Structural modelling of composite beams with application to wind turbine rotor blades

\[(EIv'')'' = q - \rho A \ddot{v}\]

Philippe Jacques Couturier
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Preface

This thesis is submitted in partial fulfilment of the Ph.D. degree from the Technical University of Denmark. The work was done as part of a collaboration between Siemens Wind Power A/S and the Department of Mechanical Engineering at the Technical University of Denmark. The work has been performed in the period of November 2012 to January 2016 under the supervision of Professor, Dr. Techn. Steen Krenk as main supervisor, Associate Professor Jan Høgsberg as co-supervisor, and Chief Engineer Jesper W. Staerdahl as advisor at Siemens. I owe my deepest gratitude to Steen Krenk, Jan Høgsberg, and Jesper W. Staerdahl for their guidance and support and for openly sharing their knowledge on mechanics of structures and wind turbines with me.

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Kgs. Lyngby, January 2016

Philippe Jacques Couturier
Abstract

The ever changing structure and growing size of wind turbine blades put focus on the accuracy and flexibility of design tools. The present thesis is organized in four parts - all concerning the development of efficient computational methods for the structural modelling of composite beams which will support future growth in the rotor size.

The first part presents a two-node beam element formulation, based on complementary elastic energy, valid for fully coupled beams with variable cross-section properties. The element stiffness matrix is derived by use of the six equilibrium states of the element corresponding to tension, torsion, bending, and shear. This approach avoids the need for explicit interpolation of kinematic variables and provides a direct locking-free formulation. The formulation includes a consistent representation of distributed loads and enables recovery of the exact internal force distributions.

In the second part a formulation developed for analysis of the stiffness properties of general cross-sections with arbitrary geometry and material distribution is presented. The full six by six cross-section stiffness matrix is obtained by imposing simple deformation modes on a single layer of 3D finite elements. The method avoids the development of any special 2D theory for the stress and strain distributions and enables a simple and direct representation of material discontinuities and general anisotropy via their well-established representation in 3D elements.

The third part presents an extension of the 3D cross-section analysis by an efficient Finite Element modelling approach for thin and thick-walled sections which substantially reduces the meshing effort. The approach is based on discretizing the walls of the section using a single layer of displacement based elements with the layers represented within the elements. A postprocessing scheme is also presented to recover interlaminar stresses via equilibrium equations of 3D elasticity derived in the laminate coordinate system.

In the final part of the thesis a flexible method for analysing two types of instabilities associated with bending of thin-walled prismatic beams is presented. First, the flattening instability from the Brazier effect is modelled by representing the cross-section by two-dimensional non-linear co-rotating beam elements with imposed in-plane loads proportional to the curvature. Second, the bifurcation instability from longitudinal stresses is modelled with a Finite Strip buckling analysis based on the deformed cross-section. The analysis is well suited for early stages of design as it only requires a simple 2D line mesh of the cross-section.
**Resumé**

En fortsat ændring af strukturen og en voksende størrelse af vindmøllevejinger sætter fokus på nøjagtighed og fleksibilitet af designværktøjer. Denne afhandling er organiseret i fire dele - alle omhandlende udvikling af effektive beregningsmetoder til strukturel modellering af kompositbjælker, hvilke vil understøtte den fremtidige vækst af rotorsterrelse.


Den tredje del præsenterer en udvidelse af 3D-tværsnitsanalysen via en effektiv fremgangsmåde baseret på elementmetoden til tyk- og tyndvæggede tværsnit, som reducerer behovet for meshing betydeligt. Fremgangsmåden er baseret på diskretisering af væggene i tværsnittet ved brug af et enkelt lag af flytningbaserede elementer, hvor materialelagene er repræsenteret inden i elementerne. Et postprocesseringssystem er præsenteret til genskabelse af interlaminare spændinger via ligevægtsligninger for 3D-elasticitet udledt i laminatets koordinatsystem.

Publications

Journal papers


Conference papers


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Chapter 1

Introduction

The wind turbine industry has seen a rapid expansion over the past twenty years. The industry now comprises many large well established global manufacturers which install turbines onshore and offshore. In order for wind turbines to remain a viable energy source in the future when government subsidies are uncertain, manufacturers need to reduce the cost of energy of their product to that of alternative power sources. A common solution used in the industry to reduce the cost of energy is to introduce larger rotors. The reason behind this continuous increase in the size of the blade is two fold: first larger rotors can capture more energy at low wind sites, and second are the economy of scale factors such as increasing the energy produced for every foundation and electrical connection and reducing the maintenance hours per megawatt produced. The offshore wind farm and service boat shown in Fig. 1.1 give an appreciation of the size of modern wind turbines which now have rotors exceeding 150 m in diameter.

