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The Rearrangement Process in a Two-Stage Broadcast Switching Network

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Abstract—This paper considers the rearrangement process in the two-stage broadcast switching network presented by F. K. Hwang and G. W. Richards in IEEE TRANSACTIONS ON COMMUNICATIONS, October 1985. By defining a certain function it is possible to calculate an upper bound on the number of connections to be moved during a rearrangement. When each inlet channel appears twice, the maximum number of connections to be moved is found. For a special class of inlet assignment patterns in the case where each inlet channel appears three times, the maximum number of connections to be moved is also found. In the general case, an upper bound is given when the number of outlets at each second-stage switch is kept below a certain bound.

I. THE KNOWN PROPERTIES OF THE NETWORK

The network to be considered here (see Fig. 1) is identical to the one presented in [1], and it is described by the three parameters $n_1$, $n_2$, and $M$ where $n_1$ is the number of inlet channels at each first-stage switch, $n_2$ is the number of outlets at each second-stage switch, and $M$ is the number of times each inlet channel appears in the first stage.

The number of crosspoints in the network divided by the number of crosspoints in the corresponding rectangular switch is called the reduced number of crosspoints and is given by

$$C_{red} = M \left( \frac{1}{n_1} + \frac{1}{n_2} \right). \quad (1.1)$$

To minimize $C_{red}$, the fraction $M/n_2$ has to be made as close to zero as possible but the rearrangement requirement puts a lower bound on the fraction.

Hall's theorem on a system of distinct representatives [2] ensures that the network is rearrangeable, if and only if, the following condition is fulfilled.

The Rearrangement Condition: For any $n \leq n_2$, there are at least $n$ first-stage switches containing appearance of any $n$ inlet channels.

To ensure that the $n_1^2$ inlet channels are effectively rotated in the $M$ blocks the following condition is assumed to be fulfilled.

The Pair Condition: No pair of inlet channels appears on the same first-stage switch more than once throughout the $Mn_1$ first-stage switches.

All the inlet assignment patterns presented in [1] fulfill the pair condition, but instead of working with some explicit patterns, it is more advantageous in a general approach just to assume the pair condition to be fulfilled.

II. AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT BY MEANS OF THE FUNCTION $S_{M,n_1}$

As it will be seen later, the rearrangement condition as well as an upper bound on the number of connections to be moved during a rearrangement can be given by means of a function whose values are in general unknown. The function will be called $S_{M,n_1}$, and it is defined by means of another function $T_{M,n_1}$. Let $E$ be a subset of the set of inlet channels. Then,

$$T_{M,n_1}(E) := \text{The number of first-stage switches having elements from } E \text{ amongst its inlet channels}. \quad (2.1)$$

Now $S_{M,n_1}$ is defined by

$$S_{M,n_1}(n) := \min \{ T_{M,n_1}(E) \mid E \text{ has } n \text{ elements} \}. \quad (2.2)$$

$S_{M,n_1}$ is an increasing function, it depends on the inlet assignment pattern chosen, and $S_{M,n_1}(n)$ denotes the smallest number of first-stage switches that $n$ inlet channels can appear on.

In terms of $S_{M,n_1}$, we have

The network is rearrangeable if and only if

$$n \leq S_{M,n_1}(n) \text{ for all } n \leq n_2.$$ 

This means that the optimal choice for $n_2$ is

$$n_2 = \max \{ n \mid n \leq S_{M,n_1}(n) \}. \quad (2.3)$$

An upper bound on the number of connections to be moved during a rearrangement can be found in terms of a sequence...
The definition is taken from the multiplication table of a group. 

Result 2.1: Let \( r \) denote the integer with the property: \( s_i \leq n_2 - 1 \) and \( s_{i+1} \geq n_2 \). Then the number of connections to be moved during a rearrangement will never exceed \( r \).

Proof: To prove the result, consider the rearrangement algorithm given in [1]. It is characterized by the blocking relationship tree and its associated levels. The development of the tree stops at the first level where an idle switch arises. See Fig. 2. The important fact, given in [1], is that the number of connections to be moved is one less than the number of inlet switches.

