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On the difference between permutation polynomials over finite fields

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Abstract

The well-known Chowla and Zassenhaus conjecture, proven by Cohen in 1990, states that if \( p > (d^2 - 3d + 4)^2 \), then there is no complete mapping polynomial \( f \) in \( \mathbb{F}_p[x] \) of degree \( d \geq 2 \). For arbitrary finite fields \( \mathbb{F}_q \), a similar non-existence result is obtained recently by İşık, Topuzoğlu and Winterhof in terms of the Carlitz rank of \( f \).
Cohen, Mullen and Shiue generalized the Chowla-Zassenhaus-Cohen Theorem significantly in 1995, by considering differences of permutation polynomials. More precisely, they showed that if \( f \) and \( f + g \) are both permutation polynomials of degree \( d \geq 2 \) over \( \mathbb{F}_p \), with \( p > (d^2 - 3d + 4)^2 \), then the degree \( k \) of \( g \) satisfies \( k \geq 3d/5 \), unless \( g \) is constant. In this article, assuming \( f \) and \( f + g \) are permutation polynomials in \( \mathbb{F}_q[x] \), we give lower bounds for \( k \) in terms of the Carlitz rank of \( f \) and \( q \). Our results generalize the above mentioned result of Işık et al. We also show for a special class of polynomials \( f \) of Carlitz rank \( n \geq 1 \) that if \( f + x^k \) is a permutation over \( \mathbb{F}_q \), with \( \gcd(k + 1, q - 1) = 1 \), then \( k \geq (q - n)/(n + 3) \).

1 Introduction

Let \( \mathbb{F}_q \) be the finite field with \( q = p^r \) elements, where \( r \geq 1 \) and \( p \) is a prime. Throughout we assume \( q \geq 3 \). We recall that \( f \in \mathbb{F}_q[x] \) is a permutation polynomial over \( \mathbb{F}_q \) if it induces a bijection from \( \mathbb{F}_q \) to \( \mathbb{F}_q \). If \( f(x) \) and \( f(x) + x \) are both permutation polynomials over \( \mathbb{F}_q \), then \( f \) is called a complete mapping. We refer the reader to [11] for a detailed study of complete mapping polynomials over finite fields. Their use in the construction of mutually orthogonal Latin squares is described, for instance, in [9]. For various other applications, see [10, 12, 13, 14]. The paper [8] lists some recent work on complete mappings.


**Theorem 1.** If \( d \geq 2 \) and \( p > (d^2 - 3d + 4)^2 \), then there is no complete mapping polynomial of degree \( d \) over \( \mathbb{F}_p \).

A significant generalization of this result was obtained by Cohen, Mullen and Shiue [6] in 1995, and gives a lower bound for the degree of the difference of two permutation polynomials in \( \mathbb{F}_p[x] \) of the same degree \( d \), when \( p > (d^2 - 3d + 4)^2 \).

**Theorem 2.** Suppose \( f \) and \( f + g \) are monic permutation polynomials over \( \mathbb{F}_p \) of degree \( d \geq 3 \), where \( p > (d^2 - 3d + 4)^2 \). If \( \deg(g) = k \geq 1 \), then \( k \geq 3d/5 \).

An alternative invariant, the so-called Carlitz rank, attached to permutation polynomials, was used by Işık, Topuzoğlu and Winterhof [8] recently to obtain a non-existence result, similar to that in Theorem 1. The concept of Carlitz rank was first introduced in [1]. We describe it here briefly. The interested reader may see [16] for details.

By a well-known result of Carlitz [2] that any permutation polynomial over
\[ F_q, \text{ with } q \geq 3 \text{ is a composition of linear polynomials } ax + b, \ a, b \in F_q, \ a \neq 0, \] and \[ x^{q^2-2}, \] any permutation \( f \) over \( F_q \) can be represented by a polynomial of the form
\[
P_n(x) = \left( \ldots \left( (a_0 x + a_1)^{q-2} + a_2 \right)^{q-2} \ldots + a_n \right)^{q-2} + a_{n+1}, \tag{1.1}
\]
for some \( n \geq 0 \), where \( a_i \neq 0 \), for \( i = 0, 2, \ldots, n \). Note that \( f(c) = P_n(c) \) holds for all \( c \in F_q \), however this representation is not unique, and \( n \) is not necessarily minimal. Accordingly the authors of [1] define the \textit{Carlitz rank} of a permutation polynomial \( f \) over \( F_q \) to be the smallest integer \( n \geq 0 \) satisfying \( f = P_n \) for a permutation \( P_n \) of the form (1.1), and denote it by \( \text{Crk}(f) \).

