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Spin-Caloritronic Batteries

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The thermoelectric performance of a topological energy converter is analyzed. The H-shaped device is based on a combination of transverse topological effects involving the spin: the inverse spin Hall effect and the spin Nernst effect. The device can convert a temperature drop in one arm into an electric power output in the other arm. Analytical expressions for the output voltage, the figure of merit (ZT), and energy-converting efficiency are reported. We show that the output voltage and the ZT can be tuned by the geometry of the device and the physical properties of the material. Importantly, contrary to a conventional thermoelectric device, here a low electric conductivity may, in fact, enhance the ZT value, thereby opening a path to strategies in optimizing the figure of merit.

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I. INTRODUCTION

Conventional thermoelectric (TE) energy converters can be used for recycling waste heat through the Seebeck effect converting the heat current into electric power, or, reversely, be used for TE cooling through the Peltier effect [1,2]. The efficiency of TE can be characterized by the dimensionless figure of merit \[ ZT = (S^2\sigma T/\kappa), \] where \( S \) is the Seebeck coefficient, \( T \) indicates absolute temperature, and \( \sigma(\kappa) \) is the electrical (thermal) conductivity. \( \kappa \) has contributions from both electrons and phonons. To optimize the efficiency, \( S \) and \( \sigma \) should be maximized, and \( \kappa \) has to be minimized. However, \( \sigma \) usually has a similar dependence on external parameters as \( \kappa \). For example, decreasing disorder leads to a larger electrical conductivity, but also \( \kappa \) tends to increase at the same time. Increasing \( \sigma \) by a higher charge carrier concentration is usually counteracted by a decreasing Seebeck coefficient \( S \). The conventional strategies to optimize the \( ZT \) are based on an attempt to control the electrical conductivity and thermal conductivity separately: One tries to find a material in which electrical conductivity is high but the thermal conductivity (mostly due to phonons) is low. Owing to the mutual interdependence of the three coefficients \( (S, \sigma, \kappa) \), it is a daunting challenge to achieve simultaneous optimization in a single material [4]. In the last 20 years, strategies have focused on breaking this entanglement [5], giving a doubling of the efficiency of the laboratory materials. By careful nano-engineering, it is possible to design devices which have a high electrical conductance and a low thermal conductance (see, e.g., Ref. [6]), but the scalability of these devices is challenging. In spite of the progress, the efficiency of TE devices still remains too low for widespread applications.

Spin caloritronics [4,7–12], which is an extension and combination of spintronics and the conventional thermoelectrics, has recently emerged as a research area. Here, a particular focus is on the interplay between a temperature gradient and spins, and effects are discovered which provide a promising platform for improving the thermoelectric performance. Energy converters based on spin caloritronics are devised and have, conceptually, advantages over the conventional TE devices. The spins, which behave essentially as an angular momentum, can be manipulated or affected by external magnetic field, ferromagnetic materials, and spin-orbit coupling (SOC). The heat, on the other hand, is mainly carried by phonons which do not carry angular momentum. Therefore, the two main components of spin caloritronics can, in principle, be controlled independently. This is a great advantage and may lead to high efficiencies for an appropriately designed energy converter.

The spin Seebeck effect has been investigated earlier as the driving mechanism in an energy converter [13,14].
dependent effects, i.e., the SNE and ISHE, rather than the TE devices, the mechanisms involved here are two spin-an electrical power output. In contrast to the conventional battery, where the temperature difference is converted into a voltage drop along the left arm due to the inverse spin Hall effect (ISHE). We show that the voltage drop can be expressed as \( \Delta \) expressed as
\[
\Delta V_{\text{ISHE}} = \frac{2e}{h} \alpha_{xy} \Delta T
\]
where \( \alpha_{xy} \) is the thermal conductivity, \( S_r \) is the Seebeck coefficient, and \( \sigma_r \) is the Nernst coefficient, respectively. In an open circuit, there is no charge-current density in the \( x \) direction, i.e., \( j^x \) = 0. Therefore, the electrochemical potential difference \( \partial_x \mu^e \) is determined by the spin electrochemical potential difference \( \partial_x \mu^h \) and the temperature gradient \( \partial_x T \), leading to
\[
\begin{bmatrix}
    j^x_r \\
    j^y_r
\end{bmatrix}
= \begin{bmatrix}
    \sigma_r & \theta_{SHr} \sigma_r & S_r \sigma_r \\
    -\theta_{SHr} \sigma_r & \sigma_r & \frac{2e}{h} \alpha_{xy} \\
    S_r \sigma_r T & -\frac{2e}{h} \alpha_{xy} \sigma_r T & \kappa_r + S_r \sigma_r T
\end{bmatrix}
\times \begin{bmatrix}
    -\partial_x \mu^h / e \\
    -\partial_x \mu^h / 2e \\
    -\partial_x T
\end{bmatrix},
\]
(1)

where the subscript “\( r \)” refers to the right arm, \( \theta_{SHr} = \sigma_{SHr} / \sigma_r \) is the spin Hall angle, and \( \sigma_{SHr} \) is the spin Hall conductivity. \( \kappa_r \), \( S_r \), and \( \alpha_{xy} \) are the thermal conductivity, Seebeck coefficient, and spin Nernst coefficient, respectively.

The spin electrochemical potential \( \mu^h \) is determined by the spin-diffusion equation [20, 21]

\[
\nabla^2 \mu^h = \frac{(\mu^h / \lambda_e)^2}{\partial_t
\]
where \( \lambda_e = \sqrt{D_e \tau_{SF}} \) is the spin-diffusion length, \( \tau_{SF} \) is the spin-flip relaxation time [20], and \( D_e = \mu m^* v_F^2 / 2 \) is the charge diffusion constant determined by mobility \( \mu \), the effective mass \( m^* \), and the Fermi velocity \( v_F = 5.336 \times 10^5 \) m/s. The spin-flip relaxation time in MoS\(_2\) is found to be larger than nanoseconds (10–100 ns) from both theory [22] and experiments [23–25]. We use \( \mu = 400 \) cm\(^2\) V\(^{-1}\) s\(^{-1}\) [26] and \( m^* = 0.54 \) m [27] for the hole. Thus, the spin-diffusion length of monolayer MoS\(_2\) is found to be in the range of 6–60 \( \mu \)m. Since \( \lambda_e \) is a good quantum number [28], a relatively longer spin-relaxation length can be expected coinciding with the experimental observations.

