Integrating load-balancing into multi-dimensional bin-packing problems

Trivella, Alessio; Pisinger, David

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Integrating load-balancing into multi-dimensional bin-packing problems

Technical University of Denmark

Alessio Trivella
David Pisinger

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1. Recall the bin-packing problem
2. Integration of load-balancing: problem definition
3. MIP model for balancing a single bin
4. MIP model for joint packing and balancing
5. A multi-level local-search heuristic
6. Computational results
Multi-dimensional Bin-Packing problem (MBP)

Instance
- Set of D-dimensional rectangular-shaped boxes
  \[ V = \{1, \ldots, n\}. \text{ Box } i \text{ has width } = w_{i,d} \text{ in dimension } d \]
- Identical bins with width \( = W_d \) in dimension \( d \)

Problem
Orthogonally insert all boxes into the bins avoiding overlapping and using as few bins as possible. Rotations are not allowed

Applications
Shipping and transportation industry, filling up containers, loading trucks etc. Most real-world problems have \( D \leq 3 \), but all results hold for any dimension
MIP model for the MBP

\[ \begin{align*}
\text{min} & \quad N \\
\text{s.t.} & \quad \sum_{d \in D} (l_{ijd} + l_{jdi}) + p_{ij} + p_{ji} \geq 1 \\
& \quad x_{id} - x_{jd} + W_d l_{ijd} \leq W_d - w_id \\
& \quad x_{id} \leq W_d - w_id \\
& \quad a_i - a_j + n p_{ij} \leq n - 1 \\
& \quad 1 \leq a_i \leq N \\
\var : & \quad a_j, N \in \mathbb{N}, \; x_{id} \in \mathbb{R}^+, \; l_{ijd}, p_{ij} \in \{0, 1\} \\
\end{align*} \]

\[ \begin{align*}
& \quad \forall i < j \in V \\
& \quad \forall i \neq j \in V, \; d \in D \\
& \quad \forall i \in V, \; d \in D \\
& \quad \forall i \neq j \in V \\
& \quad \forall i \in V \\
& \quad \forall i, j \in V, \; d \in D
\end{align*} \]

Difficult to solve in practice due to many:
- symmetries
- big-M constraints
Load-Balanced MBP (LB-MBP)

**Instance**
MBP instance + density of items $\rho_i$ (or mass)

**Problem**
Arrange items into the minimum number of bins, in such a way that the barycenters of the loaded bins fall as close as possible to an ideal point (e.g. the center of the bin or center of its base)

**Applications**
Transport (ship, truck, aircraft’s cargo): a good position of the center of mass increases the safety and efficiency of the travel, minimizing the waste of fuel
Objective function

Minimize the total imbalance over:
- used bins
- dimensions

\[
\begin{align*}
\text{Optimal barycenter} & : \text{imbalance} = 3 \\
\text{Actual barycenter} & : \text{imbalance} = 0 \\
\text{imbalance} = 2.5 \\
\text{total imbalance} & = 5.5
\end{align*}
\]
Balancing a single bin

Assume to have a set $V$ of items which fit into a single bin:

$$\min \sum_{d \in D} k_d \left( r_d + s_d \right)$$

s.t. : $r_d - s_d = B^\text{opt}_d - \frac{1}{M} \left( \sum_i m_i \left( x_{id} + \frac{w_{id}}{2} \right) \right)$  \quad \forall d \in D

$$\sum_{d \in D} (l_{ijd} + l_{jid}) \geq 1$$  \quad \forall i < j \in V

$$x_{id} - x_{jd} + W_d l_{ijd} \leq W_d - w_{id}$$  \quad \forall i \neq j \in V, \; d \in D

$$x_{id} \leq W_d - w_{id}$$  \quad \forall i \in V, \; d \in D

var : $x_{id}, \; r_d, \; s_d \in \mathbb{R}^+$  \quad l_{ijd} \in \{0, 1\}$

To solve the LB-MBP we could:

