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A Sufficient Condition on Convex Relaxation of AC Optimal Power Flow in Distribution Networks

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Abstract—This paper proposes a sufficient condition for the convex relaxation of AC Optimal Power Flow (OPF) in radial distribution networks as a second order cone program (SOCP) to be exact. The condition requires that the allowed reverse power flow is only reactive or active, or none. Under the proposed sufficient condition, the feasible sub-injection region (power injections of nodes excluding the root node) of the AC OPF is convex. The exactness of the convex relaxation under the proposed condition is proved through constructing a group of monotonic series with limits, which ensures that the optimal solution of the SOCP can be converted to an optimal solution of the original AC OPF. The efficacy of the convex relaxation to solve the AC OPF is demonstrated by case studies of an optimal multi-period planning problem of electric vehicles (EVs) in distribution networks.

Index Terms—AC optimal power flow (AC OPF), convex relaxation, convexity, electric vehicle (EV), power distribution network, second order cone program (SOCP).

I. NOMENCLATURE

Sets:
- $\mathcal{E}$ set of edges of the whole network
- $\mathcal{E}_i$ set of edges of the subtree from node $i$
- $\mathcal{N}$ set of nodes of the whole network
- $\mathcal{N}_i$ set of nodes of the subtree from node $i$
- $\mathcal{N}^+$ subset of nodes, excluding node 0, named as plus nodes
- $\mathcal{N}^+_i$ subset of nodes of the subtree, excluding node $i$

Parameters:
- $k_{\text{ta}}$ EV availability
- $\pi_i$ Conventional consumption, complex number
- $c_i$ energy price
- $\overline{\tau}_i$ EV charging power upper limit, active power
- $r$ (square of the magnitude) line current upper limit
- $\overline{\tau}$ real part of $\tau$
- $\overline{s}$ imaginary part of $\tau$
- $\overline{\eta}$ upper limit of the sub-injection
- $\underline{s}$ lower limit of the sub-injection
- $x$ reactance
- $v_0$ (square) voltage of node 0, real number
- $v_{i}$ (square) voltage of plus nodes, lower limit
- $v_{i}$ (square) voltage of plus nodes, upper limit
- $z$ impedance, $z = r + jx$
- $\overline{S}$ line capacity (absolute value) upper limit

Variables:
- $e$ EV charging power, active power
- $i$ (square of the magnitude) line current
- $p, p_i$ real part of $s$
- $p_0$ real part of $s_0$
- $q, q_i$ imaginary part of $s$
- $s, s_i$ injection of plus nodes ($i > 0$), i.e. sub-injection
- $s_0$ injection of node 0, complex number
- $v, v_{i}$ (square) voltage of plus nodes, real number, $i > 0$
- $\hat{v}_{i}$ linear approximation of (square) voltage
- $P$ reverse line flow, active part
- $\hat{P}$ linear approximation of the reverse line flow, active part
- $\hat{Q}$ linear approximation of the reverse line flow, reactive part
- $\hat{Q}$ linear approximation of the reverse line flow, reactive part
- $s$ reverse line flow, complex number
OPTIMAL power flow (OPF) is a very important optimization problem widely used in power engineering applications, such as congestion management, economic operation and control. The OPF considers the economic aspect of the power system components, and models the system at the steady state. There are two types of OPF, i.e. AC OPF [1] and DC OPF [2], resulting from AC and DC power flow models. The DC OPF is a linear problem, and can be solved very efficiently and robustly by many commercial optimization solvers such as GAMS/CPLEX [3]. However, the solution of the DC OPF is only an approximation of the power system status. It is far from accurate if the R/X ratio is high in the studied system.

The AC OPF is non-convex in its original form and is an NP-hard problem. General-purpose nonlinear programming (NLP) solvers can be used to solve the AC OPF. A number of dedicated methods were developed to solve the AC OPF problems in the last two decades, such as the trust region interior point algorithm [4], [5], Lagrangian method [6], and primal-dual interior point method [7]. However, these methods normally obtain a locally optimal solution and it is not possible to know how far it is from the global optimum.

