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Abstract

We prove a general result on graph factors modulo $k$. A special case says that, for each natural number $k$, every $(12k - 7)$-edge-connected graph with an even number of vertices contains a spanning subgraph in which each vertex has degree congruent to $k$ modulo $2k$.

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1 Introduction

Jaeger [7], [8] generalized Tutte’s 3-flow conjecture to the following conjecture which he called the \textit{circular flow conjecture}:

If $k$ is an odd natural number, and $G$ is a $(2k - 2)$-edge-connected graph, then $G$ has an orientation such that each vertex has the same indegree and outdegree modulo $k$.

This conjecture does not extend to the case where $k$ is an even number (because a vertex of odd degree cannot be balanced modulo an even number) also not in the weak version where we replace the edge-connectivity $2k - 2$ by a larger function of $k$. However, the weak version becomes true also when

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Let $k$ be a natural number, and let $G$ be a $(2k^2 + k)$-edge-connected graph with $n$ vertices $v_1, v_2, \ldots, v_n$. Let $d_i$ be an integer for each $i = 1, 2, \ldots, n$ such that the sum of all $d_i$ is congruent ($\mod k$) to the number of edges of $G$. Then $G$ has an orientation such that each $v_i$ has outdegrees $d_i$ modulo $k$. In [9] the quadratic bound $(2k^2 + k)$ was improved to the linear bound $3k - 3$.

This orientation result has applications to instances of the graph decomposition conjecture in [3] that, for every tree $T$, every graph of large edge-connectivity (depending on $T$ only) has an edge-decomposition into copies of $T$ (provided the size of $T$ divides the size of the graph), see [12], [13].

It also implies the $(2 + \epsilon)$-flow conjecture by Goddyn and Seymour. In [12] an application of the weak 3-flow conjecture to the $(2 + \epsilon)$-flow conjecture was discussed. However, as pointed out by a referee of an early version of the present paper, the general orientation result in [12] implies the $(2 + \epsilon)$-flow conjecture in its full strength by the discussion in Section 9.2 in [15]. In [14] it is shown precisely which flow values can be used in the $(2 + \epsilon)$-flow conjecture. Prior to that, the $(2 + \epsilon)$-flow conjecture had been verified first for planar graphs [6] and then for graphs on a fixed surface [16]. Apart from these results not much was known about the conjecture, as pointed out in [17].

In this paper we reformulate the orientation result in the weak circular flow conjecture as a factor result for bipartite graphs and derive the special case mentioned in the Abstract. This special case is related to results of Alon, Friedland, and Kalai [1] concerning non-empty subgraphs where all degrees are divisible by $k$. Those results are based on edge-densities only, whereas the results in the present paper need some kind of connectivity as well. To illustrate the different nature of the results in [1] and the present note, a special result in [1] (see also [2]) says that every graph with $n$ vertices and at least $2n + 1$ edges contains a non-empty subgraph in which each vertex has degree divisible by 3. Such a subgraph may be small as it may contain vertices of degree zero. By replacing edge-density by edge-connectivity we obtain subgraphs where all vertices have positive degrees, all divisible by 3. Specifically, every 29-edge-connected graph with an even number of vertices
has a spanning subgraph in which each vertex has degree 3 modulo 6.

2 Graph factors modulo \(k\)

The terminology and notation are the same as in [12] which are essentially the same as in [4], [5], [10]. In the present paper, however, a graph may have multiple edges (but no loops).

**Theorem 1** Let \(k\) be a natural number, and let \(G\) be a \((3k−3)\)-edge-connected bipartite graph with \(n\) vertices \(v_1, v_2, \ldots, v_n\) and with partite sets \(A, B\). Let \(d_i\) be an integer for each \(i = 1, 2, \ldots, n\) such that the sum of all \(d_i\) where \(v_i\) is in \(A\) is congruent \((\mod k)\) to the sum of all \(d_i\) where \(v_i\) is in \(B\). Then \(G\) has a spanning subgraph \(H\) such that

\[
d(v_i, H) \equiv d_i (\mod k)
\]

for \(i = 1, 2, \ldots, n\).

