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Published in:
Physical Review E

Link to article, DOI:
10.1103/PhysRevE.77.026206

Publication date:
2008

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Transitions from phase-locked dynamics to chaos in a piecewise-linear map

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(Received 27 October 2007; published 11 February 2008)

Recent work has shown that torus formation in piecewise-smooth maps can take place through a special type of border-collision bifurcation in which a pair of complex conjugate multipliers for a stable cycle abruptly jump out of the unit circle. Transitions from an ergodic to a resonant torus take place via border-collision fold bifurcations. We examine the transition to chaos through torus destruction in such maps. Considering a piecewise-linear normal-form map we show that this transition, by virtue of the interplay of border-collision bifurcations with period-doubling and homoclinic bifurcations, can involve mechanisms that differ qualitatively from those described by Afraimovich and Shilnikov.

DOI: 10.1103/PhysRevE.77.026206 PACS number(s): 05.45.Gg, 05.45.Pq

I. INTRODUCTION

Many problems in engineering and applied science lead us to consider piecewise-smooth maps. Examples of such systems include relay and pulse-width modulated control systems [1,2], mechanical systems with dry friction or impacts [3], and managerial or economic systems with well-defined intervention thresholds [4].

As a parameter is varied, the fixed point for the Poincaré map of such a system may move in phase space and collide with the boundary between two smooth regions. When this happens, the Jacobian matrix can change abruptly, leading to a special class of nonlinear dynamic phenomena known as border-collision bifurcations [5–9].

A simple type of border-collision bifurcation consists in the direct transition from one periodic orbit into another with the same period. However, more complicated phenomena are also possible, including period-multiplying bifurcations, multiple-choice bifurcations, and direct transition from periodicity to chaos [2,10–15]. Border-collision-related bifurcations also include corner-collision, sliding, and grazing bifurcations [16–18].

Piecewise-smooth systems can also display quasiperiodic behavior. In a series of recent publications [19–22] we have shown that border-collision bifurcations can lead to the birth of an invariant torus associated with quasiperiodic or phase-locked periodic dynamics. This transition resembles the well-known Neimark-Sacker bifurcation in several respects. However, rather than through a continuous crossing of a pair of complex-conjugate multipliers of the periodic orbit through the unit circle, the border-collision bifurcation involves a jump of the multipliers from the inside to the outside of this circle. We have also demonstrated the existence of a special type of border-collision bifurcation in which a stable periodic orbit arises simultaneously with a quasiperiodic or phase-locked invariant torus [19,23].

Along with the period-doubling route and various types of intermittency transitions, the formation and subsequent destruction of a two-dimensional torus is one of the classic routes to chaos in dissipative systems. Before breakdown, the resonance torus typically loses its smoothness in discrete points through folding (or winding) of the involved manifolds, and this loss of smoothness then spreads to the entire torus surface through local (i.e., saddle-node) or global (i.e., homoclinic or heteroclinic) bifurcations.

The basic theorem for the destruction of a two-dimensional torus in smooth dynamical systems was proved by Afraimovich and Shilnikov [24], and three possible routes for the appearance of chaotic dynamics were described. The generic character of these processes has since been confirmed numerically as well as experimentally for wide classes of both continuous- and discrete-time systems [25–27].

The purpose of the present paper is to investigate some of the mechanisms that are involved in the transitions from phase-locked periodic dynamics to chaos in nonsmooth maps. With this purpose we follow the bifurcations that take place as the point of operation for a piecewise-linear normal-form map leaves the 1:4 resonance tongue along three different routes. We show that the interplay between period-doubling, border-collision, and homoclinic bifurcations leads to transitions that are qualitatively different from those of Afraimovich and Shilnikov. In particular, we consider a route...
in which a homoclinic bifurcation first destroys the resonance torus while leaving the original stable-node cycle. This node subsequently undergoes a period-doubling bifurcation combined with a simultaneous border-collision bifurcation for the appearing subharmonic, and chaos arises. Other routes involve regions of coexistence of periodic and chaotic oscillations or of different chaotic oscillations. The paper discusses the specific features of these routes and outlines some of the characteristic differences between the routes followed in smooth and nonsmooth maps.

