Synthesis of nonuniformly spaced arrays using a general nonlinear minimax optimisation method

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Published in:
I E E E Transactions on Antennas and Propagation

Publication date:
1976

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
This comes from the fact that surface current distribution is a dipole instead of a Huygens source. In fact, there is a constant proportionality between (19) and (31). As \( \Omega \) increases, the angle \( \theta \), which sees the main lobe and first few sidelobes, gets smaller so the term \( 1 - \cos \theta \) approaches zero yielding zero far-field cross polarization provided polarization purity is achieved at the aperture plane.

CONCLUSIONS

All these considerations lead us to the conclusion that the cross polarization performance of reflector antenna systems largely depends on the feed system and \( \lambda/D \) ratio. For this reason, superior performance of Cassegrainian or front-fed paraboloids with respect to polarization purity is closely related to the feed design. Thus this paper puts an end to the controversy recently arisen (7)-(9) about the immunity of these two types of antenna systems against depolarization effects. A detailed treatment may be required to see to what extent other factors may influence the cross polarization performance of reflector antennas.

REFERENCES

certain regions the corresponding weights should be chosen greater than 1.

Our problem is now to minimize with respect to \( x \) the maximum error \( F(x) \) defined by

\[
F(x) = \| f(x) \| = \max_{1 \leq j \leq m} | f_j(x) | \tag{4}
\]
to achieve the desired minimax solution to the pattern synthesis problem.

It should be pointed out that we consider the general case where the residuals are nonlinear functions of the design parameters. Note also that no restriction has been put on the type of patterns considered; both real- and complex-valued patterns are covered.

In the next section we intend to minimize (4) by developing an iterative technique with sure convergence properties.

III. NONLINEAR MINIMAX OPTIMIZATION

The iterative method for minimization of (4) is based on successive linear approximations to the nonlinear residuals (3). At the \( k \)th stage of the iteration let \( F_k(h) \) be defined through

\[
F_k(h) = \| f(x_k) + B_k h \| \tag{5}
\]

where \( B_k \) is an approximation to the derivative matrix \( \frac{\partial f_j}{\partial x_i} \) found by means of the Broyden updating formula [5] without extra calculations of the residuals. Thus, for small values of \( |h| \), \( F_k(h) \) is an approximation to \( F(x_k + h) \). The increment \( h_k \) to the current point \( x_k \) is chosen by solving the constrained linear minimax problem

\[
F_k(h_k) = \min_{\| h \| \leq \gamma_k} [F_k(h)] \tag{6}
\]

which may be set up as a linear programming problem and solved by any suitable linear programming routine. However, for greater efficiency we use a method [6] similar to the exchange algorithm for solving systems of linear equations in the minimax sense. The point \( (x_k + h_k) \) is accepted as the next current point if the decrease in the objective function \( F \) exceeds a small multiple of the decrease in the linear approximation, more precisely if

\[
F(x_k) - F(x_k + h_k) \geq \alpha [F(x_k) - F_k(h_k)] \tag{7}
\]

\( \alpha = 0.01 \). Otherwise we let \( x_{k+1} = x_k \).

The bound \( \gamma_k \) is adjusted during the iterations according to the following strategy. If the decrease in the objective function \( F \) is poor compared to the decrease predicted by the linear approximation \( F_k \), then the step length may be too large. Consequently, if (7) is not satisfied with \( \alpha = 0.1 \), we let \( \gamma_{k+1} = 0.7 \gamma_k \). Alternatively the step length may be unnecessarily small if the agreement between each residual \( f_j \) and its linear approximation is very good at the point \( x_k + h_k \). Therefore we test the inequality

\[
| f(x_k + h_k) - [f(x_k) + B_k h_k] | \leq \frac{1}{2} [F(x_k) - F(x_k + h_k)] \tag{8}
\]

and if it holds we let \( \gamma_{k+1} = 2 \gamma_k \). This is sensible since Taylor's formula states that the term on the left-hand side of (8) is of the order \( \| h_k \|^2 \), whereas the term on the right-hand side is of the order \( \| h_k \| \). In all other cases we let \( \gamma_{k+1} = \gamma_k \). The various constants governing the optimization strategy have been determined through extensive numerical experiments. It was found, however, that the choice of the constants was not critical.

In order that the matrices \( B_k \) are good approximations to the derivative matrices, it is important that the directions \( h_k \) satisfy some linear independence conditions. This is ensured by introducing "special iterations" as suggested by Powell [7]. A description of this is beyond the scope of this paper, details may be found in [7] and [8].

An important feature of the algorithm is that it possesses sure convergence properties. Madsen [9] has shown that if the problem under consideration is nonsingular the final rate of convergence is super-linear. This means that if the optimum solution is denoted by \( x^* \) the inequality

\[
| x_{k+1} - x^* | \leq e_k | x_k - x^* | \tag{9}
\]

where \( e_k \to 0 \), will be fulfilled.

