On the scattering cross section of passive linear arrays

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Published in:
IEEE Transactions on Antennas and Propagation

Publication date:
1973

Document Version
Publisher's PDF, also known as Version of record

Citation (APA):
The geometry of the array and of the parameters of the matching matrix. A general formula is derived for the scattering cross section of a passive n-element linear array consisting of isotropic radiators. When all the reactivities are tuned out and scattering in the mirror direction is investigated, it is found that \( A_{sr} \), the relative scattering cross section is equal to the square of the maximum gain the array can produce. As a consequence, for forward scattering in the limiting case of zero spacing between the elements, \( A_{sr} = n^4 \).

The cross section of a scattering object having cylindrical symmetry is defined as follows

\[
A_\delta(\theta_1, \theta_2) = 4\pi r^2 \frac{\text{power density at distant point } r \text{ in direction } \theta_1}{\text{power density of plane wave incident from direction } \theta_1}.
\]

For an array one is more interested in the relative scattering cross section

\[
A_{sr}(\theta_1, \theta_2) = A_\delta(\theta_1, \theta_2) / A_\delta(\theta_1, \theta_1) = A_r(\theta_1, \theta_2).
\]

where \( A_{sr} \) is the scattering cross section of the individual radiator. In this Communication we shall express the cross section of a scattering object having cylindrical symmetry, in terms of the geometry of the array and of the parameters of the matching matrix. A general formulation for an arbitrary three-dimensional array is certainly possible, but we shall, for simplicity, restrict the investigation to linear arrays consisting of isotropic radiators. In that particular case there are several theorems available making
it easier to evaluate the results. We shall also assume that the radiator considered has zero scattering cross-section when open-circuited so a single equivalent circuit may be used valid both for reception and reradiation.

Consider now the array of Fig. 1 where \( Z_n \) is a linear passive matching network connected to the terminals of the array. The element positions are \( d_1, d_2, \ldots, d_n \) measured from an arbitrarily chosen zero.

The voltage induced in a single element may be written in the form

\[
e = C_1 S_{in}^{1/2} e^{jkd_1 \cos \theta_1}
\]

where \( S_{in} \) is the incident power density and \( C_1 \) is a constant the value of which is of no interest in the present context. Taking the phase zero at the chosen zero point, the excitation of the \( i \)th element is as follows

\[
e_i = C_1 S_{in}^{1/2} e^{jkd_1 \cos \theta_1}
\]

which can also be written as

\[
e = C_1 S_{in}^{1/2} f(\theta_1)
\]

where the \( n \)-dimensional vectors \( e \) and \( f \) are defined as

\[
e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \quad \text{and} \quad f(\theta) = \begin{bmatrix} \exp jk d_1 \cos \theta \\ \exp jk d_2 \cos \theta \\ \vdots \\ \exp jk d_n \cos \theta \end{bmatrix}.
\]

The current flowing in the elements will depend on the self- and mutual-impedances of the array and on the matching network but can always be expressed in the form

\[
i = \hat{Y} e
\]

where \( \hat{Y} \) is an \( n \times n \) admittance matrix and \( i \) is an \( n \)-dimensional vector

\[
i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}.
\]

The power density produced at the distance \( r \) by a single radiator of current \( i_1 \) is

\[
S(r) = C_3 \frac{4 \pi r^2}{4 \pi r^2} \left| f(\theta_1) i \right|^2.
\]

Substituting for \( f \) from (7) and using (5) we get

\[
S(r) = \frac{C_3^2 C_1 S_{in}}{4 \pi r^2} \left| f(\theta_1) \hat{Y} f(\theta_1) \right|^2.
\]

Hence the relative scattering cross section (2) is of the form

\[
A_n = R_n \frac{1}{2} \left| f(\theta_1) \hat{Y} f(\theta_1) \right|^2
\]

where \( R_n \) is the self-resistance of the individual elements and the self-reactance is assumed to be zero. Note that \( A_n \) equals unity for \( n = 1 \).

Let us now investigate the special case when the matching network is chosen so as to tune out all the mutual reactances that is

\[
Y = R^{-1}
\]

where \( R \) is the real part of the impedance matrix of the array. Choosing further

\[
\theta_3 = \pi - \theta_1
\]

that is evaluating the scattering in the mirror direction, we get

\[
A_n = \left| f^*(\theta_1) R \hat{Y} f(\theta_1) \right|^2.
\]

The expression in the bracket may be recognized to be the maximum gain an array can produce in the \( \theta_1 \) (or \( \pi - \theta_1 \)) direction. Thus

\[
A_n(0, \pi - \theta_1) = G_{max}(\theta_1).
\]

An interesting limiting case is \( \theta_1 = 0 \) when the distance between the elements tends to zero. Then, according to Uzkov [1] the maximum gain is equal to \( n^2 \) leading to

\[
A_n(0, \pi) = n^4
\]

Equations (16) and (17) prove not only that linear arrays may have “supergain” qualities as scatterers but also that they can be “supergain” as “receivers” from the \( \theta_1 \) direction and reradiators in the \( \theta_2 \) direction at the same time. It is easy to account for the latter property. A plane wave incident from the \( \theta_1 \) direction will induce equal voltage amplitudes and just the right phase for radiating in the \( \pi - \theta_1 \) direction. According to the travelling wave theorem of Bloch et al. [2] this voltage distribution will give the maximum gain.

It is more difficult to devise a simple argument for explaining the “receiving” properties of the array but perhaps it is only due to our unfamiliarity with the subject. One usually looks at transmission and reception separately and perhaps when they occur simultaneously there is little benefit in looking at them separately. A separation though is possible. One can express the power going into the array (and necessarily reradiated if all the matching elements are reactive) as follows

\[
P = \frac{1}{2} \text{Re} e^{*i} = \frac{1}{2} \text{Re} e^{*i} \hat{Y} e = \frac{1}{2} C_3^2 S_{in} \text{Re} f^*(\theta_1) \hat{Y} f(\theta_1).
\]

Choosing now \( \hat{Y} \) in the form of (13) it may be seen that \( P \) becomes proportional to \( G_{max}(\theta_1) \). The gain in the mirror direction is also \( G_{max}(\theta_1) \); so according to this alternative derivation of (16) both “reception” and reradiation are proportional to \( G_{max} \). Note, however, that this relationship is no longer valid when scattering in other than mirror direction is considered.

Acknowledgment

The author wishes to thank Dr. techn. J. Bach Andersen for a critical reading of the manuscript and for a number of valuable discussions.

References