The representation of \( f \) as in (1.1) enables approximation of \( f \) by a fractional transformation in the following sense.

For \( 0 \leq k \leq n \), consider
\[
R_k(x) = \frac{\alpha_{k+1} x + \beta_{k+1}}{\alpha_k x + \beta_k}, \tag{1.2}
\]
where \( \alpha_0 = 0, \alpha_1 = a_0, \beta_0 = 1, \beta_1 = a_1 \), and
\[
\alpha_k = a_k \alpha_{k-1} + \alpha_{k-2} \quad \text{and} \quad \beta_k = a_k \beta_{k-1} + \beta_{k-2} \tag{1.3}
\]
for \( k \geq 2 \). The set
\[
\mathcal{O}_n = \left\{ x_k : x_k = \frac{-\beta_k}{\alpha_k}, \ k = 1, \ldots, n \right\} \subset \mathbb{P}^1(F_q) = F_q \cup \{\infty\} \tag{1.4}
\]
is called the \textit{set of poles} of \( f \). The elements of \( \mathcal{O}_n \) may not be distinct.

It can easily be verified that
\[
f(c) = P_n(c) = R_n(c) \quad \text{for all } c \in F_q \setminus \mathcal{O}_n. \tag{1.5}
\]

Obviously, this property is particularly useful when \( \text{Crk}(f) \) is small with respect to the field size. The values that \( f \) takes on \( \mathcal{O}_n \) can also be expressed in terms of \( R_n \), see [16]. In case \( \alpha_n = 0 \), i.e., the last pole \( x_n = \infty \), \( R_n \) is linear. Following the terminology of [8], we define the \textit{linearity} of \( f \in F_q[x] \) as \( \mathcal{L}(f) = \max_{a,b \in F_q} |\{c \in F_q : f(c) = ac + b\}|. \) Intuitively \( \mathcal{L}(f) \) is large when \( f \) is a permutation polynomial of \( F_q \) of \( \text{Crk}(f) = n \), \( R_n \) is linear, and \( n \) is small with respect to \( q \).

Now we are ready to state the main result of [8]. We remark that the Theorems 1 and 2 hold over prime fields only, while the Theorem 3 is true for any finite field.
Theorem 3. If $f(x)$ is a complete mapping over $\mathbb{F}_q$, and $\mathcal{L}(f) < \lfloor(q+5)/2\rfloor$, then $\text{Crk}(f) \geq \lfloor q/2 \rfloor$.

The purpose of this note is to obtain a lower bound for the degree of the difference between two permutation polynomials, analogous to Theorem 2, generalizing Theorem 3. In what follows we assume that $f$ and $f + g$ are permutation polynomials over $\mathbb{F}_q$, where $g \in \mathbb{F}_q[x]$ has degree $k$ with $1 \leq k < q - 1$. We give lower bounds for $k$ in terms of $q$ and the Carlitz rank of $f$, see Theorems 2.1 and 3.1 below.

## 2 Degree of the difference of two permutation polynomials

Let $f$ be a permutation polynomial over $\mathbb{F}_q$, $q \geq 3$, with $\text{Crk}(f) = n \geq 1$. Suppose that $f$ has a representation as in (1.1) and the fractional linear transformation $R_n$ in (1.2), which is associated to $f$ as in (1.5) is not linear, in other words $\alpha_n$ in (1.3) is not zero. We denote the set of all such permutations by $C_{1,n}$, i.e., the set $C_{1,n}$ consists of all permutation polynomials over $\mathbb{F}_q$, satisfying $\text{Crk}(f) = n \geq 1$ and $\alpha_n \neq 0$. Clearly $\mathcal{L}(f) \leq n + 2$, if $f \in C_{1,n}$. We note that permutations $f \in \mathbb{F}_q[x]$ with $\alpha_n = 0$ behave very differently. For instance, there are examples of complete mappings over $\mathbb{F}_q$ of Carlitz rank 4 for infinitely many values of $q$. Indeed, the condition on the linearity of $f$ in Theorem 3 corresponds to the case $\alpha_n = 0$. Therefore, we only consider permutations in $C_{1,n}$.

