the situation where results (e.g. $S_x$) from $n$ sets of analyses or experiments must be fitted with a function of $p$ independent variables. The results in the vector, $\bar{y}$, can be expressed as a function of some dimensionless variables, $X$, with coefficients, $\bar{b}$, and a fitting residual, $\bar{e}$, as

$$
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix} =
\begin{bmatrix}
1 & X_{12} & X_{31} & \ldots & X_{1p} \\
1 & X_{22} & X_{23} & \ldots & X_{2p} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & X_{n2} & X_{n3} & \ldots & X_{np}
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix} +
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_n
\end{bmatrix}
$$

(4.12)

For example, $y_1$ contains the results from case No. 1 (the value of either $S_x$, $B_x$, $S_y$ or $B_y$), $X_{1j}$ contain the independent variables from Table 4.1 evaluated by the heating parameters.

Table 4.1: Table of parameters based on dimensionless variables.

<table>
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<th>$v$</th>
<th>$\sigma_y / \rho$</th>
<th>$\sqrt{\frac{\sigma_y}{\rho_0 v}}$</th>
<th>$T_{\text{max}} / h^2$</th>
<th>$\frac{T_{\text{max}} c_p \rho \lambda}{\sigma_y h^2}$</th>
<th>$h / v$</th>
<th>$\frac{\rho \lambda}{\sigma_y c_p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{\text{max}}$</td>
<td>$\frac{T_{\text{max}} \sigma_y \rho \lambda}{\sigma_y h^2}$</td>
<td>$\frac{T_{\text{max}} c_p \rho \lambda}{\sigma_y h^2}$</td>
<td>$h / v$</td>
<td>$\frac{\rho \lambda}{\sigma_y c_p}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Dimensionless variables are dimensionally dependent if one is equal to the other raised to any power. Thus, for example $\frac{1}{v}$ and $v^2$ are equivalent properties. Further, as the present problem always concerns the same torch and the same type of material the variables relating to heat conduction and stress analysis can be neglected. This leaves the mixtures of the three heating parameters which—from a point of view of dimensional analysis—must be governing for the forming process. Eleven independent parameters consisting of $T_{\text{max}}$, $h$, and $v$ are identified from seventeen dimensionless parameters as shown in Table 4.1. If relations between the heating parameters and the linearised plastic strains exist, they must be composed of combinations of those terms.

The variables, along with $T_{\text{max}}$, $h$, and $v$, are raised to various powers (positive and negative), and the correlation between the variables and the linearised plastic strains is calculated numerically (by means of (Lee, 1998)). All in all about 100 variables are tested for correlation with the linearised plastic strains, and the variables with the numerically largest correlation are chosen for further treatment by multivariate analysis.
Chapter 4. Empirical Relations

### Figure 4.2: Graph and coefficients for $S_x$.

<table>
<thead>
<tr>
<th>$S_x$</th>
<th>$b$</th>
<th>$X$</th>
<th>Corr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$4.37 \cdot 10^{-5}$</td>
<td>$1$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$-7.87 \cdot 10^{-9}$</td>
<td>$\sqrt{\frac{1}{h^2}}$</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>$-3.37 \cdot 10^{-6}$</td>
<td>$\sqrt{\frac{1}{hv}}$</td>
<td>-0.74</td>
</tr>
<tr>
<td></td>
<td>$2.58 \cdot 10^{-13}$</td>
<td>$\frac{T_{\text{max}}}{h^2}$</td>
<td>-0.54</td>
</tr>
<tr>
<td></td>
<td>$-2.36 \cdot 10^{-8}$</td>
<td>$T_{\text{max}}$</td>
<td>-0.90</td>
</tr>
<tr>
<td></td>
<td>$8.62 \cdot 10^{-5}$</td>
<td>$\sqrt{\frac{1}{h^2}}$</td>
<td>-0.79</td>
</tr>
<tr>
<td></td>
<td>$1.07 \cdot 10^{-13}$</td>
<td>$T_{\text{max}} h^2$</td>
<td>-0.88</td>
</tr>
</tbody>
</table>

### Figure 4.3: Graph and coefficients for $B_x$.

<table>
<thead>
<tr>
<th>$B_x$</th>
<th>$b$</th>
<th>$X$</th>
<th>Corr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-3.71 \cdot 10^{-4}$</td>
<td>$1$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$-4.23 \cdot 10^{-6}$</td>
<td>$\sqrt{\frac{1}{h^2}}$</td>
<td>-0.88</td>
</tr>
<tr>
<td></td>
<td>$1.49 \cdot 10^{-4}$</td>
<td>$\sqrt{\frac{1}{hv}}$</td>
<td>-0.88</td>
</tr>
<tr>
<td></td>
<td>$-2.93 \cdot 10^{-7}$</td>
<td>$\frac{T_{\text{max}}}{h^2}$</td>
<td>-0.85</td>
</tr>
<tr>
<td></td>
<td>$-1.01 \cdot 10^{-14}$</td>
<td>$\frac{T_{\text{max}}}{h^2}$</td>
<td>-0.82</td>
</tr>
<tr>
<td></td>
<td>$7.56 \cdot 10^{-8}$</td>
<td>$(T_{\text{max}} - 300)^2 v$</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>$3.83 \cdot 10^{-4}$</td>
<td>$T_{\text{max}} h^2$</td>
<td>0.85</td>
</tr>
</tbody>
</table>

### Figure 4.4: Graph and coefficients for $S_y$.

<table>
<thead>
<tr>
<th>$S_y$</th>
<th>$b$</th>
<th>$X$</th>
<th>Corr.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1.34 \cdot 10^{-4}$</td>
<td>$1$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$-2.02 \cdot 10^{-10}$</td>
<td>$\frac{T_{\text{max}}}{h^2}$</td>
<td>-0.87</td>
</tr>
<tr>
<td></td>
<td>$2.52 \cdot 10^{-5}$</td>
<td>$T_{\text{max}} v$</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>$8.39 \cdot 10^{-7}$</td>
<td>$\sqrt{\frac{T_{\text{max}}}{h^2}}$</td>
<td>-0.84</td>
</tr>
<tr>
<td></td>
<td>$-5.78 \cdot 10^{-6}$</td>
<td>$\frac{1}{h}$</td>
<td>-0.60</td>
</tr>
<tr>
<td></td>
<td>$-1.76 \cdot 10^{-6}$</td>
<td>$T_{\text{max}}$</td>
<td>-0.54</td>
</tr>
</tbody>
</table>
4.2 Fitting of Functions

$T_{\text{max}}$, $h$, and $v$, according to case No. 1. The coefficients, $\bar{b}$, can then be found from the ordinary least squares method:

$$(X^T X)\bar{b} = X^T \bar{y} \quad (4.13)$$

If $(X^T X)^{-1}$ exists, $\bar{b}$ is expressed as

$$\bar{b} = (X^T X)^{-1} X^T \bar{y} \quad (4.14)$$

The terms from Table 4.1 are tested for correlation with the linearised strains, and by altering the terms and coefficients of the fitting polynomials, functions with as few terms as possible are found. By inspecting the fit quality, an assessment of whether the dependence on the variables is too strong or too weak is made. Thus, a term like $(T_{\text{max}} - 300)^2$ in Figure 4.3 seems appropriate, and even though some of the terms in the polynomials have small correlation values, they prove to be essential to obtain a proper fit. Figures 4.2 to 4.5 show the polynomials, $X$, and their coefficients, $\bar{b}$, for each of $S_x$, $B_x$, $S_y$, and $B_y$ along with bar charts which evaluate the fitted polynomials and the linearised strains. As seen, the fitted polynomials yield a fairly precise representation of the data from cases 1 to 27, which indicates that the dimensionless parameters found in Section 4.1 are governing for the physical problem.

### 4.2.2 Interpolation

To spare the effort of performing a dimensional analysis followed by multivariate analysis another method is available. Instead of fitting a polynomial to the data set, parabolic interpolation through the data points is applicable.
The data set comes from 3x3x3 variations of the heating parameters, which means that there are three results in each ‘direction’. Therefore, a parabolic interpolation between the data can be used, and for simplicity the shape function of a twenty-noded element is adopted.

With reference to Figure 4.6 the s-direction corresponds to values of velocity, v, the t-direction corresponds to thicknesses, h, and the r-direction corresponds to different temperatures, $T_{\text{max}}$. The ‘element nodes’ are numbered ‘i, j, k,…, z, a, b’ and each of the test cases are numbered from 1 to 27. It should be noted that some of the results are discarded, as there are only 20 ‘nodes’ for results. Those are Nos. 5, 11, 13, 14, 15, 17, and 23. The rest of the cases correspond to the node numbering as shown in Table 4.2 and Figure 4.6.

The benefit of using this method is that the interpolation is exact at the points where data is given (as opposed to MVA). Only 20 nodes (results) are used, which is both a benefit and a drawback, as fewer simulations are required to obtain data for interpolations, at the cost of precision. Further, the method is less general than MVA as it cannot use more (or

---

**Table 4.2: Corresponding element nodes and test cases.**

<table>
<thead>
<tr>
<th>Node</th>
<th>Case No.</th>
<th>Node</th>
<th>Case No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>1</td>
<td>s</td>
<td>6</td>
</tr>
<tr>
<td>j</td>
<td>7</td>
<td>t</td>
<td>2</td>
</tr>
<tr>
<td>k</td>
<td>9</td>
<td>u</td>
<td>22</td>
</tr>
<tr>
<td>l</td>
<td>3</td>
<td>v</td>
<td>26</td>
</tr>
<tr>
<td>m</td>
<td>19</td>
<td>w</td>
<td>24</td>
</tr>
<tr>
<td>n</td>
<td>25</td>
<td>x</td>
<td>20</td>
</tr>
<tr>
<td>o</td>
<td>27</td>
<td>y</td>
<td>10</td>
</tr>
<tr>
<td>p</td>
<td>21</td>
<td>z</td>
<td>16</td>
</tr>
<tr>
<td>q</td>
<td>4</td>
<td>a</td>
<td>18</td>
</tr>
<tr>
<td>r</td>
<td>8</td>
<td>b</td>
<td>12</td>
</tr>
</tbody>
</table>
4.2 Fitting of Functions

Figure 4.7: Comparison of FEM results and interpolation fit for $S_x$.