II. PIECEWISE-LINEAR NORMAL-FORM MAP

It is well known that dynamical phenomena related to border-collapse bifurcations can be examined by means of a piecewise-linear approximation to the Poincaré map in the neighborhood of the border-crossing fixed point, expressed in the convenient normal form [6,7,10]

$$ F_1(x,y) \rightarrow \begin{cases} F_1(x,y), & x \leq 0, \\ F_2(x,y), & x \geq 0, \end{cases} $$ (1)

where

$$ F_1(x,y) = \left( \begin{array}{c} \tau_x x + y + \mu \\ -\delta_x y \end{array} \right), \quad F_2(x,y) = \left( \begin{array}{c} \tau_y x + y + \mu \\ -\delta_y y \end{array} \right), \quad (x,y) \in \mathbb{R}^2. $$

In this representation, the phase plane is divided into two regions $L = \{(x,y): x \leq 0, y \in \mathbb{R}\}$ and $R = \{(x,y): x > 0, y \in \mathbb{R}\}$. $\tau_x$ and $\delta_x$ denote the trace and the determinant, respectively, of the Jacobian matrix $J_L$ in the half-plane $L$, and $\tau_y$ and $\delta_y$ are the trace and determinant of the Jacobian matrix $J_R$ in the region $R$. It is clear that

$$ J_L = \begin{pmatrix} \tau_x & 1 \\ -\delta_x & 0 \end{pmatrix}, \quad J_R = \begin{pmatrix} \tau_y & 1 \\ -\delta_y & 0 \end{pmatrix}. $$

The stability of the fixed point for the map (1) is determined by the eigenvalues of the corresponding Jacobian matrix $\lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\delta} \right)$. These fixed points at the two sides are given by

$$ \left( \frac{\mu}{\chi_L(1)} - \frac{\mu \delta_L}{\chi_L(1)}, \frac{\mu}{\chi_R(1)} - \frac{\mu \delta_R}{\chi_R(1)} \right), $$

with $\chi(1)$ representing the value of the characteristic polynomial $\chi(\lambda) = \lambda^2 - \tau \lambda + \delta$ for $\lambda = 1$ in the considered half-plane.

As the parameter $\mu$ of the map (1) is varied from negative to positive values, the fixed point of (1) moves from $L$ to $R$ and a border-collision occurs at $\mu = 0$. Let us choose the parameters such that $\delta_L < 1$ and $\delta_R > 1$. The conditions

$$ \chi_L(1) \chi_R(1) > 0, $$

$$ -1 - \delta_L < \tau_L < 1 + \delta_L, \quad -2 \sqrt{\delta_R} < \tau_R < 2 \sqrt{\delta_R}, $$ (2)

then ensure that the fixed point is attracting for $\mu < 0$ and a spiral repeller for $\mu > 0$.

In the present analysis we have assumed the following values for the determinants: $\delta_L = 0.5$ and $\delta_R = 1.6$. For $\mu < 0$ the map (1) then has a single nontrivial stable fixed point with a negative $x$ coordinate. When $\mu$ changes sign, the $x$ coordinate of the fixed point also changes sign and the fixed point abruptly loses stability as a pair of complex-conjugate eigenvalues of the Jacobian matrix jumps from the inside to the outside of the unit circle; i.e., the stable focus transforms abruptly into an unstable focus. If the parameters $\tau_L$ and $\tau_R$ of the map (1) are varied within the range delineated by (2), one can observe a large variety of dynamical phenomena associated with the interplay between homoclinic bifurcations and different forms of border-collapse bifurcations [19]. Figure 1 shows the chart of dynamical modes (two-parameter bifurcation diagram) in the parameter plane $(\tau_L, \tau_R)$ for positive values of $\mu$. Inspection of this chart reveals the presence of a dense set of periodic tongues. The main resonance tongues are marked with the corresponding rotation numbers. In the 1:4 tongue, for instance, the system retraces itself after four iterations of the map (1). Between the tongues there are parameter combinations that lead to quasiperiodicity and chaos. The white regions $\Pi_{\omega,1}$ and $\Pi_{\omega,2}$ are domains where the trajectories of the map diverge to infinity for all initial conditions.