Figure 1.1: Offshore wind farm. Photo: Courtesy of Siemens.
Chapter 1. Introduction

The choice of rotor diameter is limited by multiple constraints such as the fatigue life, weight, cost, installation limitations, and clearance between the blade tip and the tower during operation. Challenges associated with larger rotors are easily understood from scaling laws which indicate that the power that a turbine can extract from the wind varies with the square of the rotor diameter, while the aerodynamic and gravity based loads of structurally similar rotors scale with the cube and fourth power of the rotor diameter, respectively [17]. Higher loads on the rotor also have a compound effect whereby the connecting components, such as the bearings, have to increase in size. In order to provide maximum aerodynamic performance while limiting the blade mass increase, rotors must be designed with complex structures made up of composite fiber materials. The materials commonly used in the fabrication of rotors are fiberglass polyester and fibreglass epoxy. However, in an effort to achieve the desired stiffness without the weight penalty of using more material, some manufacturers are turning to more expensive materials such carbon fiber and glass fibers with higher tensile modulus.

The design of bigger and lighter rotors pushes the material and structure to their limits which puts focus on the accuracy of the design tools. At the same time, rapidly developing technologies and shorter time between blade generations require the blade modelling approach to easily accommodate geometry and material updates from previous designs. For most of the design process, full three-dimensional Finite Element analysis is too computationally expensive and does not lend itself well to design-space exploration and the analysis of numerous load scenarios. However, because the cross-section dimensions of rotor blades are much smaller than their overall length and that the cross-sections retain their integrity, beam models can be used to accurately and effectively predict their behaviour. Using beam elements, the complex 3D behaviour of each blade can be modelled using only $10^2 - 10^3$ degrees of freedom compared to $10^5 - 10^6$ DOF when using shell elements.

The basic process used to construct a reduced model of the 3D composite structure using beam elements is shown in Fig. 1.2. The first step to reduce the dimensionality of a blade is to calculate the mechanical properties associated with the individual beam cross-sections. The associated cross-section analysis tool must be able to analyse thin and thick-walled cross-sections with isotropic and general anisotropic materials generally found in wind turbine blades. The second step is to use the cross-section properties to obtain the beam elements stiffness matrices. The modelling approach in this step should account for geometry and material variations along the blade span. The beam analysis is also often supplemented by an analysis which accounts for large non-linear deformations such a local buckling.

The objective of the present work is to develop efficient computational methods for the structural modelling of composite beam such as wind turbine rotor blades. The developed theories should facilitate the design of wind turbine blades using anisotropic materials and complex geometry to generate desired displacement char-
acteristics to enable further growth in the rotor size. The thesis consists of an extensive summary covering the main aspects of the theories developed and associated representative numerical simulations, followed by four journal papers, denoted [P1] to [P4], which cover the presented material in greater detail and four conference papers, denoted [C1] to [C4], which document additional applications of the theories. The extended summary is organized as follows: First, a beam element formulation which is valid for fully coupled beams with variable cross-section properties presented in [P4] is summarized in Chapter 2. The formulation includes a consistent representation of distributed loads and enables recovery of the exact internal force and moment distributions. Next, a formulation developed for analysis of the properties of general cross-sections with arbitrary geometry and material distribution presented in [P1] is summarized in Chapter 3. The analysis is based on imposing simple deformation modes on a single layer of 3D finite elements. The cross-section analysis is extended in Chapter 4 by an efficient Finite Element modelling approach presented in [P2] for thin and thick-walled sections which substantially reduces the meshing effort by discretizing the walls of the section using a single layer of displacement based elements with the layers represented within the elements. Finally, Chapter 5 summarizes a flexible method presented in [P3] for analysing two types of instabilities of thin-walled beams namely, flattening instability from the Brazier effect, and bifurcation instability from longitudinal stresses. The last section of the extended summary, Chapter 6, presents concluding remarks based on the main results.
Chapter 2
Equilibrium based beam element