We now prove our main theorem.

**Theorem 2.1.** Let $f$ and $f + g$ be permutation polynomials over $\mathbb{F}_q$, where $f \in C_{1,n}$ and the degree $k$ of $g \in \mathbb{F}_q[x]$ satisfies $1 \leq k < q - 1$. Then

$$nk + k(k-1)\sqrt{q} \geq q - \nu - n,$$

where $\nu = \gcd(k, q - 1)$.

**Proof.** Since $f \in C_{1,n}$, there exist $a, b, d \in \mathbb{F}_q$, such that $f(z) = R_n(z)$ for $z \in \mathbb{F}_q \setminus \mathcal{O}_n$, where

$$R_n(z) = \frac{az + b}{z + d}.$$

The fact that $ad - b \neq 0$ follows from (1.3).

The polynomial $f(z) + g(z)$ can be represented by $G_n(z) = R_n(z) + g(z)$ for $z \in \mathbb{F}_q \setminus \mathcal{O}_n$. Since $f + g$ is a permutation over $\mathbb{F}_q$, the map $G_n$ is injective on $\mathbb{F}_q \setminus \mathcal{O}_n$. 

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For \( u \in \mathbb{F}_q \) and
\[
G_n(z) = \frac{az + b}{z + d} + g(z) = u \tag{2.2}
\]
we set
\[
H_n(x) = G_n(x - d) = \frac{ax - \tilde{b}}{x} + h(x) = u.
\]
where \( \tilde{b} = ad - b \neq 0 \) and \( h(x) = g(x - d) \). Note that \( H_n(x) = u \) for some nonzero \( x \in \mathbb{F}_q \) if and only if \( z \neq -d \) is a solution of Equation (2.2). Let \( S \) be the set of pairs \( (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \) such that
\[
S = \{ (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* : x \neq y \ \text{and} \ H_n(x) = H_n(y) \}.
\]
Denote the value set of \( H_n \) by \( V_{H_n} \), i.e.,
\[
V_{H_n} = \{ u \in \mathbb{F}_q : \exists x \in \mathbb{F}_q \ \text{with} \ H_n(x) = u \}.
\]
Suppose that the cardinality \( |S| \) of \( S \) is \( \mu \). For \( u \in V_{H_n} \), we consider the inverse image; \( H_n^{-1}(u) = \{ x \in \mathbb{F}_q : H_n(x) = u \} \) and put \( n_u = |H_n^{-1}(u)| \). We remark that \( 0 \notin H_n^{-1}(u) \) and that \( x \in H_n^{-1}(u) \) if and only if \( x \) is a root of the polynomial
\[
xh(x) + (a - u)x - \tilde{b}.
\tag{2.3}
\]
This shows that for any \( u \in V_{H_n} \) we have \( n_u \leq k + 1 \) as the polynomial in Equation (2.3) has degree \( k + 1 \). We then conclude that
\[
\mu = \sum_{u \in V_{H_n}} n_u(n_u - 1) \leq (k + 1) \sum_{u \in V_{H_n}} (n_u - 1) \tag{2.4}
\]
If there exist \( n_u \) distinct elements \( x \) with \( H_n(x) = u \), then there exist \( n_u \) distinct elements \( z \) with \( G_n(z) = u \). Since \( G_n(z) \) is injective on \( \mathbb{F}_q \setminus \mathcal{O}_n \), this shows that \( n_u - 1 \) distinct elements \( z \) lie in the set of poles \( \mathcal{O}_n \). In particular, by Equation (2.4) and the fact that \( -d \in \mathcal{O}_n \) we conclude that
\[
n \geq |\mathcal{O}_n| \geq 1 + \sum_{u \in V_{H_n}} (n_u - 1) \geq 1 + \frac{\mu}{k + 1}. \tag{2.5}
\]
Therefore in order to obtain a lower bound for \( k \) in terms of \( q \) and \( n \), it is sufficient to determine \( \mu \) in relation to \( q \) and \( k \).