As shown in Fig. 1(b), we divide the right (left) arm into three regions. Owing to different boundary conditions

II. SYSTEM AND COMPUTATIONAL DETAILS

For a temperature gradient along the right arm \( [x \) direction in Fig. 1(a)], the spin current density \( j^x_r \) along the \( y \) direction and the charge- (heat-) current density \( j^y_r \) along the \( x \) direction in the right arm are given in the linear-regime as [7, 16–19]

\[
\begin{bmatrix}
    \sigma_r & \theta_{SHr} \sigma_r & S_r \sigma_r \\
    -\theta_{SHr} \sigma_r & \sigma_r & \frac{2e}{h} \alpha_{xy} \\
    S_r \sigma_r T & -\frac{2e}{h} \alpha_{xy} \sigma_r T & \kappa_r + S_r \sigma_r T
\end{bmatrix}
\times \begin{bmatrix}
    -\partial_x \mu^h / e \\
    -\partial_x \mu^h / 2e \\
    -\partial_x T
\end{bmatrix},
\]
(2)

2015, we studied the spin Nernst effect (SNE) and proposed an H-shaped device [Fig. 1(a)] based on monolayer group-VI dichalcogenides (TMDCs) [15] to generate pure spin currents. Because of the SOC in the material and the SNE, a pure transverse spin current can be produced when applying a temperature gradient in the right arm of the device. The spin current can be injected into the left arm through the horizontal bridge. The injected spin current can be converted into a voltage drop along the left arm due to the inverse spin Hall effect (ISHE). We show that the voltage drop can be expressed as \( \Delta V_{\text{ISHE}} = -\left( \sigma_{\text{SH}} / \sigma^2 \right) (2e / h) \alpha_{xy} \Delta T \), where \( \sigma_{\text{SH}} \) is spin Hall conductivity. In this paper, we show that this device can also function as a two-dimensional thermal battery, where the temperature difference is converted into an electrical power output. In contrast to the conventional TE devices, the mechanisms involved here are two spin-dependent effects, i.e., the SNE and ISHE, rather than the conventional Seebeck effect. We evaluate the expected device performance, the energy-converting efficiency, and the figure of merit \( ZT \). We show that the output voltage and the \( ZT \) can be tuned by the geometrical shape and material parameters. We believe that this flexibility in controlling the \( ZT \) can be utilized in realistic applications.
along the $y$ direction for regions $\Omega_{R2}$ and $\Omega_{R1}(\Omega_{R3})$, the
temperature gradient in each region instead of the entire
right arm is assumed uniform in the linear-response regime.
The total temperature difference between the ends of
the right arm is $\Delta T = (L - L_{1}/2)(\partial_{x_{1}}T + \partial_{x_{3}}T) + L_{1}\partial_{x_{2}}T$, where
$\partial_{x_{1}}T$ is derived to be the same as $\partial_{x_{3}}T$ (see Appendix A for a
detailed discussion). For fixed boundaries-in the open-circuit case, the spin current flowing in each
direction will be balanced by a backflow of spin current in the
opposite direction, which leads to zero spin current and
spin accumulation at these boundaries. The heat current
$J_{x}^{Q} = \int_{w_{l}}^{w_{r}} J_{x}^{Q} dy$ is uniform in the entire right arm. Thus, the
boundary conditions are

$$
j_{y}^{S}(y = -w_{r}) = 0, \quad i = 1, 2, 3,
$$

$$
j_{y}^{S}(y = 0) = j_{y}^{S},
$$

$$
j_{y}^{S}(y = 0) = 0, \quad i = 1, 3,
$$

$$
J_{x_{1}}^{Q} = J_{x_{2}}^{Q} = J_{x_{3}}^{Q},
$$

(3)

where $j_{yb}^{S}$ is the spin current density in the bridge region
and is determined below. The bridge is assumed to be
shorter than the spin-flip length so that the spin current
can be viewed as spatially independent. With these
conditions, the spin accumulation $j_{yb}^{S}$ and the temperature
gradients $\partial_{x_{i}}T$ in each region are linear functions of the
temperature difference $\Delta T$ and the spin current $j_{yb}^{S}$ in
the bridge (see Appendix A for a detailed discussion). The heat current becomes

$$
J_{x}^{Q} = \left( -\kappa_{w_{r}} + 2\xi_{r} \tanh \frac{w_{r}}{2\lambda_{r}} \right) \frac{\Delta T}{L}
+ \frac{4\lambda_{r}^{2}}{\eta \lambda_{r}} \frac{\Lambda_{y}^{S}}{\Delta T} j_{yb}^{S},
$$

(4)

where $\Theta = \frac{\theta_{SH}}{\lambda_{r}} + 1$, $\xi_{r} = -\{[\delta_{SH}, \sigma_{S} + (2e/\hbar)\alpha_{S}^{S}]T/2e\}$, and $\xi_{r} = \{[\delta_{SH}, \sigma_{S} + (2e/\hbar)\alpha_{S}^{S}]2\lambda_{r}\}.\frac{\Lambda_{y}^{S}}{\Theta_{S}}$.

When a spin current is injected into the left arm through
the bridge, a charge current $j_{y}^{S}$ is induced along the $x$
direction owing to the ISHE, which, in turn, reduces the
spin current $j_{yb}^{S}$ due to the spin Hall effect (SHE). In the linear-response regime,

$$
\left( \frac{j_{y}^{S}}{\Delta T} \right) = \sigma_{l} \left( \frac{1}{\theta_{SH}} \right) \left( -\theta_{SH} \frac{\Delta T}{L} \right),
$$

(5)

where $\sigma_{l}$ is the electrical conductivity of the left arm, $\mu_{1}^{S} = (\mu_{1} + \mu_{1})/2$ is the electrochemical potential, and $\mu_{1}^{S}$ means the spin electrochemical potential of the left arm.

In the linear-response regime, the induced voltage drop in
each region can be assumed to be uniform, which yields
$\Delta V = (L_{1} - L/2e)(\partial_{x_{1}}\mu_{1}^{S} + \partial_{x_{3}}\mu_{1}^{S}) - (L_{1}/e)\partial_{x_{2}}\mu_{1}^{S}$, where

$\Delta V = V|_{x=0} - V|_{x=L}$. Analogously, the spin accumulation $\mu_{1}^{S}$ also obeys the spin-diffusion equation, i.e.,
$\nabla^{2}\mu_{1}^{S} = \mu_{1}^{S}/\lambda_{1}^{2}$, where $\lambda_{1}^{2}$ is the spin-diffusion length
of the left arm. By using the boundary condition $j_{y}^{S}(y = d + w_{r}) = 0$ (all regions $\Omega_{1}, \Omega_{2}, \Omega_{3}$), $j_{y}^{S}(y = d) = 0$ (regions $\Omega_{1}$ and $\Omega_{3}$), $j_{y}^{S}(y = d) = j_{y}^{S}$ (region $\Omega_{2}$), and the uniform charge current $J_{x}^{Q} = \int_{w_{l}}^{w_{r}} J_{x}^{Q} dy$ in the entire
left arm, $\mu_{1}^{S}/\partial_{x_{1}}\mu_{1}^{S}$ can be expressed as linear functions of $\Delta V$ and $j_{yb}^{S}$. The relation between the charge current $J_{c}$ and
the voltage drop along the left arm becomes (details can be found in Appendix B)

$$
J_{x}^{Q} = \int_{w_{l}}^{w_{r}} J_{x}^{Q} dy = \frac{\Delta V}{L} \left( \frac{w_{l}}{2\lambda_{l}} \right) + \frac{L_{1}}{L} \frac{\theta_{SH}^{S}}{\lambda_{l}} \left( \frac{w_{l}}{2\lambda_{l}} \right) \frac{2e}{\hbar} j_{yb}^{S}.
$$

(6)

To obtain an optimal output, spin coherence should be
preserved in the bridge. The SOC is the main source of
spin relaxation in a material. Nevertheless, the $s_{2}$ is a good
quantum number in the TMDCs. In addition, owing to the
strong spin and valley coupling at the valence-band edges,
only atomic scale magnetic scatters lead to spin flip [28].
In the case of a short bridge operating in the ballistic regime,
the spins are expected to be conserved. We also assume that
the spin-diffusion length is larger than the length of the
bridge such that there is no spin accumulation in the bridge, $\mu_{1}^{S} = \mu_{1}^{S} = \mu_{1}^{S}$.