Depending on the parameter values, we observe a variety of different scenarios.

(i) If the values of the parameters $\tau_L$ and $\tau_R$ are chosen within a tongue of periodicity, then an attracting closed invariant curve softly arises from the fixed point as the parameter $\mu$ crosses the bifurcation point at $\mu = 0$. This invariant curve is formed by the unstable manifolds of a saddle cycle and the points of the corresponding saddle and stable cycles. Figure 2 displays the bifurcation diagram (a) and the phase portrait (b) for a point $(\tau_L, \tau_R)$ in the 1:4 tongue.

(ii) If we choose the parameters $\tau_L$ and $\tau_R$ in a region of quasiperiodicity, the stable fixed point for $\mu < 0$ turns into an unstable focus point on the $R$ side and quasiperiodic behavior arises. Figure 3 presents the bifurcation diagram for such a transition.

(iii) If $\tau_L$ or $\tau_R$ are varied within the region (2) for positive values of $\mu$, more complicated bifurcation phenomena are
saddle. fp is the unstable fixed point. of the stable period-4 cycle and dashed lines show the period-4
−0.05 to 0.05. Solid lines to the right in the diagram mark the points
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−0.05 to 0.05. The diagram shows a direct transition from a stable
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connection
curve which typically takes the form of a saddle-node con-
bifurcation in which a quasiperiodic attractor
nected with the transitions from periodic to quasiperiodic
phenom-
possible in the transition from phase-locked dynamics to

quasiperiodicity and vice versa. In particular, these phenom-
ela include the border-collision fold bifurcation that is con-
nected with the transitions from periodic to quasiperiodic
dynamics and a modified variant of the multiple-attractor
definition in which a quasiperiodic attractor (or a mode-
locked periodic orbit) arises together with one (or more)
stable cycles [23].

III. TRANSITIONS FROM PHASE-LOCKED DYNAMICS
TO CHAOS

In each resonance tongue with the rotation number r:q
(see Fig. 1) the map displays an attracting closed invariant
curve which typically takes the form of a saddle-node con-
nection [Fig. 2(b)]. The unstable manifold of the period-q


FIG. 2. Birth of a stable closed invariant curve from a stable
fixed point in a border-collision bifurcation. τ_L =0.25 and τ_R
=−0.25. (a) Bifurcation diagram as the parameter μ varies from
−0.05 to 0.05. Solid lines to the right in the diagram mark the points
of the stable period-4 cycle and dashed lines show the period-4

FIG. 3. Bifurcation diagram as the parameter μ varies from
−0.05 to 0.05. The diagram shows a direct transition from a stable
period-1 focus cycle to a quasiperiodic orbit. The other parameters are
τ_L =0.14, τ_R =−0.25, δ_L =0.5, and δ_R =1.6.
saddle connects to the period-q node, thus forming a closed
attracting curve. For other parameter values, the closed in-
vARIANT curve may be associated with a pair of saddle and

focus cycles of similar periodicity [26].

In a couple of recent papers [19,28] we have demonstrated
that under variation of the parameters, this closed

invariant curve is destroyed through a homoclinic bifurca-

tion. However, the stable and saddle cycles may continue to
exist after the torus destruction. With further change of the
parameters, these cycles then merge and disappear in a
border-collision fold bifurcation. As a result, between the
curves of homoclinic bifurcation and of border-collapse fold
bifurcation there is a region of multistability, on the bound-
aries of which one can observe transitions with hysteresis.
Using a dc-dc power converter as an example of a piecewise-
smooth system, we have shown experimentally that the hys-

teretic transitions observed for the piecewise-linear normal-
form map actually occur in practical systems [19].