The convex relaxation method for solving the AC OPF was first presented in [8] as a second-order cone programming (SOCP) for radial networks and in [9] as a semidefinite programming (SDP) for meshed networks. For meshed networks, [10] analyzed the exactness of the convex relaxation. However, the analysis is limited to resistive networks.

For radial networks, [11]–[13] have proposed several sufficient conditions for the convex relaxation to be exact. In [11], [12], it is proposed that if just one of any two connected nodes has a lower active power bound and no node has a lower reactive power bound, the convex relaxation is exact. However, these sufficient conditions may not be practical. In [13], the authors propose that if the upper bounds of the active and reactive power are not too large, the convex relaxation will be exact. The conclusion of [13] is promising as it allows a certain amount of feed-in power from renewables; however, the constraints of the AC OPF formulation in [13] miss the thermal and line capacity limits. The line capacity limit may lead to inexactness of the convex relaxation of the AC OPF as shown in [14]. Besides, the thermal and line capacity limits are necessary in many practical applications with renewable energies or flexible demands present, such as the congestion management applications in [15], [16]. The details of the convex relaxation for the AC OPF can be found in [17], [18].

This paper proposes a sufficient condition, denoted as C1 and described in section IV, such that the convex relaxation of the AC OPF with all practical constraints is exact. The main contributions of this paper can be summarized as follows: (a) Prove that under condition C1, the feasible sub-injection region (feasible power injections to the nodes excluding the root node) of the AC OPF is the same as the feasible sub-injection region of the SOCP, and both are convex. (b) Relax the requirement of the cost function from being strictly increasing to non-decreasing. (c) Prove C1 is a sufficient condition leading to exact convex relaxation of the AC OPF with current, voltage and line capacity constraints. (d) Analyze the sufficient condition for exact convex relaxation of OPF with inter-temporal constraints, and demonstrate the efficacy of the convex relaxation for solving the AC OPF of multi-period energy planning of electrical vehicles (EVs).

The paper is organized as follows. Section III discusses the convexity of the sub-injection region under condition A1, which is straightforward but more conservative than C1. Section IV analyzes the exactness of the convex relaxation under condition C1. In section V, the AC OPF with multi-period energy planning of EVs is described and analyzed. Case studies of the AC OPF with EV planning are presented and discussed in section VI, followed by conclusions.

III. CONVEXITY AND CONVEX RELAXATION OF AC OPF IN DISTRIBUTION NETWORKS

A. Convex Relaxation

A few assumptions are made for the convexity analysis and convex relaxation of AC OPF in distribution networks. Only radial networks are considered, i.e. the network $(N,E)$ is a tree. Node 0 is chosen to be the slack node, i.e. voltage $v_0$ is fixed and given (normally 1 p.u.). Node 0 is also the root of the tree graph and the direction of each edge is pointed towards the root. In all practical applications in distribution networks, the voltage limits of each node are $1 \pm \alpha \%$ p.u., where $\alpha$ can be a number between 4 and 10 depending on the standards. Therefore, it is reasonable to assume that $\nabla > v_0$ and $\nabla < v_0$ (assume that $v_i = 1$ p.u.).

The original AC OPF can be written as (1)-(9). The function $f_0$ is listed separately in the objective function because there is a special requirement for $f_0$ (see discussion in section IV.A). Constraints (2)-(5) describe the line flow through a branch flow model. Constraints (6)-(7) are the thermal and line capacity limits. Constraint (8) is the limit of the generator output or the demand, and (9) is the voltage limits.

\[
\text{OPF: } \min_{s_i, \hat{S}_f, \lambda} f_0(\text{Re}(s_i)) + f_1(\text{Re}(s)) ,\tag{1}
\]
\[
s = s_f + \sum_{i \in N} (S_{i0} - z_{i0}i_{i0}), \forall (i, j) \in E , \tag{2}
\]
\[
0 = s_f + \sum_{i \in N} (S_{i0} - z_{i0}i_{i0}) , \tag{3}
\]
\[
v_i - v_j = 2 \text{Re}(\bar{v}_i S_j) - \|i\|_2^2, \forall (i, j) \in E , \tag{4}
\]
\[
i_j = \|i\|_2^2, \forall (i, j) \in E , \tag{5}
\]
\[
i_j \leq T_j, \forall (i, j) \in E , \tag{6}
\]
The sub-injection. Proposition 1 will be proven after the proof of the following lemma.