With known $\mu_{1}^{S} = \mu_{1}^{S} = \mu_{1}^{S} = \mu_{1}^{S}$, the spin current $j_{yb}^{S}$ can be
determined as a function of the temperature gradient $\Delta T$ of
the right arm and the voltage drop $\Delta V$ generated in the
left arm [see Eq. (C1)]. Then, the relation between various
currents and effective forces can be summarized as

$$
\begin{align*}
\left( \frac{J_{c}}{J_{Q}} \right) &= G_{H} \left( \frac{1}{\Pi_{H}} \right) \left( \frac{\Delta V}{\Delta T} \right) \\
G_{H} &= (J_{c}/\Delta V)_{\Delta T=0}.
\end{align*}
$$

(7)

$G_{H} = (J_{c}/\Delta V)_{\Delta T=0}$ is the effective charge conductance of
the system, $K_{H} = -(J_{Q}/\Delta T)_{\Delta T=0}$ is the effective
heat conductance for an open electric circuit, $A_{H} = -(J_{c}/\Delta T)_{\Delta V=0}$
represents nonlocal Nernst conductance, $\Pi_{H} = (J_{Q}/J_{c})_{\Delta T=0}$ is a
nonlocal Peltier coefficient, and $S_{H} = (\Delta V/\Delta T)_{J_{c}=0}$ denotes a
nonlocal Seebeck coefficient of the system. Here, “nonlocal” is used because of the
spatial decoupling of the heat current $J_{Q}$ in the right arm
and charge current $J_{c}$ in the left arm. For an ordinary Peltier
coefficient and Seebeck coefficient, the four parameters
($J_{Q}, J_{c}, \Delta T, \Delta V$) are defined in the same spatial region.
Explicit expressions for the various coefficients ($G_{H}, K_{H},$ $A_{H}, \Pi_{H}, S_{H}$) are given in Eq. (D2).
III. RESULTS AND DISCUSSION

A. The voltage output

In the open-circuit case, \( J_c = 0 \) and the voltage drop is \( V_{\text{open}} = (A_H/G_H)|\Delta T| \). \( V_{\text{open}} \) depends on the widths of the arms of the device, as shown in Figs. 2(a) and 2(b). A maximum value is attained for a certain range of the geometric parameters (the dark red regions). In the two limits of \( w_l \to 0 \) or \( w_l \to \infty \), \( V_{\text{open}} \) tends to zero, as expected. In the latter case, spin coherence is not preserved. At a fixed \( w_l/\lambda_l \), \( V_{\text{open}} \) varies monotonically with \( w_r/\lambda_r \) tending to a constant value [see Fig. 2(a)]. There is no explicit and severe restriction on the width of right arm \( (w_r) \) for optimizing \( V_{\text{open}} \) by constraining only the ratio of \( w_r/\lambda_r \).

Figures 2(c) and 2(d) show the variation of \( V_{\text{open}} \) with different material parameters. A larger \( V_{\text{open}} \) can be obtained by increasing the \( \alpha_y^S \) of the right arm and the spin Hall angle \( \theta_{\text{SHH}} \). Consider now a varying dilute nonmagnetic disorder in the left arm, which strongly affects the longitudinal conductivity, while the spin Hall conductivity \( \sigma_{\text{SHH}} \) is essentially unchanged (because the spin Hall effect is of topological origin and is protected against such disorder, as long as spin coherence is maintained). Changing the doping, thus, provides a technologically viable way to optimize the output voltage in the device. The spin-diffusion length of the left arm, however, will also be reduced with increasing doping level owing to the decreasing mobility. Thus, one should ensure \( w_l \) is of the order of the spin-relaxation length when optimizing the output voltage through doping dilute nonmagnetic disorder into the left arm. This demand of the length of left arm can be guaranteed since the lithography resolution can already reach 25 nm [32].

On the other hand, the impact of varying the thermal conductivity \( \kappa_r \) is insignificant [inset in Fig. 2(c)]. We also observe that even in the absence of the SNE, there is still nonzero \( V_{\text{open}} \) [Fig. 2(c)], which can be ascribed to the combination of the SHE and Seebeck effect (the extra term \( \theta_{\text{SHH}}, \sigma_{\text{SHH}} \)) in Eq. (2) in the right arm. The extra term has the following meaning. When a temperature gradient is applied to the right arm, an electric field is induced along the direction of the temperature gradient owing to the conventional Seebeck effect. The generated electric field will induce a transverse spin current through the SHE, which is superpositioned to the one generated via the SNE. This superposition explains the finite \( V_{\text{open}} \) even at zero \( \alpha_y^S \).

Finally, the spin current injected into the left arm induces \( V_{\text{open}} \) along the arm direction. From this perspective, the combined effect can be viewed as a generalized SNE.

B. The figure of merit \( ZT_H \) of the H-shaped device

Figure 1(c) shows the equivalent circuit for the proposed device. The output power \( P \) of the device is

\[
P = (V_{\text{open}} - R_H J_c) J_c = J_c \frac{A_H}{G_H} |\Delta T| - J_c^2 R_H,
\]

where \( R_H \) is the internal resistance of the SNE-based device, and \( R_H J_c^2 \) is the Joule heating produced by the electric current flowing through the internal resistance. Based on Eq. (7), the averaged heat current \( J_Q \) in the right arm can be given as a function of \( J_c \),

\[
J_Q = \frac{A_H}{G_H} T J_c + K_H |\Delta T| = e_H T J_c + K_H |\Delta T|.
\]

Compared to the formula for the conventional TE generator (the charge Seebeck effect) [3], the term due to the Joule heating is absent in Eq. (9). This makes sense since there is no charge current flowing along the right arm. Thus, the power-conversion efficiency \( \eta_{\text{SNE}} \) can be obtained as a function of \( J_c \):

\[
\eta_{\text{SNE}}(J_c) = \frac{P}{J_Q} = \frac{J_c \frac{A_H}{G_H} |\Delta T| - J_c^2 R_H}{\frac{A_H}{G_H} T J_c + K_H |\Delta T|}.
\]

The maximum efficiency is reached at the optimal \( J_c^{\text{opt}} \) given by

\[
J_c^{\text{opt}} = \frac{|\Delta T| \frac{A_H}{G_H}}{R_H + R_{\text{load}}^{\text{opt}}}, \quad R_{\text{load}}^{\text{opt}} = R_H \sqrt{1 + (ZT)_H}
\]