In the present paper we are interested in mechanisms of
torus breakdown that relate to the transition from resonance
behavior to chaotic dynamics. With this purpose we shall
follow the bifurcations that take place as we leave the 1:4
resonance tongue of our normal-form map along three differ-
ent directions in parameter space.

It is well known that the resonance tongues in piecewise-
smooth systems are bounded by border-collapse fold bifur-
cation curves [2,15,28]. As illustrated in Fig. 1, the 1:4 reso-

TABLE 1: Parameter values of the section


figure 4 shows a close-up of the chart of dynamical
modes emphasizing this part of the 1:4 tongue. Here, the two
border-collision curves are indicated by N_C. 1 and 2 are ho-
moclinic bifurcation curves, N_1 is a smooth period-doubling
bifurcation curve, and N_div delineates the boundary of diver-
gent behavior. The arrows marked A, B, and C represent the
directions in which we shall study the transitions in detail.

IV. PERIOD-DOUBLING ROUTE

Let us first analyze what happens when moving from the
inside to the outside of the resonance tongue through the
period-doubling bifurcation curve N_ along the direction A.
Results of a bifurcation analysis for the section
{(τ_L, τ_R): 0.95 ≤ τ_L ≤ 1.1, τ_R =−1} are presented in Figs. 5

and 6.

Figure 5(a) shows the bifurcation diagram obtained
through direct simulation, and Fig. 5(b) displays the corre-

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sponding diagram as obtained by following the periodic orbits. To better illustrate the latter transition, a magnified section of Fig. 5 is presented in Fig. 6. The variation of the multipliers for the period-4 cycle for this section is shown in Fig. 6. At the point \( L = L^* \approx 1.0236 \) the largest multiplier of the period-4 cycle in absolute value crosses through \(-1\). The loss of stability of the period-4 cycle is not accompanied by the appearance of a stable period-8 cycle. Rather a whole family of unstable cycles arise through the border-collision bifurcation at \( L^* \) [see Figs. 5(b) and 6(a)].

A. Repeated folding of the stable and unstable manifolds

Let us consider the characteristics of the bifurcational behavior shown in Figs. 5 and 6 in more detail in order to understand the mechanism of the transition between mode locking and chaos. Before the transition, the system displays a closed invariant curve that is the union of the unstable manifold of the period-4 saddle cycle and the points of the stable-focus period-4 cycle. Inspection of Fig. 7 shows that the unstable manifold of the saddle fixed point has an infinite number of the linear pieces that fold at specific corner points before converging on the stable fixed points.
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As shown by Banerjee and Grebogi [10], the unstable manifolds of the map (1) fold at every intersection with the \( x \) axis and the image of a fold point is also a fold point. The stable manifolds fold at every intersection with the \( y \) axis, and every preimage of a fold point is also a fold point. This creates the succession of folds in the invariant manifolds seen in Fig. 7.

Following [30–32] we denote the borderline as

\[
LC_{-1} = \{(x, y) : x = 0, y \in \mathbb{R}\}.
\]

The image of this line under one iteration of the map \( F \), relation (1), is called the fold line \( LC_0 \) [32]

\[
LC_0 = F(LC_{-1}) = \{(x, y) : y = 0\}.
\]

The image of \( LC_0 \) is also a fold line:

\[
LC_1 = F(LC_0) = \left\{(x, y) : y = -\frac{\delta}{\tau}(x - \mu)\right\}.
\]

There will be two lines denoted by \( LC_1 \), one obtained by using the trace and determinant for the left half (region \( L \)) and the other obtained by using these parameters for the right half (region \( R \)).

To illustrate the role of these intersections, Fig. 8 shows the phase portrait of the map for \( \tau_R = -0.98 \) and \( \tau_L = 0.86 \) where we have drawn the lines \( LC_{-1}, LC_0, \) and \( LC_1 \). It is evident from this figure that the unstable manifolds between the saddle points 1 and 2 cross the borderline \( LC_{-1} \) at five points \( A^+ \) and are folded on the fold line \( LC_0 \), producing five corner points \( A^- \). The forward images of these points are again corner points. The images of the points \( A^+ \) lie on the line \( LC_1 \) and are denoted as \( A^- \). Analogically, one segment of the unstable manifold between the points 3 and 4 of the saddle cycle crosses the line \( LC_{-1} \) at the point \( A^- \), and then folds on \( LC_0 \), producing the corner point \( A^- \). The image of this point on the line \( LC_1 \) is denoted as \( A^- \) in Fig. 8.