A Fortran implementation of the algorithm described above along with another minimax algorithm that requires evaluation of the exact first-order derivatives [10] is available in Madsen and Schjærg-Jacobsen [11]. The user has to specify initial values of \( \lambda \) and a general upper bound \( \Lambda \) on the step length. In all examples presented in the following \( \lambda_0 = \Lambda = 0.1 \) is used.

IV. SYNTHESIS OF ARRAYS BY SPACING VARIATION

Consider a linear symmetrical broad-side array with \( N \) elements as an example (Fig. 1). The radiation pattern is given by

\[
g_a(\theta) = 2 \sum_{i=1}^{N/2} a_i \cos (2\pi \xi_i u), \quad u = \sin \theta, \; N \text{ even} \tag{10a}
\]

\[
g_o(\theta) = 1 + 2 \sum_{i=1}^{N-1/2} a_i \cos (2\pi \xi_i u), \quad u = \sin \theta, \; N \text{ odd} \tag{10b}
\]

where \( a_i \) is the (normalized) excitation coefficient and \( \xi_i \) the distance (in wavelengths) from the array center. We now attempt to synthesize arrays with minimum equal sidelobes by varying the element positions rather than the excitation coefficients. This problem clearly represents a nonlinear minimax problem, whereas finding the excitation coefficients for given (unequal) element spacings is a linear problem that may be solved by well-known linear programming techniques.

To minimize the maximum sidelobe level define \( n \) design parameters \( x_i \) such that

\[
\xi_i = x_i + \xi_{i-1},
\]

\[
i = 1, 2, \ldots, n \quad \left\{ \begin{array}{ll}
N - 2 \quad & N \text{ even} \tag{11a} \\
N - 3 \quad & N \text{ odd} \tag{11b}
\end{array} \right.
\]

with \( \xi_0 = 0 \) and \( \xi_{N+1} = (\lambda/4)(N - 1) \). Hence, \( x_i \) represents the element spacings, except when \( i = 1 \) for \( N \) even in which case \( x_1 \)

\(^2\) By this term we mean that \( x_k \) will converge to a stationary point of \( F \).
is the distance from the array center to element 1. Note that the total length is a fixed constant by this definition.

The residuals in (3) are defined through

$$f_j = g_0(\theta_j)/g_0(0), \quad j = 1, 2, \cdots, m$$  \hspace{1cm} (12)

where the $m$ sample points $\theta_j$ are chosen such that

$$0^\circ < \theta_1 < \theta_2 < \cdots < \theta_j < \cdots < \theta_m = 90^\circ.$$  \hspace{1cm} (13)

This corresponds to the choice $w_j = 1$ and $P_D(\theta_j) = 0$ in (3).

V. DOLPH-CHEBYSHEV ARRAYS AS TEST CASES

In order to demonstrate the method on cases where the exact minimax solutions are unknown, we first consider optimization of Dolph–Chebyshev arrays by variation of element spacings. With the element spacings being 142, the normalized excitation coefficients for such arrays may be calculated by the formulas given by Stegen [12]. For six- and eight-element arrays we get

$$(a_1, a_2, a_3) = (1.0, 0.7768, 0.5406), \quad N = 6$$

and

$$(a_1, a_2, a_3, a_4) = (1.0, 0.8751, 0.6603, 0.5799), \quad N = 8$$

corresponding to a sidelobe level of $1/R = 0.1$ or $20 \log (1/R) = -20$ dB.

To be able to exactly reproduce the $l/2$ spacing in an optimization process we have to sample the pattern at the angles $\theta_i$ where the optimum pattern attains its maximum sidelobe levels. These angles are calculated by the formula

$$\theta_i = \frac{180}{\pi} \sin^{-1} \left[ \frac{2}{\pi} \cos^{-1} \left( \frac{\cos \left( \frac{p\pi}{N-1} \right)}{\cosh \left( \frac{1}{N-1} \right)} \right) \right],$$

where $p = 0, 1, \cdots, \frac{N}{2}$, $N$ even

$$p = 0, 1, \cdots, \frac{N+1}{2}$, $N$ odd.  \hspace{1cm} (14)

We also insert additional sample points at essentially $5^\circ$ spacing so that for $N = 6$ we sample at the angles $(\theta_1, \theta_2, \cdots, \theta_m)$ given by

$$(21.11, 25.0, 31.43, 35.0, 40.0, \cdots, 56.30, \cdots, 90.0), \quad m = 15,$$

and for $N = 8$ at the angles given by

$$(15.32, 20.0, 22.52, 30.0, 37.84, 40.0, 45.0, \cdots, 60.40, \cdots, 90.0), \quad m = 16.$$  \hspace{1cm} (15)

In Table I(a) and (b) the number of pattern evaluations required to exceed a specified accuracy $\delta$ is given for different starting values of the element spacings. $\delta$ is defined by

$$\delta = \max |x_i - x_i^*|$$

where $x_i^* = (0.25, 0.50)$ for $N = 6$ and $x_i^* = (0.25, 0.50, 0.50)$ for $N = 8$. From Fig. 2 it is seen that the starting points chosen represent patterns that are considerably different from that of the optimum Dolph–Chebyshev pattern.