Figure 4.8: Comparison of FEM results and interpolation fit for $B_x$.

Figure 4.9: Comparison of FEM results and interpolation fit for $S_y$.

Figure 4.10: Comparison of FEM results and interpolation fit for $B_y$. 
Written in terms of case numbers, heating parameters, and $S_x$ as an example this becomes

\begin{align*}
S_{x, \text{interpolated}} &= (S_{x,1}(1 - \hat{v})(1 - h)(1 - \hat{T}_{\text{max}})(-\hat{v} - \hat{h} - \hat{T}_{\text{max}} - 2) \\
&+ S_{x,7}(1 + \hat{v})(1 - h)(1 - \hat{T}_{\text{max}})(\hat{v} - \hat{h} - \hat{T}_{\text{max}} - 2) \\
&+ S_{x,9}(1 + \hat{v})(1 + h)(1 - \hat{T}_{\text{max}})(\hat{v} + \hat{h} - \hat{T}_{\text{max}} - 2) \\
&+ S_{x,3}(1 - \hat{v})(1 + h)(1 - \hat{T}_{\text{max}})(-\hat{v} + \hat{h} - \hat{T}_{\text{max}} - 2) \\
&+ S_{x,10}(1 - \hat{v})(1 - h)(1 + \hat{T}_{\text{max}})(-\hat{v} - \hat{h} + \hat{T}_{\text{max}} - 2) \\
&+ S_{x,25}(1 + \hat{v})(1 - h)(1 + \hat{T}_{\text{max}})(\hat{v} - \hat{h} + \hat{T}_{\text{max}} - 2) \\
&+ S_{x,27}(1 + \hat{v})(1 + h)(1 + \hat{T}_{\text{max}})(\hat{v} + \hat{h} + \hat{T}_{\text{max}} - 2) \\
&+ S_{x,21}(1 - \hat{v})(1 + h)(1 + \hat{T}_{\text{max}})(-\hat{v} + \hat{h} + \hat{T}_{\text{max}} - 2)) / 8 \\
&+ (S_{x,4}(1 - \hat{v}^2)(1 - \hat{h})(1 - \hat{T}_{\text{max}}) + S_{x,8}(1 + \hat{v})(1 - \hat{T}_{\text{max}}) \\
&+ S_{x,6}(1 - \hat{v}^2)(1 + \hat{h})(1 - \hat{T}_{\text{max}}) + S_{x,2}(1 - \hat{v})(1 - \hat{h}^2)(1 - \hat{T}_{\text{max}}) \\
&+ S_{x,22}(1 - \hat{v}^2)(1 - \hat{h})(1 + \hat{T}_{\text{max}}) + S_{x,26}(1 + \hat{v})(1 - \hat{h}^2)(1 + \hat{T}_{\text{max}}) \\
&+ S_{x,24}(1 - \hat{v}^2)(1 + \hat{h})(1 + \hat{T}_{\text{max}}) + S_{x,20}(1 - \hat{v})(1 - \hat{h}^2)(1 + \hat{T}_{\text{max}}) \\
&+ S_{x,10}(1 - \hat{v})(1 - \hat{h})(1 - \hat{T}_{\text{max}}^2) + S_{x,16}(1 + \hat{v})(1 - \hat{h})(1 - \hat{T}_{\text{max}}^2) \\
&+ S_{x,18}(1 + \hat{v})(1 + \hat{h})(1 - \hat{T}_{\text{max}}^2) + S_{x,12}(1 - \hat{v})(1 + \hat{h})(1 - \hat{T}_{\text{max}}^2)) / 4
\end{align*}

Eq. (4.15) gives the interpolation function in general terms, where $r$, $s$, and $t$ are natural coordinates in the range $[-1;1]$ and $u_{i,j,...,b}$ are the values of either $S_x$, $B_x$, $S_y$, or $B_y$ in each of the test cases.
where \( \hat{T}_{\text{max}} \), \( \hat{h} \), and \( \hat{v} \) are natural coordinate versions of \( T_{\text{max}} \), \( h \), and \( v \):

\[
\hat{T}_{\text{max}} = \frac{T_{\text{max}} - 500}{200} \cdot 2 - 1 \quad \hat{T}_{\text{max}} \in [-1; 1]
\]

\[
\hat{h} = \frac{h - 0.010}{0.010} \cdot 2 - 1 \quad \hat{h} \in [-1; 1]
\]

\[
\hat{v} = \frac{v - 0.005}{0.010} \cdot 2 - 1 \quad \hat{v} \in [-1; 1]
\]

The results of this interpolation are shown in Figures 4.7 to 4.10. By comparison of MVA and parabolic interpolation it is seen that the latter reproduces the fitted data more precisely. In the next section, it is investigated whether one of the methods predicts the results for yet unknown heating parameters better than the other.

### 4.3 Prediction and Comments

Additional cases are carried out to test the ability of the polynomials to predict—by interpolation—the plastic strains for yet uncalculated heating parameters. The extra cases are the same as in Table 3.4, which is shown again in Table 4.3 for convenience.

Figures 4.11 and 4.12 show the predictions along with actual calculations. Generally, the fit is acceptable, except for the value predicted by MVA for case No. 5, which is 54% too large.

It may be concluded that the variation with input parameters is sufficiently consistent (smooth) so that interpolation in a database is possible.

Looking at the results used for linearisation in Appendix B, reveals that some of the cases have not reached a steady level in the centre of the heating path where the strains are evaluated. As a verification of this, the strains from the individual cases are discussed on the basis of the graphs in the appendix. As shown in Figure 3.15, the strains are evaluated in the centre of the heating path \((x=0)\), so that it is crucial that the strains have in fact reached the quasi-steady state in that position. According to the convergence tests (figures 3.11 and 3.12), this should be the case. However, when the longitudinal distribution of strains in Figures B.1 to B.27 (the upper right pane) is considered this is sometimes not the case. Especially, cases No. 10 and 19 show a large variation along the heated line, and as

<table>
<thead>
<tr>
<th>Case No.</th>
<th>( v ) [mm/s]</th>
<th>( h ) [mm]</th>
<th>( T_{\text{max}} ) [(^\circ)C]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>12.5</td>
<td>1</td>
<td>550</td>
</tr>
<tr>
<td>7.5</td>
<td>12.5</td>
<td>2</td>
<td>650</td>
</tr>
<tr>
<td>7.5</td>
<td>12.5</td>
<td>3</td>
<td>650</td>
</tr>
<tr>
<td>7.5</td>
<td>12.5</td>
<td>4</td>
<td>650</td>
</tr>
</tbody>
</table>

Table 4.3: Heating parameters for additional test cases.
both cases have low velocity it suggests that velocity is the cause of the problems. This is further supported by the lack of a quasi-steady state in cases 28 to 32, which all have very low velocities. At low velocities, the heat will accumulate in front of the torch and lead to a slowly increasing (i.e. not constant) temperature.

Another possible cause of the deviation is that the process of linearisation is questionable. When the plastic strain penetration is small—which is the case at high velocities and of thick plates—the plastic strain distribution is poorly represented by a straight line as seen in e.g. Figure B.6. However, in Section 3.6 the linearised strains are applied to elastic analyses which yield deformation in good accordance with those from the plastic analyses, so this seems not to be a severe problem.

In conclusion, slightly longer heating paths must be employed in future simulations if reliable data should be deducted, and results from linearisation of low velocity cases should be used with care.
Chapter 5

Heating Line Generation

In the previous chapters, it is investigated how the response to heating in a certain position can be simulated, both by use of direct elasto-plastic and simplified elastic analyses. However, calculating the result of heating is of little use if it is not known where to heat the plate. Therefore, a scheme for the prediction of heating patterns and parameters is proposed in the following.

A straightforward method is to use an optimisation procedure, in which the artificial temperature field analysis (ATFA) from Section 3.6 is employed to vary the position and the strength of the heating lines. However, this proves to converge poorly, other methods are suggested in the following.

Two methods for predicting the heating patterns are investigated. One is based on information on the principal bending directions of the target surface by use of differential geometry, and the other is based on the principal membrane strain directions from ‘mapping’ the target surface onto a flat surface. The two methods are not mutually exclusive, as common practice in manual line heating is to obtain bending first (by fast source movement) followed by shrinkage (by slower movement).

5.1 Determining Heating Paths

As suggested by Ueda et al. (1994a), the heating paths may be determined from information on the principal compressive strain direction: Heating along a line will often result in more bending in a direction perpendicular to the heating path than along the heating path, see Table 3.5 p. 38.

There is a mixture of both bending and shrinkage in a heated region, but when the heating paths are traced, a choice must be made between principal directions from either bending or
shrinkage separately, as the information comes from two different methods which can hardly be combined. Instead they can be applied in succession, so that the lines for bending are determined first followed by of the shrinkage lines.

5.1.1 Bending Paths

If the target surface is given as a function (e.g. a spline surface), the principal bending vectors, \( \bar{v}_{1,2} \), can be calculated from the eigenvalue problem stated in standard differential geometry literature (e.g. Lipschutz (1969)):

\[
\begin{bmatrix} L & M \\ M & N \end{bmatrix} - \kappa \begin{bmatrix} E & F \\ F & G \end{bmatrix} = 0
\] (5.1)

where \( \kappa \) is the eigenvalue (principal curvature), and \( \{E, F, G\} \) and \( \{L, M, N\} \) are the coefficients of the first and second fundamental forms, I and II, respectively.