Beam elements are used to model slender structures where one length dimension is much greater than the other two. They are widely used in many branches of structural analysis, such as in the design of bridges, helicopter rotor blades, and spacecraft parts. They have also become the workhorse in the wind turbine industry to obtain fast and accurate predictions of the natural frequencies, deflections, and the overall dynamic behaviour. For an accurate representation, the beam model must reproduce the elastic energy of the 3D structure. This can only be achieved if one accounts for the variations in elastic cross-section properties from the anisotropic composite structure of varying geometry in the spanwise direction as well as accounting for the governing kinematic behaviours, e.g. deformation mode coupling and transverse shear deformation. The lower mould used in the fabrication of a rotor blade illustrating the size and complexity in the outer geometry of a wind turbine rotor blade is shown in Fig. 2.1.

Derivation of the stiffness matrix of beam elements is often based on the kinematics of the beam represented using a number of displacement shape functions. The latter are obtained by integration of the differential equations of the beam kinematics. This introduces complications when treating non prismatic beams with varying cross-sections, deformation mode coupling, and transverse shear deformation. Errors in the shape functions also affect the accuracy of the representation of distributed loads in the form of equivalent nodal forces.

Figure 2.1: Lower mould used in the fabrication of a 75 m long wind turbine blade. Photo: Courtesy of Siemens.
Here a brief description is given of the beam element developed in [P4] which is valid for a fully coupled beam with variable cross-section properties. The method builds on the formulation first presented by Krenk [31] for plane curved and non-homogeneous beams and the complementary energy based beam element presented in Krenk [32]. The beam stiffness matrix is obtained by use of the six equilibrium states of the element corresponding to tension, torsion, bending, and shear. The equilibrium states are defined explicitly, whereby there is no need to solve differential equations, and avoids explicit interpolation of kinematic variables. This provides a direct locking-free formulation. Also presented are the specific formulas for the representation of internally distributed loads by equivalent nodal forces. The approach also enables the recovery of the exact internal force and moment distributions by use of the stationarity of the complementary energy.

2.1. Beam statics description

In the present three dimensional beam formulation the beam element of length \( l \) is located in a \([x_1, x_2, x_3]\) coordinate system with the \( x_3 \) axis along the beam, and the \( x_1 \) and \( x_2 \) axes defining a cross-section plane, as shown in Fig. 2.2(a) for the case of a prismatic beam. The beam can have varying cross-sections along \( x_3 \), however the beam element is straight in its undeformed configuration. The added complexity of including an initial curvature in the beam element is not warranted for most beam structures, including wind turbine rotor blades, as their small to moderate curvatures are adequately captured using a discretization with several straight beam elements.

The static state of a beam is defined by three forces and three moments at each cross-section plane along the longitudinal axis \( x_3 \). Theses forces are grouped together in the force vector \( \mathbf{q}(x_3) = [Q_1(x_3) \ Q_2(x_3) \ Q_3(x_3) \ M_1(x_3) \ M_2(x_3) \ M_3(x_3)]^T \). The forces are defined in terms of the in-plane stresses on the cross-section \([\sigma_{33}, \sigma_{31}, \sigma_{32}]\) whereby the axial force \( Q_3 \) and the transverse shear forces \( Q_1 \) and \( Q_2 \) are defined.

![Figure 2.2: (a) Coordinate system, (b) section forces and moments. From [P1].](image)
2.1 Beam statics description

by the area integrals

\[ Q_1 = \int_A \sigma_{31} dA, \quad Q_2 = \int_A \sigma_{32} dA, \quad Q_3 = \int_A \sigma_{33} dA, \quad (2.1) \]

and the torsional moment \( M_3 \) and the two bending moments \( M_1 \) and \( M_2 \) are defined as

\[ M_1 = \int_A \sigma_{33} x_2 dA, \quad M_2 = -\int_A \sigma_{33} x_1 dA, \quad M_3 = \int_A (\sigma_{32} x_1 - \sigma_{31} x_2) dA. \quad (2.2) \]

The internal force and moment components are illustrated in Fig. 2.2(b).