**Lemma 1:** If $A_1$ holds, then $S^+ = \hat{S}^+$, i.e. the sub-injection region is equal to the convex-relaxed one.

$$\hat{S} = \{s_0, s : (2)-(4), (6)-(10)\}$$

It is obvious that $\forall s \in S^+$, then $s \in \hat{S}^+$ because of the relaxation. Therefore, there is $S^+ \subseteq \hat{S}^+$. Then we only need to prove $\hat{S}^+ \supseteq S^+$.

Let $(S^{(k)}, v^{(k)}, i^{(k)}, s_0) = (S^{(0)}, v^{(0)}, i^{(0)}, s_0), \forall k \in \mathbb{Z}^+$, and $v_0^{(k)} = v_0$. Then in the $k$-th ($k > 0$) iteration, apply a Forward-Backward sweep algorithm as described below.

**Forward:** For $i = n, n-1, n-2, \ldots, 1$, and $(i, j) \in \mathcal{E}$, apply sequentially,

$$s_y^{(i)} = s_y + \sum_{k=0}^{\infty} \left(S_{b_{y}}^{(i)} - z_{b_{y}}i_{b_{y}}^{(k)} - z_{b_{y}}j_{b_{y}}^{(k)}\right)$$

$$\geq s_y + \sum_{k=0}^{\infty} \left(S_{b_{y}}^{(i-1)} - z_{b_{y}}i_{b_{y}}^{(k-1)} - z_{b_{y}}j_{b_{y}}^{(k-1)}\right)$$

$$= S_y^{(i-1)}$$

$$i_y^{(i)} = \frac{S_y^{(i)} + S_y^{(i-1)}}{v_y^{(i)} + v_y^{(i-1)}} \leq \frac{S_y^{(i-1)} + S_y^{(i-2)}}{v_y^{(i-2)}} \leq i_y^{(i-1)}.$$

**Backward:** For $i = 1, 2, 3, \ldots, n-1, n$, and $(i, j) \in \mathcal{E}$, apply,

$$s_0^{(i)} = -\sum_{k=0}^{\infty} \left(S_{b_{0}}^{(i-1)} - z_{b_{0}}i_{b_{0}}^{(k)} - z_{b_{0}}j_{b_{0}}^{(k)}\right)$$

$$\leq -\sum_{k=0}^{\infty} \left(S_{b_{0}}^{(i-1)} - z_{b_{0}}i_{b_{0}}^{(k-1)} - z_{b_{0}}j_{b_{0}}^{(k-1)}\right).$$

$$= s_0^{(i-1)}$$
\[ v^{(k)}_i = v^{(k)}_j + 2 \text{Re}(\Sigma_{y'\neq y} S_{y'j}) - \left| z_{ij} \right| I_{y}^{(k)} \geq v^{(k-1)}_j + 2 \text{Re}(\Sigma_{y'\neq y} S_{y'j}) - \left| z_{ij} \right| I_{y}^{(k-1)}. \]  

(14)

The above algorithm not only shows how to update the variables iteratively, but also ensures that the obtained series are monotonic.

In order to show the monotone of the series, let \( k = 1 \). Since \( \forall i > 0 \), \( j \) is unique such that \( (i, j) \in E \), the subscription \( j \) is sometimes omitted in branch variables or parameters for brevity. In the forward sweep, when \( i = n \), the inequality in (11) holds because it is a leaf node. Because \( s_x \leq 0 \) due to A1, \( S_x^{(0)} \leq 0 \). Therefore, the first inequality in (12) holds due to (11). Because \( v^{(1)} = v_{s} \), \( \forall k \leq 0 \), the second inequality in (12) holds. According to (10), \( i^{(0)}_x \geq \left| S_x^{(0)} \right| / v_{s}^{(0)} \), the third inequality in (12) holds as well. When \( i = n-1 \), the inequality in (11) holds because (11) and (12) holds for any of its child nodes (if any) with the relabeling shown in Fig. 1. Noticing that for any of its child nodes (if any), \( S_x^{(0)} \leq 0 \), \( z_{x,x+1} \geq 0 \) and \( i^{(0)}_x \geq 0 \), there is \( S_x^{(0)} \). With (11) and \( S_x^{(0)} \), the first inequality in (12) holds. Similar to the analysis for \( i = n \), the second and third inequality in (12) holds for \( i = n-1 \). Similarly, it can be verified that (11) and (12) hold for the rest \( i \). The analysis of (13) is the same as (11).

