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Published in:
IEEE Transactions on Communications

Link to article, DOI: 10.1109/26.2774

Publication date: 1988

Document Version
Publisher's PDF, also known as Version of record

Citation (APA):

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The Rearrangement Process in a Two-Stage Broadcast Switching Network

Søren B. Jacobsen

Abstract—This paper considers the rearrangement process in the two-stage broadcast switching network presented by F. K. Hwang and G. W. Richards in IEEE Transactions on Communications, October 1985. By defining a certain function it is possible to calculate an upper bound on the number of connections to be moved during a rearrangement. When each inlet channel appears twice, the maximum number of connections to be moved is found. For a special class of inlet assignment patterns in the case where each inlet channel appears three times, the maximum number of connections to be moved is also found. In the general case, an upper bound is given when the number of outlets at each second-stage switch is kept below a certain bound.

I. THE KNOWN PROPERTIES OF THE NETWORK

The network to be considered here (see Fig. 1) is identical to the one presented in [1], and it is described by the three parameters $n_1$, $n_2$, and $M$ where

- $n_1$ is the number of inlet channels at each first-stage switch,
- $n_2$ is the number of outlets at each second-stage switch, and
- $M$ is the number of times each inlet channel appears in the first stage.

The number of crosspoints in the network divided by the number of crosspoints in the corresponding rectangular switch is called the reduced number of crosspoints and is given by

$$C_{red} = M(1/n_1 + 1/n_2).$$

To minimize $C_{red}$, the fraction $M/n_2$ has to be made as close to zero as possible but the rearrangement requirement puts a lower bound on the fraction.

Hall's theorem on a system of distinct representatives [2] ensures that the network is rearrangeable, if and only if, the following condition is fulfilled.

**The Rearrangement Condition:** For any $n \leq n_2$, there are at least $n$ first-stage switches containing appearance of any $n$ inlet channels.

To ensure that the $n_1^2$ inlet channels are effectively rotated in the $M$ blocks the following condition is assumed to be fulfilled.

**The Pair Condition:** No pair of inlet channels appears on the same first-stage switch more than once throughout the $Mn_1$ first-stage switches.

All the inlet assignment patterns presented in [1] fulfill the pair condition, but instead of working with some explicit patterns, it is more advantageous in a general approach just to assume the pair condition to be fulfilled.

II. AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT BY MEANS OF THE FUNCTION $S_{M,n_1}$

As it will be seen later, the rearrangement condition as well as an upper bound on the number of connections to be moved during a rearrangement can be given by means of a function whose values are in general unknown. The function will be called $S_{M,n_1}$, and it is defined by means of another function $T_{M,n_1}$. Let $E$ be a subset of the set of inlet channels. Then,

$$T_{M,n_1}(E) := \text{The number of first-stage switches having elements from } E \text{ amongst its inlet channels.}$$

Now $S_{M,n_1}$ is defined by

$$S_{M,n_1}(n) := \min \{ T_{M,n_1}(E) | E \text{ has } n \text{ elements} \}.$$  \hspace{1cm} (2.2)

$S_{M,n_1}$ is an increasing function, it depends on the inlet assignment pattern chosen, and $S_{M,n_1}(n)$ denotes the smallest number of first-stage switches that $n$ inlet channels can appear on.

In terms of $S_{M,n_1}$, we have

The network is rearrangeable if and only if

$$n \leq S_{M,n_1}(n) \text{ for all } n \leq n_2.$$  \hspace{1cm} (2.3)

This means that the optimal choice for $n_2$ is

$$n_2 = \max \{ n | n \leq S_{M,n_1}(n) \}.$$  \hspace{1cm} (2.3)

An upper bound on the number of connections to be moved during a rearrangement can be found in terms of a sequence...
The definition is taken from multiplication table of a group. Result assignment pattern induces a latin square of order inlet assignment patterns where the induced latin square is the that the switches in each of the be numbered 0, the number of first-stage switches that have appeared in the number of inlet channels that have appeared so far, and to see how many first-stage switches they necessarily have appearance on. Sooner or later a level is reached where so many inlet channels have appeared, that the number of switches they demand, exceed the number of busy lines, which is, at most, $n_2 - 1$. When this occurs, at least one switch is idle.

The sequence $\{s_k\}$ is defined so that $s_k$ is a lower bound on the number of first-stage switches that have appeared in the first $k$ levels of the blocking relationship tree. This fact is explained in Fig. 3. Therefore, an idle switch must arise in the first level where $s_k \geq n_2$, which according to the definition of $r$ is level $r + 1$. This concludes the proof of Result 2.1.