Now the idea is to step through all levels of the blocking relationship tree, and in each level, keep an eye on the total number of inlet channels that have appeared so far, and to see how many first-stage switches they necessarily have appearance on. Sooner or later a level is reached where so many inlet channels have appeared, that the number of switches they demand, exceed the number of busy lines, which is, at most, \( n_2 - 1 \). When this occurs, at least one switch is idle.

The sequence \( \{s_k\} \) is defined so that \( s_k \) is the lower bound on the number of first-stage switches that have appeared in the first \( k \) levels of the blocking relationship tree. This fact is explained in Fig. 3. Therefore, an idle switch must arise in the first level where \( s_k \geq n_2 \), which according to the definition of \( r \) is level \( r + 1 \). This concludes the proof of Result 2.1.

To use Result 2.1 on a given network, it is sufficient to know \( S_{M,n_2}(n) \) for \( n = 1, 2, \ldots, n_2 + 1 \). In the case \( M = 2 \), \( S_{M,n_2}(n) \) is independent of \( n_1 \) and for \( n \leq 6 \) it takes the following values:

\[
S_2(1) = 2, \quad S_2(2) = 3, \quad S_2(3) = 4, \quad S_2(4) = 4, \quad S_2(5) = 5, \quad S_2(6) = 5.
\]

This means that the optimal choice for \( n_2 \) is 5, as given in [1].

The sequence \( \{s_k\} \) takes for \( k \leq 3 \), the values

\[
s_1 = 2, \quad s_2 = S_2(3) = 4, \quad s_3 = S_2(5) = 5 = n_2
\]

from which we see that the integer \( r \) defined in Result 2.1 equals 2 proving.

Result 2.2: The number of connections to be moved during a rearrangement will never exceed 2, when \( M = 2 \).

Fig. 2 shows a situation where two connections must be moved. This means that the maximum number of connections to be moved is two.

III. THE MAXIMUM NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT FOR A SPECIAL CLASS OF INLET ASSIGNMENT PATTERNS WHEN \( M = 3 \)

In this section, we consider the case \( M = 3 \). Fix an inlet assignment pattern and let the switches in each of the 3 blocks be numbered 0, 1, \ldots, \( n_1 - 1 \). We will show that this inlet assignment pattern induces a latin square of order \( n_1 \). Define the \( n_1 \times n_1 \) matrix \( Z \) by

\[
\text{ij'te element in } Z \text{ is the number of the switch in block 3, containing the common element of switch i from block 1 and switch j from block 2.}
\]

This definition is taken from [3] and the pair condition ensures that \( Z \) is a latin square. We restrict the calculation of \( S_{M,n_2} \) to inlet assignment patterns where the induced latin square is the multiplication table of a group. Result 4.1 applied to the case

![Diagram showing the development of the blocking relationship tree. Here the tree stops in level 3 and the rearrangement is done by moving channel 11 to switch 13 and channel 1 to switch 12.](image)
$M = 3$ together with the upper bound $U_3$ given in Appendix A yields

$$S_{3,n_1}(1) = 3, \quad S_{3,n_1}(2) = 5, \quad S_{3,n_1}(3) = 6,$$

$$S_{3,n_1}(5) = 8, \quad S_{3,n_1}(7) = 9, \quad S_{3,n_1}(10) = 11,$$

$$S_{3,n_1}(4) = 6 \text{ or } 7, \quad S_{3,n_1}(6) = 8 \text{ or } 9, \quad S_{3,n_1}(8) = 9 \text{ or } 10,$$

$$S_{3,n_1}(9) = 10 \text{ or } 11, \quad S_{3,n_1}(11) = 11 \text{ or } 12,$$

$$S_{3,n_1}(13) = 12 \text{ or } 13 \quad S_{3,n_1}(14) = 12 \text{ or } 13.$$

We will now determine when $S_{3,n_1}(4) = 6$. If $S_{3,n_1}(4) = 6$ then there must exist four inlet channels appearing in two switches in block 1, in two switches in block 2, and in two switches in block 3. (If all four inlet channels appeared on the same switch in one of the blocks then they would appear on four different switches in the remaining two blocks.) In the language of latin squares this means we can find four entries appearing in two rows and two columns so that these four entries contain only two different element which we call $x_1$ and $x_2$. Since the latin square corresponds to the multiplication table of a group, the two rows correspond to two group elements $a_1$, $a_2$, and the two columns correspond to two group elements $b_1$, $b_2$ so that the multiplication table for $\{a_1, a_2\} \times \{b_1, b_2\}$ is

$$\begin{array}{cccc}
  b_1 & b_2 \\
  a_1 & x_1 & x_2 \\
  a_2 & x_3 & x_4 \\
\end{array}$$