3 These examples were run in double precision Fortran IV on an ICL 1906S computer.
choosing good starting points in optimization problems where local optima may occur.

The general trend in Table I is, however, that convergence to the minimax optimum is obtained to a high degree of accuracy with a very limited number of pattern evaluations. The test with \( N \) ranging from 4 to 15. For each \( N \) we start the optimization, i.e., all excitation coefficients \( a_i \) are chosen equal to unity. From a technical point of view, uniform excitation is much simpler to realize than the Dolph–Chebyshev excitation or any other tapered excitation. Again the element spacings as defined by (11) are the optimization parameters. The patterns are sampled in steps of 0.5\(^\circ\), \( \theta_1 \) being chosen sufficiently small to ensure that all sidelobes are included. If we take the case with \( N = 15 \), \( \theta_1 \) has been chosen to 9\(^\circ\) and consequently the number of sample points \( n = 163 \).

A series of pattern synthesis examples has been carried out with \( N \) ranging from 4 to 15. For each \( N \) we start the optimization from a \( \lambda/2 \) spaced array\(^4\) and continue the iterations until

\[
\| a_k \| < 10^{-6} \| x_{n+1} \|
\]

which produces sidelobes that are identical to approximately 10 significant digits.

Results are presented in Table II(a) for \( N \) even and in Table II(b) for \( N \) odd. The number of required antenna pattern evaluations \( N_{\text{pat}} \) is seen to be very moderate, one pattern evaluation corresponding to one evaluation of (12) for \( j = 1, 2, \ldots, m \), i.e.,

\[\text{evaluation of (10) for } m + 1 \text{ values of } \theta.\]

For \( N = 15 \) the optimization required a total CPU time of 5.6 s using double precision Fortran IV on an IBM 370/175 machine. Of this time, 1.8 s were spent computing the residuals (12), and 2.4 s were spent solving the linear subproblems (6). Fig. 3 shows the optimized radiation patterns as well as the initial patterns for \( N \) even, and Fig. 4 shows the corresponding patterns for \( N \) odd.

To give an impression of the overall convergence, the maximum value of the residuals (12) is plotted in Fig. 5 as a function of the number of pattern evaluations for \( N = 14 \) and 15. The plateau on both curves at the beginning is a result of initially approximating the derivative matrix \( \mathbf{B}_k \) by finite differences. This requires \( n + 1 \) pattern evaluations in which improvement on the maximum residual is not attempted.

Some comments on the results are in order. In Table II the position of the first null for the optimized patterns as found by inspection of Figs. 3 and 4 is compared to the position of the first null for a \( \lambda/2 \) spaced Dolph–Chebyshev array with the same number of elements, same total length, and same sidelobe level [13]. It is interesting to observe that the optimized patterns, although presumably not being expressible in terms of Chebyshev polynomials, exhibit in general main beam widths very close to those of the conventional arrays.

The initial patterns for \( N \) even have, by obvious reasons, nulls in the direction \( \theta = 90^\circ \). In all cases considered, however, an extra sidelobe developed in that direction during the optimization, such that the total number of (equal) sidelobes became \( n + 1 \). The initial patterns for \( N \) odd have, of course, a sidelobe in the direction \( \theta = 90^\circ \). During the optimization this particular sidelobe is shifted in position and the final pattern ends up with \( n + 1 \) sidelobes.

These observations agree with fundamentals from minimax approximation theory according to which solu-

<table>
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<tr>
<th>Table II Optimized of Unequally Spaced Symmetric N-Element Linear Broadside Arrays</th>
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<td>(a) ( N ) even</td>
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(16)
Initial patterns (dashed) and optimized patterns (solid) of unequally spaced symmetric N-element linear broadside arrays, N even.

Fig. 3.

Initial patterns (dashed) and optimized patterns (solid) of unequally spaced symmetric N-element linear broadside arrays, N odd.

Fig. 4.

Number of pattern evaluations.

Fig. 5. Maximum residual (sidelobe level) as function of number of pattern evaluations for N = 14 and 15.