To use Result 2.1 on a given network, it is sufficient to know $S_{M,n_1}(n)$ for $n = 1, 2, \ldots, n_2 + 1$. In the case $M = 2$, $S_{M,n_1}$ is independent of $n_1$ and for $n \leq 6$ it takes the following values:

\[
S_2(1) = 2, \quad S_2(2) = 3, \quad S_2(3) = 4, \quad S_2(4) = 4, \quad S_2(5) = 5, \quad S_2(6) = 5.
\]

This means that the optimal choice for $n_2$ is 5, as given in [1]. The sequence $\{s_k\}$ takes for $k \leq 3$, the values

\[
s_1 = 2, \quad s_2 = s_2(3) = 4, \quad s_3 = s_2(5) = 5 = n_2
\]

from which we see that the integer $r$ defined in Result 2.1 equals 2 proving.

Result 2.2: The number of connections to be moved during a rearrangement will never exceed $2r$.

Fig. 2 shows a situation where two connections must be moved. This means that the maximum number of connections to be moved is two.

**III. The Maximum Number of Connections to be Moved during a Rearrangement for a Special Class of Inlet Assignment Patterns when $M = 3$**

In this section, we consider the case $M = 3$. Fix an inlet assignment pattern and let the switches in each of the 3 blocks be numbered 0, 1, $\ldots$, $n_1 - 1$. We will show that this inlet assignment pattern induces a latin square of order $n_1$. Define the $n_1 \times n_1$ matrix $Z$ by

ij'te element in $Z$ is the number of the switch in block 3, containing the common element of switch $i$ from block 1 and switch $j$ from block 2.

This definition is taken from [3] and the pair condition ensures that $Z$ is a latin square. We restrict the calculation of $S_{3,n_1}$ to inlet assignment patterns where the induced latin square is the multiplication table of a group. Result 4.1 applied to the case

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14 \\
15 & 16 & 17 & 18 & 19 \\
20 & 21 & 22 & 23 & 24
\end{bmatrix}
\]

Fig. 3. Inlet channel $a_0$ is present at $s_1 = S_{M,n_1}(1)$ first-stage switches. None of them are idle meaning that $s_1 + 1$ channels are present. They have appearance on at least $s_1 = S_{M,n_1}(1)$ first-stage switches. If none of them are idle we now have a total of $s_1 + 1$ channels present, etc.
When \( n_1 \) and \( M \) grow, it becomes very time consuming to calculate the values of \( S_{M,n_1} \). It would, therefore, be advantageous if upper and lower bounds could be given. In Appendix A we prove the following.

Result 4.1: For all inlet assignment patterns fulfilling the pair condition the following estimate is valid:

\[
S_{M,n_1}(n) \geq G_M(n) \text{ for any } n \leq n_1^2
\]  

(4.1)

where

\[
G_M(n) = \begin{cases} 
(p + 1)M - 1 & \text{when } n^2 + 1 \leq n \leq n^2 + p \v1{p+1}M \text{ when } n^2 + p + 1 \leq n \leq (p + 1)^2.
\end{cases}
\]  

(4.2)

Since \( G_M(n) \leq S_{M,n_1}(n) \), the network is rearrangeable as long as \( n_2 \leq \max \{n|n \leq G_M(n)\} \). But (4.2) gives that max \( \{n|n \leq G_M(n)\} = M(M + 1) - 1 \) so

\[
n_2 = M(M + 1) - 1
\]  

(4.3)

well known from [1].

To get an upper bound on the number of connections to be moved during a rearrangement the following sequence \( \{ \mathcal{g}_k \} \) [compare to (2.4)] is defined

\[
\mathcal{g}_1 := G_M(1) \text{ and } \mathcal{g}_{k+1} = G_M(\mathcal{g}_k + 1).
\]  

(4.4)

\( \mathcal{g}_k \) has the following two obvious properties:

1) \( \mathcal{g}_k \leq \mathcal{g}_k \) for any \( k \) and \( \mathcal{g}_k \) is therefore a lower bound on the number of first-stage switches that have appeared in the first \( k \) levels of the blocking relationship tree.

2) Let \( m \) be the integer with the property \( \mathcal{g}_m = n_2 - 1 \) and \( \mathcal{g}_{m+1} \geq n_1 \). Then \( m \) is an upper bound on the number of connections to be moved during a rearrangement.

It is now easy to verify the following.