This yields $a_1 = x_1b_1^{-1}$ and $a_2 = x_2b_1^{-1}$ and therefore $x_1b_1^{-1}b_2 = x_2$ and $x_2b_1^{-1}b_1 = x_3$ implying $(b_1^{-1}b_2)^2 = 1$. Since an element of order 2 exist, if and only if, 2 divides the order of the group, we have proved

$$S_{3,n_1}(4) = \begin{cases} 6 \text{ when 2 divides } n_1 \\ 7 \text{ else} \end{cases}$$

In Appendix A, similar methods are used to find $S_{3,n_1}(n)$ when $n = 11$ and 13. The remaining values can be calculated the same way. We therefore have the following.

**Result 3.1:** Assume $M = 3$ and $n_1 \geq 4$. For all inlet assignment patterns where the induced latin square is the multiplication table of a group, we have:

1) $S_{3,n_1}$ takes the values in Table I.

2) The optimal choice for $n_1$ (assuming $S_{3,n_1}(14) = 13$ for all inlet assignment patterns), and the maximum number of connections to be moved during a rearrangement are the numbers given in Table II.

When $n_1$ is a multiple of four or five, it is easy to construct states where a rearrangement requires 3 connections to be moved. Figs. 4 and 5 show a state where 4 (5) connections has to be moved. The upper bounds given in Table II are therefore the maximum number of connections to be moved.

Can the results in this section be extended so that they include arbitrary latin squares? The answer is no. Consider the following latin square:

$$\begin{array}{cccc}
  0 & 1 & 2 & 3 \\
  1 & 0 & 3 & 4 \\
  2 & 4 & 0 & 1 \\
  3 & 2 & 4 & 0 \\
  4 & 3 & 1 & 2 \\
\end{array}$$

Select the four entries in the two upper rows and the two left columns and conclude that $S_{3,n_1}(4) = 6$ in this case where 2 does not divides $n_1 = 5$.

IV. AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT WHEN $M$ IS ARBITRARY

When $n_1$ and $M$ grow, it becomes very time consuming to calculate the values of $S_{M,n_1}$. It would, therefore, be advantageous if upper and lower bounds could be given. In Appendix A we prove the following.

**Result 4.1:** For all inlet assignment patterns fulfilling the pair condition the following estimate is valid:

$$S_{M,n_1}(n) \leq GM(n) \quad \text{for all } n \leq n_1^2$$

where

$$GM(n) = \begin{cases} (p+1)M - 1 \text{ when } p^2 + 1 \leq n \leq p^2 + p \\ \frac{(p+1)M}{p^2} \text{ when } p^2 + p + 1 \leq n \leq (p+1)^2 \end{cases}.$$}

Since $GM(n) \leq S_{M,n_1}(n)$, the network is rearrangeable as long as $n_2 = \max \{n | n \leq GM(n)\}$. But (4.2) gives that max $\{n | n \leq GM(n)\} = M(M + 1) - 1$ so

$$n_2 = M(M + 1) - 1$$

well known from [1].

To get an upper bound on the number of connections to be moved during a rearrangement the following sequence $\{g_k\}$ [compare to (2.4)] is defined

$$g_1 := GM(1) \quad \text{and} \quad g_{k+1} = GM(g_k + 1).$$

$g_k$ has the following two obvious properties:

1) $g_k \leq g_4$ for any $k$ and $g_k$ is therefore a lower bound on the number of first-stage switches that have appeared in the first $k$ levels of the blocking relationship tree.

2) Let $m$ be the integer with the property $g_m = n_2 - 1$ and $g_{m+1} \geq n_1$. Then $m$ is an upper bound on the number of connections to be moved during a rearrangement. It is now easy to verify the following.