1) Find the smallest number of bins
2) Balance each bin to optimality

...but the packing and balancing phases are not linked together!
MIP model for the LB-MBP

\[
\min \ N \ C + \sum_{d=1}^{D} \sum_{j=1}^{N} K_d \left( r_{jd} + s_{jd} \right)
\]

s.t. : \[
\sum_{d=1}^{D} \left( l_{ijd} + l_{jid} \right) + p_{ij} + p_{ji} \geq 1 \quad \forall i < j
\]
\[
x_{id} - x_{jd} + W_d \ l_{ijd} \leq W_d - w_{id} \quad \forall i, j, \forall d
\]
\[
a_i - a_j + n \ p_{ij} \leq n - 1 \quad \forall i, j
\]
\[
x_{id} \leq W_d - w_{id} \quad \forall i
\]
\[
1 \leq a_i \leq N \quad \forall i
\]
\[
n (c_{ij} - 1) \leq a_i - j \leq n (1 - c_{ij}) \quad \forall i, j
\]
\[
1 - (n + 1) (1 - \delta_{ij}) \leq a_i - j \leq -1 + (n + 1) (1 - \gamma_{ij}) \quad \forall i, j
\]
\[
c_{ij} + \gamma_{ij} + \delta_{ij} = 1 \quad \forall i, j
\]
\[
m_i \ W_d \left( c_{ij} - 1 \right) \leq e_{ijd} - m_i \ (x_{id} + w_{id} / 2) \leq m_i \ W_d \left( 1 - c_{ij} \right) \quad \forall i, j, \forall d
\]
\[
m_i \ W_d \left( c_{ij} - 1 \right) \leq \alpha_{ijd} - m_i \ \left( W_{d}^{opt} - r_{jd} + s_{jd} \right) \leq m_i \ W_d \left( 1 - c_{ij} \right) \quad \forall i, j, \forall d
\]
\[
e_{ijd} \leq c_{ij} \ W_d \ m_i \quad \forall i, j, \forall d
\]
\[
\alpha_{ijd} \leq c_{ij} \ W_d \ m_i \quad \forall i, j, \forall d
\]
\[
\sum_{i=1}^{N} e_{ijd} = \sum_{i=1}^{N} \alpha_{ijd} \quad \forall j, \forall d
\]

var: \( a_j, N \in \mathbb{N}, \quad x_{id}, r_{jd}, s_{jd}, e_{ijd}, \alpha_{ijd} \in \mathbb{R}^+, \quad l_{ijd}, p_{ij}, c_{ij}, \gamma_{ij}, \delta_{ij} \in \{0, 1\} \)
Sequential vs. joint problem

3D instance with 18 items, $\rho_i = 1$, $B^{opt} = (5, 5, 0)$

**Sequential problem**

<table>
<thead>
<tr>
<th>bin 1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>8</th>
<th>10</th>
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<tr>
<td>bin 2</td>
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<td>11</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>bin 3</td>
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<td>12</td>
<td>14</td>
<td>17</td>
<td></td>
<td></td>
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<tr>
<td>bin 4</td>
<td>13</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

Optimal 3DBPP: uses 4 bins

<table>
<thead>
<tr>
<th>bin</th>
<th>$B_x$</th>
<th>$B_y$</th>
<th>$B_z$</th>
<th>$f_{bin}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bin 1</td>
<td>5.00</td>
<td>5.00</td>
<td>4.18</td>
<td>4.18</td>
</tr>
<tr>
<td>bin 2</td>
<td>6.12</td>
<td>5.00</td>
<td>4.44</td>
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<tr>
<td>bin 3</td>
<td>5.00</td>
<td>5.00</td>
<td>4.38</td>
<td>4.38</td>
</tr>
<tr>
<td>bin 4</td>
<td>5.00</td>
<td>5.00</td>
<td>3.32</td>
<td>3.32</td>
</tr>
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</table>

$\text{f}_{\text{coord}}$ | 1.12 | 0.00 | 16.32 | 17.44 |

**Joint problem**

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<tr>
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<th>5</th>
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</thead>
<tbody>
<tr>
<td>bin 2</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>bin 3</td>
<td>9</td>
<td>11</td>
<td>14</td>
<td>17</td>
</tr>
<tr>
<td>bin 4</td>
<td>13</td>
<td>15</td>
<td>16</td>
<td>18</td>
</tr>
</tbody>
</table>

Different 3DBPP solution

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<th>$B_z$</th>
<th>$f_{bin}$</th>
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<tr>
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<td>5.00</td>
<td>3.64</td>
<td>3.64</td>
</tr>
<tr>
<td>bin 3</td>
<td>5.00</td>
<td>5.00</td>
<td>4.59</td>
<td>4.59</td>
</tr>
<tr>
<td>bin 4</td>
<td>5.00</td>
<td>5.00</td>
<td>3.32</td>
<td>3.32</td>
</tr>
</tbody>
</table>

$\text{f}_{\text{coord}}$ | 0.00 | 0.00 | 15.77 | 15.77 |

10% improvement!

But running time is 4 vs. 132 seconds. In general the joint model cannot be solved for instances larger than 15-20 items.
Heuristic load balancing

We now develop a heuristic algorithm to solve large instances.