**B. Period-doubling border-collision route**

Based on this structure of the stable and unstable manifolds, the main stages of the torus transformation can be understood. As the trace \( \tau \) increases, at the point \( \tau = 0.99 \) the first homoclinic bifurcation occurs (or homoclinic contact by analogy with the homoclinic tangency in smooth maps) [see Fig. 9(a)]. With further increase in the value of \( \tau \), the stable and unstable manifolds of the period-4 saddle cycle intersect transversally to form a homoclinic structure [see Fig. 9(b)]. The intersection of the two manifolds implies the existence of a Smale horseshoe and, therefore, of an infinite number of high-periodic orbits [26]. After the homoclinic tangency, the attractor of the map is still the period-4 node, but the torus no longer exists.

In the meantime the multipliers of the stable period-4 cycle change as shown in Fig. 10. At \( \tau = 0.9744 \), the complex-conjugate multipliers merge on the negative real line and the fixed point becomes a flip attractor. Following this, the two negative real eigenvalues move away from each other, and at \( \tau = 1.0236 \) one of the eigenvalues reach the unit circle on the negative real line. This marks a smooth period-doubling bifurcation, and it is known that beyond this point the torus cannot exist.

Since the map is linear on each side of the border at \( x = 0 \), the period-8 cycle produced in the period doubling instantly moves away and one of its points collides with the border. This leads to the abrupt transition to an eight-band strange attractor.

Figure 11(a) shows the phase portrait after the period-doubling bifurcation. The phase portrait contains the points of the unstable cycles with the periods 4, 8, 12, 16, 24, 32, and 40. As mentioned above, these cycles arise through a border-collision bifurcation occurring at the same parameter.
value as the period doubling, $\tau_L = \tau^*_L$. Figure 11(b) illustrates the phase portrait of the map for the single-band strange attractor.

It is possible that the period doubling takes place after a second homoclinic intersection. When that happens, we observe a hard transition to chaos through the period-doubling bifurcation. This transition is observed when the system leaves the resonance tongue along the section that crosses the line of the second homoclinic tangency $2$ and the curve of the period doubling bifurcation $N$. The bifurcation diagram in Fig. 12 illustrates such a transition when we change the parameter $\tau_L$ along the direction $C$ (Fig. 4). To better illustrate the bifurcation transitions we do not include all the branches of the bifurcation diagram, but only a magnified view of one of the four.

FIG. 9. (Color online) Torus destruction through homoclinic bifurcation before the smooth period doubling. (a) Phase portrait of the map near the first homoclinic contact (the analog of a homoclinic tangency in smooth maps), $\tau_L = 0.99$. (b) Homoclinic intersections of the unstable and stable manifolds of the saddle period-4 cycle, $\tau_L = 1.018$. The multipliers of the stable period-4 cycle are real and negative.

FIG. 10. Locus of the multipliers of the stable period-4 cycle. For $0.9744 > \tau_L > 0.5030$, the two multipliers are complex conjugated. At $\tau_L = 1.0236$, the period-4 cycle undergoes a period doubling.

FIG. 11. (Color online) Phase portrait of the map after a period-doubling bifurcation. (a) Eight-band strange attractor for $\tau_L = 1.035$. On the phase portrait one can see the points of the unstable cycles with the periods 4, 8, 12, 16, 24, 32, and 40 (see also the bifurcation diagram in Fig. 5). (b) Single-band strange attractor for $\tau_L = 1.0725$.

First, the torus is destroyed through the homoclinic bifurcation. With further increase of $\tau_L$ the second homoclinic tangency occurs.