In the backward sweep, when \( i = 1 \), the inequality holds in (14) because \( v^{(0)}_s = v_{s} \), (11) and (12). When \( i = \lambda \), 1, the inequality in (14) holds for \( i = 2, 3, \ldots, n-1, n \), respectively. The second equation in (11), (13) and (14) holds because (2)-(4) hold for \( (s, S_x^{(0)}, v^{(0)}, i^{(0)}, s_y^{(0)}) \).

Because of the iterative nature of (11)-(14), it can be verified that they hold for \( k = 2, 3, 4, \ldots \). Therefore, there exists the following monotonic series.

\[ \{S_x^{(k)}; S_x^{(0)} \leq S_x^{(1)} \leq S_x^{(2)} \leq \ldots \leq 0 \} \]

\[ \{i^{(k)}; i^{(0)} \geq i^{(1)} \geq i^{(2)} \geq \ldots \geq 0 \} \]

\[ \{s_y^{(k)}; s_y^{(0)} \leq s_y^{(1)} \leq s_y^{(2)} \geq \ldots \geq 0 \} \]

\[ \{v^{(k)}; v^{(0)} \leq v^{(1)} \leq v^{(2)} \leq \ldots \leq v_{s} \} \]

In the last series, \( v_{s} \) is a bound, because \( 2 \text{Re}(\Sigma_{y'\neq y} S_{y'j}) \) is negative. Since all the above infinite monotonic series have bounds, they have limits. Denote \( S' = \lim_{k \to +\infty} S^{(k)}, i' = \lim_{k \to +\infty} i^{(k)}, s'_y = \lim_{k \to +\infty} s_y^{(k)} \) and \( v' = \lim_{k \to +\infty} v^{(k)}. \)

It can be verified that \((s, S', v', i', s'_y)\) fulfills the line flow constraints (2)-(5) and \( s \) satisfies (8). It can also be shown that \((S', v', i')\) satisfies constraints (6), (7) and (9) because of the assumption of \((S^{(0)}, v^{(0)}, i^{(0)})\) and the relation between \((S^{(0)}, v^{(0)}, i^{(0)})\) and \((S', v', i')\). Hence, \((s, s_y') \in S' \) and 

IV. EXACTNESS ANALYSIS

A. Exactness of Convex Relaxation

In [13], a definition of the exactness of the convex relaxation is given: as the SOCP is exact if every of its optimal solutions satisfies the nonlinear line flow constraint (5). This requires (10) to be active at the optimal point and \( f_{s} \) to be strictly increasing. However, this requirement can be relaxed.

It can be seen from the proof of Lemma 1 that any optimal point of the SOCP can be converted to a feasible point to the original AC OPF. In order to use this Lemma, the definition of the convex relaxation exactness is modified to an intuitive one: the SOCP is exact if the gap between the optimal values of the AC OPF and its convex relaxation is zero. It is shown below that \( f_{s} \) is only required to be non-decreasing for the SOCP to be exact.

Proposition 2: When \( f_{s} \) is non-decreasing, the SOCP is exact if A1 holds.

Proof: Assume that \((s, s_y')\) is an optimal solution of the SOCP and the constructed feasible point of the AC OPF is \((s, s_y')\), where \( s_y' \) is the constructed feasible injection at node 0 using the Forward-Backward sweep method described in Section III.B. It is obvious that \( f_{s} (\text{Re}(s_y')) + f_{s} (\text{Re}(s')) \leq f_{s} (\text{Re}(s_y''')) + f_{s} (\text{Re}(s'')) \), because \((s', s_y'')\) is an optimal point of the SOCP. It is known from the construction process that \( s_y'' \leq s_y' \). Therefore, if \( f_{s} \) is non-decreasing, there is

\[ f_{s} (\text{Re}(s_y')) + f_{s} (\text{Re}(s')) \geq f_{s} (\text{Re}(s_y''')) + f_{s} (\text{Re}(s'')) \]

Therefore, they are equal. It is obvious that \((s', s_y'')\) is an optimal solution of the AC OPF and it has the same optimal value. This ends the proof of Proposition 2.