We can re-write the equation \( H_n(x) = H_n(y) \) as
\[
y(xh(x) - \tilde{b}) - x(yh(y) - \tilde{b}) = 0.
\]
Note that $x - y$ is a factor of $y(\chi h(x) - \tilde{b}) - x(yh(y) - \tilde{b})$. We want to find an absolutely irreducible factor over $\mathbb{F}_q$ of the polynomial in two variables of degree $k + 1$ defined by

$$\frac{y(\chi h(x) - \tilde{b}) - x(yh(y) - \tilde{b})}{x - y},$$

or equivalently defined by

$$xy \frac{h(x) - h(y)}{x - y} + \tilde{b}.$$ (2.6)

We recall that a rational function $\ell(x)/t(x) \in \mathbb{F}_q(x)$ is called exceptional over $\mathbb{F}_q$ if the polynomial $\Theta_{\ell/t}$, defined by

$$\Theta_{\ell/t} = \frac{t(Y)\ell(X) - t(X)\ell(Y)}{X - Y}$$

has no absolutely irreducible factor in $\mathbb{F}_q[X,Y]$. By Theorem 5 of [4], $\ell/t$ is a permutation over $\mathbb{F}_q$ if it is an exceptional function over $\mathbb{F}_q$. In particular, $t(\alpha) \neq 0$ for all $\alpha \in \mathbb{F}_q$. Now we put $\ell/t = (\chi h(x) - \tilde{b})/x$, and conclude that the rational function in (2.6) has an absolutely irreducible factor $p(x,y)$ over $\mathbb{F}_q$. We note that $\tilde{b}$ is not zero and hence $p(x,y)$ is a factor different from $x - y$. Moreover we assume without loss of generality that $p(x,y)$ is separable; otherwise we can replace $p(x,y)$ with a separable polynomial of smaller degree.

Consider the curve $\mathcal{X}$ whose affine equation is given by $p(x,y)$ of degree $\varrho \leq k + 1$. Then by [7, Theorem 9.57] the number of rational points $N(\mathcal{X})$ in $\text{PG}(2,q)$ of $\mathcal{X}$ is bounded by

$$N(\mathcal{X}) \geq q + 1 - (\varrho - 1)(\varrho - 2)\sqrt{q} \geq q + 1 - k(k - 1)\sqrt{q}.$$  

We denote by $P(X,Y,Z)$ the homogenized polynomial of $p(x,y)$, i.e.,

$$P(X,Y,Z) = Z^\varrho p \left( \frac{X}{Z}, \frac{Y}{Z} \right).$$

In order to find the number of affine solutions $(x : y : 1)$ such that $xy \neq 0$ and $x \neq y$, we proceed as follows. From Equation (2.6) we have that $P(X,Y,Z)$ is a divisor of the homogeneous polynomial

$$XYZ^{k-1} \left( \frac{h(X/Z) - h(Y/Z)}{X - Y} \right) + \tilde{b}Z^{k+1}.$$ (2.7)
Hence we conclude that there is no affine solution \((x : y : 1)\) of \(P(X, Y, Z)\) with \(xy = 0\). We now estimate the number of rational points of \(X\) at infinity, i.e., the points of the form \((x : y : 0)\) for \(x, y \in \mathbb{F}_q\). By Equation (2.7) the point \((x : y : 0)\) is on \(X\) only if
\[
xy \frac{x^k - y^k}{x - y} = 0.
\]
This holds only if \((x : y : 0) = (0 : 1 : 0), (1 : 0 : 0)\) or \(x^k = y^k\) for some \(x, y \in \mathbb{F}_q^*\). Since \(\nu = \gcd(k, q - 1)\), the equality \(x^k = y^k\) is satisfied if and only if \(x/y\) is an \(\nu\)-th root of unity in \(\mathbb{F}_q\). Hence there exist at most \(\nu + 2\) rational points of \(X\) lying at infinity.