FIG. 2. (a) The voltage drop \( V_{\text{open}} \) as a function of \( w_l/\lambda_l \) and \( w_r/\lambda_r \). (b) The voltage drop \( V_{\text{open}} \) as a function of \( w_r/\lambda_l \) and \( w_r/\lambda_r \). (c) \( V_{\text{open}} \) versus the spin Nernst coefficient \( \alpha_y^S \) at two different thermal conductivities. Inset: \( V_{\text{open}} \) versus thermal conductivity. (d) \( V_{\text{open}} \) versus the spin Hall angle \( \theta_{\text{SHH}} \) of the left arm at different temperature differences at the two ends of the right arm. Here, \( \theta_{\text{SHH}} = 0.83 \), \( S_r = 250 \mu \text{V/K} \) [29], \( (L_1/L) = 0.5 \), \( T = 300 \text{ K} \), and \( |\Delta T| = 4 \text{ K} \). Parameters \( w_l/\lambda_l = 6 \), \( \lambda_l/\lambda_r = 1.0 \), \( \alpha_y^S = 0.18 \alpha_y^H \) \( \left[ \alpha_y^H = K_H/8\pi \right] \) [15], \( J_c = 20 \text{ W/mK} \) [30], \( \theta_{\text{SHH}} = 0.83 \), and \( \sigma_{\text{SHH}} = 1.16 \times 10^{-2} \text{e}^2 \text{h}^{-1} \) [31] are fixed in the other three figures. Here, all material parameters are taken for a monolayer MoS2.
and has the value
\[ n_{\text{SNE}}^{\text{max}} = \frac{|\Delta T|}{T} \sqrt{\frac{1 + (ZT)_{\text{H}}} {1 + (ZT)_{\text{H}}^2}}, \]  
(12)

This is a monotonically increasing function of the figure of merit \((ZT)_{\text{H}}\). The \(ZT\) value for the present device is
\[ (ZT)_{\text{H}} = \frac{(A_{\text{H}})^2}{K_{\text{H}}} T = \frac{(S_{\text{H}})^2 G_{\text{H}}}{K_{\text{H}}} T, \]  
(13)

where \(S_{\text{H}}\) is the effective Seebeck coefficient of the \(H\)-shaped device. The \(ZT\) has a similar expression as that of a conventional energy converter. Using the explicit expressions for \((ZT)_{\text{H}}\) given in Eq. (D14), we can find the optimal value of the ratio \((ZT)_{\text{H}}\) and the load resistance. It is remarkable that the thermal properties (i.e., thermal conductivity) have little impact on the output voltage. It is interesting to note that contrary to the conventional TE energy converter, a low charge conductivity enhances the \(ZT_{\text{H}}\) here. This makes it possible to optimize the electrical conductivity, thermal conductivity, and Seebeck coefficient simultaneously in a single material.

\[ ZT_{\text{H}} = \frac{(S_{\text{H}})^2}{K_{\text{H}}} T, \]  
(14)

In summary, we study the performance of a two-dimensional energy generator based on the concerted effect of the SNE and ISHE. We find that the performance depends not only on the properties of the materials and the geometry but also on the matching of the load resistance. It is interesting to note that contrary to the conventional TE energy converter, a low charge conductivity enhances the \(ZT_{\text{H}}\). Here, it makes it possible to optimize the electrical conductivity, thermal conductivity, and Seebeck coefficient simultaneously in a single material. In addition, the heat current in the right arm and the charge current in the left arm are spatially decoupled, which excels the conventional TE. The properties of the material in different arms can be manipulated independently.

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**APPENDIX A: LINEAR TRANSPORT PROPERTIES IN THE RIGHT ARM**

The linear equation in the right arm of the \(H\)-shape is given in Eq. (1). Because there is no charge-current density in the \(x\) direction, i.e., \(j_x = 0\), the charge electrochemical potential difference \(\partial_x \mu_x\) in the \(x\) direction is found to be
\[ -\partial_x \mu_x / e = \theta_{\text{SH}r} \partial_x \mu_x / 2e + S_r \partial_x T, \]  
which produces...
arm is divided into three regions \( \Omega_{1,2,3} \) (see the main text), and the temperature gradient of each region is assumed to be uniform (namely, \( \nabla T = 0 \)) and labeled as \( \partial_x T \), where \( i = 1, 2, 3 \) indicates the corresponding region. Hence, one can find

\[
T_3 - T_4 = \frac{L - L_1}{2} \partial_x T, \\
T_2 - T_3 = L_1 \partial_x T, \\
T_1 - T_2 = \frac{L + L_1}{2} \partial_x T, \tag{A4}
\]

where \( T_4, T_3, T_2, T_1 \) represent the temperatures for \( x = 0, (L - L_1)/2, (L + L_1)/2, L \), respectively. In addition, \( \Delta T = T_1 - T_4 \) (or \( T_{\text{cold}} - T_{\text{hot}} \)) is the temperature difference of the two ends of the right arm. It is intuitive to obtain

\[
\Delta T = \frac{L - L_1}{2} (\partial_x T + \partial_x T) + L_1 \partial_x T. \tag{A5}
\]

For the bound boundaries in the open-circuit case, the spin current-density conservation at the boundaries \( y = 0, -w_r \) gives \( j^y_S(y = -w_r) = 0 \) (all regions) and \( j^y_S(y = 0) = 0 \) (regions \( \Omega_{R1,3} \)) but \( j^y_S(y = 0) = j^y_S \) in region \( \Omega_{R2} \). \( j^y_S \) is an undetermined parameter (the concrete formula is determined following) denoting the spin current density of the bridge region in the \( y \) direction. Thus, we obtain

\[
-A_{ri} + B_{ri} = \xi_r (\partial_i T), \quad \text{where } i = 1, 3, \\
A_{rj} e^{(w_r/\lambda_j)} - B_{rj} e^{-(w_r/\lambda_j)} = \xi_r \partial_j T, \quad \text{where } j = 1, 2, 3, \\
A_{r2} - B_{r2} - \xi_r (\partial_x T) = \frac{\lambda_r 4 e^2}{\hbar} j^y_S, \tag{A6}
\]

Meanwhile, the heat current (\( J^y_x = \int_{-w_r}^0 j^y_x dy \)) conservation at the boundaries \( x_1 = (L - L_1)/2, x_2 = (L + L_1)/2 \), giving \( J^y_1 |_{x_1} = J^y_0 |_{x_1} \) and \( J^y_1 |_{x_2} = J^y_0 |_{x_2} \). Combining with Eq. (A3) yields

\[
(A_{r1} - A_{r2}) (1 - e^{(w_r/\lambda_1)}) + (B_{r1} - B_{r2}) (1 - e^{-(w_r/\lambda_1)}) = \frac{\kappa_{wr} w_r}{\xi_r} \partial_x T - \partial_x T, \\
(A_{r2} - A_{r3}) (1 - e^{(w_r/\lambda_2)}) + (B_{r2} - B_{r3}) (1 - e^{-(w_r/\lambda_2)}) = \frac{\kappa_{wr} w_r}{\xi_r} \partial_x T - \partial_x T. \tag{A7}
\]