Result 4.2: If \( n_2 \leq M(M + 1) - 1 \) then the number of
Fig. 4. In this network having a total of 49 inlet channels, a request is made for channel 8 and four connections have to be moved.

connections to be moved during a rearrangement will for \( M \leq 2^l \) never exceed the numbers given in Table III.

In Appendix C it is proven that:

Result 4.3: If \( n_2 \leq M(M + 1) - 1 \), then the number of connections to be moved during a rearrangement will never exceed \( 3 + \lfloor (l/\ln 2)\ln(M^2\ln M) \rfloor \) where \( \lfloor x \rfloor \) is the smallest integer bigger than or equal to \( x \).

Result 4.3 is not the best obtainable but it shows that the number of connections to be moved grow at most logarithmic in \( M \).

V. THE INLET ASSIGNMENT PATTERN AND FINITE GEOMETRY

In this section, results from the theory of finite geometries is used to examine the inlet assignment pattern.

According to the definition [4, p. 251], a geometric \( k \) net is a set of points together with a set of lines appearing in \( k \) different parallel classes such that:

1) each point belongs to exactly one line of each parallel class
2) if \( l_1 \) and \( l_2 \) are lines of different parallel classes, then \( l_1 \) and \( l_2 \) have exactly one point in common
3) there are at least two points on each line.
Consider a network with \( n_1 \), \( M \), and an inlet assignment pattern given. The \( n_1^2 \) inlet channels correspond to the set of points. Each of the \( M \) blocks corresponds to a parallel class, and each switch in a block corresponds to a line in this parallel class. Since each inlet channel is present exactly once in each block, 1) is fulfilled. The pair condition ensures that 2) is fulfilled, and for \( n_1 \geq 2 \) the inlet assignment pattern given is a geometric \( M \)-net of order \( n_1 \).

This connection to \( k \) net can be used to find the highest possible value of \( M \) before the pair condition is violated. For \( n_1 \leq 9 \), we have (14, ch. 8).

The function \( S_{M,n_1} \) is known in general to depend on the inlet assignment pattern chosen. To be more precise an equivalence relation is introduced. Let \( P_1 \) and \( P_2 \) be two inlet assignment patterns in the same network, i.e., \( M \) and \( n_1 \) is fixed. Then,

\[
P_1 \sim P_2 \iff S_{M,n_1} P_1 = S_{M,n_1} P_2.
\]

The equivalence relation splits the set of inlet assignment patterns for the network into classes and in order to obtain the best network an inlet pattern that makes \( S_{M,n_1} \) as big as possible has to be chosen.

When \( M = 2 \), there is for any \( n_1 \) only one class, and it is therefore impossible to improve the network by using inlet patterns different from the one used in the Fig. 2.

When \( M = 3 \), the results in Section III prove that \( S_{S_{M,n_1}} \) depends on \( n_1 \) and for \( n_1 \geq 5 \) there are in general more than one class. From lemma 1 in [3], it can be seen that for \( n_1 \) a prime all inlet assignment patterns made from the subarrays given in [1] are contained in only one class.

For a general \( M \) and \( n_1 \), it seems very difficult to determine the classes. Then it seems more practical to find an useful upper bound \( U_M \) on \( S_{M,n_1} \), which can be used to decide whether or not a given inlet pattern makes \( S_{M,n_1} \) big enough. In [4] and [5], geometric nets are used to construct projective planes, and it is not unlikely that methods and results there can be helpful in finding an useful upper bound.

VI. CONCLUSIONS

In this paper, an upper bound on the number of connections to be moved during a rearrangement in a two-stage broadcast switching network is found. In general, the bound is given in terms of the function \( S_{M,n_1} \), which means that when the values of \( S_{M,n_1} \) are known, then the upper bound is easily calculated.

When \( M = 2 \) the function \( S_{M,n_1} \) is independent of \( n_1 \) and of the inlet assignment pattern, and two is the maximum number of connections to be moved during a rearrangement.

When \( M = 3 \) the function \( S_{M,n_1} \) depends on \( n_1 \). For a special class of inlet patterns the values of \( S_{M,n_1} \) are found, and the optimal choice for \( n_1 \) is 11 when \( n_1 \) is a multiple of 4, it is 12 when \( n_1 \) is a multiple of 5 but not 4, and it is 13 in the other cases. The maximum number of connections to be moved during a rearrangement is 3 when \( n_1 \) is a multiple of 4 or 5. When \( n_1 \) is not a multiple of 4 or 5, the maximum number of connections to be moved is 5 and when \( n_1 \) is a prime it is 4.