Bezout’s theorem implies that there are at most \(k + 1\) rational points \((x : y : z)\) of \(X\) with \(x = y\), since the degree of \(X\) is at most \(k + 1\).

This shows that the cardinality \(\mu\) of the set \(S\) satisfies
\[
\mu \geq q + 1 - k(k - 1)\sqrt{q} - (\nu + k + 2).
\]
Note that we subtract \(\nu + k + 2\) instead of \(\nu + k + 3\). This is because of the point \((1 : 1 : 0)\). If \((1 : 1 : 0)\) is on \(X\) then it is taken into account twice. If it is not on \(X\) then we do not have to exclude it as a point at infinity. Therefore, \(\Crk(f) = n\) satisfies
\[
n \geq 1 + \frac{1}{k+1}(q + 1 - k(k - 1)\sqrt{q} - (\nu + k + 2)) = \frac{1}{k+1}(q - k(k - 1)\sqrt{q} - \nu),
\]
by (2.5), which implies the desired result.

For \(k = 1\) (and hence \(\nu = 1\)) we obtain Theorem 3, i.e., the main result in [8].

**Corollary 2.2.** Let \(f \in \mathcal{C}_{1,n}\). If \(n < (q - 1)/2\), then \(f\) is not a complete mapping.

**Remark 2.3.** We note that the bound given in (2.1) is non-trivial only when \(q \geq k(k - 1)\sqrt{q} + k + \nu + 1\).

### 3 The case \(g(x) = cx^k\)

Throughout this section we focus on the monomials \(g(x) = cx^k \in \mathbb{F}_q[x]\) and \(f \in \mathcal{C}_{1,n}\), where \(x_n \in \mathcal{O}_n\) in (1.4) satisfies \(x_n = 0\). In this particular case, the
lower bound in (2.1) can be simplified significantly when \( \gcd(k + 1, q - 1) = 1 \).

Let \( C_{2,n} \) be the set of \( f \in C_{1,n} \) such that the last pole \( x_n \) of \( f \) is zero.

**Theorem 3.1.** Let \( f(x) \) and \( f(x) + cx^k \) be permutation polynomials over \( \mathbb{F}_q \), where \( f \in C_{2,n}, \ 1 \leq k < q - 1, \ c \in \mathbb{F}_q^* \). Put \( m = \gcd(k + 1, q - 1) \). Then

\[
k(n + 3) + (k - 1)(m - 1)\sqrt{q} \geq q - n.
\]

In particular, if \( m = 1 \), then \( k \geq (q - n)/(n + 3) \).

**Proof.** The condition \( x_n = 0 \) implies that \( \beta_n \) in (1.3) is zero. Hence we have \( R_n(x) = \frac{ax + b}{x} \) for some \( a, b \in \mathbb{F}_q \), with \( b \neq 0 \). That is, for \( x \in \mathbb{F}_q \) \( \setminus \mathcal{O}_n \) we can represent \( f + cx^k \) by \( G_n(x) = R_n(x) + cx^k \).

We proceed as in the proof of Theorem 2.1. The equation \( G_n(x) = u \) for some \( u \in \mathbb{F}_q \) becomes

\[
\frac{ax + b}{x} + cx^k = u.
\]

Then for some \( x, y \in \mathbb{F}_q^* \), we have \( G_n(x) = G_n(y) \) if and only if the equation

\[
cx^k + \frac{b}{x} = cy^k + \frac{b}{y},
\]

or equivalently the equation

\[
x^k - y^k = \frac{b}{c} \left( \frac{x - y}{xy} \right) \tag{3.1}
\]

holds.