An example of non-decreasing function \( f_{s} \) is shown in Fig. 2. In fact, \( f_{s} \) can be zero or constant, which is a special case of non-decreasing. In that case, the SOCP is equivalent to the AC OPF in terms of having the same sub-injection region and the same objective value for every feasible point. The significance of this argument can be seen from the EV charging application presented in [20], [21]. The charging service provider employs an OPF problem to make an optimal charging plan for the EVs with the network constraint information received from the DSO. The optimal planning considers only the cost of EV charging; therefore, the cost function \( f_{s} \) is zero in this application. The authors of [20], [21] use an iterative method to solve the ACOPF problem, which is slow and complicated. However,
er, employ the method proposed in this paper, this application can be solved more efficiently (see section V and VI).

\[ \Delta |S_{ij}| = |S_{ij}^{(i)}| - |S_{ij}^{(0)}| = (P_{ij}^{(i)})^2 + (Q_{ij}^{(i)})^2 - (P_{ij}^{(0)})^2 - (Q_{ij}^{(0)})^2 \]
\[ = (\Delta P_{ij})(P_{ij}^{(i)} + P_{ij}^{(0)}) + (\Delta Q_{ij})(Q_{ij}^{(i)} + Q_{ij}^{(0)}) \leq (\Delta P_{ij})(2\hat{P}_i) + (\Delta Q_{ij})(2\hat{Q}_i) \]
\[ = 2(\sum_{(k, j) \in e} -r_{ij}\Delta u_i)\hat{P}_i + 2(\sum_{(k, j) \in e} -s_{ij}\Delta u_j)\hat{Q}_j \]
\[ = 2\sum_{(k, j) \in e} -\Delta u_i(r_{ij}\hat{P}_i + s_{ij}\hat{Q}_j) \leq 0 \]
\[ \Delta |S_{ij}| = |S_{ij}^{(i)}| - |S_{ij}^{(0)}| \leq \Delta |S_{ij}^{(i)}| \leq 0 \]

Equations (19)-(21) can be verified sequentially, i.e. after verify all of them for \( i = n \), start over again to verify (19)-(21) for \( i = n-1 \). For node 0, there is,
\[ \Delta x_0 = x_0^{(i)} - x_0^{(0)} = \sum_{n \in N^i} z_i (i_{ni}^{(i)} - i_{ni}^{(0)}) \]
\[ \leq 0 \]
For \( i = 1, 2, 3, \ldots, n-1 \), and \( (i, j) \in E \),
\[ \Delta v_i = v_i^{(i)} - v_i^{(0)} = \Delta v_i + 2Re(\hat{S}_{ij}) - |\hat{S}_{ij}|^2 \Delta u_i \geq 0 \]

Therefore, the voltage is increasing, which supports the first inequality in (21). The first inequality in (20) is due to the property of \( \hat{S}_{ij} = \hat{P}_i + j\hat{Q}_j \). The second inequality in (20) is due to (17).

It is not difficult to verify that for \( k \geq 2, (19)-(23) \) are valid. Notice that \( \hat{v}^{(i)} \leq \hat{v} \leq \bar{v} \), the new series will be,
\[ \hat{S}^{(i)} : S_{ij}^{(0)} \leq S_{ij}^{(i)} \leq \hat{S}_{ij} \leq \bar{S}_{ij} \]
\[ |\hat{v}^{(i)}| : |v| \geq |v^{(i)}| \geq |v^{(0)}| \geq 0 \]
\[ |\hat{s}^{(i)}| : |s| \geq |s^{(i)}| \geq |s^{(0)}| \geq 0 \]
\[ |\hat{v}^{(i)}| : v^{(i)} \leq v^{(0)} \leq v^{(2)} \leq \hat{v} \]

Therefore, the above series have limits and the limits satisfy the constraints of the OPF. Moreover, the SOCP is exact. This ends the proof of Proposition 3.

C. Discussion on Sufficient Condition A1, B1 and C1

The sufficient condition proposed in [13], named as B1, is rewritten in this paper.