Once the eigenvalues (principal curvatures) have been established, the corresponding eigenvectors (principal curvature directions) can be calculated from

\[
\begin{bmatrix} L & M \\ M & N \end{bmatrix} - \kappa_1 \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (5.2)

where \( \kappa_1 \) is chosen to be the numerically largest principal curvature and \( \bar{v} \) is the corresponding eigenvector. As this system of equations is singular (the determinant of the coefficient matrix is equal to zero (5.1)) is reduces to

\[(L - \kappa_1 E)v_1 + (M - \kappa_1 F)v_2 = 0 \quad \text{or} \]
\[
\bar{v}_1 = s \left( - \frac{M - \kappa_1 F}{L - \kappa_1 E}, 1 \right), \quad s \in \mathbb{R}
\] (5.4)

If, for instance, the surface, \( \bar{x} \), is given as a function of \( x \) and \( y \) as \( \bar{x} = x\bar{e}_1 + y\bar{e}_2 + f(x, y)\bar{e}_3 \) then the coefficients of the first and second fundamental form are simply

\[
E = 1 + (f_{xx})^2
\]
\[
F = f_{xx} f_{xy}
\]
\[
G = 1 + (f_{yy})^2
\]
\[
L = \frac{f_{xx}}{\sqrt{1 + (f_{xx})^2 + (f_{yy})^2}}
\]
\[
M = \frac{f_{xy}}{\sqrt{1 + (f_{xx})^2 + (f_{yy})^2}}
\]
\[
N = \frac{f_{yy}}{\sqrt{1 + (f_{xx})^2 + (f_{yy})^2}}
\] (5.5)
where \( \frac{d}{dx} \) denotes differentiation with respect to \( x \).

When the vector field for the principal directions of curvature has been found, it must be traced into heating lines. As mentioned, heating produces more bending perpendicularly to the heating direction than along it, and therefore the bending paths can be traced perpendicularly to the field. It can be carried out as outlined in the next section, with the addition that the sign of the principal curvature (positive or negative) suggests whether one or the other side should be heated.

### 5.1.2 Shrinkage Path

According to Table 3.5 of \( S_y/S_x \) ratios, the largest shrinkage direction is not necessarily perpendicular to the heating path. On the contrary, the heating path must be along the principal membrane direction at low temperatures and perpendicular to it at high temperatures. The principle of tracing the lines is, however, the same and therefore only the high temperature (perpendicular) case is dealt with here.

As mentioned in Chapter 2, there are various methods for mapping the curved plate onto a plane. Despite the local nature of the deformations induced by line heating, not knowing the position of the lines in advance implies the use of averaging (not local) mapping methods. Differential geometry can be used, provided that a function for the target surface is known, or elastic finite element analysis can be applied as follows.

To compute the in-plane strains, a model of a flat plate can be forced into shape by means of specified \( z \)-direction (out-of-plane) deflections, which means that the nodes will move only in this direction and not in the \( x \)- or \( y \)-direction. However, for deflections larger than the magnitude of the plate thickness, this results in too large membrane strains, so that this non-linear geometric effect must be included. This will—on the other hand—introduce deflections in the \( x \)- and \( y \)-directions of the nodes. Thus, the specified \( z \)-deflections become slightly wrong as the fixed \( z \)-deflection is now specified for slightly altered \( x \)- and \( y \)-coordinates. This dilemma can be solved by reversing the process. If the forced deformation is applied to an originally \emph{curved} plate, the plate is guaranteed to become perfectly flat. Even though there is an in-plane deflection the required \( z \)-deflection of a node does not change—the value must still be zero. It should be noted, that the found membrane strains are now the negative of the ones required.

From the strain field of the elastic analysis the principal direction, \( \theta_p \), and the principal strains, \( \varepsilon_{1,2} \), can be computed from the \( x \)- and \( y \) components of the strain field, \( \varepsilon_x, \varepsilon_y \), and \( \varepsilon_{xy} \). For convenience, \( \varepsilon_1 \) is the most negative (compressive) of the two. As information on \( z \) components of the strain is not needed and this is assumed to be zero the principal direction
is found from the plane strain formulation:

\[
\theta_p = \frac{1}{2} \arctan\left( \frac{\epsilon_{xy}}{\epsilon_x - \epsilon_y} \right) \tag{5.6}
\]

\[
\epsilon_{1,2} = \frac{1}{2} \left( \epsilon_x + \epsilon_y \pm \sqrt{(\epsilon_x - \epsilon_y)^2 + \epsilon_{xy}^2} \right) \tag{5.7}
\]

Eq. (5.6) may be the direction of either of the two perpendicular principal directions in the plane. As the direction for \( \epsilon_1 \) is needed, \( \epsilon_1 \) and \( \epsilon_2 \) can be compared with (5.8) to find the principal strains based on the just found angle, \( \theta_p \), and thus establish which of the strains belongs to the direction:

\[
\epsilon_{\text{test}} = \frac{1}{2} \left( (\epsilon_x + \epsilon_y) + (\epsilon_x - \epsilon_y) \cos 2\theta_p + \epsilon_{xy} \sin 2\theta_p \right) \tag{5.8}
\]

If the found \( \epsilon_{\text{test}} \) is equal to \( \epsilon_1 \) the angle, \( \theta_p \), used in (5.8) was in fact the wanted. Otherwise it belongs to the other principal strain, and \( \frac{\pi}{2} \) may be added due to the nature of ‘arctan’ used in (5.6).

This evaluation of the principal strains and angles is made in all nodes of the finite element model and an addition of 90° is made to find the heating line directions field. A visualisation of the directions and shrinkage magnitudes calculated by the shrinkage method for a parabolic plate \( (z^2 = a(x^2 + y^2)) \) is presented in Figure 5.1. The length of the arrows represents the strain magnitude, and a threshold is applied so that only the areas with the 30% largest membrane strains are designated for heating.

Before tracing the strain directions into actual heating paths, a requirement for the distance between the heating lines must be determined. Since a heating line can only produce a certain amount of bending or shrinkage the distance can be based on an inverse proportionality between required strain and distance. If much shrinkage or bending is required, the distance between the lines should be small. This can be expressed in the following way: The displacement from the heating line (that is shrinkage strain, \( S_y \), multiplied by zone width, \( w^p \)) should be equal to the elastic analysis strain, \( \epsilon_1 \), multiplied by an unknown width, \( d_L \), which is the distance between the lines of interest:

\[
d_L = \frac{S_y}{\epsilon_1} w^p \tag{5.9}
\]

Finally, the heating lines can be traced. Figure 5.2 shows the vector field converted into heating lines as calculated by ANSYS with threshold and distance criterion applied. This is the cause of the pattern of alternating short and long heating lines as the distance should increase towards the middle of the plate. The edges of the plate are not perfectly straight, which suggests how the plate should be cut prior to heat treatment. The heating pattern agrees well with e.g. Ueda et al. (1994b, p. 246).

Even though this method has not yet been verified by experiments, it is a first step towards a rational way to determine the position of the heating lines, and as such it is not important whether it is perfectly correct, as it can be amended according to further investigations. This is certainly an interesting area to be dealt with in the future.
5.2 Future Work

Besides the refinement of the above methods, a few things are needed to accomplish the task of automatically deriving line heating information.

5.2.1 Amount of Heating

A method is still lacking for finding the velocity and temperature corresponding to the heating paths above. This can be done by a soft zone method suggested in Ueda et al. (1994a):

The zones at the heating lines are assigned a Young’s modulus thousand times smaller than the surrounding material in an elastic numerical model. Thus, the strains will be

Figure 5.1: Heating line directions and magnitudes.

Figure 5.2: Heating pattern on parabolic plate as calculated by ANSYS.
concentrated in the soft zones and yield information on how large strains each heating line must produce. Subsequently, a comparison to a database (from experiments or numerical analyses) will reveal which parameters produce the strains in the soft zones.

Again, this is a mere suggestion as it needs further development. However, it seems feasible as the method of Ueda et al. is used (with modifications) by the IHI shipyard in Japan.

5.2.2 Final Check by ATFA

Once the heating parameters (temperature and velocity) have been determined for each of the heating line candidates, it can be checked by means of the combination of artificial temperature field analysis (ATFA as described in Section 3.6) and the data from Table 3.3 p. 32, whether the target shape was obtained to a satisfactory degree of accuracy. If not, a means of reiteration of the above procedure must be devised. Otherwise, the data can be sent to immediate processing in the workshop.
Chapter 6

Experiments

A number of experiments are carried out to establish a basis for validation of the assumed heat flux distribution and of the deformations calculated numerically. All experiments are done at Seoul National University, where a numerically controlled gantry crane equipped with an oxyacetylene torch can move about in a given pattern. Along with the torch are also a contact-type measuring probe, a water cooling system and a magnetic distance controller. The torch is a type for gas cutting (\(<YS>\) 3051 No. 7). A photograph of the machine is shown in Figure 6.1. Temperatures are measured with thermo couples connected to a Hewlett-Packard 34970A data acquisition unit with a 16-channel scanner for thermocouples. The thermocouples are created by attaching thermocouple extension cable to the plate by means of resistance welding (i.e. by discharging a large electrical capacitor). For details, see Appendix D.

6.1 Calibration of Thermocouples

First, it must be ensured that the thermocouples are calibrated after the crude assembly, where the welding process may introduce impurities and thus cause the thermocouples to be inaccurate. A plate is prepared with 14 thermocouples arranged along an 11 cm long line on the bottom side of a plate. To produce an even temperature as a basis of the calibration, the plate is insulated with mineral wool and then heated approximately 30 cm from the thermo couples. A photograph of the insulated plate is shown in Figure 6.2. The plate is heated for half an hour so it reaches a temperature of 200 °C. Then the plate—still being insulated—is allowed to cool down. Figure 6.3 shows the time history of the temperatures, and during cooling they are quite the same. After almost 2 hours the insulation is removed from the upper side. Figure 6.4 shows the error, \(e_j^T\), at each channel, \(j\), as a function of the average temperature, \(\bar{T}\), over all the channels.

\[ e_j^T = \frac{T_j - \bar{T}}{\bar{T}} \quad (6.1) \]
Figure 6.1: *Heating and measuring machine, iCALM, at Seoul National University.*

Figure 6.2: *Plate insulation for calibration. The thermocouples are under the piece of mineral wool to the left on the bottom side and the cables.*
6.1 Calibration of Termocouples

Figure 6.3: Temperature as a function of time during calibration.

Figure 6.4: Temperature percentage deviation as a function of average temperature. 1 hour between the dots.
where $T_j$ is the temperature measured in the $j^{\text{th}}$ channel. As seen in Figure 6.4, the temperature error is smaller than ±2% half an hour after the heating has ceased. This error may very well be due to temperature differences in the plate despite the insulation.

This leads to the conclusion that the same error (if any) is made when all thermocouples are welded. Although a reference temperature is not measured the relative differences are small, so it is most likely that the measurement of temperatures is precise. According to the specifications for the acquisition unit, the instrument error alone is smaller than 1 °C.