The statics of the beam can be described by six equilibrium modes, namely the homogeneous states of extension, torsion, bending, and shear. The six static states are illustrated in Fig. 2.3. From equilibrium considerations, a beam element without external loads can only support constant internal axial force \( Q_3 \), shear forces \( Q_1 \) and \( Q_2 \) and torsional moment \( M_3 \), while the bending moments \( M_1 \) and \( M_2 \) can vary linearly with the shear force as gradient. This allows the distribution of internal forces of the six equilibrium modes to be defined by the internal forces \( \mathbf{q}_0 = [Q_0^1 \ Q_0^2 \ Q_0^3 \ M_0^1 \ M_0^2 \ M_0^3]^T \) at the center of the beam as

\[ \mathbf{q}(x_3) = \mathbf{T}(x_3) \ \mathbf{q}_0, \quad (2.3) \]

where the matrix \( \mathbf{T}(x_3) \) is a 6 by 6 transformation matrix which varies linearly with the axial position in the beam \( x_3 \). In this format, the magnitude of the forces and moments are defined solely by \( \mathbf{q}_0 \), while the interpolation matrix \( \mathbf{T}(x_3) \) provides the spatial variation of the forces and moments. The six equilibrium states do not capture effects with a local character and lengthwise variation in the beam, e.g. local buckling. These non-linear effects are treated in Chapter 5.

Figure 2.3: Six equilibrium modes: (a) Tension, (d) torsion, (b,e) bending, (c,f) shear. From [P1].
2.2. Beam flexibility matrix and stiffness matrix

Methods to calculate the strain energy for linear elastic beam elements can be grouped in two categories, namely classic stiffness methods and complimentary energy based methods. In the stiffness methods, one integrates the kinematic field and the associated cross-section stiffness 6 by 6 matrices $D$ over the element’s length. The diagonal of the stiffness matrix $D$ contains the common strength of materials stiffness parameters, namely the shear stiffness about both in-plane axes, the extensional stiffness, the bending stiffness about both in-plane axes, and the torsion stiffness, while the off-diagonal entries contain the coupling terms, e.g. bend-twist coupling. This method requires an interpolated kinematic field often obtained using shape functions which are approximated by assuming a prismatic beam. In the present beam formulation, the strain energy of the beam element is calculated using a complementary energy approach based on integration of the static field description and the associated cross-section flexibility matrix $C = D^{-1}$,

$$W_e = \frac{1}{2} \int_{-l/2}^{l/2} q(x_3)^T C q(x_3) \, dx_3 = \frac{1}{2} q_0^T H q_0 . \quad (2.4)$$

The equivalent strain energy expression on the right of (2.4) is expressed in terms of the mid-section forces $q_0$ by (2.3) and the beam flexibility matrix $H$ corresponding to the six equilibrium modes defined by the integral

$$H = \int_{-l/2}^{l/2} T(x_3)^T C T(x_3) \, dx_3 . \quad (2.5)$$

Calculating the elastic energy from the static field is independent of the beam configuration, thereby allowing an exact lengthwise integration of the potential strain energy in beams with general and varying cross-section properties. An explicit and compact form for $H$ for the special case of prismatic beam elements, i.e. where the flexibility matrix $C$ is constant, follows from carrying out the integration,

$$H = l \begin{bmatrix}
C_{11} + \frac{l^2}{12} C_{55} & C_{12} - \frac{l^2}{12} C_{54} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} - \frac{l^2}{12} C_{45} & C_{22} + \frac{l^2}{12} C_{44} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix} . \quad (2.6)$$

It is seen how the beam flexibility matrix is a function of the cross-section properties as well as the length $l$ of the beam. The explicit relation in (2.6) between the beam flexibility matrix $H$ and the cross-section flexibility matrix $C$ will be used in Chapter 3 to calculate the full 6 by 6 set of stiffness coefficients of general cross-sections.
The displacements of a beam accommodating linear bending and constant extension and torsion are described in terms of six degrees of freedom at each of its two end-nodes defined as $A$ and $B$. The 12 nodal displacement components are ordered in the vector $v = [u_A^T, \varphi_A^T, u_B^T, \varphi_B^T]^T$, where the column vectors defining the rotation and translation with respect to each axis are defined as $u^T = [u_1, u_2, u_3]$ and $\varphi^T = [\varphi_1, \varphi_2, \varphi_3]$, respectively. The strain energy of the beam can be written in terms of the nodal displacements and rotations $v$ as

$$W_e = \frac{1}{2} v^T K v,$$

where the 12 by 12 element stiffness matrix $K$ is obtained from the inverse of the 6 by 6 equilibrium mode flexibility matrix $H$ and an appropriate transformation of the mid-section forces and nodal displacements via the end point values of the internal force distribution matrix $G = [-T(-l/2)^T, T(l/2)^T]^T$,

$$K = G H^{-1} G^T.$$ (2.8)

The resulting matrix is symmetric and is only a function of the section properties and the length of the beam.