**Result 4.2:** If $n_2 \leq M(M + 1) - 1$ then the number of
connections to be moved during a rearrangement will for
M = 21 never exceed the numbers given in Table III.
In Appendix C it is proven that:
Result 4.3: If \( n_2 \leq M(M + 1) - 1 \), then the number of
connections to be moved during a rearrangement will never
exceed \( 3 + \lceil (\ln 2)\ln^2 M / \ln M \rceil \) where \( \lceil x \rceil \) is the smallest
integer bigger than or equal to \( x \).
Result 4.3 is not the best obtainable but it shows that the
number of connections to be moved grow at most logarithmic
in \( M \).

V. THE INLET ASSIGNMENT PATTERN AND FINITE GEOMETRY
In this section, results from the theory of finite geometries is
used to examine the inlet assignment pattern.

According to the definition [4, p. 251], a geometric \( k \) net is
a set of points together with a set of lines appearing in \( k 
\)
different parallel classes such that
1) each point belongs to exactly one line of each parallel
class
2) if \( l_1 \) and \( l_2 \) are lines of different parallel classes, then \( l_1 
\)
and \( l_2 \) have exactly one point in common
3) there are at least two points on each line.
TABLE II
THE MAXIMUM NUMBER OF OUTLETS AT EACH SECOND STAGE SWITCH AND THE MAXIMUM NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT

<table>
<thead>
<tr>
<th>n1</th>
<th>Optimal value for n2</th>
<th>The maximum number of connections to be moved during a rearrangement</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>4 or 5</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>2, 3, or 5</td>
<td>13</td>
<td>4</td>
</tr>
</tbody>
</table>

TABLE III
AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT WHEN n1 ≤ M(M + 1) - 1

<table>
<thead>
<tr>
<th>m</th>
<th>The upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

Consider a network with n1, M, and an inlet assignment pattern given. The n1 inlet channels correspond to the set of points. Each of the M blocks corresponds to a parallel class, and each switch in a block corresponds to a line in this parallel class. Since each inlet channel is present exactly once in each block, 1) is fulfilled. The pair condition ensures that 2) is fulfilled, and for n1 ≥ 2 the inlet assignment pattern given is a geometric M-net of order n1.

This connection to k net can be used to find the highest possible value of M before the pair condition is violated. For n1 ≤ 9, we have (4, ch. 8).

The function S_{M,n1} is known in general to depend on the inlet assignment pattern chosen. To be more precise an equivalence relation is introduced. Let P1 and P2 be two inlet assignment patterns in the same network, i.e., M and n1 is fixed. Then,

P1 - P2 ≡ S_{M,n1}, P1 = S_{M,n1}, P2.

The equivalence relation splits the set of inlet assignment patterns for the network into classes and in order to obtain the best network an inlet pattern that makes S_{M,n1} as big as possible has to be chosen.

When M = 2, there is for any n1 only one class, and it is therefore impossible to improve the network by using inlet patterns different from the one used in the Fig. 2.

When M = 3, the results in Section III prove that S_{i,n1} depends on n1 and for n1 ≥ 5 there are in general more than one class. From lemma 1 in [3], it can be seen that for n1 a prime all inlet assignment patterns made from the subarrays given in [1] are contained in only one class.

For a general M and n1, it seems very difficult to determine the classes. Then it seems more practical to find an useful upper bound U_M on S_{M,n1}, which can be used to decide whether or not a given inlet pattern makes S_{M,n1} big enough. In [4] and [5], geometric nets are used to construct projective planes, and it is not unlikely that methods and results there can be helpful in finding an useful upper bound.

VI. CONCLUSIONS
In this paper, an upper bound on the number of connections to be moved during a rearrangement in a two-stage broadcast switching network is found. In general, the bound is given in terms of the function S_{M,n1}, which means that when the values of S_{M,n1} are known, then the upper bound is easily calculated.

When M = 2 the function S_{M,n1} is independent of n1, and the optimal choice for n1 is 11 when n1 is a multiple of 4, it is 12 when n1 is a multiple of 5 but not 4, and it is 13 in the other cases. The maximum number of connections to be moved during a rearrangement is 3 when n1 is a multiple of 4 or 5. When n1 is not a multiple of 4 or 5, the maximum number of connections to be moved is 5 and when n1 is a prime it is 4.

In the case where M is arbitrary the pair condition is used to find a lower bound G_M on S_{M,n1}, and this lower bound yields that the number of connections to be moved during a rearrangement grows, at most, logarithmic as a function of M when the number of outlets at each second-stage switch is not exceeding M(M + 1) - 1.