B1: The positive linear approximation of the reverse power based on the upper limit of the sub-injection \( \hat{S} \) fulfills (24), and the linear approximation of the voltage based on the sub-injection \( s \) fulfills (18).
\[ A_i, A_j, A_s, A_u, v_i > 0, \forall i \sim i_s, i_t > 0, \]

where \( i_s \) is the parent node of \( i_{s+1} \).

\[ A_i = I - \frac{2}{\gamma_i} \left( \tilde{P}(\tilde{p})^T + \tilde{Q}(\tilde{q})^T \right), \]

\[ u_i = \left( \frac{r_i}{x_i} \right), \]

symbol \( a^\ast \) means max \((a, 0)\).

It can be seen that \( A_1 \) is a special case of \( B_1 \) given that \( r, x > 0 \). According to \( A_1 \), \( \tilde{p} \leq 0 \). There are \( (\tilde{p}(\tilde{p}))^T = 0 \) and \( (\tilde{q}(\tilde{q}))^T = 0 \). Therefore, \( A_i = I \) and the left side of (24) equals to \( u_i \), which is strictly positive since \( r, x > 0 \). Equation (18) is also satisfied because \( v \leq 0 < v \).

\[ A_1 \] is also a special case of \( C_1 \). Since \( \tilde{p} \leq 0 \), there is \( s \leq 0 \).

Therefore, \( \tilde{S} \leq 0 \) and the left side of (17) is always negative. Equation (18) is satisfied because \( v \leq 0 < v \). Comparing to \( A_1 \), \( C_1 \) allows active reverse power flow or reactive reverse power flow while \( A_1 \) doesn’t allow any reverse power flow.

The differences between \( B_1 \) and \( C_1 \) are analyzed below. First of all, it should be emphasized that \( B_1 \) is a sufficient condition for the exactness of the SOCP without line flow constraints while \( C_1 \) is proposed for the case with line flow constraints.

Secondly, for \( B_1 \) to be a sufficient condition, the cost function \( z_0 \) is required to be strictly increasing, while for the case with \( C_1 \), it only needs to be non-decreasing.

Thirdly, there is no requirement of the impedance for \( C_1 \) (and \( A_1 \) as well), but it is required that \( r, x > 0 \) for \( B_1 \). This is because the ‘greater than’ symbol is used in (24).

At last, \( B_1 \) does not imply \( C_1 \) and vice versa. According to \( B_1 \), the reverse power flow has an obvious upper limit, \( \tilde{P}(\tilde{p}) \leq \frac{V}{2r} \) and \( \tilde{Q}(\tilde{q}) \leq \frac{V}{2x} \), due to (24) and the definition of \( A_i \). However, according to \( C_1 \), if the active power flow is very high, the allowed reactive reverse power flow will also be very high, without an explicit upper limit. On the other hand, \( B_1 \) may allow both active and reactive reverse power flow at the same time. However, \( C_1 \) does not allow both active and reactive reverse power flow at the same time.

V. AC OPF FOR MULTI-PERIOD EV ENERGY PLANNING

In this section, the AC OPF with EV charging planning over multi-period is described considering the line flow and voltage limits.

**OPF-EV:**

\[
\begin{align*}
\text{min } & c, s, v, \lambda, g \\
\text{s.t.} & \\
S_{\text{op}} & = s_0 + \sum_{\lambda \in \lambda_0} (S_{\text{op}} - z_{\text{a},1\lambda}) , \forall (i, j) \in \mathcal{E}, t \in \mathcal{T} \\
0 & = s_0 + \sum_{\lambda \in \lambda_0} (S_{\text{op}} - z_{\text{a},1\lambda}) , \forall t \in \mathcal{T}
\end{align*}
\]