### 2.3 Distributed loads and internal force recovery

An extension of the equilibrium element formulation was presented in [P4] to include distributed loads whereby equivalent nodal forces and internal force and moment distributions based on the properties of the element can be calculated. In the formulation, the exact distribution of the internal forces and moments in $q(x_3)$ are defined by a set of internal forces $\tilde{q}(x_3) = [\tilde{Q}_x(x_3) \ \tilde{Q}_y(x_3) \ \tilde{Q}_z(x_3) \ \tilde{M}_x(x_3) \ \tilde{M}_y(x_3) \ \tilde{M}_z(x_3)]^T$ that are in equilibrium with the distributed load and an additional homogeneous part expressed in terms of the mid-section forces $q_0$,

$$q(x_3) = q(x_3) + T(x_3) q_0.$$ (2.9)

Although the internal forces $\tilde{q}(x_3)$ can be chosen to represent equilibrium with the external forces associated with any combination of support conditions at the nodes, it is convenient for numerical implementation to choose these as corresponding to simple boundary conditions, e.g. representing a cantilever beam. Following [P4], the total complementary energy of the beam expressed in terms of the internal forces $q(x_3)$ can be written in a compact form as

$$W_e = -\frac{1}{2} v^T K v + v^T r + \text{const},$$ (2.10)

where $K$ is the element stiffness matrix (2.8), while the equivalent nodal forces $r$ on the element are given by the vector

$$r = G H^{-1} h - g.$$ (2.11)
where the end point values corresponding to the forces $\tilde{q}(x_3)$ are contained in the 12-component vector $g = [-\tilde{q}(-l/2)^T, \tilde{q}(l/2)^T]^T$, while the vector $h$ is defined by

$$h = \int_{-l/2}^{l/2} T(x_3)^T C(x_3) \tilde{q}(x_3) \, dx_3. \quad (2.12)$$

This procedure determines the 12 components of the equivalent nodal force vector $r$ from the external distributed loads and the distribution of stiffness properties along the beam element.

A key point in the formulation presented in [P4] is that it enables recovery of the internal force and moment distributions $q(x_3)$ in the beam element from the general Finite Element results of the nodal displacements $v$. By use of the stationarity of the complementary energy, the static variables $q_0$, defining the homogeneous part of the internal forces (2.9), are obtained from

$$q_0 = H^{-1}(G^T v - h). \quad (2.13)$$

The force and moment distributions $q(x_3)$ then follow directly upon substitution of $q_0$ into (2.9) with the previously determined internal force distribution $\tilde{q}(x_3)$. Note that the procedure does not require the use of ad-hoc assumptions about the distribution of forces. Hereby the internal load distributions are exact, apart from any approximation introduced by numerical evaluation of the integrals.

2.4. Numerical examples

The capacity of the equilibrium formulation to include anisotropy and distributed loads for non-homogeneous beam elements has been illustrated by several examples in [P4]. Here, the main results are summarized based on two selected examples. In particular, the capability of representing deformation mode coupling and the accurate representation of complex distributed loads and their associated internal force distribution for real industrial type structures are considered.

**Composite box beam**

This first example is a benchmark problem investigated experimentally by Chandra et al. [6] associated with the static deflection of a composite cantilever box beam that exhibits bend-twist coupling via the placement of the unidirectional graphite fibers in the lamina not parallel to the beam axis. The cantilever is described in terms of its length $l$, width $w$, height $h$, the wall thickness $t$, and the fiber orientation $\alpha$, as shown in Fig. 2.4. Applications of structural coupling via the use of off-axis fibers include load alleviation in wind turbine blades through bend-twist coupling and the increase in the wing divergence speed in the X-29 Forward-Swept Wing Flight Demonstrator by Grumman [34,36]. Results for the twist and the bending slope at
2.4 Numerical examples

Figure 2.4: Composite thin-walled box beam. From [P4].

the middle of the beam under a tip torque for three different fiber orientations using a single beam element are shown in Fig. 2.5 and Fig. 2.6, respectively. Furthermore, the results obtained experimentally by Chandra et al. [6] and by several authors who have used this example to test their beam formulation [15, 42, 43] are shown. Good agreement has been found between the results obtained by the present formulation and the results of a 3D Finite Element model. As explained in [P4], good agreement was also obtained with the experimental results. Intrinsic to this problem, is the calculation of the cross-section stiffness properties which have been obtained using the method presented in [P1] and summarized in Chapter 3.