### 6.2 Heat Flux from Gas Torch

#### 6.2.1 Experimental Set-up

The first set of experiments is designed to evaluate the true heat flux distribution from a gas torch. The idea is to heat a plate with a stationary torch and measure the temperature profile at a given time, after which the profile is compared to FE simulations to establish the corresponding heat flux distribution. The same plate as during calibration is used. It is 11.5 mm thick and the thermocouples are positioned at the distances from the centre of the torch shown in Table 6.1 (two extra channels are added since the calibration). Again, the thermocouples are on the opposite side of the plate, as the wires would otherwise melt.

A total of 12 experiments is carried out, in which the torch clearance is varied between 40, 50, 70, and 100 mm with three repetitions each. The temperature profiles for a temperature increase of 400 °C are shown in Figure 6.5. The curves represent the rise in temperature from the initial state (which varies due to the cooling period between the experiments) to the time, when the maximum temperature is increased by precisely 400 °C. This time is on average 15.1, 16.3, 28.3, and 74.6 s, respectively. One of the results from the experiments with a torch clearance of 70 mm is discarded, as it does not fit the other two results.

#### 6.2.2 Temperature to Heat Flux Conversion

To find the heat flux distribution which resulted in the measured temperatures, an optimisation procedure is carried out on a FE temperature analysis. As indicated in Figure 6.6 the heat flux profile on the upper side is varied to fit the measured temperatures on the lower side. This is done with 14 heat fluxes as design variables, the temperature error at each

<table>
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<th>TC No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<td>90</td>
<td>110</td>
<td>140</td>
<td>170</td>
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*Table 6.1: Position of thermocouples from the torch centre.*
Figure 6.5: Temperature profiles.
Figure 6.6: Optimising heat flux distribution to fit measured temperatures.

Figure 6.7: Clearance 40 mm. (A) Measured and optimised temperature profiles. (B) Optimised heat flux distribution and equivalent Gaussian distribution.

Figure 6.8: Clearance 50 mm. (A) Measured and optimised temperature profiles. (B) Optimised heat flux distribution and equivalent Gaussian distribution.
6.2 Heat Flux from Gas Torch

The optimisation procedure leads to a temperature distribution for a torch clearance of 40 mm, which agrees well with the measured data, as shown in Figure 6.7(A). The flux itself is shown in Figure 6.7(B) along with a graph of a Gaussian distribution with a total heat power, $Q_{\text{tot}}$, of 3000 W and a torch width, $r_{\text{torch}}$, (as defined by (3.3) p. 12) of 4 cm. Figures 6.8(A) and (B) show the same results, but for a 50 mm clearance. The drop in the heat flux in the centre of the torch can be interpreted as a stagnation point of the torch jet flow, as the heat transfer between a fluid and a solid relates to the velocity of the fluid. A similar result is found by Tomita et al. (1998), who measured the velocity field from a much smaller torch and converted it into heat flux by means of computational fluid dynamics, see Figure 6.10.

The optimised flux distribution can be integrated, on the assumption that it is sectionally
linear (see Figure 6.9), to determine the total power absorbed by the plate.

\[
Q_{tot} = \sum_j Q_j = \sum_j \int_0^{2\pi} \int_{r_{j-1}}^{r_j} \left\{ \frac{Q''_j - Q''_{j-1}}{r_j - r_{j-1}} (r - r_{j-1}) + Q''_{j-1} \right\} r \, dr \, d\theta
\]

\[
= 2\pi \sum_j \left\{ \frac{Q''_j - Q''_{j-1}}{r_j - r_{j-1}} \int_{r_{j-1}}^{r_j} r^2 - rr_{j-1} \, dr + Q''_{j-1} \int_{r_{j-1}}^{r_j} r \, dr \right\}
\]

\[
= \pi \sum_j \left\{ \frac{Q''_j - Q''_{j-1}}{r_j - r_{j-1}} \left( 2r_j^3 - 3r_j^2 r_{j-1} + r_{j-1}^3 \right) / 3 + Q''_{j-1} (r_j^3 - r_{j-1}^3) \right\}
\]

(6.2)

where \( Q_j \) is the total heat power in section \( j \), \( r_j \) is the \( j \)th distance from the torch centre and \( Q''_j \) is the individual heat flux from the optimisation.

By use of (6.2), the effective heat power for a clearance of 40 mm is calculated to be 2850 W and similarly for the clearance of 50 mm to be 2830 W. It is not possible to find the heat distribution from clearances of 70 and 100 mm, as the heating times are too long. Thus, the relation between heat flux distribution and temperature distribution is too diffuse for the optimisation procedure to converge. Using a shorter heating time (and hence lower temperatures) should ensure convergence for 70 and 100 mm clearances as well.

### 6.3 Validation of Structural Analysis

Another set of experiments is designed to evaluate the correctness of the numerically derived deflections in Chapter 3.

#### 6.3.1 Experimental Set-up

Heating is performed on seven plates of the dimensions 600x600 mm in two parallel lines at 1/3 and 2/3 of the width of the plate, see Figure 6.11. The thicknesses and torch velocities are in Table 6.2. The torch is kept at a distance of 40 mm from the plates so that the true

<table>
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<th>Plate</th>
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<td>3.0</td>
</tr>
<tr>
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<tr>
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<td>3.0</td>
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<td>13.0</td>
<td>4.0</td>
</tr>
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<td>12.8</td>
<td>5.0</td>
</tr>
<tr>
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<td>13.0</td>
<td>4.0</td>
</tr>
<tr>
<td>7</td>
<td>13.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Table 6.2: Forming experiments numbering.
6.3 Validation of Structural Analysis

Figure 6.11: Experiment layout.

Figure 6.12: Measuring points. All figures in [mm].

Figure 6.13: Measured deflections from (A) experiment 2 and (B) experiment 3. Not to scale!
heat flux distribution is known. Before and after forming a plate, it is measured with the contact-type probe, so that relative deflections at each point are available. Measurements are carried out at the points indicated in Figure 6.12.

Unfortunately, only experiment No. 3 (the thinnest plate) yields deflections large enough to be measured by the contact probe. All the other plates deform less than the accuracy of the measuring unit—compare Figures 6.13(A) and (B). The deformation of the plate from experiment 3 is shown in Table 6.3.

### 6.3.2 Numerical Model and Comparison

A model similar to experiment No. 3 is made in ANSYS. It consists of 28740 elements and 34334 nodes (see Figure 6.14) and is thus very large compared to the test cases in Chapter 3. It takes about a week to solve the problem on the same computer as the test cases.
6.3 Validation of Structural Analysis

Figure 6.15: Measured and simulated temperature profiles.

Figure 6.16: Contour plots of numerical simulation results and experimental data.
Temperatures are measured on the bottom side during the experiments, and a comparison between those and simulated temperatures is given in Figure 6.15. The (A) graph shows the temperature as a function of time at a fixed point on the bottom side—both measured and simulated. The (B) graph shows the temperature profile perpendicular to the heating path at the time of maximum temperature on the bottom side. There is good agreement between the measured and simulated temperatures, although the simulated plate cools a little too quickly (either due to surface heat loss or conduction). The agreement between simulated and measured deflection described below means that the difference in cooling rate does not influence the results. According to the simulation, the maximum temperature on the heated side, $T_{max}$, is 635 °C.

The deformations calculated by an elasto-plastic analysis prove mainly to be governed by pure bending about the heating lines, as this geometrically linear analysis does not account for the influence of the membrane stresses on the out-of-plane deflections. Including non-linear geometric effects allows for this coupling between membrane and bending modes, but including the non-linear effects directly in the elasto-plastic problem will make the problem insurmountable for the available computers. To solve this problem, the plastic strains are applied to an artificial temperature field analysis (as described in Section 3.6), which is so much simpler that the non-linear effects can be handled easily. The strain distribution from the elasto-plastic analysis and the linearised strains are presented in Figure B.33 in the Appendix.

For comparison of the two surfaces from experiment 3 and the numerical analysis, a contour plot is made in Figure 6.16. It shows the plate from above, with contours representing the out-of-plane deflections of the plate—each contour is 0.75 mm apart. The two deformation surfaces are aligned (rotated and translated) to account for rigid body motions. The measured deformations are the short lines (as the measuring grid does not extend to the edges of the plate) and the numerical results are the long lines. It is seen that the results from the experiment and the numerical analyses are very similar indeed. On the left half of the plot, the contours are nicely aligned. On the right half, the contours do not fit as well due to a twist of the plate in the experiment and stronger bending along the heating line. However, the results are all in all very encouraging.

Though only one deformation experiment was successful, it demonstrates that the numerical model is capable of simulating the actual behaviour of a steel plate being heated.
Chapter 7

Conclusions

A number of investigations are carried out during the research on line heating. The conclusions reached are as described below.

- A numerical finite element model for the investigation of the behaviour of a plate subjected to local heating is built on the following assumptions/restrictions:
  - The material is isotropic and without residual stresses
  - Plasticity is modelled by an isotropic hardening law
  - Heat loss by convection and radiation are included
  - The temperatures are lower than 700 °C (i.e. no phase transition of the steel).

- A parameter study is carried out with this model to obtain relations between heating parameters and resulting strains. The results of the parameters study are further analysed to find a physical explanation of the mechanisms of the process.

- The strain field caused by the heat treatment can be linearised to consist of bending and shrinkage contributions along and perpendicularly to the heating path. Hence, the strain field can be described by four parameters only.

- Empirical relations between the heating parameters and the linearised strains are derived. This gives continuous functions which can be used to predict the plastic strains for heating parameters not yet calculated. The prediction is fairly accurate as seen by comparing with direct calculations.

- A method for simplified elastic analysis which yields results equivalent to the fully elastoplastic analysis is developed. This can be used for fast prediction of the deformations of a plate of any shape subjected to given plastic strains.

- A sensitivity analysis shows that the plastic strains are very sensitive to changes in materials properties and torch modelling.
• The effects of heating near an edge are investigated. The conclusion is that the plastic strains are almost unaffected by the edge if heating is carried out from the edge towards the interior of the plate.

• A rational method for predicting the position of the heating lines is proposed and implemented. As yet, the method is very simple but being rational it provides a basis for further development/refinement so that it may take account for residual stresses in the plate and anisotropy.

• A method of ‘soft zones’ which can convert the information on heating line positions into heating parameters is suggested.