\[
\begin{align*}
\mathbf{v}_a - \mathbf{v}_p &= 2 \text{Re} (\mathbf{z}_{s,0}) - \int_0^T \mathbf{i}_{0, \gamma} (i, j) \mathcal{E}, t \in \mathcal{T} \\
i_{\gamma} &= \frac{1}{\mathbf{v}_a} [\mathbf{v}(i, j) \in \mathcal{E}, t \in \mathcal{T} \\
\mathbf{v}_a & \leq \mathbf{v}, \forall (i, j) \in \mathcal{E}, t \in \mathcal{T} \\
|\mathbf{s}_a| & \leq \mathbf{v}_a, \forall (i, j) \in \mathcal{E}, t \in \mathcal{T} \\
\text{Re}(s_{\gamma}) &= -\sum_{i \in \mathcal{V}} e_{\gamma} - \text{Re}(b_{\gamma}), \forall (i, j) \in \mathcal{E}, t \in \mathcal{T} \\
\sum_{\gamma \in \mathcal{V}} e_{\gamma} &= d_{\gamma}, \forall \mathcal{V} \\
0 & \leq e_{\gamma} \leq c_{\gamma} a_{\gamma}, \forall \mathcal{V}, t \in \mathcal{T}
\end{align*}
\]

The objective function is to minimize the total charging cost for the EVs. The EV charging related constraints (33)-(36) are all linear. Only (35) is coupling the multiple periods, which is to satisfy the total charged energy required by each EV. Constraint (36) is to limit the charging power, which will lead to the limit of \( s_\gamma \) through (33). Because of (29), the OPF-EV problem is nonconvex, which is very hard to solve. Similar to the method employed in the single period OPF, (29) can be relaxed. The corresponding SOCP is written below.

**SOCP-EV:**

\[
\begin{align*}
(25) & \text{ s.t. } (26)-(28), (30)-(36), \text{ and} \\
i_{\gamma} & \geq \frac{1}{\mathbf{v}_a} [\mathbf{v}(i, j) \in \mathcal{E}, t \in \mathcal{T}].
\end{align*}
\]

**Proposition 4:** The SOCP-EV is exact if \( C_1 \) holds.

\[ g = \sum_{\gamma \in \mathcal{V}} e_{\gamma} \text{Re}(s_{\gamma}). \]

\[ \sum_{\gamma \in \mathcal{V}} e_{\gamma} \geq d_{\gamma}, \forall \mathcal{V}, t \in \mathcal{T} \]
VI. Case Studies

Case studies were conducted using the Danish driving pattern and the Bus 4 distribution system of the Roy Billinton Test System (RBTS) [23] and the IEEE 123 node feeder [24]. The details of the case studies are presented in this section.

A. Grid Data

The single line diagram of the Bus 4 distribution network is shown in Fig. 3. Line segments and nodes of the feeder one are labeled in Fig. 3, among which L2-1, L4-3, L6-5, L8-7, L9-7, L11-10, and L12-10 refer to the transformers connecting the corresponding load points. Notice that the labelling follows the rules mentioned in section III.B. The study is focused on this feeder because it has the most diversity among all the feeders: 5 residential load points with different peak conventional demands and two commercial load points. The detailed data of these load points are listed in Table I. The reactive power consumption is assumed to be 10% of the active power consumption for each load point. The peak conventional demands of residential customers occur at 18:00 when people come home and start cooking (shown in Fig. 6).

### Table I

<table>
<thead>
<tr>
<th>load points</th>
<th>customer type</th>
<th>peak conv. load / point (kW)</th>
<th>peak conv. load / point (kVar)</th>
<th>number of customers per point</th>
</tr>
</thead>
<tbody>
<tr>
<td>node 2,4,6,8</td>
<td>residential</td>
<td>886.9</td>
<td>88.69</td>
<td>200</td>
</tr>
<tr>
<td>node 9</td>
<td>residential</td>
<td>813.7</td>
<td>81.37</td>
<td>200</td>
</tr>
<tr>
<td>node 11,12</td>
<td>commercial</td>
<td>671.4</td>
<td>67.14</td>
<td>10</td>
</tr>
</tbody>
</table>

Voltage limits are set to be ±5% p.u., then the limits of the square voltage will be $v = 0.95^2$ and $v = 1.05^2$. Assume that $v_0 = 1$. The line parameters including resistance, reactance and line capacity limits can be seen in Table II. The prices for the EV charging are the predicted day-ahead market system price, as shown in Fig. 4.