Figure 2.5: Twist at mid span of box beam under tip torque. From [P4].

Figure 2.6: Bending slope at mid span of box beam under tip torque. From [P4].
Wind turbine blade

This example considers a static and internal force recovery analysis of a 75m long wind turbine blade manufactured by Siemens Wind Power A/S, illustrated in Fig. 2.7. This structure has large variations in the cross-section properties along its length from changes in the outside geometry as well as from changes in the structural layout. The blade is loaded with a static distributed load representing the lift during normal operation. The relative error of the in-plane tip displacement from the distributed load obtained with the different discretizations shown in Fig. 2.7 relative to a reference calculated using 75 elements are shown in Fig. 2.8(a). A relative error within 1% is achieved for both in-plane displacement components when using four or more elements. This illustrates how a few straight elements are adequate to capture the variation in material properties as well as the curvature of a wind turbine rotor blade. The distributions of the internal shear force $Q_2$ and the moment $M_1$ recovered are shown in Fig. 2.8(b). An excellent agreement has been found between both the shear and the moment distributions obtained using only two elements and the distributions obtained using a refined mesh with 75 elements. The internal force recovery method captures the force continuity between the elements and the stress free blade tip.

Figure 2.7: Rotor blade model discretization. From [P4].

Figure 2.8: Distributed lift force: (a) Static tip deflection, (b) moment and shear force. From [P4].
Irrespective of the beam theory used to model a beam structure, the cross-section stiffness properties constitute an essential part of the beam model. In the wind energy community, cross-section analysis tools must be able to cope with advanced anisotropic material distributions and complex geometries including multi-web designs and thin and thick walled sections. For example, the use of fiber reinforced materials can introduce couplings at the material constitutive level and laminate level which can lead to deformation mode coupling, e.g. bend-twist coupling. Furthermore, the use of materials with dissimilar properties in the lamina, e.g. different Poisson’s ratio, can lead to interlaminar stresses which can result in delamination of the laminates.

The classic cross-section analysis approach is based on integrating the stress distribution associated with an imposed strain field, while assuming that the section does not exhibit in-plane deformations. This assumption however breaks down for many composite beams as the in-plane and out-of-plane deformation of the cross-section, referred to as warping, affects the stress field [19, 29]. Cross-section procedures which include warping can generally be grouped in two categories based on whether a centerline approach or a Finite Element discretization is used [8, 19, 29]. The centerline based approaches require little meshing effort and computational time and provide satisfactory results for many thin-walled cross-sections. However, their underlying assumptions which simplify the analysis may limit the accuracy depending on the level of material anisotropy and on the cross-section geometry. Because of these limitations, approaches which model beams with complex geometries and general anisotropic materials rely on Finite Element discretization. Two such methodologies that have been shown to provide accurate stiffness matrices for most engineering structures are the theories developed by Giavotto et al. [14, 16], called non-homogeneous anisotropic beam section analysis (ANBA) which was recently revised by Morendini et al. [37], and that of Hodges et al. [5, 21, 47], called variational asymptotic beam sectional analysis (VABS). Both the VABS and the ANBA theories reduce the inherent three dimensional nature of the problem to a two dimensional form.

In [P1] a method for analysing the cross-section stiffness properties of elastic beams
Chapter 3. Cross-section flexibility and stiffness analysis

with arbitrary cross-section geometry and material distribution was presented. The method builds on the concept of six equilibrium states of a beam previously used for the analysis of cross-sections made of orthotropic materials [24, 28, 30]. The analysis procedure avoids the use of any special 2D theory by analysing a thin slice of the beam using a single layer of 3D finite elements with cubic lengthwise displacement interpolation. The full 6 by 6 cross-section stiffness matrix is evaluated from six independent equilibrium deformation modes corresponding to extension, torsion, homogeneous bending, and homogeneous shear, generated by imposing suitable displacement increments across the beam slice. Several examples validating and illustrating the theory have been presented in [P1], [C1], [C2], and [C3]. The main points of the theory are summarized in the following.