• Experiments to evaluate the validity of the numerical simulations and to find the true heat flux distribution from a gas torch are carried out. The findings are that the numerical simulations—despite the assumptions—accurately find the same deformations as measured during the experiments. Further, it turns out to be possible to establish the flux distribution from measured temperatures on a plate.
References


Tvergaard, V., Effect of Plasticity on Post-Buckling Behaviour, Notes for CISM and INTIA lectures, Department of Solid Mechanics, Technical University of Denmark, 1985.


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Appendix A

Differential Equations for Plastic v. Kármán Plates

In this chapter the differential equations of a plate with elastic-plastic materials properties are derived. The kinematics of the plate is based on the v. Kármán plate theory, and the behaviour of the material is based on the $J_2$ flow theory with isotropic hardening. The goal is to develop an efficient finite difference formulation to be used for the simulation of line heating.

Consider an infinitely small patch of a thin plate as depicted in Figure A.1. The patch subjected to forces and moments must be in equilibrium, which is ensured by means of force and moment projections. The following derivation is based on the v. Kármán plate theory as described in Timoshenko & Woinowsky-Krieger (1959) and Pedersen & Jensen (1983). In the following, $u_\alpha$ denote in-plane deflections and $w$ denotes out-of-plane deflection. Elastic-plastic constitutive relations are from Tvergaard (1997) and partly from Tvergaard (1985).

Figure A.1: Definition of forces and moments on an infinitesimally small patch.
A.1 Equilibrium Equations

First all forces as defined in Figure A.1 are projected onto the main axes, $x_1$, $x_2$ and $x_3$. As regards projection onto the $x_3$-axis, the deflection of the patch has to be considered, as deflections are assumed to be relatively large.

\[ x_1: \quad \frac{\partial N_{11}}{\partial x_1} dx_1 dx_2 + \frac{\partial N_{21}}{\partial x_2} dx_2 dx_1 = 0 \quad \text{(A.1)} \]
\[
\begin{align*}
x_2: & \quad \frac{\partial N_{22}}{\partial x_2} dx_2 dx_1 + \frac{\partial N_{12}}{\partial x_1} dx_1 dx_2 = 0 \\
x_3: & \quad \frac{\partial N_{13}}{\partial x_1} dx_1 dx_2 + \frac{\partial N_{23}}{\partial x_2} dx_2 dx_1 + q dx_1 dx_2
\end{align*}
\]

\[ \begin{aligned}
- N_{11} \frac{\partial w}{\partial x_1} dx_2 + \left( N_{11} + \frac{\partial N_{11}}{\partial x_1} dx_1 \right) \left( \frac{\partial w}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2} dx_1 \right) dx_2 \\
- N_{22} \frac{\partial w}{\partial x_2} dx_1 + \left( N_{22} + \frac{\partial N_{22}}{\partial x_2} dx_2 \right) \left( \frac{\partial w}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2} dx_2 \right) dx_1 \\
- N_{21} \frac{\partial w}{\partial x_1} dx_1 + \left( N_{21} + \frac{\partial N_{21}}{\partial x_1} dx_1 \right) \left( \frac{\partial w}{\partial x_1} + \frac{\partial^2 w}{\partial x_1 x_1} dx_1 \right) dx_1 \\
- N_{12} \frac{\partial w}{\partial x_2} dx_2 + \left( N_{12} + \frac{\partial N_{12}}{\partial x_2} dx_2 \right) \left( \frac{\partial w}{\partial x_2} + \frac{\partial^2 w}{\partial x_1 x_2} dx_1 \right) dx_2 = 0
\end{aligned} \quad \text{(A.3)} \]

Eq. (A.1) yields

\[ \frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{21}}{\partial x_2} = 0 \quad \text{or} \quad N_{\alpha 1, \alpha} = 0 \quad \text{(A.4)} \]

Eq. (A.2) yields

\[ \frac{\partial N_{22}}{\partial x_2} + \frac{\partial N_{12}}{\partial x_1} = 0 \quad \text{or} \quad N_{\alpha 2, \alpha} = 0 \quad \text{(A.5)} \]

Eq. (A.3) yields

\[
\begin{aligned}
\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q & \\
+ \frac{\partial N_{11}}{\partial x_1} \frac{\partial w}{\partial x_1} + N_{11} \frac{\partial^2 w}{\partial x_1^2} dx_1 + \frac{\partial N_{11}}{\partial x_1} \frac{\partial^2 w}{\partial x_1^2} & \\
+ \frac{\partial N_{22}}{\partial x_2} \frac{\partial w}{\partial x_2} + N_{22} \frac{\partial^2 w}{\partial x_2^2} dx_2 + \frac{\partial N_{22}}{\partial x_2} \frac{\partial^2 w}{\partial x_2^2} & \\
+ \frac{\partial N_{21}}{\partial x_1} \frac{\partial w}{\partial x_1} + N_{21} \frac{\partial^2 w}{\partial x_1 x_1} dx_1 + \frac{\partial N_{21}}{\partial x_1} \frac{\partial^2 w}{\partial x_1 x_1} & \\
+ \frac{\partial N_{12}}{\partial x_2} \frac{\partial w}{\partial x_2} + N_{12} \frac{\partial^2 w}{\partial x_1 x_2} dx_1 + \frac{\partial N_{12}}{\partial x_2} \frac{\partial^2 w}{\partial x_1 x_2} & \\
& = 0
\end{aligned}
\]
Neglecting higher order terms gives

\[
\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q + \frac{N_{11}}{\partial x_1} + \frac{N_{11}}{\partial x_2} + \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial N_{21}}{\partial x_1} = 0
\]

\[+ N_{21} \frac{\partial^2 w}{\partial x_1 \partial x_2} + \frac{\partial N_{12}}{\partial x_1} \frac{\partial w}{\partial x_1} + N_{12} \frac{\partial^2 w}{\partial x_1 \partial x_2} = 0\]

\[
\nabla
\]

\[N_{\alpha3\alpha} + q + N_{\alpha\beta} w_{\alpha\beta} + N_{\alpha3\alpha} w_{\beta} = 0\] (A.6)

Secondly, all moments acting on the patch are taken about the main axes:

\[
x_1: \quad \frac{\partial M_{12}}{\partial x_1} dx_1 dx_2 + \frac{\partial M_{22}}{\partial x_2} dx_2 dx_1 + N_{23} dx_1 \frac{dx_2}{2} \cdot 2 + \frac{\partial N_{23}}{\partial x_2} dx_2 dx_1 \frac{dx_2}{2} = 0\] (A.7)

\[
x_2: \quad -\frac{\partial M_{21}}{\partial x_2} dx_2 dx_1 - \frac{\partial M_{11}}{\partial x_1} dx_1 dx_2 - N_{13} dx_2 \frac{dx_1}{2} \cdot 2 - \frac{\partial N_{13}}{\partial x_1} dx_1 dx_2 \frac{dx_1}{2} = 0\] (A.8)

\[
x_3: \quad 2N_{12} dx_2 \frac{dx_1}{2} \cdot \frac{\partial N_{12}}{\partial x_1} dx_1 dx_2 - 2N_{21} dx_1 \frac{dx_2}{2} - \frac{\partial N_{21}}{\partial x_2} dx_2 dx_1 \frac{dx_2}{2} = 0\] (A.9)

Eqs. (A.7), (A.8), and (A.9) leads to

\[
\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} + N_{23} + \frac{\partial N_{23}}{\partial x_2} \frac{2}{2} = 0
\]

\[
\frac{\partial M_{21}}{\partial x_2} + \frac{\partial M_{11}}{\partial x_1} + N_{13} + \frac{\partial N_{13}}{\partial x_1} \frac{2}{2} = 0
\]

\[
2N_{12} - 2N_{21} + \frac{\partial N_{12}}{\partial x_1} \frac{dx_1}{2} - \frac{\partial N_{12}}{\partial x_2} \frac{dx_2}{2} = 0
\]

Again, neglecting higher order terms gives

\[
\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} + N_{23} + \frac{\partial N_{23}}{\partial x_2} \frac{2}{2} = 0
\]

\[
\frac{\partial M_{21}}{\partial x_2} + \frac{\partial M_{11}}{\partial x_1} + N_{13} + \frac{\partial N_{13}}{\partial x_1} \frac{2}{2} = 0
\]

\[
\nabla
\]

\[M_{\alpha3\alpha} + N_{\beta3} = 0\] (A.10)

and

\[N_{12} = N_{21}\] (A.11)
Eqs. (A.4), (A.5), and (A.11) inserted in (A.6) yield
\[
\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q + \frac{\partial N_{11}}{\partial x_1} \frac{\partial w}{\partial x_1} + N_{11} \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial N_{22}}{\partial x_2} \frac{\partial w}{\partial x_2} + N_{22} \frac{\partial^2 w}{\partial x_2^2} \\
- \frac{\partial N_{11}}{\partial x_1} w + N_{12} \frac{\partial^2 w}{\partial x_1 \partial x_2} - \frac{\partial N_{22}}{\partial x_2} \frac{\partial w}{\partial x_2} + N_{12} \frac{\partial^2 w}{\partial x_1 \partial x_2} = 0
\]
\[
\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q + N_{11} \frac{\partial^2 w}{\partial x_1^2} + N_{22} \frac{\partial^2 w}{\partial x_2^2} + 2N_{12} \frac{\partial^2 w}{\partial x_1 \partial x_2} = 0
\]

or
\[N_{\alpha 3, \alpha} + q + N_{\alpha \beta w, \alpha \beta} = 0\] (A.12)

By differentiation of each of the two equations represented by (A.10) by \(x_1\) and \(x_2\), respectively, and addition of them \(M_{\alpha \beta, \alpha \beta} + N_{33, \beta} = 0\) is obtained. Insertion of this in (A.12) leads to
\[M_{\alpha \beta, \alpha \beta} = N_{\alpha \beta w, \alpha \beta} + q\] (A.13)

which is the general equation of equilibrium of a plate with relatively large deflections, i.e. the v. Kármán equations. To use this in the case of plasticity, the incremental form is needed, which gives (by differentiation with a time-like parameter):
\[\dot{M}_{\alpha \beta, \alpha \beta} = \dot{N}_{\alpha \beta w, \alpha \beta} + N_{\alpha \beta \dot{w}, \alpha \beta} + \dot{q}\] (A.14)

where \(\dot{\cdot}\) denotes differentiation with respect to the parameter.