VII. DISCUSSION

If the element spacing in unequally spaced arrays gets too small, unwanted coupling between the elements may arise. The closest spacing in the optimized arrays presented in Table II is seen to be 0.37362 which occurs when N = 15. It is therefore relevant to try to develop minimax optimization methods which allow for constraining the optimization parameters to be larger than a certain value, or more general, to lie between preassigned limits $a_i$ and $b_i$

$$a_i \leq x_i \leq b_i, \quad i = 1, 2, \cdots, n. \quad (17)$$

The algorithm presented in this paper seems to be particularly suitable for such an extension since constraints of the type (17) are inherently used when solving the linearized subproblems (6).

The present algorithm may also be applied to sidelobe minimization with thinned arrays. However, it is to be remembered that with n variables only $n + 1$ sidelobes are likely to be equalized in the minimax optimum. Since the number of sidelobes in thinned arrays is much larger than $n + 1$, one can easily imagine that there exist many different combinations of element spacings that produce $n + 1$ equal sidelobes. In other words, extreme caution has to be taken against local minima. Actually we have reconsidered the 19/2 aperture array with 9 elements (i.e., three variable element spacings) which Lo and Lee [14] optimized by "total enumeration." By sampling the pattern in
145 equidistantly spaced sample points in the interval \(0.052632 \leq u \leq 1.94737\), and starting out from their best solution \(x = (0.5, 2.0, 1.5)\), virtually no improvement was obtained. This was to be expected since the initial pattern contains already about 10 almost equal sidelobes [14, Fig. 3]. Choosing other starting points our algorithm converged to a number of local minima.

VIII. CONCLUSION

The general minimax antenna pattern synthesis problem has been formulated and a new nonlinear optimization algorithm with super-linear final convergence is proposed for its solution. Not requiring derivative evaluation, the proposed method is easily applied to synthesis problems with complicated pattern calculations. In this paper, Dolph–Chebyshev arrays were considered for testing purposes and also applications to sidelobe minimization in unequally spaced linear arrays with up to 15 elements by spacing variation demonstrate that the approach is very efficient and reliable. The optimized patterns are compared to conventional Chebyshev patterns and are shown to be comparable in terms of main beamwidth for the same sidelobe level. Local minima in conjunction with sidelobe minimization have been discussed.

It is believed that the present method will find wide-spread future applications in solving pattern synthesis problems that hitherto have been approached by cut-and-try methods.

ACKNOWLEDGMENT

Jesper E. Hansen is gratefully acknowledged for valuable discussions and suggestions. The authors also wish to thank Birgitte Holle for making the artwork.

REFERENCES


Horizontally Polarized Waves in Inhomogeneous Media—Energy Conservation and Reciprocity Relationships

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Abstract—Full wave solutions for the electromagnetic fields of a horizontally polarized wave propagating through an inhomogeneous ionized medium are derived using a generalized WKB method. Both the electron density and the collision frequency of the horizontally stratified media are assumed to vary and special attention is given to permittivity profiles with critical coupling regions. The reflection and transmission coefficients and the characteristic surface impedance for an inhomogeneous layer of finite thickness are computed as functions of the transverse wave number for various permittivity profiles. Excitation of both propagating and evanescent waves are considered. For some special permittivity profiles considered, closed form analytical solutions for the electromagnetic fields are known. Computations derived from these solutions are in good agreement with those obtained using the generalized WKB method. The results are also shown to satisfy energy conservation and reciprocity relationships in electromagnetic theory.

I. INTRODUCTION

Considerable effort has been made to derive full wave solutions for the electromagnetic fields in horizontally stratified media in terms of mathematical functions tabulated in handbooks [1]–[6]. While they do not provide solutions to the general problem in which the complex permittivity of the media is assumed to vary arbitrarily, they provide an important basis for the analysis of electromagnetic waves in inhomogeneous media. Thus using the Green’s function technique, for instance, the solutions are formulated in terms of integral equations involving comparison functions that are chosen from the list of permittivity profiles for which closed form analytical solutions are known [7].

The method employed in this paper to compute horizontally polarized electromagnetic fields in an ionized medium with varying electron density and collision frequency profiles is based on the conversion of Maxwell’s equations into a set of loosely coupled first-order differential equations for the wave amplitudes [8]. Special attention is given to permittivity profiles with critical coupling regions where the familiar WKB approach fails.

The reflection and transmission coefficients and the characteristic surface impedance for a horizontally stratified layer of finite thickness is computed as a function of the transverse wave number and excitations of both propagating and evanescent waves are considered.

The reciprocity and realizability relationships are formulated and the numerical results are in agreement up to at least four significant figures. The computed values for the reflection and transmission coefficients and the characteristic surface impedance are also shown to be in good agreement with those derived from closed form analytical solutions for special permittivity profiles.

II. FORMULATION OF THE PROBLEM

The electromagnetic fields due to an electric line source \(J(\psi) = J(x, y, z)\), parallel to a horizontally stratified dielectric \(\varepsilon(z)\) can be expressed in terms of a complete spectrum of horizontally polarized waves (see Fig. 1). Employing the Fourier