Finally, the close connection between the inlet assignment pattern and finite geometry is considered.

APPENDIX A
THE CALCULATIONS OF S_{2,n1}
We first find an upper bound U_1 for S_{2,n1}. Let for 1 ≤ n ≤ 14, U_1 be defined by Table V.

Result A.1: Assume n1 ≥ 4. For all inlet assignment patterns where the induced latin square is the multiplication table of a group we have S_{2,n1}(n) ≤ U_1(n), n = 1, 2, ..., 13. S_{2,n1}(14) ≤ 13 when the induced latin square...
We only prove the lemma in the case where \( |C_j \cap C_l| \geq 3 \) for \( j = 2, 3 \). Then there are three possibilities to consider: 1) \( ab \) is the unknown element, i.e., we do not know whether or not \( ab \in \{1, a, b, c\} \). 2) \( b^2 \) is the unknown, or 3) \( cb \) is the unknown element.

Assume that \( ab \) is the unknown element: Then \( b^2 = a \) or \( b^2 = 1 \). If \( b^2 = a \) then \( b \) has order 4 and the subgroup generated by \( b \) is a subgroup of order 4. If \( b^2 = 1 \) we have \( cb = a \) implying \( cb^2 = ab \). But then \( a^2 = 1, b^2 = 1, c = ab = ba \) and therefore \( \{1, a, b, c\} \) is the Klein Four Group \( \{2, 3\} \).

Assume that \( b^2 \) is the unknown element: Then \( ab = c \). Multiplying by \( a \) from left gives \( b = ac \). Now \( cb = 1 \) or \( cb = a \). If \( cb = 1 \) then \( ab = acb = b^3 \) implying that \( b \) has order four. If \( cb = a \) then \( a^2 = acb = b^3 \) and then \( ab = ba = c \) implying that \( \{1, a, b, c\} \) is the Klein Four Group.

Assume that \( cb \) is the unknown element; then \( ab = c \). Multiplying by \( b \) from left gives \( a = cb \). Now \( cb \) is the unknown element. Therefore \( b^2 = 1 \) or \( b^2 = a \). If \( b^2 = 1 \) then \( a = cb = a \) implying that \( \{1, a, b, c\} \) is the Klein Four Group and if \( b^2 = a \) then \( b \) has order 4. Since we have now covered all cases the proof of lemma A.1 is completed.

We now proceed with the calculation of \( S_{3,n} \). The fact that no element appears more than once in a row (column) will be used without comment.

For \( n = 11 \) we only use the result from the standard group theory, see for example [6, chap. 1].6. Let \( p \) denote the greatest prime that divides \( n \). Then \( G \) has an element of order \( p \) and, therefore, a cyclic subgroup of order \( p \). When \( p \) is odd, we get the result by choosing the rows, columns, and entries among the elements of this subgroup. If \( n \) is a power of 2, \( G \) is a two-group and therefore it has a subgroup of order 4, which is either \( Z_4 \) or \( Z_2 \times Z_2 \). Conclude by choosing rows, columns, and entries among this subgroup. When \( n \) is a power of 3, \( G \) is a three-group and therefore it contains a subgroup of order 9 (either \( Z_9 \) or \( Z_3 \times Z_3 \)). Finally, if \( n = 2^{2k} \), then \( G \) has a subgroup of order \( 2^k \) and a subgroup of order \( 3^k \). Leaving only \( n_1 = 6 \) is unknown. But \( Z_6 \) and \( Z_9 \) are the only groups of order 6 and \( U_3 \) is known to be an upper bound in these cases.

For \( n = 14 \) and the latin square equal to \( Z_4 \times Z_4 \), the author has only been able to select 14 entries in a way that proves \( S_{3,14}(11) = 14 \). When the latin square corresponds to a group of the form \( Z_4 \times \cdots \times Z_4 \) it is therefore in doubt whether \( S_{3,14}(11) = 14 \) or 14.

Result 4.1 in the case \( M = 3 \) and Result A.1 determine \( S_{3,n}(n) \) when \( n = 1, 2, 3, 5, 7, 10 \). Since the calculation of \( S_{3,n}(n) \) proceeds the same way for \( n = 4, 6, 8, 9, 11, 12, 13, 14 \), we only consider the two most difficult cases \( n = 11 \) and \( n = 13 \) here. We start with the following.