The coefficients \( A_{r1}, B_{r1}, \partial_x T \) can be proved to be equal to \( A_{r3}, B_{r3}, \partial_x T \), namely, the spin electrochemical potential distribution and temperature gradient in the region \( \Omega_{R1} \) is equal to that in region \( \Omega_{R3} \). The following is the detail. Based on Eq. (A6), we can have

\[
A_{r1} = \frac{\xi_r}{1 + e^{(w_r/\lambda_1)}} (\partial_x T), \quad B_{r1} = \frac{\xi_r}{1 + e^{(w_r/\lambda_1)}} e^{(w_r/\lambda_1)} (\partial_x T), \\
A_{r3} = \frac{\xi_r}{1 + e^{(w_r/\lambda_3)}} (\partial_x T), \quad B_{r3} = \frac{\xi_r}{1 + e^{(w_r/\lambda_3)}} e^{(w_r/\lambda_3)} (\partial_x T). \tag{A8}
\]

The relations in Eq. (A7) give rise to

\[
(A_{r1} - A_{r3})(1 - e^{(w_r/\lambda_2)}) + (B_{r1} - B_{r3})(1 - e^{-(w_r/\lambda_2)}) = \kappa_{wr} \partial_x T - \partial_x T. \tag{A9}
\]

Taking \( A_1, B_1, A_3, B_3 \) in Eq. (A8) into the above equation, we get

\[
2\xi_r \frac{e^{(w_r/\lambda_2)} - 1}{(e^{(w_r/\lambda_2)} + 1)} (\partial_x T - \partial_x T) = \kappa_{wr} \partial_x T - \partial_x T. \tag{A10}
\]

Owing to \( [2\xi_r \xi_r (e^{(w_r/\lambda_2)} - 1)/(e^{(w_r/\lambda_2)} + 1)] \neq \kappa_{wr} \), we can obtain

\[
\partial_x T = \partial_x T \Rightarrow \left\{ \begin{array}{l} A_{r1} = A_{r3}, \\
B_{r1} = B_{r3}. \end{array} \right. \tag{A11}
\]

After some algebra, one obtains six equations with six independent coefficients:
\[ A_{r1} - \frac{\xi_r}{1 + e^{(w_r/\lambda_r)}} \partial_{x1} T = 0, \quad B_{r1} + \frac{\xi_r}{1 + e^{(w_r/\lambda_r)}} e^{(w_r/\lambda_r)} \partial_{x1} T = 0, \quad -e^{(w_r/\lambda_r)} A_{r2} + e^{-(w_r/\lambda_r)} B_{r2} = -\xi_r \partial_{x2} T, \]

\[ A_{r2} - B_{r2} - \xi_r (\partial_{x2} T) = \frac{\lambda_r 4 e^2}{\Theta \sigma_r \hbar} j_y^k, \quad \frac{L - L_1}{2} (\partial_{x1} T + \partial_{x3} T) = \Delta T - L_1 \partial_{x2} T, \]

\[ \kappa_r w_r (\partial_{x2} T - \partial_{x1} T) = \xi_r [(A_{r1} - A_{r2})(1 - e^{(w_r/\lambda_r)}) + (B_{r1} - B_{r2})(1 - e^{-(w_r/\lambda_r)})]. \]

(A12)

Finally, we obtain the parameters

\[ \partial_{x1} T = \frac{\Delta T}{L - L_1} P_r, \quad \partial_{x2} T = \frac{\Delta T}{L} + \frac{(L - L_1) P_r}{L \xi_r}, \quad A_{r1} = \frac{\xi_r \Delta T}{(1 + e^{(w_r/\lambda_r)}) L} - \frac{P_r L_1}{(1 + e^{(w_r/\lambda_r)}) L}, \]

\[ B_{r1} = -\frac{\xi_r \Delta T}{(1 + e^{-(w_r/\lambda_r)}) L} + \frac{P_r L_1}{(1 + e^{-(w_r/\lambda_r)}) L}, \quad A_{r2} = \frac{\xi_r \Delta T}{(1 + e^{(w_r/\lambda_r)}) L} + \frac{2 e^2 (1 - \coth \frac{w_r}{\lambda_r}) \xi_r}{\Theta \sigma_r \hbar} j_y^k + \frac{P_r (L - L_1)}{(1 + e^{(w_r/\lambda_r)}) L}, \]

\[ B_{r2} = -\frac{\xi_r \Delta T}{(1 + e^{-(w_r/\lambda_r)}) L} - \frac{2 e^2 (1 - \coth \frac{w_r}{\lambda_r}) \xi_r}{\Theta \sigma_r \hbar} j_y^k - \frac{P_r (L - L_1)}{(1 + e^{(w_r/\lambda_r)}) L}. \]

(A13)

where \( P_r = \left( \frac{4 e^2}{\hbar} \right) \left( \frac{1}{\Theta \sigma_r} \right) (\kappa_r w_r \coth(w_r/2 \lambda_r) - 2 \xi_r \sigma_r) \} j_y^k. \) Thus, the solutions of the spin-diffusion equation for the region \( \Omega_{r1}(\Omega_{r3}) \) and \( \Omega_{r1} \) are

\[ \mu_{r1}^S = \mu_{r3}^S = -\frac{\xi_r \sinh \frac{w_r + 2 y}{2 \lambda_r}}{L \cosh \frac{w_r}{\lambda_r}} \Delta T + \frac{4 e^2}{\hbar \sinh \frac{w_r}{\lambda_r}} j_y^k \Theta \sigma_r \left( \frac{1}{1 + \cosh \frac{w_r}{\lambda_r}} \right) \left( \begin{array}{c} \kappa_r w_r \coth(w_r/2 \lambda_r) - 2 \xi_r \sigma_r \sinh \frac{w_r}{\lambda_r} \end{array} \right), \]

\[ \Delta T = \frac{2 \xi_r \Delta T}{\lambda_r} \coth \frac{w_r}{\lambda_r} \Delta T \sinh \frac{w_r}{\lambda_r} \frac{(1 + \cosh \frac{w_r}{\lambda_r}) \kappa_r w_r - 2 \xi_r \sigma_r \sinh \frac{w_r}{\lambda_r} \xi_r \cos \frac{w_r}{\lambda_r} \xi_r \coth \frac{w_r}{\lambda_r} \). \]

(A14)

Thus,

\[ \mu_{r2}^S \bigg|_{y=0} = -\frac{\xi_r \tanh \frac{w_r}{\lambda_r}}{L} \Delta T + \left[ \frac{- \coth \frac{w_r}{\lambda_r} \xi_r}{\Theta \sigma_r} + \frac{\lambda_r \xi_r \xi_r (L - L_1) \tan \frac{w_r}{2 \lambda_r}}{\Theta \xi_r w_r \coth \frac{w_r}{\lambda_r} + 2 \xi_r \coth \frac{w_r}{\lambda_r} \coth \frac{w_r}{\lambda_r}} \right] \frac{4 e^2}{\hbar} j_y^k. \]

(A15)

Taking \( A_1, B_1, \partial_{x1} T \) into Eq. (A3), we can determine the heat current \( j_y^Q \),

\[ j_y^Q = \left( -\kappa_r w_r + 2 \xi_r \coth \frac{w_r}{2 \lambda_r} \right) \frac{\Delta T}{L} + \frac{4 e^2 L_1 \xi_r \lambda_r \tanh \frac{w_r}{2 \lambda_r}}{\hbar \Theta \sigma_r} j_y^k. \]