In the case where \( M \) is arbitrary the pair condition is used to find a lower bound \( G_M \) on \( S_{M,n_1} \), and this lower bound yields that the number of connections to be moved during a rearrangement grows, at most, logarithmic as a function of \( M \) when the number of outlets at each second-stage switch is not exceeding \( M(M + 1) - 1 \).

Finally, the close connection between the inlet assignment pattern and finite geometry is considered.

APPENDIX A

THE CALCULATIONS OF \( S_{M,n_1} \)

We first find an upper bound \( U_1 \) for \( S_{M,n_1} \). Let for \( 1 \leq n \leq 14 \), \( U_1 \) be defined by Table V.

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>( p^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>highest possible</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>( p^2 + 1 )</td>
<td></td>
</tr>
</tbody>
</table>

In the first stage before the pair condition is violated.

TABLE II

<table>
<thead>
<tr>
<th>The maximum number of connections to be moved during a rearrangement</th>
<th>Optimal value for ( n_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1 ) is a multiple of 4</td>
<td>11</td>
</tr>
<tr>
<td>( n_1 ) is a multiple of 5 but not 4</td>
<td>12</td>
</tr>
<tr>
<td>( n_1 ) is not a multiple of 4 or 5</td>
<td>13</td>
</tr>
<tr>
<td>( n_1 ) is not a multiple of 2, 3 or 5</td>
<td>13</td>
</tr>
</tbody>
</table>

TABLE III

<table>
<thead>
<tr>
<th>An upper bound on the number of connections to be moved during a rearrangement when ( n_1 \leq M(M + 1) - 1 )</th>
<th>( M )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>the upper bound ( m )</td>
<td>( M = 2 )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( M = 3 )</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>( M = 5 )</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>( M = 10, 14 )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>( M = 15, 21 )</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

TABLE IV

<table>
<thead>
<tr>
<th>The maximum number of times each inlet channel can appear in the first stage before the pair condition is violated</th>
<th>( n_1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>( p^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of ( M )</td>
<td>highest possible</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>( p^2 + 1 )</td>
</tr>
</tbody>
</table>
TABLE V

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>U3(n)</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

corresponds to a group not isomorphic to groups of the form $Z_3 \times \cdots \times Z_3$.

Proof: Assume $1 \leq n \leq 13$. By explicitly selecting the rows, columns, and entries in the latin square the way described in [3, Sect. II], we get the result when the latin square is the multiplication table of $Z_n$ (the cyclic group of order $n$). For the groups $Z_2 \times Z_2$, $Z_3 \times Z_2$, and $Z_5$ (the permutation group of three elements), it is also easy by explicit selection to show that $U_1$ is upper bound in these cases.

To prove the result in the general case, assume that $G$ is an arbitrary finite group of order $n$. In the rest of the proof, we will use the result from the standard group theory, see for example [6, chap. 1]. Let $p$ denote the greatest prime that divides $n$. Then $G$ has an element of order $p$ and, therefore, a cyclic subgroup of order $p$. When $p = 5$, we get the result by choosing the rows, columns, and entries among the elements of this subgroup. If $n$ is a power of 2, $G$ is a two-group and therefore it has a subgroup of order 4, which is either $Z_4$ or $Z_2 \times Z_2$. Conclude by choosing rows, columns, and entries among this subgroup. When $n_1$ is a power of 3, $G$ is a three-group and therefore it contains a subgroup of order 9 (either $Z_9$ or $Z_3 \times Z_3$). Finally, if $n = 2^3$, then $G$ has a subgroup of order 8 and a subgroup of order 3 leaving only $n_1 = 6$ as unsolved. But $Z_6$ and $S_3$ are the only groups of order 6 and $U_1$ is known to be an upper bound in these cases.

For $n = 14$ and the latin square equal to $Z_2 \times Z_2$, the author has only been able to select 14 entries in a way that proves $S_{n_1}(14) \leq 14$. When the latin square corresponds to a group of the form $Z_2 \times Z_3 \times Z_2$, it is therefore in doubt whether $S_{n_1}(14) = 14$.

Result 4.1 in the case $M = 3$ and Result A.1 determine $S_{n_1}(n)$ when $n = 1, 2, 3, 5, 7, 10$. Since the calculation of $S_{n_1}(n)$ proceeds the same way for $n = 4, 6, 8, 9, 11, 12, 13, 14$, we only consider the two most difficult cases $n = 11$ and $n = 13$ here. We start with the following.