### A.2 Strains

As the plate is subjected to large deformations, the strain measure \(\epsilon = \frac{\Delta x}{x_0}\) must be discussed in detail. On the assumption that the strain may be divided into three contributing parts, respectively, from bending \(\tilde{\epsilon}_{\alpha \beta}\), from membrane deflection \(e_{\alpha \beta}\), and from temperature \(\epsilon^T_{\alpha \beta}\) the following is obtained
\[\epsilon_{\alpha \beta} = \tilde{\epsilon}_{\alpha \beta} + e_{\alpha \beta} + \epsilon^T_{\alpha \beta}\] (A.15)

From standard plate theory using the Kirchhoff assumption, the following relation between curvature, \(w_{\alpha \beta}\), and strain is obtained:
\[\tilde{\epsilon}_{\alpha \beta} = -x_3 w_{\alpha \beta}\] (A.16)

Regarding the membrane contribution to the strain the strip of a plate deformed as in Figure A.2 can be considered. Its original length \(dx_1\) is elongated to a new length \(dl\), which
may be described in terms of the two perpendicular sides of a triangle. Thus, \( dl \) is written as

\[
dl \cong \sqrt{(dx_1 + \partial u_1/\partial x_1 dx_1)^2 + (\partial w/\partial x_1 dx_1)^2}
\]

\[
= dx_1 \sqrt{1 + \left( \frac{\partial u_1}{\partial x_1} \right)^2 + 2 \frac{\partial u_1}{\partial x_1} + \left( \frac{\partial w}{\partial x_1} \right)^2}
\]

(A.17)

Expansion by Taylor series gives

\[
dl \cong dx_1 \left\{ 1 + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + 2 \frac{\partial u_1}{\partial x_1} + \left( \frac{\partial w}{\partial x_1} \right)^2 \right] \right\}
\]

\[- \frac{1}{8} \left[ \left( \frac{\partial u_1}{\partial x_2} \right)^2 + 2 \frac{\partial u_1}{\partial x_1} + \left( \frac{\partial w}{\partial x_1} \right)^2 \right]^2 \}
\]

Neglecting second order terms except \( \partial w/\partial x_1 \) leads to

\[
dl \cong dx_1 \left( 1 + \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial w}{\partial x_1} \right)^2 \right)
\]

(A.18)
which for $e_{11}$ yields

$$
e_{11} = \frac{dx_1}{dx_1} \left( 1 + \frac{\partial u}{\partial x_1} + \frac{1}{2} \left( \frac{\partial u}{\partial x_1} \right)^2 \right) dx_1 - dx_1 = u_{1,1} + \frac{1}{2} w_{1,1}^2$$  \hspace{1cm} (A.19a)

In the same manner the strain in the $x_2$-direction gives

$$e_{22} = u_{2,2} + \frac{1}{2} w_{2,2}^2 \hspace{1cm} (A.19b)$$

As shown in (Timoshenko & Woinowsky-Krieger, 1959, p. 385), the shear strain due to bending and shearing is

$$e_{12} = \frac{1}{2} (u_{1,2} + u_{2,1} + w_{,1} w_{,2}) \hspace{1cm} (A.19c)$$

Thus, (A.19a), (A.19b) and (A.19c) may be written as

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + w_{\alpha} w_{,\beta}) \hspace{1cm} (A.20)$$

In general the strain due to temperature, $e^T_{\alpha\beta}$, may be expressed as

$$e^T_{\alpha\beta} = \delta_{\alpha\beta} \hat{\alpha} \Delta T$$

where

- $\delta_{\alpha\beta}$ is Kronecker’s delta,
- $E$ is the linear modulus of elasticity (Young’s modulus),
- $\hat{\alpha}$ is the linear coefficient of thermal expansion,
- $\Delta T$ is the difference between a reference temperature and the heated condition,
- $\nu$ is Poisson’s ratio

All in all the strain in (A.15) may be expressed as

$$\epsilon_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + w_{\alpha} w_{,\beta}) - x_3 w_{,\alpha\beta} + \delta_{\alpha\beta} \hat{\alpha} \Delta T \hspace{1cm} (A.21)$$

On incremental form this equation is

$$\dot{\epsilon}_{\alpha\beta} = \frac{1}{2} (\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} + \dot{w}_{\alpha} w_{,\beta} + w_{,\alpha} \dot{w}_{,\beta}) - x_3 \dot{w}_{,\alpha\beta} + \delta_{\alpha\beta} \left( \hat{\alpha} \Delta T + \dot{\hat{\alpha}} \right) \hspace{1cm} (A.22)$$

### A.3 Constitutive Relations

A relationship between strain and stress in the material has to be established. Using $J_2$ flow theory for the plane stress case gives

$$\sigma_{\alpha\beta} = \hat{L}_{\alpha\beta\gamma} \epsilon_{\gamma\delta} \hspace{1cm} (A.23)$$
where
\[ \hat{\sigma}_{\alpha\beta} \] denote plane stress variables (except for \( \hat{\alpha} \)),
\[ \hat{\sigma}_{\alpha\beta} \] are incremental stresses,
\[ \hat{L}_{\alpha\beta\gamma\delta} \] are plane stress elastic-plastic relations between strains and stresses,
\[ \hat{\epsilon}_{\gamma\delta} \] are incremental strains

(A.24)

The elastic-plastic relation, \( \hat{L}_{\alpha\beta\gamma\delta} \), may be expressed as the sum of the elastic and the plastic contribution. Thus,

\[ \hat{L}_{\alpha\beta\gamma\delta} = \hat{L}_{\alpha\beta\gamma\delta} - \hat{\mu} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} \]

where
\[ \hat{L}_{\alpha\beta\gamma\delta} = \frac{E}{1 - \nu^2} \left\{ \frac{1}{2} (1 - \nu) (\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) + \nu \delta_{\alpha\beta} \delta_{\gamma\delta} \right\} \]

is Hooke’s law for plane stress
\[ \hat{\mu} = \begin{cases} \frac{E}{E_t} & \text{for } \hat{m}_{\alpha\beta} \hat{\epsilon}_{\alpha\beta} \geq 0 \\ 0 & \text{for } \hat{m}_{\alpha\beta} \hat{\epsilon}_{\alpha\beta} < 0 \end{cases} \]

\[ \mu = \frac{E}{1 + \nu} - \frac{E}{E_t} - \frac{1 - 2\nu}{3} \]

\[ \hat{m}_{\alpha\beta} = m_{\alpha\beta} - m_{33} \hat{L}_{3333} \]

\[ L_{ijkl} = L_{ijkl} - \mu m_{ij} m_{kl} \]

\[ m_{ij} = \sqrt{3} s_{ij} \]

\[ \hat{L}_{ijkl} = \frac{E}{1 + \nu} \left[ \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kl} \right] \]

\[ s_{ij} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{3} \]

\[ \sigma_e = \sqrt{\frac{3}{2} s_{ij} s_{ij}} \quad \text{(assuming a v. Mises yield surface)} \]

(A.25)

\[ E_t \] is the tangent stiffness
Insertion of the strain definition (A.22) and the constitutive relations (A.23) in the equilibrium equations (A.14) still remains. This is done by using the following relations:

\[ N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} dx_3 \]  

(A.26)

simply integrating over the current stress state.

\[ \tilde{N}_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{L}_{\alpha\beta\gamma\delta} \dot{\varepsilon}_{\gamma\delta} dx_3 \]

\[ = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \tilde{L}_{\alpha\beta\gamma\delta} - \hat{\mu} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} \right) \dot{\varepsilon}_{\gamma\delta} dx_3 \]

\[ = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \tilde{L}_{\alpha\beta\gamma\delta} - \hat{\mu} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} \right] \left[ \frac{1}{2} \left( \dot{u}_{\gamma\delta} + \dot{u}_{\delta\gamma} + w_{\gamma} \dot{w}_{\delta} + w_{\delta} \dot{w}_{\gamma} \right) \right. \]

\[ - x_3 \dot{w}_{\gamma\delta} + \delta_{\gamma\delta} \left( \dot{\alpha} \Delta T + \dot{\alpha} \hat{T} \right) \right] dx_3 \]

\[ = \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} dx_3 \]

\[ = \tilde{L}_{\alpha\beta\gamma\delta} (\dot{\varepsilon}_{\gamma\delta} + \dot{\varepsilon}_{\gamma\delta}^T) h - (\dot{\varepsilon}_{\gamma\delta} + \dot{\varepsilon}_{\gamma\delta}^T) \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} dx_3 \]

which leads to

\[ \tilde{N}_{\alpha\beta} = (\dot{\varepsilon}_{\gamma\delta} + \dot{\varepsilon}_{\gamma\delta}^T) \left( h \tilde{L}_{\alpha\beta\gamma\delta} - \mathcal{M}_{\alpha\beta\gamma\delta}^{(1)} \right) - \hat{\varepsilon}_{\gamma\delta} \mathcal{M}_{\alpha\beta\gamma\delta}^{(1)} \]

(A.27)

where

\[ \mathcal{M}_{\alpha\beta\gamma\delta}^{(1)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} x_3 dx_3 \]

(A.28)

\[ M_{\alpha\beta} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{L}_{\alpha\beta\gamma\delta} \dot{\varepsilon}_{\gamma\delta} x_3 dx_3 \]

\[ = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \tilde{L}_{\alpha\beta\gamma\delta} - \hat{\mu} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} \right) (\dot{\varepsilon}_{\gamma\delta} - \dot{\varepsilon}_{\gamma\delta} + \dot{\varepsilon}_{\gamma\delta}^T) x_3 dx_3 \]

\[ = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} x_3 dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \tilde{L}_{\alpha\beta\gamma\delta} x_3 dx_3 \]

\[ + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\varepsilon}_{\gamma\delta} \hat{m}_{\alpha\beta} \hat{m}_{\gamma\delta} x_3 dx_3 \]
A.3 Constitutive Relations

\[ \dot{M}_{\alpha\beta} = \dot{w}_{\gamma\delta} \left( \frac{h^3}{12} \hat{h}_{\alpha\beta} - \hat{M}^{(2)}_{\alpha\beta\gamma\delta} \right) + \left( \dot{e}_{\gamma\delta} + \dot{e}_{T\gamma\delta} \right) \hat{M}^{(1)}_{\alpha\beta\gamma\delta} \]  

(A.29)

For insertion in (A.14), \( \dot{M}_{\alpha\beta} \) has to be differentiated with respect to \( \alpha\beta \).