Between the point of the second homoclinic bifurcation $\tau^*_L$ and the point $\tau^*_L$ of the period-doubling bifurcation, the stable period-4 cycle coexists with the single-band strange attractor. At the point $\tau^*_L$ we can observe the abrupt transition to an eight-band strange attractor through the period-doubling bifurcation.

As the parameter $\tau_L$ increases from the value $\tau^*_L$ the eight-band strange attractor merges with the saddle period-4 cycle and disappears in a border-collision bifurcation. After this point, only the single strange attractor exists. In the region $\tau_L < \tau^*_L < \tau^*_M$ the eight-band and single-band strange attractors coexist as shown in Fig. 13. On the boundary of the second homoclinic tangency hard transition occurs from the single-band strange attractor to a stable period-4 cycle.

FIG. 12. Hard transition to chaos through period-doubling and homoclinic bifurcations, $\tau_K = -0.82$. $\tau^*_H$ is the point of the second homoclinic contact, and $\tau^*_L$ is the point of the period-doubling bifurcation.
Band and single-band strange attractors coexist. Transitions from phase-locked dynamics to...
A hard transition from a periodic to a strange attractor takes place at

with a strange attractor. We conclude that hard hysteretic
torus destruction leading from phase-locked dynamics to
mechanisms for the transition from torus to chaos in smooth
mechanisms, giving birth to chaos.

Of the unit circle. With further parameter variation, the torus
can arise through a special type of
border-collision bifurcation in which a pair of complex-
conjugated multipliers for a stable periodic orbit jumps out
of the unit circle. With further parameter variation, the torus
may be destroyed through a number of different mecha-
nisms, giving birth to chaos.

Afraimovich and Shilnikov have proposed three possible
mechanisms for the transition from torus to chaos in smooth
maps. In this paper we investigated three specific routes of
torus destruction leading from phase-locked dynamics to
chaos in piecewise-smooth maps. Using the appropriate
piecewise-linear normal-form map as a tool, we showed that
the routes to chaos in nonsmooth maps may display signifi-
cant differences from the mechanisms described by Afrai-
movich and Shilnikov.

In one of the routes reported in this paper, a homoclinic
intersection first destroys the torus. In the absence of the
torus, the stable node undergoes a period doubling,
immediately followed by a border collision that gives birth to the
chaotic orbit. In another route, the first homoclinic tangency
is followed by a second homoclinic tangency, which gives
birth to a single-band strange attractor. But the stable peri-
odic orbit persists. At a different parameter value, this peri-
odic orbit undergoes a period-doubling bifurcation, again
immediately followed by a border collision. This creates a
different multiband chaotic orbit. If the orbit before period
doubling was period \( n \), the strange attractor has \( 2n \) bands.
The multiband attractor is destroyed at a border-collision
fold bifurcation, where we see a hard transition from one
chaotic orbit to another.

In the third route, the first homoclinic tangency is fol-
lowed by a second homoclinic tangency and a strange attrac-
tor is born. This attractor coexists with the stable periodic
orbit for some parameter interval. At a specific parameter
value, the stable node (or focus) collides with the saddle on
the border and both are destroyed through a border-collision
fold bifurcation.

**VI. CONCLUSIONS**

Many systems of interest in physics, engineering, and
other sciences display discontinuities that lead to a dynami-
cal description in terms of piecewise-smooth maps. In such
systems, quasiperiodic or phase-locked resonant behavior on
the surface of a torus can arise through a special type of
border-collision bifurcation in which a pair of complex-
conjugated multipliers for a stable periodic orbit jumps out
of the unit circle. With further parameter variation, the torus
may be destroyed through a number of different mecha-
nisms, giving birth to chaos.

Afraimovich and Shilnikov have proposed three possible
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**ACKNOWLEDGMENTS**

The work was supported by the Russian Foundation for
Basic Research (Grant No. 06-01-00811) and by the Danish
Natural Science Foundation through the Center for Model-
ling, Nonlinear Dynamics, and Irreversible Thermodynamics
(MIDIT). S. D. acknowledges support from the Council for
Scientific and Industrial Research, India.

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