(A16)

**APPENDIX B: THE TRANSPORT EQUATION FOR THE LEFT ARM IN THE LINEAR-RESPONSE REGIME**

When reaching equilibrium, the charge- and spin current densities in the left arm can be written as

\[ \left( \begin{array}{c} j_x^c \\ j_y^c \\ j_x^S \\ j_y^S \end{array} \right) = \sigma_i \left( \begin{array}{c} 1 \\ -\Theta \Sigma_i \end{array} \right) \left( \begin{array}{c} -\partial_{y} \mu_i^c / e \\ -\partial_{y} \mu_i^S / 2 e \end{array} \right), \]

(B1)

leading to

\[ \Delta V_1 = -\frac{L - L_1}{2} \left( \frac{\partial_{y} \mu_i^c}{e} \right), \]

\[ \Delta V_2 = -L_1 \left( \frac{\partial_{y} \mu_i^c}{e} \right), \]

\[ \Delta V_3 = -\frac{L - L_1}{2} \left( \frac{\partial_{y} \mu_i^S}{e} \right), \]

(B3)

where \( \Delta V_1, \Delta V_2, \) and \( \Delta V_3 \) represent the voltage drops developed in each corresponding region, respectively.
\[ \Delta V = V_{x=0} - V_{x=L} = \Delta V_1 + \Delta V_2 + \Delta V_3 \] is the total voltage drop induced in the left arm and is found to be

\[ \Delta V = \frac{L_1 - L}{2e} (\partial_x \mu_{i1}^c + \partial_x \mu_{i3}^c) - \frac{L_1}{e} \partial_x \mu_{i2}^c. \] \hspace{1cm} (B4)

Analogously, the spin electrochemical potential $\mu_{i1}^c$ (i.e. $1$, $2$, $3$ is the region index) also obeys the spin-diffusion equation ($\partial_x^2 \mu_{i1}^c = (\mu_{i1}^c/\lambda_i^c)\), which yields

\[ 2\theta_{SHi} \lambda_i \partial_x \mu_{i1}^c + A_{i1} e^{-(w_1+d/\lambda_i)} = B_{i1} e^{(w_1+d/\lambda_i)}, \]

where $i = 1, 2, 3$;

\[ 2\theta_{SHi} \lambda_i \partial_x \mu_{i1}^c + A_{i1} e^{-d/\lambda_i} = B_{i1} e^{d/\lambda_i}, \]

where $i = 1, 3$;

\[ \frac{\theta_{SHi}}{e} \partial_x \mu_{i2}^c + \frac{\sigma_{i1}}{2e \lambda_i} (A_{i1} e^{-(d/\lambda_i)} - B_{i1} e^{d/\lambda_i}) = \frac{2e^2}{\hbar} j_{yb}^S. \] \hspace{1cm} (B6)

Similarly, the spin current density conservation at the boundaries $y = w_1 + d, d$ produce $j_{yi}^S (y = d + w_1) = 0$ (all regions $\Omega_1, \Omega_2, \Omega_3$) and $j_{yi}^S (y = d) = 0$ (for regions $\Omega_1$ and $\Omega_2$) but $j_{yi}^S (y = d) = j_{yb}^S$ (for region $\Omega_2$). Thus, we obtain

\[ j_{yi}^S \left|_{x_1} \right. = j_{yi}^S \left|_{x_2} \right. = 0, \]

and

\[ j_{yi}^S \left|_{x_2} \right. = j_{yi}^S \left|_{x_3} \right. = 0, \]

where $EX^- = e^{\pm (d/\lambda_i)} (e^{(w_1/\lambda_i)} - 1)$.

The coefficients $A_{i1}, B_{i1}, \partial_x \mu_{i1}^c$ can be proved to be equal to $A_{i3}, B_{i3}, \partial_x \mu_{i3}^c$. Namely, the spin electrochemical potential distribution and temperature gradient in the region $\Omega_1$ is similar to that in region $\Omega_3$. We show the details below. According to Eq. (B6), we have

\[ \frac{\theta_{SHi}}{e} \partial_x \mu_{i1}^c + \frac{1}{2e \lambda_i} (A_{i1} e^{-(w_1+d/\lambda_i)} - B_{i1} e^{(w_1+d/\lambda_i)}) = 0, \]

\[ \frac{\theta_{SHi}}{e} \partial_x \mu_{i3}^c + \frac{1}{2e \lambda_i} (A_{i1} e^{-(d/\lambda_i)} - B_{i1} e^{d/\lambda_i}) = 0, \]

\[ \frac{\theta_{SHi}}{e} \partial_x \mu_{i3}^c + \frac{1}{2e \lambda_i} (A_{i3} e^{-(w_1+d/\lambda_i)} - B_{i3} e^{(w_1+d/\lambda_i)}) = 0, \]

\[ \frac{\theta_{SHi}}{e} \partial_x \mu_{i3}^c + \frac{1}{2e \lambda_i} (A_{i3} e^{-(d/\lambda_i)} - B_{i3} e^{d/\lambda_i}) = 0. \] \hspace{1cm} (B9)

This leads us to

\[ A_{i1} = - \frac{2 e^{(w_1+d/\lambda_i)} \theta_{SHi} \lambda_i}{1 + e^{(w_1/\lambda_i)}} \partial_x \mu_{i1}^c, \]

\[ B_{i1} = \frac{2 e^{-(d/\lambda_i)} \theta_{SHi} \lambda_i}{1 + e^{(w_1/\lambda_i)}} \partial_x \mu_{i1}^c, \]

\[ A_{i3} = - \frac{2 e^{(w_1+d/\lambda_i)} \theta_{SHi} \lambda_i}{1 + e^{(w_1/\lambda_i)}} \partial_x \mu_{i3}^c, \]

\[ B_{i3} = \frac{2 e^{-(d/\lambda_i)} \theta_{SHi} \lambda_i}{1 + e^{(w_1/\lambda_i)}} \partial_x \mu_{i3}^c. \] \hspace{1cm} (B10)

From Eq. (B8), we obtain

\[ \frac{2w_i}{\theta_{SHi}} (\partial_x \mu_{i3}^c - \partial_x \mu_{i1}^c) = (A_{i1} - A_{i3}) EX^- + (B_{i1} - B_{i3}) EX^+. \] \hspace{1cm} (B11)

Taking $A_{i1}, B_{i1}, A_{i3}, B_{i3}$ in Eq. (B10) into the above equation leads to
Because of the inequality $2\theta_{SH}\lambda_i\cos(d/\lambda_i)\tan(w_i/2\lambda_i) \neq -(w_i/\theta_{SH})$, we have

$$\partial_{\lambda_i}\mu_i = \partial_{\lambda_i}\mu_3 \Rightarrow \begin{cases} A_{11} = A_{13}, \\ B_{11} = B_{13}. \end{cases}$$