Although the Poisson ratio, \( \nu \), for mild steels varies approximately 50% in the temperature range from 0 to 700°C, it is chosen to keep \( \nu \) constant. This ensures that only \( E \), \( E_t \), and \( s_{\alpha\beta} \) vary through the scope of the plate, and thus the integral of \( \dot{M}_{\alpha\beta} \) differentiated by \( \alpha\beta \) is as follows:

\[ \dot{M}_{\alpha\beta\gamma\delta} = \dot{w}_{\gamma\delta} \left( \frac{h^3}{12} \hat{h}_{\alpha\beta} - \hat{M}^{(2)}_{\alpha\beta\gamma\delta} \right) + \dot{w}_{\gamma\delta} \left( \frac{h^3}{12} \hat{h}_{\alpha\beta\gamma\delta\alpha\beta} - \hat{M}^{(2)}_{\alpha\beta\gamma\delta\alpha\beta} \right) + \left( \dot{e}_{\gamma\delta\alpha\beta} + \dot{e}_{T\gamma\delta\alpha\beta} \right) \hat{M}^{(1)}_{\alpha\beta\gamma\delta} + \left( \dot{e}_{\gamma\delta} + \dot{e}_{T\gamma\delta} \right) \hat{M}^{(1)}_{\alpha\beta\gamma\delta\alpha\beta} \]  

(A.30)

From (A.30) it is obvious that \( \hat{h}_{\alpha\beta\gamma\delta} \) and \( \hat{M}^{(i)}_{\alpha\beta\gamma\delta\alpha\beta} \) must be found. The former is simply found by differentiation:

\[ \dot{L}_{\alpha\beta\gamma\delta} = \frac{E_{\alpha\beta}}{1 - \nu^2} \left\{ \frac{1}{2} (1 - \nu) \left( \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma} \right) + \nu \delta_{\alpha\beta} \delta_{\gamma\delta} \right\} \]  

(A.31)

The latter is found by rearranging part of the integrand of \( \hat{M}^{(i)}_{\alpha\beta\gamma\delta} \) by means of MAPLE V:\n
\[ \hat{m}_{\alpha\beta\gamma\delta} = \frac{k_1}{3 (E^* - 1) (1 + \nu^*)} \left\{ \frac{k_2}{E_{\alpha\beta} - \frac{s_{33} \hat{L}_{\alpha\beta\gamma\delta} \nu^*}{1 + \nu^*}} \right\} \left\{ \frac{k_3}{E_{\gamma\delta} - \frac{s_{33} \hat{L}_{\delta\gamma\delta} \nu^*}{1 + \nu^*}} \right\} \left\{ \frac{k_4}{2 (1 + \nu)^2 \left( E_{\alpha\beta} + \frac{2 \nu - 1}{3} \right)} \right\} \left\{ \frac{k_5}{\sigma_{\epsilon}^* \frac{1 + \nu^*}{1 + \nu} - \frac{3 s_{33}^2 (E^* - 1)}{2 (1 + \nu) \left( E_{\alpha\beta} + \frac{2 \nu - 1}{3} \right)}} \right\} \]  

(A.32)

where \( \nu^* = \frac{\nu}{1 - 2\nu} \) and \( E^* = \frac{E}{E_t} \).

\(^1\)Computer program which derives algebraic solutions from user input
It is also necessary to derive $\dot{\gamma}_{\delta \alpha \beta}$, used in (A.33). These are

$$\dot{\gamma}_{\delta \alpha \beta} = \frac{1}{2} \left( \dot{u}_{\gamma \alpha \beta \delta} + \dot{u}_{\delta \alpha \beta \gamma} + \dot{w}_{\alpha \beta \gamma} w_{\gamma \delta} + \dot{w}_{\gamma \alpha \beta \gamma} + \dot{w}_{\alpha \beta \gamma} \dot{w}_{\gamma \delta} + \dot{w}_{\gamma \alpha \beta \gamma} \dot{w}_{\gamma \delta} + \dot{w}_{\gamma \alpha \beta \gamma} \dot{w}_{\gamma \delta} \right) \quad (A.34)$$

$$\dot{\epsilon}_{\gamma \delta \alpha \beta} = \delta_{\delta \beta} \left( \dot{\alpha}_{\alpha \beta} \Delta T + \dot{\alpha} T_{, \alpha \beta} + \dot{\alpha} T_{, \alpha \beta} \right) \quad (A.35)$$
Inserting (A.26), (A.27), and (A.33) along with the definitions in (A.25), (A.28), and (A.31) in the equilibrium equations (A.14) yields a fourth order differential equation in $u_\alpha$, $w$, and $T$, which is to be solved numerically.

\[ \ddot{w}_{\alpha \beta \gamma \delta} \left( \frac{h^3}{12} \hat{\mathcal{L}}_{\alpha \beta \gamma \delta} - \hat{\mathcal{M}}^{(2)}_{\alpha \beta \gamma \delta} \right) + \dot{w}_{\gamma \delta} \left( \frac{h^3}{12} \hat{\mathcal{L}}_{\alpha \beta \gamma \delta ; \alpha \beta} - \hat{\mathcal{M}}^{(2)}_{\alpha \beta \gamma \delta ; \alpha \beta} \right) 
+ (\dot{\epsilon}_{\gamma \delta} + \dot{\epsilon}_\gamma^{\mathcal{T} ; \alpha \beta} ) \hat{\mathcal{M}}^{(1)}_{\alpha \beta \gamma \delta} + (\dot{\epsilon}_{\gamma \delta} + \dot{\epsilon}_\gamma^{\mathcal{T}} ) \hat{\mathcal{M}}^{(1)}_{\alpha \beta \gamma \delta ; \alpha \beta} 
= \left[ (\dot{\epsilon}_{\gamma \delta} + \dot{\epsilon}_\gamma^{\mathcal{T}} ) \left( h\hat{\mathcal{L}}_{\alpha \beta \gamma \delta} - \hat{\mathcal{M}}^{(1)}_{\alpha \beta \gamma \delta} \right) - \dot{w}_{\gamma \delta} \hat{\mathcal{M}}^{(1)}_{\alpha \beta \gamma \delta} \right] w_{\gamma \delta} + \left[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} dx_3 \right] \dot{w}_{\alpha \beta} + \dot{q} \]  

(A.36)

As defined earlier, the strain components $\dot{\epsilon}_{\gamma \delta}$, $\dot{\epsilon}_\gamma^{\mathcal{T} ; \alpha \beta}$ are

\[ \dot{\epsilon}_{\gamma \delta} = \frac{1}{2} \left( \ddot{u}_{\gamma \delta} + \ddot{u}_{\delta ; \gamma} + \ddot{w}_{\gamma} w_{\delta} + \ddot{w}_{\gamma} w_{\delta} \right) \]  

(A.37)

\[ \dot{\epsilon}_\gamma^{\mathcal{T} ; \alpha \beta} = \delta_{\gamma \delta} \left( \dddot{\alpha} T + \dddot{\alpha} T \right) \]  

(A.38)

By use of the summation rule for the index notation, (A.36) represents one equation only, although it is very long indeed. The terms in (A.34)–(A.38) marked with dots are the only variables to solve for in each iteration of the solution scheme. Since dotted terms are not multiplied, the equation is linear in each increment.

The solution of the equation can be reached by means of the finite difference method by discretisation of the above systems of equations. This is, however, not done in favour of the ease of using a ready-made finite element code like ANSYS.
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Appendix B

Numerical Results

For a description of the panes in each figure, the figures below are referred to. Above the line (A to F) the graphs show how the plastic strain $x$, $y$, and $z$-components are distributed along and perpendicularly to the heating path on the top and bottom sides of the plate. Thus, pane A holds $\varepsilon^p_x$, evaluated perpendicularly to the centre of the heating path, while B holds the same property, evaluated along the middle of the heating path. Panes C, D, E, and F are the same but for $\varepsilon^p_y$ and $\varepsilon^p_z$, respectively. The G pane shows the $\varepsilon^p$ distribution through the thickness in the very centre of the heating line. Pane H shows the maximum temperature ‘measured’ for each time step. Finally, the I and J panes show how the averaged plastic strains are linearised. The dash lines are the original plastic data, while the thick straight line represents the linearisation in either $\{S_x, B_x\}$ or $\{S_y, B_y\}$.
Figure B.1: Test case 1. Final plastic strain in $x$, $y$, and $z$-directions for $x$, $y$, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^{\text{p},x,y}$. $h=10$ [mm], $v=5$ [mm/s], $T_{\text{max}}=500$ [°C].
Figure B.2: Test case 2. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\tilde{\varepsilon}_{x,y}$. $h=15$ [mm], $v=5$ [mm/s], $T_{\text{max}}=500$ [$^\circ$C].
Figure B.3: Test case 3. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=20$ [mm], $v=5$ [mm/s], $T_{\text{max}}=500$ [°C].
Figure B.4: Test case 4. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\epsilon}_p^{x,y}$. $h=10$ [mm], $v=10$ [mm/s], $T_{max}=500 \, ^\circ C$. 
Figure B.5: Test case 5. Final plastic strain in $x$, $y$, and $z$-directions for $x$, $y$, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}^p$. $h=15$ [mm], $v=10$ [mm/s], $T_{max}=500$ [°C].
Figure B.6: Test case 6. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=20$ [mm], $v=10$ [mm/s], $T_{\text{max}}=500$ [$^\circ$C].
Figure B.7: Test case 7. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^{p}_{x,y}$. $h=10$ [mm], $v=15$ [mm/s], $T_{max}=500$ [$^\circ$C].
Figure B.8: Test case 8. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\bar{\varepsilon}_{x,y}$. $h=15$ [mm], $v=15$ [mm/s], $T_{\text{max}}=500$ [$^\circ$C].
Appendix B. Numerical Results