It is possible to characterize feasible packings by means of a set of Interval Graphs (Fekete-Schepers).

\[ G_x \quad G_y \]
Properties (1/2)

**Theorem 1**

If $D$ graphs $G_d$, $d \in D$, are obtained from a packing, then the following conditions are fulfilled:

- $P_1$: Each $G_d$ is an interval graph
- $P_2$: $\bigcap_d G_d = \emptyset$
- $P_3$: The stable sets of $G_d$ have total weight less than the $d$-dimension of the bin

**Definition**

Let $G$ be an undirected graph. An orientation $\Phi$ of $G$ is called transitive orientation (TRO) if:

$$(a, b) \in \Phi \land (b, c) \in \Phi \implies (a, c) \in \Phi$$
**Theorem 2**

If $G$ is an interval graph, then its complement $\overline{G}$ is transitively orientable.

**Theorem 3**

Let $G_d$, $d \in D$ be $D$ graphs satisfying $P_1$, $P_2$, $P_3$, and call $\Phi = (\Phi_d)_{d \in D}$, where $\Phi_d$ is a transitive orientation of $\overline{G_d}$.

The function $p^\Phi : V \rightarrow \mathbb{R}_0^{+D}$ defined by:

$$p_d^\Phi(v) = \begin{cases} 0 & \text{if } \nexists u \in V : (u, v) \in \Phi_d \\ \max\{p_d^\Phi(u) + w_d(u) \mid (u, v) \in \Phi_d\} & \text{otherwise} \end{cases}$$

produces a packing.
How many transitive orientations?

Different transitive orientations produce different packings

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How many TROs?
From graph theory: number of TROs of a graph is \( \prod_{i=1}^{k} r_i! \) where \( r_i \) is the number of vertices of particular substructures

Is it possible to find TROs?
From graph theory: we can characterize them all TROs of a graph (it’s complicated though)

Local Search
We define a best-improvement local search exploring a quadratic neighborhood of TROs. For each TRO:
- go back to the corresponding packing
- evaluate the load balancing
Example for a 2D case

Items have different densities, bin = 5x5, $B_{opt} = (2.5, 2.5)$

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Local search at graph level

Purpose: improve cases where the number of TROs is limited

How: modifying the structure itself of the graphs:
- Consider interval graphs $G_d$
- Add or remove edges using specific rules (Crainic et al.)

If new graphs correspond to a packing, then start local search on TROs
Local search at bin-packing level (1/2)

Purpose
Exploit the balancing potential of having a large number of bin-packings solutions with the same number of bins

How
Iteratively repack and rebalance n-tuples of weakly balanced bins using *variable-depth neighborhood search* (VDNS)
Local search at bin-packing level (2/2)

Define a *k*-neighborhood as the set of all bin-packing solutions obtained by repacking at most *k* bins

VDNS algorithm

1. Assign imbalance scores to the bins
2. Select *k* bins using *roulette wheel selection*
3. Repack the bins using a heuristic for MBP
4. If *k* bins are still used, balance them
5. If balancing is improved: save solution and update scores

*k* is dynamically adjusted:
if no solutions are found after *n* iterations: $k = k + 1$
Results for 3D Bin-Packing instances (1/2)

Optimal barycenter \( \left( \frac{W_1}{2}, \frac{W_2}{2}, \frac{W_3}{2} \right) \): center of the bin

<table>
<thead>
<tr>
<th>cl.</th>
<th>size</th>
<th>LB</th>
<th>bins</th>
<th>( \rho_i = 1 )</th>
<th>( \rho_i \sim U(1, 2) )</th>
<th>( \rho_i \sim U(1, 6) )</th>
</tr>
</thead>
<tbody>
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<td>Init.</td>
<td>Bin</td>
<td>VDNS</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>24.12</td>
<td>25.64</td>
<td>16.48</td>
<td>0.91</td>
<td>0.011</td>
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<tr>
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Results are average over 25 instances, running time is < 5-10s
Results for 3D Bin-Packing instances (2/2)

Optimal barycenter \( \left( \frac{W_1}{2}, \frac{W_2}{2}, 0 \right) \): center of the base

<table>
<thead>
<tr>
<th>cl.</th>
<th>size</th>
<th>LB</th>
<th>bins</th>
<th>( \rho_i = 1 )</th>
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<td><strong>44.37</strong></td>
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</tbody>
</table>

- Results are average over 25 instances, running time is < 5-10s
- LLB is a lower bound obtained from “liquifying” the items
Thank you.