After rearrangement, we obtain six equations with six independent coefficients,

$$\frac{\theta_{SH}\sigma_l e}{2e}\partial_{\lambda_i}\mu_i^2 + \frac{\sigma_l}{2e\lambda_i}(A_{12}e^{-(d/\lambda_i)} - B_{12}e^{d/\lambda_i}) = \frac{2e}{h} f^S_{yb},$$

$$A_{11} = -\frac{2e}{1 + e^{(w_i/\lambda_i)}}(A_{12}e^{w_i/\lambda_i} - B_{12}e^{-(w_i/\lambda_i)}),$$

$$\frac{2w_i}{\theta_{SH}}(\partial_{\lambda_i}\mu_i^2 - \partial_{\lambda_i}\mu_i^3) - (A_{11} - A_{12})e^{-(d/\lambda_i)}(e^{w_i/\lambda_i} - 1) = (B_{11} - B_{12})e^{(d/\lambda_i)}(e^{w_i/\lambda_i} - 1),$$

which produce

$$\partial_{\lambda_i}\mu_i = -\frac{e\Delta V}{L} - \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

$$\partial_{\lambda_i}\mu_i^3 = -\frac{e\Delta V}{L} - \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

$$A_{11} = -\frac{2e}{L} + \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

$$B_{11} = -\frac{2e}{L} + \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

$$A_{12} = -\frac{2e}{L} + \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

$$B_{12} = -\frac{2e}{L} + \frac{\frac{2e}{2}\frac{\lambda_i^2}{e}\theta_{SH}\lambda_i L_1}{L(w_i\sigma_l \coth\frac{w_i}{2\lambda_i} + \theta_{SH}^2\lambda_i \sigma_l)},$$

where $P_t = (4e^2/h)\{((\theta_{SH}\lambda_i^2) L_1)/(1 + e^{(w_i/\lambda_i)})\} \times (w_i\sigma_l \coth(w_i/2\lambda_i) + 2\theta_{SH}^2\lambda_i \sigma_l)\}.f^S_{yb}$.

Owing to $d \ll w_i$, here we can approximate $w_i + d \approx w_i$.

The charge current $J_{S\mu}^c$ in Eq. (B7) and the spin electrochemical potential $\mu_{S\mu}^c$ are given by, respectively,

$$J_{S\mu}^c = \Delta V_{\sigma_l} \left(\frac{w_i}{\lambda_i} + 2\theta_{SH}^2\lambda_i \coth\frac{w_i}{2\lambda_i}\right) \frac{2e}{h} f^S_{yb},$$

$$\mu_{S\mu}^c = \frac{2e}{L} \frac{\sinh\frac{2w_i}{2\lambda_i}}{\cosh\frac{w_i}{2\lambda_i}} \Delta V_{\lambda_i} \left(\frac{\cosh\frac{w_i}{2\lambda_i}}{\sinh\frac{w_i}{2\lambda_i}} \right)^2 \frac{\cosh\frac{w_i}{2\lambda_i}}{\sinh\frac{w_i}{2\lambda_i}} \Delta V_{\lambda_i} \left(\frac{\cosh\frac{w_i}{2\lambda_i}}{\sinh\frac{w_i}{2\lambda_i}} \right)^2 \frac{4e^2\lambda_i}{\sigma_l} f^S_{yb}.$$
where

\[
\eta_r = \Theta \coth \left( \frac{w_r}{2\lambda_r} \left( -\kappa_r w_r \coth \frac{w_r}{2\lambda_r} + 2 \right) \right),
\]

\[
\tau_i = \coth \left( \frac{w_i}{2\lambda_i} \left( \frac{\theta_{SHi}}{\lambda_i} + 2 \right) \right),
\]

\[
\xi_r^i = -\frac{(\Theta \sigma_s \sigma_r + \frac{2e\sigma_s^2}{\sigma_i})^2}{\Theta \sigma_r} \frac{\lambda_i}{\Delta T}. \quad (C2)
\]

For simplicity, we introduce a parameter \( \Xi \) as

\[
\Xi = \frac{\lambda_i \coth \frac{w_i}{2\lambda_i} + \lambda_i \coth \frac{w_i}{2\lambda_i}}{\Theta \sigma_r} - \frac{(L - L_1) \lambda_i}{\eta_r L \sigma_r} - \frac{(L - L_1) \lambda_i}{\tau_i L \sigma_i}. \quad (C3)
\]

Hence, spin current \( j_{yb}^S \) can be written as

\[
\frac{2e}{\hbar} j_{yb}^S = -\frac{\theta_{SHi} \lambda_i \tanh \frac{w_i}{2\lambda_i}}{L \Xi} \Delta V - \frac{\xi_r \tanh \frac{w_r}{2\lambda_r}}{2eL \Xi} \Delta T. \quad (C4)
\]

**APPENDIX D: THE FORMULAS OF FIGURE OF MERIT ZTH IN THE H-SHAPED DEVICE**

The heat current \( J_Q \) [i.e., \( J_Q^0 \) in Eq. (A16)] in the right arm and charge current \( J_c \) [namely, \( J_c^0 \) in Eq. (B15)] in the left arm are found to be expressed as a linear function of temperature difference \( \Delta T \) (voltage drop \( \Delta V \)) in the right (left) arm and spin current density \( j_{yb}^S \) in the bridge region, respectively, whereas \( j_{yb}^S \) can be given as a linear function of \( \Delta T \) and \( \Delta V \) in Eq. (C4). Hence, the \( J_Q \) (\( J_c \)) is also written as the linear function of \( \Delta T \) and \( \Delta V \),

\[
J_Q = \left( -\kappa_r w_r + 2\xi_r \frac{\xi_r}{\lambda_r} \tanh \frac{w_r}{2\lambda_r} - \frac{L_1 \xi_r \lambda_i \tanh \frac{w_r}{2\lambda_i}}{L^2 \Theta \sigma_r \Xi} \right) \Delta T + \frac{L_1 \theta_{SHi} \lambda_i \tanh \frac{w_r}{2\lambda_i}}{2eL^2 \Xi} \Delta V,
\]

\[
J_c = \left( \frac{\sigma_i (w_i + 2\theta_{SHi} \lambda_i \tanh \frac{w_i}{2\lambda_i})}{L} - \frac{L_1 \theta_{SHi} \lambda_i ^2 \tanh \frac{w_r}{2\lambda_i}}{L^2 \Xi} \right) \Delta V - \frac{L_1 \theta_{SHi} \lambda_i \tanh \frac{w_r}{2\lambda_i}}{2eL^2 \Xi} \Delta T. \quad (D1)
\]

Here, we can define the effective charge conductance \( G_H = (J_c / \Delta V)_{\Delta T = 0} \), thermal conductance \( K_H = -(J_Q / \Delta T)_{J_c = 0} \), and the Peltier coefficient \( \Pi_H = (J_Q / J_c)_{\Delta T = 0} \), the “Nernst signal” \( S_H = (\Delta V / \Delta T)_{J_c = 0} \), the Nernst conductance \( A_H = -(J_c / \Delta T)_{\Delta V = 0} \).