Figure B.9: Test case 9. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_p^{x,y}$. $h=20$ [mm], $v=15$ [mm/s], $T_{max}=500$ [$^\circ$C].
Figure B.10: Test case 10. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\varepsilon_{x,y}^{p}$. $h=10$ [mm], $v=5$ [mm/s], $T_{\text{max}}=600$ [$^\circ$C].
Figure B.11: Test case 11. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=15$ [mm], $v=5$ [mm/s], $T_{\text{max}}=600$ [°C].
Figure B.12: Test case 12. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^{p}_{x,y}$. $h=20$ [mm], $v=5$ [mm/s], $T_{\text{max}}=600$ [$^\circ$C].
Figure B.13: Test case 13. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=10$ [mm], $v=10$ [mm/s], $T_{max}=600$ [$^\circ$C].
Figure B.14: Test case 14. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^p_{x,y}$. $h=15$ [mm], $v=10$ [mm/s], $T_{\text{max}}=600$ [$^\circ$C].
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Figure B.15: Test case 15. Final plastic strain in $x$, $y$, and $z$-directions for $x$, $y$, and $z$-components. Linearised data compared to averaged plastic strain, $\varepsilon_{x,y}^p$. $h=20$ [mm], $v=10$ [mm/s], $T_{\text{max}}=600$ [°C].
Figure B.16: Test case 16. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=10$ [mm], $v=15$ [mm/s], $T_{\text{max}}=600$ [$^\circ$C].
Figure B.17: Test case 17. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. h=15 [mm], v=15 [mm/s], $T_{max}=600 \degree C$. 
Figure B.18: Test case 18. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_p^{x,y}$. $h=20$ [mm], $v=15$ [mm/s], $T_{\text{max}}=600$ [°C].
Figure B.19: Test case 19. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\varepsilon_{p,x,y}^{\text{avg}}$. $h=10$ [mm], $v=5$ [mm/s], $T_{\text{max}}=700 \degree$C.
Figure B.20: Test case 20. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=15$ [mm], $v=5$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.21: Test case 21. Final plastic strain in $x$, $y$, and $z$-directions for $x$, $y$, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=20$ [mm], $v=5$ [mm/s], $T_{\text{max}}=700$ [$^\circ$C].
Figure B.22: Test case 22. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}$. $h=10$ [mm], $v=10$ [mm/s], $T_{\text{max}}=700$ [$^\circ$C].
Figure B.23: Test case 23. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=15$ [mm], $v=10$ [mm/s], $T_{max}=700$ [$^\circ$C].
Figure B.24: Test case 24. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\varepsilon_{p,x,y}^\text{h}$. $h=20$ [mm], $v=10$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.25: Test case 25. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=10$ [mm], $v=15$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.26: Test case 26. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=15$ [mm], $v=15$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.27: Test case 27. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\varepsilon_{x,y}^p$. $h=20$ [mm], $v=15$ [mm/s], $T_{\text{max}}=700$ [$^\circ$C].
Figure B.28: Slow case 1. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^{p}_{x,y}$. $h=10$ [mm], $v=2$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.29: Slow case 2. Final plastic strain in \( x \), \( y \), and \( z \)-directions for \( x \), \( y \), and \( z \)-components. Linearised data compared to averaged plastic strain, \( \bar{\varepsilon}_{p,x,y} \). \( h=15 \) [mm], \( v=2 \) [mm/s], \( T_{\text{max}}=700 \) [\(^\circ\)C].
Figure B.30: Slow case 3. Final plastic strain in x-, y-, and z-directions for x-, y-, and z-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{p,x,y}$. $h=20$ [mm], $v=2$ [mm/s], $T_{\text{max}}=700$ [°C].
Figure B.31: Slow case 4. Final plastic strain in \( x \)-, \( y \)-, and \( z \)-directions for \( x \)-, \( y \)-, and \( z \)-components. Linearised data compared to averaged plastic strain, \( \hat{\varepsilon}_{p,x,y} \). \( h=20 \text{ [mm]} \), \( v=1 \text{ [mm/s]} \), \( T_{\text{max}}=700 \text{ [\degree C]} \).
Figure B.32: Slow case 5. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}_{x,y}^p$. $h=20$ [mm], $v=0.5$ [mm/s], $T_{max}=700$ [°C].
Figure B.33: Experiment. Final plastic strain in $x$-, $y$-, and $z$-directions for $x$-, $y$-, and $z$-components. Linearised data compared to averaged plastic strain, $\hat{\varepsilon}^p_{x,y}$. $h=9.7$ [mm], $v=3$ [mm/s], $T_{\text{max}}=635$ [$^\circ$C].
Appendix C

Results of ATFA

This appendix gives the results of validating the artificial temperature field analysis. For each of the 27 test cases in Chapter 3 the linearised plastic strains are applied to an elastic analysis. The following graphs show the out-of-plane deflections at the symmetry line (the heating path) and at the edge parallel to the heating path.

The plates appear to be heated from below (so that it curves downwards), and the curves showing most curvature are the ones at the heating line.
Figure C.1: Comparison of elastoplastic and pure elastic analysis. Test case 1 through 8.
Figure C.2: Comparison of elastoplastic and pure elastic analysis. Test case 9 through 16.
Figure C.3: Comparison of elastoplastic and pure elastic analysis. Test case 17 through 24.
Figure C.4: Comparison of elastoplastic and pure elastic analysis. Test case 25 through 27.
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Appendix D

A Resistance Welding Apparatus

Thermocouples are made of K type extension cable (2x0.25 mm type K from Thermocoax). The ends of the cable are welded onto the plate by resistance welding.

The welder is made of a 36 mF capacitor, $C_1$, and a large tyristor. To protect the tyristor, the peak current is restricted by a coil, $L$, with two windings, see the chart in Figure D.1. Further, the circuit consists of a small capacitor, $C_2$, to trigger the tyristor, a charging and a discharging resistance, $R_1$ and $R_2$, and a rectifier.

Figure D.2 shows thermocouples welded onto a plate.

A Hewlett-Packard 34970A data acquisition unit with a 34902A 16 channel reed multiplexer for thermocouples is used to gather temperature versus time data.
Figure D.2: Thermocouples welded onto a plate.
Strøm-Tejsen, J.
Damage Stability Calculations on the Computer DASK.

Silovic, V.
A Five Hole Spherical Pilot Tube for three Dimensional Wake Measurements.

Chomchuenchit, V.
Determination of the Weight Distribution of Ship Models.

Chislett, M.S.
A Planar Motion Mechanism.

Nicordhanon, P.
A Phase Changer in the HyA Planar Motion Mechanism and Calculation of Phase Angle.

Jensen, B.
Anvendelse af statistiske metoder til kontrol af forskellige eksisterende tilnærmelsesformler og udarbejdelse af nye til bestemmelse af skibes tonnage og stabilitet.

Aage, C.
Eksperimentel og beregningsmæssig bestemmelse af vindkræfter på skibe.

Prytz, K.
Datamatorienterede studier af planende bådes fremdrivningsforhold.

Hee, J.M.
Store sideportes indflydelse på langskibs styrke.

Madsen, N.F.
Vibrations in Ships.

Andersen, P.
Bølgeinducerede bevægelser og belastninger for skib på lægt vand.

Römeling, J.U.
Buling af afstivede pladepaneler.

Sørensen, H.H.
Sammenkobling af rotations-symmetriske og generelle tre-dimensionale konstruktioner i elementmetode-beregninger.

Fabian, O.
1980  Petersen, M.J.
Ship Collisions.

1981  Gong, J.
A Rational Approach to Automatic Design of Ship Sections.

1982  Nielsen, K.
Bølgeenergimaskiner.

1984  Nielsen, N.J.R.
Structural Optimization of Ship Structures.

1984  Liebst, J.
Torsion of Container Ships.

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Mathematical Definition of Ship Hull Surfaces using B-splines.

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Stationære skibsbølger.

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Collapse of Offshore Platforms.

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3-D Analysis of Pipelines during Laying.

1987  Holt-Madsen, A.

1989  Andersen, S.V.

1989  Rasmussen, J.
Structural Design of Sandwich Structures.

1990  Bastrup, J.
Structural Analysis of Marine Structures.

1990  Wedel-Heinen, J.
Vibration Analysis of Imperfect Elements in Marine Structures.

1991  Almlund, J.
Life Cycle Model for Offshore Installations for Use in Prospect Evaluation.

1991  Back-Pedersen, A.
Analysis of Slender Marine Structures.
1992 Bendiksen, E.
Hull Girder Collapse.

1992 Petersen, J.B.
Non-Linear Strip Theories for Ship Response in Waves.

1992 Schalck, S.
Ship Design Using B-spline Patches.

1993 Kierkegaard, H.
Ship Collisions with Icebergs.

1994 Pedersen, B.
A Free-Surface Analysis of a Two-Dimensional Moving Surface-Piercing Body.

1994 Hansen, P.F.
Reliability Analysis of a Midship Section.

1994 Michelsen, J.
A Free-Form Geometric Modelling Approach with Ship Design Applications.

1995 Hansen, A.M.
Reliability Methods for the Longitudinal Strength of Ships.

1995 Branner, K.
Capacity and Lifetime of Foam Core Sandwich Structures.

1995 Schack, C.
Skrogudvikling af hurtiggående færger med henblik på sødygtighed og lav modstand.

1997 Simonsen, B.C.
Mechanics of Ship Grounding.

1997 Olesen, N.A.
Turbulent Flow past Ship Hulls.

1997 Riber, H.J.
Response Analysis of Dynamically Loaded Composite Panels.

1998 Andersen, M.R.
Fatigue Crack Initiation and Growth in Ship Structures.

1998 Nielsen, L.P.
Structural Capacity of the Hull Girder.

1999 Zhang, S.
The Mechanics of Ship Collisions.

1999 Birk-Sørensen, M.
Simulation of Welding Distorsions of Ship Sections.
1999  Jensen, K.
Analysis and Documentation of Ancient Ships.

2000  Wang, Z.
Hydroelastic Analysis of High Speed Ships.

2000  Petersen, T.
Wave Load Prediction—a Design Tool.

2000  Banke, L.
Flexible Pipe End Fitting.

2000  Simonsen, C.D.
Rudder, Propeller and Hull Interaction by RANS.

2000  Clausen, H.B.
Plate Forming by Line Heating.