Lemma A.1: Let $G$ be a finite group of order at least 4. Let $l, a, b, c$ be four distinct elements of $G$ where $l$ denotes the identity element. Assume that $a^2 = l$. Consider the multiplication table for $\{ l, a, b, c \}$. Put for $j = 1, 2, 3, 4,

\begin{align*}
C_j := & \text{ the set of elements in column } j \\
\text{of the multiplication table} & \\
| 1 & a & b & c \\
- & a & b & c \\
- & l & a & b & c \\
- & b & l & a & c \\
\end{align*}

$\text{If } |C_j \cap C_j| \geq 3$ for $j = 2, 3$ or $j = 2, 4$, then $G$ contains a subgroup of order 4.

Proof: Since $|C_j \cap C_j| \geq 3$, we must have $ba = c$ or $ca = b$ since no element appears more than once in a row or a column. Since $a^2 = l$ this implies that $ba = c$ and $ca = b$.

The multiplication table then looks as follows:

\begin{align*}
| 1 & a & b & c \\
1 & l & a & b & c \\
1 & a & l & a & c \\
1 & b & c & l & c \\
1 & c & b & a & l \\
\end{align*}

We only prove the lemma in the case where $|C_j \cap C_j| \geq 3$ for $j = 2, 3$. Then there are three possibilities to consider: 1) $ab$ is the unknown element, i.e., we do not know whether or not $ab \in \{ l, a, b, c \}$, 2) $b^2$ is the unknown, or 3) $c$ is the unknown element.

Assume that $ab$ is the unknown element: Then $b^2 = a$ or $b^2 = l$. If $b^2 = a$ then $b$ has order 4 and the subgroup generated by $b$ is a subgroup of order 4. If $b^2 = l$ we have $c = ab$ implying $c = b^{-1} = ab$. But then $a^2 = 1$, $b^2 = 1$, $ca = ab$ and therefore $\{ l, a, b, c \}$ is the Klein Four Group ($Z_2 \times Z_2$).

Assume that $b^2$ is the unknown element: Then $ab = c$. Multiplying by $a$ from left gives $b = ac$. Now $c = 1$ implies $bc = b$. If $c = 1$ then $a = ab = b^2$ implying that $b$ has order four. If $c = 1$ then $a^2 = ab = b^2$ and then $ab = ba = c$ implying $\{ l, a, b, c \}$ is the Klein Four Group.

Assume that $c$ is the unknown element; then $ab = c$. Multiplying by $a$ from left gives $a = bc$. Now $b^2 = 1$ or $b^2 = a$. If $b^2 = 1$ then $\{ l, a, b, c \}$ is the Klein Four Group and if $b^2 = a$, $b$ has order 4. Since we have now covered all cases the proof of lemma A.1 is completed.

We now proceed with the calculation of $S_{n_1}$. The fact that no element appears more than once in a row (column) will be used without comment.

$n = 11$: Assume $S_{n_1}(11) = 11$. If the 11 inlet channels appeared on only two switches in one of the blocks then they would have to appear on at least six switches in each of the remaining two blocks and the 11 inlet channels would then appear on at least 14 switches. Because of the symmetry in rows and columns, we only have to consider the following two cases: 1) when the 11 inlet channels appear on four switches in block 1, four switches in block 2, and three switches in block 3; 2) when the 11 inlet channels appear on four switches in block 1, three switches in block 2, and four switches in block 3.

Case 1: In the language of latin squares we have 11 entries containing only three different elements $(x_1, x_2, x_3)$ and appearing in four rows and four columns. The four rows (columns) correspond to four group elements $a_1, a_2, a_3, a_4$ (or $b_1, b_2, b_3, b_4$). Since it is 11 entries containing only $x_1, x_2, x_3$, three of the rows and three of the columns contain all the elements $x_1, x_2, x_3$ and the remaining row and column contains two of these three elements. After possible renaming of the group elements we may assume that we have the following table:

\begin{align*}
| b_1 & b_2 & b_3 & b_4 \\
| a_1 & x_1 & x_2 & x_3 \\
| a_2 & x_2 & x_3 & x_1 \\
| a_3 & x_3 & x_1 & x_2 \\
| a_4 & x_1 & x_2 & x_3 \\
\end{align*}

where row 1, 2, and 3 and column 1, 2, 3, or 4 contain $x_1, x_2, x_3$ while row 4 and column 4 or 3 contain two of these three elements. Multiply the row elements by $a_1$ from the left and the column elements by $b_1$ from the right and obtain

\begin{align*}
| 1 & a & b & d \\
| 1 & l & a & b \\
| a & a & a & a \\
| b & b & c & c \\
\end{align*}

where $a := a_1^{-1}a_2 = b_2b_1^{-1}$, $b := a_1^{-1}a_3 = b_2b_3^{-1}$, $c := a_1^{-1}a_4 = b_2b_4^{-1}$, and $d := b_1b_4^{-1}$ and where it is row 4 and column 3 or 4 that have only two of the elements 1, a, b.