Lemma A.1: Let \( G \) be a finite group of order at least 4. Let \( 1, a, b, c \) be four distinct elements of \( G \) where \( 1 \) denotes the identity element. Assume that \( a^2 = 1 \). Consider the multiplication table for \( \{1, a, b, c\} \). Put for \( j = 1, 2, 3, 4, \)

\[ C_j := \{ \text{the set of elements in column } j \} \]

of the multiplication table

\[
\begin{array}{cccc}
1 & a & b & c \\
1 & a & b & c \\
a & a & ab & ac \\
b & b & ba & bc \\
c & c & ca & cb \\
\end{array}
\]

If \( \{ C_1 \cap C_j \} \geq 3 \) for \( j = 2, 3 \), then \( G \) contains a subgroup of order 4.

Proof: Since \( \{ C_1 \cap C_j \} \geq 3, \) we must have \( ba = c = ca = b \). It is also clear that \( \{ C_1 \cap C_j \} \geq 3 \) implies that \( ba = c = ca = b \).

The multiplication table then looks like:

\[
\begin{array}{cccc}
1 & 1 & a & b \\
1 & 1 & a & b \\
a & 1 & a & 1 \\
b & a & c & b \\
c & b & c & 1 \\
\end{array}
\]

where \( a := a^2, b := b^2, \) and \( c := c^2 \). G is therefore row 4 and column 3 or 4 that have only two of the elements 1, a, b. Since \( a^2 \in \{1, a, b\} \), then \( a = b^2 = 1 \) and \( c = a^{-1} \). But this implies that \( a = b^2 = 1 \).

Assume \( a^2 \in \{1, a, b\} \). Since row 2 and column 2 both contains \( a, b, \) we have \( ba = 1, ca = b, \) and \( ab = 1 \) and \( ad = b \). This implies \( b = a^{-1} \) and \( c = d = a^{-2} \). Since \( a, b, c \) are distinct \( a \) has order at least four. Use that \( b = a^{-1} \) all
three elements 1, a, a^{-1} to conclude that a has order 4 and, therefore, 4 divides n. Since a^2 \not\in \{1, a, b\}; then a^2 = 1 or a^2 = b. If a^2 = 1 then ca = b and therefore c = ba. Since row 2 contains 1, a, b, we get ad = b and therefore d = ab. The same argument is applied to row 3 gives b^2 = 1 and bd = a or b^2 = a and bd = 1. If b^2 = a then b is an element of order 4. If b = 1 then a = b^2 = b(bd) = ba = c and then \{1, a, b, c\} = \{1, a, b, d\} is the Klein Four Group.

If a^2 = b then ba = 1 or ca = 1, ba = 1 implies a^2 = 1 and b^2 = a and makes it impossible for row 4 and column 4 to contain more than one of the elements 1, a, b, c. Therefore, ca = 1 and in row 2, we get ad = 1. Then c = a^{-1} = d and \{1, a, b, c\} = \{1, a, a^{-1}, a^{-1}\} and it follows as before that a has order 4.

Assume ca \in \{1, a, b\}. Look at column 2 and conclude that ba = 1 and therefore a^2 = b. Row 3 implies that b^2 = a or bd = a, b^2 = a contradicts the assumption that row 4 and column 4 contains two of the elements 1, a, b. Therefore, bd = a. Since 1 = ba = ab, it must be column 3 that contains all the elements 1, a, b, c and by looking at row 3, we get that cb = a which implies c = a^2. Multiply bd = ab by a from the left and get d = a^3 = c. We now have the elements 1, a, a^2, a^{-1} and as before, we conclude that a has order 4.