We again consider the set \( S \) of pairs \( (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*, \ x \neq y \), where \( (x, y) \) is a solution of (3.1), and denote the cardinality of \( S \) by \( \mu \). By using the argument given in the proof of Theorem 2.1, we have \( n \geq 1 + \mu/(k + 1) \). Hence our aim now is to express \( \mu \) in terms of \( q \) and \( k \).

Applying the change of variable \( (x, y) \rightarrow (xy, y) \), Equation (3.1) becomes

\[
y^k(x^k - 1) = \frac{b(x - 1)}{cxy}.
\]

Hence we are looking for the affine points \( (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \) of the curve

\[
\mathcal{X} : y^{k+1} = \frac{b(x - 1)}{cx(x^k - 1)}. \tag{3.2}
\]
Note that in this case the solutions should not lie in the set \( \{ (\gamma^2, \gamma) | \gamma \in \mathbb{F}_q \} \). Recall that \( m = \gcd(k + 1, q - 1) \), hence the monomial \( y^{(k+1)/m} \) gives rise to a permutation over \( \mathbb{F}_q^* \). Therefore, there is one-to-one correspondence between the affine solutions \( (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \) of the curves

\[
\mathcal{Y} : \quad y^m = \frac{b(x - 1)}{cx(x^k - 1)}, \tag{3.3}
\]

and \( \mathcal{X} \) in (3.2). Equation (3.3) defines a Kummer extension. Then by using arithmetic of function fields, see [15, Proposition 3.7.3], we can estimate the number of \( \mathbb{F}_q \)-rational points of \( \mathcal{Y} \) as follows.

For the rational function field \( \mathbb{F}_q(x) \) and \( \alpha \in \mathbb{F}_q \), we denote by \((x = \alpha)\) and \((x = \infty)\) the places corresponding to the zero and the pole of \( x - \alpha \), respectively. Let \( F = \mathbb{F}_q(x, y) \) be the function field of \( \mathcal{Y} \) defined by Equation (3.3), and let \( k = p^\ell t \) with \( \gcd(p, t) = 1 \). It is clear that the places \((x = 0)\) and \((x = \alpha)\), with \( \alpha^k = 1 \) and \( \alpha \neq 1 \), are totally ramified in \( F \). In particular, this shows that the full constant field of \( F \) is \( \mathbb{F}_q \). For the place \((x = \infty)\) we have the ramification index \( e_\infty = m/\gcd(m, k) = m \), since \( m \) is a divisor of \( k + 1 \). Moreover, for \((x = 1)\) the ramification index is given by \( e_1 = m/\gcd(m, p^\ell - 1) \). Hence we conclude that the number of ramified places of \( \mathbb{F}_q(x) \) in \( F \) is at most \( k/p^\ell + 2 \) if \( \ell > 0 \) and is exactly \( k + 1 \) if \( \ell = 0 \). That is, the place \((x = 1)\) can be ramified only if \( \ell > 0 \). We consider the case \( \ell = 0 \), i.e. \( \gcd(k, p) = 1 \), where the genus of \( F \) is the largest. In this case, the ramified places are exactly

\[
(x = 0), \quad (x = \infty) \quad \text{and} \quad (x = \alpha) \quad \text{with} \quad \alpha^k = 1 \quad \text{and} \quad \alpha \neq 1.
\]

Therefore, the degree of the different divisor of \( F/\mathbb{F}_q(x) \) is \((k + 1)(m - 1)\). Then by the Hurwitz genus formula the genus \( g(F) \) of \( F \) satisfies

\[
2g(F) - 2 = -2m + (k + 1)(m - 1),
\]

which implies that \( g(F) = (k - 1)(m - 1)/2 \). By the Hasse–Weil theorem the number \( N(F) \) of \( \mathbb{F}_q \)-rational places of \( F \) is bounded by

\[
N(F) \geq q + 1 - 2g(F)\sqrt{q} = q + 1 - (k - 1)(m - 1)\sqrt{q}. \tag{3.4}
\]

We observe that the pole divisors \((x)_\infty, (y)_\infty\) of \( x, y \) are

\[
(x)_\infty = mP_\infty \quad \text{and} \quad (y)_\infty = P_0 + \sum_{\alpha^k = 1, \alpha \neq 1} P_\alpha,
\]
where $P_{\infty}, P_0, P_\alpha$ are the unique places of $F$ lying over $(x = \infty), (x = 0), (x = \alpha)$, respectively.