Assume $a^2 \notin \{ l, a, b \}$. Since row 2 and column 2 both contains 1, a, b, we have $ba = 1$, $ca = b$, and $ab = 1$ and $ad = b$. This implies $b = a^{-1}$ and $c = d = a^{-2}$. Since 1, a, b, c are distinct a has order at least four. Use that row 3 has all
three elements 1, a, a⁻¹ to conclude that a has order 4 and, therefore, 4 divides n. If ba ∈ \{1, a, b, a⁻¹\}, then \(a^2 = 1 \) or \(a^2 = b\). If \(a^2 = 1\), then \(ca = b\) and therefore \(c = ba\). Since row 2 contains 1, a, b, we get \(ad = b\) and therefore \(d = ab\). The same argument is applied to row 3 gives \(b^2 = 1\) and \(bd = a \) or \(b^2 = a\) and \(bd = 1\). If \(b^2 = a\) then \(b\) is an element of order 4. If \(b^2 = 1\) then \(a^{-1} = (ba)\) and \(b^{-1} = (a^{-1})b\). Therefore, \(d = ab\) and \(d = ba\) are the Klein Four Group.

If \(a^2 = b\) then \(ba = 1\) or \(ca = 1\). \(ba = 1\) implies \(a^2 = b\) and \(ca = a^{-1}\) and \(a^{-1} = b\). It follows as before that \(a\) has order 4.

Assume \(ca \in \{1, a, b, a^{-1}\}\). Look at column 2 and conclude that \(ba = 1\) and therefore \(a^2 = b\). Row 3 implies that \(b^2 = a\) or \(bd = a\) or \(b^2 = a\) contradicts the assumption that row 4 and column 4 to contain more than one of the elements 1, a, b. Therefore, \(ca = 1\) and in column 2, we get \(ad = 1\). Then \(c = a^{-1} = d\) and \(1, a, b, c\) \(\in \{1, a, b, d\} = \{1, a, a^{-1}\}\) and it follows as before that \(a\) has order 4.

**Case 2:** Assume that we have 11 inlet channels appearing on four switches in block 1, on three switches in block 2, and on four switches in block 3. In the induced latin square, we can find 11 entries appearing in four rows and four columns so that the 11 entries contains only four different elements. By appropriate multiplication as in the former case, we may assume that the row elements are 1, a, b, c and the column elements are 1, a, b, c. Since there are four different elements present in the 11 entries, we have \(C_1 \cap C_2 \cap C_3 \geq 3\) for \(j = 2, 3\). We assume that \(C_1 = C_2\). If \(a^2 = 1\), we conclude by lemma A.1 that 4 divides \(n\). If \(a^2 = b\), then \(ba = 1\) or \(ba = c\). Since \(ba = 1\) forces \(ca = c\), we conclude that \(ba = c\). Then \(ca = 1\) yields \(1 = bca = a^4\). Therefore, \(a\) is an element of order 4 and 4 divides \(n\). When \(a^2 = c\) the same argument yields that \(a\) has order 4.

\(n = 13\): Assume \(S_{n, 13}(13) = 12\). If the 13 inlet channels appeared on only three switches in one of the blocks this would force the inlet channels to appear on at least five switches in each of the remaining blocks. We may, therefore, assume that in the induced latin square, there exist 13 entries containing only four different elements \(x_1, x_2, x_3, x_4\) and appearing in four rows and four columns. There exists at least one row and one column each containing all the elements \(x_1, x_2, x_3, x_4\). By appropriate multiplication as in Case 1) of \(n = 11\), we may assume that the four rows and the four columns correspond to the elements 1, a, b, c. If there exist two columns both containing these four elements, we can procede exactly as in Case 2) of \(n = 11\) and conclude that 4 divides \(n\). We may, therefore, assume that \(|C_1 \cap C_2| = 3\) for \(j = 2, 3, 4\). From this we see that if \(a^2 = 1\) then by lemma A.1, we conclude that 4 divides \(n\).

\[
\begin{array}{c|ccc}
1 & a & b & c \\
\hline
1 & 1 & a & b \\
a & a & 1 & - \\
b & b & - & - \\
c & c & - & - \\
\end{array}
\]