![Fig. 3. Single line diagram of the distribution network](image)

B. EV Data

### Table III

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>EV battery size</td>
<td>25 kWh</td>
</tr>
<tr>
<td>Peak charging power</td>
<td>11 kW (3 phase)</td>
</tr>
<tr>
<td>Energy consumption per km</td>
<td>150 Wh/km</td>
</tr>
<tr>
<td>Minimum SOC</td>
<td>20%</td>
</tr>
<tr>
<td>Maximum SOC</td>
<td>85%</td>
</tr>
<tr>
<td>Average driving distance</td>
<td>40 km</td>
</tr>
</tbody>
</table>

Assume there are 1000 EVs in the network, i.e., one EV per residential customer. The key parameters of the EVs are listed in Table III. The EV availability (available for charge per hour) shown in Fig. 5 is from the driving pattern study in [25].
C. Case Study Results

The simulation is carried out using CVX, a package for specifying and solving convex programs [26], [27]. CVX is a toolbox in Matlab and it supports several SOCP solvers, such as SeDuMi, SDPT3, Gurobi and MOSEK. In this paper, CVX/MOSEK is chosen, and the platform is a personal computer (a laptop) with Intel Core i5-4310U, 2 GHz, 8 GB RAM, windows 64-bit Operation System.

CVX can transform constraints (31) and (37) in SOCP-EV to conic ones, and thereby solved by the SOCP solvers. The optimal solution of the OPF-EV can be recovered using the Forward-Backward sweep method and used to describe the operation status of the network, as shown in Fig. 6, Fig. 7 and Fig. 8.

It can be seen from Fig. 6 that most of the EV charging loads are allocated at the hour with the lowest price, i.e., hour 24. However, due to the availability and the line capacity limit, part of the loads is allocated to the hours with the second and third lowest prices. Due to small resistances, the line losses are very small, which is about 1%. Hence, it is not shown in the figure. The sharing of EV loads and conventional loads are calculated using the linear model, i.e. (15), and the losses are the gaps between the linear model and the full model.

The apparent power (magnitude) profile of three line segments is shown in Fig. 7. It can be seen that the apparent power is below or equal to the line capacity limit. Due to the heavy EV charging loads, the capacity limit of L2-1 is reached at hour 23 and hour 24. The apparent power of L3-1 is below the capacity limit at hour 24 because the voltage limit is hit first as shown in Fig. 8.

A second case study was carried out, where the EVs were allowed to produce reactive power up to 10% of its active power, to test the reactive power support function of the EVs and the validity of C1. With the feed-in of reactive power, the voltage profile is improved (see Fig. 8), and the total cost is reduced a bit because the EVs are able to consume more energy at the lowest price up to the limit of the line capacity as shown in Fig. 7.

At last, the efficiency of the SOCP-EV model was tested on distribution networks of different complexity scales. The results were tabularized in Table IV. The first simulation is the case described above. The simulation focuses on the first feeder of the network shown in Fig. 3. The 11 kV bus was deemed as the root node. Because it is a multi-period optimal planning problem, the number of scalar variables depends on the number of EVs and the number of planning periods (24 here). The optimizer only took 0.39 seconds to finish the optimization. The second simulation was done on the whole distribution network. The 33 kV bus was deemed as the root node. The optimizer took 1.56 seconds to finish the optimization.

The third simulation was done with the IEEE 123 node test feeder [24]. The number of EVs was increased to 20000 and the total optimization time is 12.09 seconds for such a complicated feeder with high penetration of EVs.
applications.

the convex relaxation to solve the AC OPF in much wider
and reactive power injections will be studied, which will allow
the sufficient condition for the applications with both active
EV planning problem with line constraints. In the future work,
exactness of the convex relaxed AC OPF and the efficacy of
active or reactive, or none. The case study demonstrates the
and proved, which allows that the reverse power flow is only
convex relaxed AC OPF in distribution networks with line
constraints as an SOCP, a weaker condition C1 is proposed
and proved, which allows that the reverse power flow is only active or reactive, or none. The case study demonstrates the
exactness of the convex relaxed AC OPF and the efficacy of
using the convex relaxed AC OPF for solving a multi-period
EV planning problem with line constraints. In the future work,
the sufficient condition for the applications with both active
and reactive power injections will be studied, which will allow
the convex relaxation to solve the AC OPF in much wider
applications.

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VIII. BIOGRAPHIES

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