\[
G_H = \frac{\sigma_i (w_i + 2\theta_{SHi} \lambda_i \tanh \frac{w_i}{2\lambda_i})}{L} - \frac{L_1 \theta_{SHi} \lambda_i ^2 \tanh \frac{w_r}{2\lambda_i}}{L^2 \Xi},
\]

\[
A_H = \frac{L_1 \theta_{SHi} \lambda_i \xi_r \tanh \frac{w_r}{2\lambda_i}}{2eL \Xi} \Delta V, \quad \Pi_H = \frac{A_H}{G_H}, \quad S_H = \frac{A_H}{G_H},
\]

\[
K_H = \frac{\kappa_r w_r - 2 \xi_r \frac{\xi_r}{\lambda_r} \tanh \frac{w_r}{2\lambda_r}}{L} + \frac{L_1 \xi_r \lambda_i \tanh \frac{w_r}{2\lambda_i}}{L^2 (\theta_{SHi} + 1) \sigma_r \Xi} - \frac{(A_H / G_H) (\lambda_i \Xi)}{L}, \quad (D2)
\]

From Eqs. (D1) and (D2), one can obtain

\[
\begin{align*}
(J_c) &= G_H \left( \frac{1}{\Delta V} \right) \left( \frac{A_H}{G_H} \right) \left( \frac{\Delta V}{\Delta T} \right) = G_H \left( \frac{1}{\Delta V} \right) \left( \frac{A_H}{G_H} \right) \left( \frac{\Delta V}{\Delta T} \right), \quad (D3)
\end{align*}
\]

Thus, the open voltage \( V_{\text{open}} \) (namely, the charge current \( J_c = 0 \)) is found to be

\[
V_{\text{open}} = \frac{A_H}{G_H} \Delta T = \frac{L_1 \lambda_i \theta_{SH} \xi_r \tanh \frac{w_r}{2\lambda_i} \tanh \frac{w_r}{2\lambda_i} \Delta T}{\frac{w_r}{L} \Xi} + \frac{\Delta V}{\Delta T} \left( \frac{L_1 \lambda_i \xi_r \tanh \frac{w_r}{2\lambda_i}}{L \Xi} \right), \quad (D4)
\]

Here, we introduce the dimensionless coefficient \( \Xi' = \Xi (\sigma_i / \lambda_i) \), and if we take the formulas of \( \xi_r \) into Eq. (C2), we can obtain

\[
V_{\text{open}} = \frac{A_H}{G_H} \Delta T = \frac{\lambda_i \theta_{SH} \xi_r \tanh \frac{w_r}{2\lambda_i} \tanh \frac{w_r}{2\lambda_i} \Delta T}{\frac{w_r}{L} \Xi} + \frac{\Delta V}{\Delta T} \left( \frac{L_1 \lambda_i \xi_r \tanh \frac{w_r}{2\lambda_i}}{L \Xi} \right), \quad (D5)
\]

The induced voltage in the left arm by the temperature difference \( \Delta T \) via the combination of the spin Nernst effect and inverse spin Hall effect is \( (A_H / G_H) \Delta T \) (\( \Delta T = T_{\text{cold}} - T_{\text{hot}} \), \( |\Delta T| = -\Delta T \)), Therefore, the voltage drop on the load (output voltage) is found to be

\[
V = \frac{A_H}{G_H} \Delta T - J_c R_H. \quad (D6)
\]

The output power \( W \) can then be represented as a function of \( J_c \).
From Eq. (D3), we can have

$$\Delta V = \frac{J_c}{G_H} + \frac{A_H}{G_H} \Delta T. \quad (D8)$$

Given that

$$J_Q = A_H T \Delta V - \left( K_H + \frac{A_H A_H}{G_H} T \right) \Delta T$$

$$= A_H T \left( \frac{J_c}{G_H} + \frac{A_H}{G_H} \Delta T \right) - \left( K_H + \frac{A_H A_H}{G_H} T \right) \Delta T$$

$$= \frac{A_H}{G_H} T J_c - K_H \Delta T$$

$$= \frac{A_H}{G_H} T J_c + K_H |\Delta T|, \quad (D9)$$

the power-conversion efficiency can also be given as a function of $J_c$,

$$\eta(J_c) = \frac{W}{J_Q} = \frac{J_c}{G_H} \frac{\Delta T}{\Delta T} - J_c^2 R_H$$

$$= \frac{\Delta T}{R_H + R_H^{\text{load}}} T J_c + K_H |\Delta T|. \quad (D10)$$

The maximum efficiency of this power-conversion scheme $\eta_{\text{max}}^{\text{SNE}}$ is reached at the optimal $J_c^{\text{opt}}$,

$$J_c^{\text{opt}} = \frac{|\Delta T|}{R_H + R_H^{\text{load}}}, \quad R_H^{\text{load}} = R_H \sqrt{1 + (ZT)_H}. \quad (D11)$$

Thus,

$$\eta_{\text{max}}^{\text{SNE}} = \frac{|\Delta T|}{T} \frac{2 + (ZT)_H - 2 \sqrt{1 + (ZT)_H}}{(ZT)_H}$$

$$= \frac{|\Delta T|}{T} \frac{1 + (ZT)_H - 1}{\sqrt{1 + (ZT)_H} + 1}. \quad (D12)$$

The value of the spin Nernst figure of merit for the ISHE scheme is

$$(ZT)_H = \frac{(A_H)^2 R_H}{K_H} T = \frac{(S_H)^2}{K_H R_H} T. \quad (D13)$$

Taking the expressions $A_H$, $R_H$, $K_H$ in Eq. (D2) into it, we can determine the ZT value of the H-shaped system

$$(ZT)_H = \frac{1}{m - 1},$$

where

$$m = \left[ -1 + \frac{L}{L_1} \Xi' \coth \left( \frac{w_1}{L_1} \right) \left( \frac{w_1}{L_1} \coth \frac{w_1}{L_1} + 2 \right) \right]$$

$$\times \left[ -1 + \frac{L}{L_1} \Xi'' \coth \left( \frac{w_r}{L_1} \right) \left( \frac{w_r}{L_1} \coth \frac{w_r}{L_1} + 2 \right) \right]. \quad (D14)$$

where $\Xi' = \Xi(\sigma_1/L_1)$, $\Xi'' = \Xi(\sigma_1/L_1)$.

### APPENDIX E: SOME COMMENTS ON THE H-SHAPED DEVICE

Our comments on applying the temperature gradient to the right arm are as follows.

1. In our conceptual study, the temperature gradient is assumed to exist only in one arm of the H-shaped device. We believe this can be achieved in experiments. For example, heater coils and laser beams have been used in experiments. For the latter, the size and position of the laser spots can be controlled precisely in experiments: The laser spots can be positioned between contacts which are about 1 μm away from each other [33]. Therefore, it should be possible to control the position of the laser beam to the right arm of the H-shaped detector.

2. Furthermore, one arm of the H-shaped detector can be made longer with a larger-sized pad so that the laser spot can be easily applied to the pad.

Our comments on having the temperature gradient in the two arms simultaneously are as follows. If the left arm unintentionally experiences a temperature gradient, the Seebeck effect may cause an even larger voltage drop at the two ends of the left arm. Moreover, the temperature gradient on the left arm is in the same direction as that in the right arm; it induces a transverse spin current in the same direction as that induced by the right-arm temperature gradient. Therefore, the left gradient does not cancel the effect due to the right temperature gradient but instead enhances the total output. To make the discussion simple and clear, we consider only the situation where the temperature is applied to the right arm.

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