We remark that the curve $Y$ defined by Equation (3.3) is of degree $k + m$ and has two points at infinity; namely $Q_1 = (1 : 0 : 0)$ and $Q_2 = (0 : 1 : 0)$. These are the only singular points of $Y$ and $Q_1$ has intersection multiplicity $m$ while $Q_2$ is an ordinary point of multiplicity $k$. Moreover, $P_{\infty}$ is the unique place corresponding to $Q_1$, and there are $k$ places corresponding to $Q_2$, which correspond to the places lying in the support of $(y)_\infty$. All the affine points in the curve $Y$ defined by Equation (3.3) are non-singular and there is a one to one correspondence between these points and the places in the function field $F$ of $Y$ which do not lie in the support of pole divisors of $x$ and $y$. Moreover, the fact that the zero divisors of $x$ and $y$ are $(x)_0 = mP_0$ and $(y)_0 = kP_{\infty}$, respectively, implies that the rational places not lying in the pole divisors correspond to points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. Therefore, Equation (3.4) implies that the number of affine points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of $Y$ is at least $q - (k - 1)(m - 1)\sqrt{q} - k$.

Now we turn our attention to the curve $X$ in Equation (3.2). We have seen that $X$ has at least $q - (k - 1)(m - 1)\sqrt{q} - k$ affine points $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. Next we estimate the number of affine points $(x, y)$ of $X$ such that $(x, y)$ is not of the form $(\gamma^2, \gamma)$ for some $\gamma \in \mathbb{F}_q$. By Equation (3.2), the affine point $(\gamma^2, \gamma)$ lies on $X$ if and only if $\gamma$ is a root of

\[T^{k+1} \sum_{i=1}^{k} T^{2i} - \frac{b}{c}.\]

Since the polynomial in Equation (3.5) has degree $3k + 1$, there can be at most $3k + 1$ such points. Hence the number $\mu$ of affine solutions $(x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ of Equation (3.2), which do not lie on the curve $x = y^2$ satisfies

$\mu \geq q - (k - 1)(m - 1)\sqrt{q} - (4k + 1)$.

Therefore $\text{Crk}(f) = n$ satisfies

$\frac{1}{k + 1} (q - (k - 1)(m - 1)\sqrt{q} - (4k + 1))$. 

\[ \square \]

**Example 3.2.** For $q = 9$, $n = 3$ and $m = 1$, the bound in Theorem 3.1 gives $k \geq 1$. Combining with Corollary 2.2 we get $k \geq 2$ as $q > 2n + 1$. Let $\zeta$ be a primitive element of $\mathbb{F}_9$ and consider the permutation polynomial $f(x) = (((x + a)^7 + b)^7 + c)^7 \in \mathbb{F}_9[x]$ of Carlitz rank 3, where $a = \zeta^5$, $b = \zeta^6$ and $c = \zeta^3$. It can be checked easily that $f(x) + x^2$ is a permutation polynomial of $\mathbb{F}_9$.
Remark 3.3. As we have seen in Example 3.2, the bound in Theorem 3.1 is weaker than the one in Theorem 2.1 for $k = 1$. The reason is the change of variable $(x, y) \rightarrow (xy, y)$ in the proof of Theorem 3.1. However, a direct calculation in this specific case is possible, and gives an alternative proof for Theorem 3, which was proven in [8]. In fact, the change of variable is not needed when $k = 1$ as Equation (3.1) becomes $xy = b$. In this case, each non-zero $x$ uniquely determines $y$, i.e., there exists $q - 1$ distinct solutions $(x, y)$ of $xy = b$. We also leave out the solutions $(x, y)$ with $x = y$. We therefore obtain $\mu = q - 2$ if $q$ is even, and $\mu = q - 3$ or $q - 1$ (depending on $b$ being square or not) if $q$ is odd. Then the fact that $n \geq 1 + \mu/2$ implies Corollary 2.2.

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