Assume that \(a, a^{-1}\) is unknown, i.e., we do not know whether or not \(a \in C_1\). Then \(ba = 1\) or \(ba = c\). If \(ba = 1\) then \(ca = b\). Then we see \(b = a^{-1}\) and \(c = a^2\). Therefore, we only have to consider the multiplication table of 1, a, a⁻¹, a²⁻¹. Since \(|C_1 \cap C_2| = 3\), we get \(a^{-1} \in \{1, a\} \) or \(a^{-1} \in \{1, a\} \)

If \(ba = c\) then \(ca = b\) or \(ca = 1\). \(ca = b\) implies \(ca^2 = c\) implying \(a^2 = 1\) and by lemma A.1, we conclude that 4 divides \(n\). \(ca = 1\) implies \(c = a^{-1}\) and \(a^{-1} = b\) and we have once more the multiplication table of 1, a, a⁻¹, a²⁻¹ from which we conclude that 4 or 5 divides \(n\).

Assume that \(ab\) is unknown. Then \(a^2 = 1\) or \(a^2 = c\). If \(a^2 = 1\) lemma A.1 yields that 4 divides \(n\). If \(a^2 = c\) then \(ca = 1\) or \(ca = b\). \(ca = 1\) implies \(a^2 = 1\) and by using \(|C_1 \cap C_2| = 3\) this implies \(b = c\) or column 3 contains the same element twice; in both cases a contradiction. Therefore, \(ca = b\) implying \(b = a^2\), i.e., \(1, a, b, c\) \(\in \{1, a, a^2\}\). As in the case \(ab\) unknown, this yields 4 divides \(n\) or 5 divides \(n\).

Since \(ab\) unknown proceeds the same way and gives the same result, it will be omitted. The calculation of \(S_{n, 13}(13)\) is therefore completed.

**Appendix B**

A LOWER ESTIMATE FOR \(S_{M, n}\)

In this Appendix, a proof of Result 4.1 is given. The \(M\) blocks in the first stage is denoted by \(B_1, B_2, \ldots, B_M\) and the \(Mn\) first-stage switches are denoted \(I_1, \ldots, I_{Mn}\).

Let \(E\) be subset of the set of inlet channels, assume that \(E\) has \(n\) elements and put

\[
k(E) := \text{The number of switches in } B \text{ having elements from } E \text{ among their inlet channels.}
\]

\[
k(E) := \min \{ |E| + 1, 2, \ldots, M \}.
\]

\[
a_i(E) := \text{The number of elements from } E \text{ appearing on } I_i.
\]

\[
a(E) := \max \{ a_i(E) | i = 1, 2, \ldots, M \}.
\]

If \(x\) is a real number \([x]\) denoted the smallest integer not less than \(x\). In this notation, a switch exist, which has at least \([n'/k(E)]\) elements from \(E\) among its inlet channels. Therefore,

\[
a(E) \geq \min \{ k(E), \frac{n}{k(E)} \}.
\]

The definition of \(k(E)\) and \(T_{M, n}(2.1)\) ensures

\[
T_{M, n}(2.1) \geq k(E)M.
\]

The pair condition gives

\[
T_{M, n}(2.1) \geq a(E)(M - 1) + k(E).
\]

From 2.2, B.1, B.2, and B.3, we get

\[
S_{M, n}(n) \geq \min \left\{ \max \left[ \frac{k(E)M}{n} \right], \frac{n}{k(E)} \right\} \times \left\lfloor \frac{M - 1}{k(E)} \right\rfloor E \text{ has } n \text{ elements.}
\]

When \(E\) runs through all subsets with \(n\) elements, \(k(E)\) runs through a subset of \(1, 2, \ldots, n\) yielding

\[
S_{M, n}(n) \geq \min \left\{ \max \left[ \frac{k(M)}{k} \right], \frac{n}{k} \right\} \times \left\lfloor \frac{M - 1}{k} \right\rfloor E \text{ has } n \text{ elements.}
\]

Put \(f_j(k) := kM\) and \(f_{x, a}(k) := \left[ \frac{n}{k} \right](M - 1) + k\). To finish the proof of Result 4.1, it is enough to show that

\[
\min \left\{ \max \left[ f_j(k), f_{x, a}(k) \right] \right\} = G_\lambda(n) \text{ for any } n \leq n^*_\lambda.
\]