Proof of Theorem 1:

For each vertex \(v_i\) in \(A\), put \(p_i = d_i\). For each vertex \(v_i\) in \(B\), put \(p_i = d(v_i, G) − d_i\). Then the sum of all \(p_i\) is congruent to the number of edges modulo \(k\). By the strengthening in [9] of Theorems 1 and 3 in [12], the edges of \(G\) can be oriented such that each vertex \(v_i\) has outdegree \(p_i\) modulo \(k\). The edges oriented from \(A\) to \(B\) can now play the role of \(H\).

So, Theorem 1 is an immediate consequence of Theorems 1 and 3 in [12] and their extension in [9]. Conversely, it is easy to derive these results (except for a weaker upper bound on the edge-connectivity needed) from Theorem 1 above because every \((2k − 1)\)-edge-connected graph \(G\) contains a spanning bipartite \(k\)-edge-connected subgraph \(H\), as pointed out in Proposition 1 in [11]. (The proof is easy: Consider a spanning bipartite subgraph with as many edges as possible. If that subgraph has a cut with fewer than \(k\) edges, then the corresponding partition of the vertex set can be used to find a spanning bipartite subgraph with more edges, a contradiction.) If we wish to orient all edges in \(G\) such that the vertices have prescribed outdegrees modulo \(k\), then we orient the edges in \(G\) but not in \(H\) at random, and then we apply Theorem 1 to \(H\) resulting in a subgraph \(H'\). All edges in \(H'\) are directed from one partite class to the other, and the edges in \(H\) but not
in $H'$ are directed in the opposite direction. By choosing the degrees in $H$ appropriately (modulo $k$), we obtain the desired orientation of $G$.

Thus we may regard Theorem 1 as a reformulation of Theorem 3 in [12] and its extension in [9]. We apply this to a result for general graphs.

**Theorem 2** Let $k$ be a natural number, and let $G$ be a $(6k-7)$-edge-connected graph with $n$ vertices. Let $d_i$ be an integer for each $i = 1, 2, \ldots, n$ such that, for any $m$ in $\{1, 2, \ldots, n-1\}$, there is a partition of $\{1, 2, \ldots, n\}$ into sets $A, B$ of cardinality $m, n - m$, respectively, such that

the sum of all $d_i$ where $i$ is in $A$ is congruent (mod $k$) to the sum of all $d_i$ where $i$ is in $B$.

Then $G$ has a spanning subgraph $H$ such that the degrees of $H$ are $d_1, d_2, \ldots, d_n$ modulo $k$.

Proof of Theorem 2:

By the above-mentioned observation in Proposition 1 in [11], $G$ has a spanning $(3k-3)$-edge-connected bipartite subgraph $M$. Then apply the partition condition of $d_1, d_2, \ldots, d_n$ where $m, n-m$ are the number of vertices in the two partite sets of $M$. Theorem 2 now follows from Theorem 1.

The partition condition of $d_1, d_2, \ldots, d_n$ is necessary because $G$ might be bipartite to begin with. Unfortunately, that condition puts a severe restriction on the applications to non-bipartite graphs. Another weakness is that we do not specify which vertices have which degrees, and therefore Theorem 2 is not really a factor result. However, special cases are about factors, for example the following.

**Theorem 3** Let $k$ be a natural number, and let $G$ be a $(12k-7)$-edge-connected graph with an even number of vertices.

Then $G$ has a spanning subgraph $H$ such that each vertex in $H$ has degree congruent to $k$ (mod $2k$).

Proof of Theorem 3: The prescribed degrees modulo $2k$ satisfy the partition condition in Theorem 2 because the number of vertices is even. Note that Theorem 3 is not true if $k$ is odd and the number of vertices is odd.
References