Case 2): Assume that we have 11 inlet channels appearing on four switches in block 1, on three switches in block 2, and on four switches in block 3. In the induced latin square, we can find 11 entries appearing in four rows and four columns so that the 11 entries contains only four different elements. By appropriate multiplication as in the former case, we may assume that the row elements are 1, a, b, c and the column elements are 1, a, b, c. Since only four different elements are present in the 11 entries, we have C_1 = C_2 = C_3 = C_4 and \{C_1 \cup C_2\} = 3 for \j = 2, 3, 4. We assume that C_1 = C_2. If a^2 = 1, we conclude by lemma A.1 that 4 divides n_1. If a^2 = b, then ba = 1 or ba = c. Since ba = 1 forces ca = c, we conclude that ba = c. Then ca = 1 forces ba = ba = a^4 and therefore, a is an element of order 4 and 4 divides n_1. When a^2 = c the same argument yields that a has order 4.

n = 13: Assume S_n(13) = 12. If the 13 inlet channels appeared on only three switches in one of the blocks the would force the inlet channels to appear on at least five switches in each of the remaining blocks. We may, therefore, assume that in the induced latin square, there exist 13 entries containing only four different elements x_1, x_2, x_3, x_4 and appearing in four rows and four columns. There exists at least one row and one column each containing all the elements x_1, x_2, x_3, x_4. By appropriate multiplication as in Case 1) of n = 11, we may assume that the four rows and the four columns correspond to the elements 1, a, b, c. If there exist two columns containing these four elements, we can proceed exactly as in Case 2) of n = 11 and conclude that 4 divides n_1. Hence, assume that \{C_1 \cup C_2\} = 3 for \j = 2, 3, 4. From this we see that if a^2 = 1 then by lemma A.1, we conclude that 4 divides n_1.

| \begin{array}{ccc} 1 & a & b \\ a & 1 & 1 \\ b & 1 & c \\ c & 1 & 1 \end{array} |

Assume that a^2 is unknown, i.e., we do not know whether or not a^2 \in C_1. Then ba = 1 or ba = c. If ba = 1 then ca = b. Then we see b = a^{-1} and c = a^{-2}. Therefore, we only have to consider the multiplication table of 1, a, a^{-1}, a^{-2}. Since \{C_1 \cup C_2\} = 3, we get a^{-1} \in \{1, a\} or a^{-2} \in \{1, a\} yielding a has order 4 or 5. The conditions \{C_1 \cup C_2\} = 3 for \j = 2, 4 give no further information.

If ba = c then ca = b or ca = 1. ca = b implies a^2 = c implying a^2 = 1 and by lemma A.1, we conclude that 4 divides n_1. ca = 1 implies c = a^{-1} and bca = 1 and we have since the multiplication table of 1, a, a^{-1}, a^{-2} from which we conclude that 4 or 5 divides n_1.

Assume that ab is unknown. Then a^2 = 1 or a^2 = c. If a^2 = 1 lemma A.1 yields that 4 divides n_1. If a^2 = c then ca = 1 or ca = b. ca = 1 implies a^2 = 1 and by using \{C_1 \cup C_2\} = 3 this implies b = c or column 3 contains the same element twice; in both cases a contradiction. Therefore, ca = b implying b = a^{-1}, i.e., \{1, a, b, c\} = \{1, a, a^{-1}, a^{-2}\}. As in the case a^2 unknown, this yields 4 divides n_1 or 5 divides n_1.

Since ac unknown proceeds the same way and gives the same result, it will be omitted. The calculation of S_n(13) is therefore completed.

APPENDIX B

A LOWER ESTIMATE FOR S_{M_1}

In this Appendix, a proof of Result 4.1 is given. The M blocks in the first stage is denoted by B_1, B_2, \ldots, B_M and the M_1 first-stage switches are denoted I_1, I_2, \ldots, I_{M_1}.

Let E be of subset of the set of inlet channels, assume that E has n elements and put

k(E) := \text{The number of switches in } B, \text{ having elements from } E \text{ among their inlet channels.}

a_j(E) := \text{The number of elements from } E \text{ appearing on } I_j.

If x is a real number \lceil x \rceil \text{ denoted the smallest integer not less than } x. \text{ In this notation, a switch exist, which has at least } \lceil n/k(E) \rceil \text{ elements from } E \text{ among its inlet channels. Therefore,}

a(E) \geq \lceil n/k(E) \rceil. \tag{B.1}

The definition of k(E) and T_{M_1}(2.1) ensures

T_{M_1}(E) = k(E).M. \tag{B.2}

The pair condition gives

T_{M_1}(E) = a(E)(M-1) + k(E). \tag{B.3}

From 2.2, B, 2.2, and B, 3, we get

S_{M_1}(n) \geq \min \{ \max \left( k(E)M, \left[ \frac{n}{k(E)} \right] \right), \ \cdot \ (M-1) + k(E) \} \text{ E has } n \text{ elements.} \tag{B.4}