Choose \(p\) as the integer having the property

\[
n = p^2 + x \text{ where } 1 \leq x \leq 2p + 1.
\]
First assume that \( 1 \leq x \leq p \). Then for \( k = 0, 1, \ldots, p - 1 \), we have
\[
f_{2,n}(p-k) = \left( \frac{p^2 + x}{p - k} \right) (M-1) + p - k \geq (p + k + 1)(M-1) + p - k \]
\[= (p+1)M + k(M - 2) - 1 \geq (p+1)M - 1. \]
Since \( f_{2,n}(p) = (p+1)M - 1 \) and \( f_{1}(k) \geq (p+1)M \) for \( k \geq p + 1 \), it is now proved that
\[
\min_k \{ \max \{ f_{1}(k), f_{2,n}(k) \} \} = (p+1)M - 1 \quad \text{when } n = p^2 + x \text{ and } 1 \leq x \leq p. \quad (B.7)
\]
Assume now that \( p + 1 \leq x \leq 2p + 1 \). Then for \( k = 0, 1, \ldots, p - 1 \), we have
\[
f_{2,n}(p-k) = \left( \frac{(p-k)(p+k+1)+k^2+k+x-p}{p-k} \right) (M-1) + p - k \geq (p+k+2)(M-1) + p - k \]
\[= (p+1)M + (k+1)(M - 2) \geq (p+1)M. \]
Since \( f_{2,n}(p+1) = (p+1)M \) and \( f_{1}(k) \geq (p+1)M \) for \( k \geq p + 1 \), it is proved that
\[
\min_k \{ \max \{ f_{1}(k), f_{2,n}(k) \} \} = (p+1)M \quad \text{when } n = p^2 + x \text{ and } p + 1 \leq x \leq 2p + 1. \quad (B.8)
\]
B.5, B.7, and B.8 concludes the proof of Result 4.1.

**Appendix C**

**Proof of Result 4.3**

In this Appendix, we prove Result 4.3. It is enough to show if \( k \geq 1 + \gamma/(\ln 2)\ln(M^3\ln M) \) then \( g_k \geq M^3 \). The following lemma will be helpful.

**Lemma C.1:** For \( k \geq 2 \) and \( M \geq 2 \) the following estimate is valid:
\[
g_k \geq M^{2-k} \left[ \frac{k^2}{\ln 2} \right] 3^{1-k}M^{-2} \]. \quad (C.1)
\]

**Proof:** We have \( g_2 = G_0(M + 1) \geq M(M + 1)^{2/3} \geq M^{5/3}(1 + 1/3M) \geq M^{5/3}(1 + 1/3M^2) \). This proves the lemma for \( k = 2 \). Assume the lemma is true for some \( k \). Then,
\[
g_{k+1} = G_0(g_k + 1) \geq M(g_k + 1)^{1/2} \]
\[\geq M \left( M^{2-k} \left[ 1 + \left( \frac{k^2}{\ln 2} \right) 3^2M^{-2} \right] ^{1/2} \right. \]
\[\cdot \left. 3^{1-k}M^{-2} \right) = M^{2-k} \left( 1 + \left( \frac{k^2}{\ln 2} \right) 3^2M^{-2} \right) \]
\[\cdot \left. 3^{1-k}M^{-2} + M^{1/2-k} \right) \]
which concludes the proof of the lemma.

Now assume \( k \geq 3 + (\gamma/\ln 2)\ln(M^3\ln M) \). By taking exp, we obtain
\[
2^{k-3} \geq M^3 \ln M.
\]
Since \( k \geq 2 \), we have \( (3^k-1)/3^{k-1} \geq 2/3 \) yielding
\[
2^{k-3} \left( \frac{3^k-1}{3^{k-1}} \right) M^3 \ln M = \frac{3^{3k-1}-1}{2 \cdot 3^{k-1}} \geq 2^{1-k} M^3 \ln M. \]
Since \( \ln(1 + x) \approx (3/4)x \) for \( x \) small, this implies
\[
\ln \left( \left[ 1 + \frac{3^k-1}{2 \cdot 3^{k-1}} \right] M^{-2} \right) \geq 2^{1-k} M^3 \ln M. \]
Take exp and multiply by \( M^{2-k} \) and obtain
\[
M^{2-k} \left( 1 + \frac{3^k-1}{2 \cdot 3^{k-1}} \right) M^{-2} \geq 2^{1-k} M^3 \ln M. \]
But,
\[
M^{2-k} \left( 1 + \frac{3^k-1}{2 \cdot 3^{k-1}} \right) M^{-2} = g_k. \]
and the proof is completed.

**References**


\[\star\]

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