When E runs through all subsets with n elements, k(E) runs through a subset of \{1, 2, \ldots, n\} yielding

S_{M_1}(n) \geq \min \{ \max \left( kM, \left[ \frac{n}{k} \right] \right), \ \cdot \ (M-1) + k \} \text{ E has } n \text{ elements.} \tag{B.5}

Put f_x(k) := kM and f_{2x}(k) := \lfloor n/k \rfloor (M-1) + k. To finish the proof of Result 4.1, it is enough to show that

\min \{ \max \{ f_x(k), f_{2x}(k) \} \} = G_\mu(n) \text{ for any } n \leq n_1^2.

Choose p as the integer having the property

n = p^2 + x \text{ where } 1 \leq x \leq 2p + 1. \tag{B.6}
First assume that $1 \leq x \leq p$. Then for $k = 0, 1, \ldots, p - 1$, we have
\[
f_{2,n}(p-k) = \left\lfloor \frac{p^2 + x}{p-k} \right\rfloor (M-1) + p - k \geq (p + k + 1)(M-1) + p - k = (p + 1)M + k(M-2) - 1 \geq (p + 1)M - 1.
\]
Since $f_{2,n}(p) = (p + 1)M - 1$ and $f_{i}(k) \geq (p + 1)M$ for $k \geq p + 1$, it is now proved that
\[
\min_{k} \{ \max \{ f_{i}(k), f_{2,n}(k) \} \} = (p + 1)M - 1 \quad \text{when } n = p^2 + x \text{ and } 1 \leq x \leq p. \quad (B.7)
\]
Assume now that $p + 1 \leq x \leq 2p + 1$. Then for $k = 0, 1, \ldots, p - 1$ we have
\[
f_{2,n}(p-k) = \left\lfloor \frac{(p-k)(p+k+1) + k^2 + k + x-p}{p-k} \right\rfloor (M-1) + p - k \geq (p + k + 2)(M-1) + p - k = (p + 1)M + (k + 1)(M-2) \geq (p + 1)M.
\]
Since $f_{2,n}(p+1) = (p + 1)M$ and $f_{i}(k) \geq (p + 1)M$ for $k \geq p + 1$ it is proved that
\[
\min_{k} \{ \max \{ f_{i}(k), f_{2,n}(k) \} \} = (p + 1)M
\]
when $n = p^2 + x$ and $p + 1 \leq x \leq 2p + 1$. \quad (B.8)

B.5, B.7, and B.8 concludes the proof of Result 4.1.

APPENDIX C

PROOF OF RESULT 4.3

In this Appendix, we prove Result 4.3. It is enough to show if $k \geq 3 + 1/(\ln 2)\ln(M^2\ln M)$ then $g_{k} \geq M^2$. The following lemma will be helpful.

Lemma C.1: For $k \geq 2$ and $M \geq 2$ the following estimate is valid:
\[
g_{k} \geq M^{2-k} \left[ 1 + \left( \sum_{l=0}^{k/2} 3^{l} \right) M^{-2} \right]. \quad (C.1)
\]

Proof: We have $g_{2} = G_{n}(M + 1) \geq M(M+1)^{1/2} \geq M^{3/2}(1 + \frac{1}{3M}) \geq M^{3/2}(1 + 1/3M^2)$. This proves the lemma for $k = 2$. Assume the lemma is true for some $k$. Then,
\[
g_{k+1} = G_{n}(g_{k} + 1) \geq M(g_{k} + 1)^{1/2} \geq M \left( M^{2-k} \left( 1 + \left( \sum_{l=0}^{k/2} 3^{l} \right) M^{-2} \right) \right)^{1/2} \geq M^{2-k} \left( 1 + \left( \sum_{l=0}^{k/2} 3^{l} \right) M^{-2} \right) \geq M^{2-k} \left( 1 + \left( \sum_{l=0}^{k/2} 3^{l} \right) \right) \geq M^{2-k} \left( 1 + \left( \sum_{l=0}^{k/2} 3^{l} \right) M^{-2} \right) = g_{k}.
\]

and the proof is completed.

REFERENCES


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