3.1. Energy equivalence and finite element representation

Consider the cross-section to be studied as extruded into a 3D straight prismatic beam slice of finite thickness, as shown in Fig. 3.1. This transformation of the 2D plane structure into a 3D beam problem allows the use of the theory of beams with no external loads presented in Chapter 2 to provide a link between the 3D structure and the cross-section properties of interest. With this approach, the core of the cross-section analysis is the calculation of the flexibility matrix $H$ representative of the beam. The cross-section flexibility matrix $C$ can then easily be calculated using the finite-length flexibility relation (2.6), and the corresponding cross-section stiffness matrix $D$ is obtained by inversion of the flexibility matrix, $D = C^{-1}$.

In the present methodology, the beam flexibility matrix $H$ is obtained using the Finite Element method, whereby the slice is discretized using 3D isoparametric finite elements. One main advantage of using 3D finite elements is that it enables a simple and direct representation of material discontinuities and general anisotropy. The deformation of the beam is described by interpolation between the displacement of $m$ nodes where the nodal displacements are contained in the column vector $v = [v_1, \ldots, v_m]^T$. Note that the static component vector conjugate to the displacement vector $v$ is the force vector $p$. The displacement field in vector components $u(x) = [u_1, u_2, u_3]^T$ in terms of the displacement of the nodes $v$ has the form

$$u(x) = N(x)v,$$

Figure 3.1: Illustration of beam slice approach.
where $N(x)$ are the shape functions corresponding to the nodal displacements. Suitable interpolation functions must be used to capture the deformations associated with prismatic beams. As such, the beam displacement field with respect to the axial direction $x_3$, which varies at most as a third degree polynomial, is captured using a single layer of elements in the axial direction with a lengthwise Hermitian interpolation. In-plane discretization must be chosen to adequately capture the in-plane and out-of-plane warping associated with the geometry and material distribution of the cross-section of interest. Note that the approach can accommodate displacement based elements with any shape, any interpolation function, and any anisotropic material distribution in the cross-section plane. This flexibility is used in Chapter 4 to present an efficient Finite Element modelling approach for thin and thick-walled sections.

The cross-section analysis procedure consists in solving the Finite Element problem corresponding to six independent equilibrium states. The six independent equilibrium states are chosen as the deformation modes corresponding to extension, twist, bending, and shear. Each of the states are obtained by imposing appropriate displacements on the end-sections of the beam slice. The deformation modes are illustrated in Fig. 3.2 for the case of a square orthotropic cross-section where the undeformed slice is sketched using dotted lines. The extension deformation mode illustrated in Fig. 3.2(a) is described by an elongation of the beam. The twist deformation mode illustrated in Fig. 3.2(b) is defined by a constant rate of twist about the axial coordinate. The assumption of constant rate of twist corresponds to assuming homogeneous St. Venant torsion with identical cross-section warping along the beam. The two bending deformation modes illustrated in Fig. 3.2(c) are characterized by a constant bending curvature. In the extension, twist, and bending deformation modes no resultant shear forces $Q_1$ and $Q_2$ occur, which leads to iden-

![Figure 3.2: Deformation modes: (a) Extension, (b) twist, (c) bending, (d) shear, where $\alpha = 1, 2$. From [P1].](image)
tical transverse contraction, i.e. each cross-section in the beam deforms identically. In contrast, the two shear deformation modes illustrated in Fig. 3.2(d) are characterized by a transverse displacement increment which results in the presence of internal shear forces $Q_1$ and $Q_2$. From equilibrium considerations, these two modes will also have linearly varying bending moments, which leads to in-plane contractions that vary with the axial position $x_3$ in the beam slice. A 3D illustration of the six deformation modes in connection with an example of a circular section are shown later in Fig. 3.5.

A key point in the analysis procedure is the extraction of the six by six flexibility matrix from the general 3D results. Following [P1], the nodal displacements $\mathbf{v}$ and nodal forces $\mathbf{p}$ associated with the resolved 3D Finite Element problem of the six independent equilibrium states $j = 1, \ldots, 6$ are stored in the column vectors $\tilde{\mathbf{v}}_j$ and $\tilde{\mathbf{p}}_j$. From the nodal forces, the statically equivalent internal forces at the center of the beam $\mathbf{q}_0$ can be calculated and stored in the column vectors $\tilde{\mathbf{q}}_0$. The three set of vectors are grouped in the following three matrices

$$
\mathbf{V} = [\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_6], \quad \mathbf{P} = [\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_6], \quad \mathbf{R} = [\tilde{\mathbf{q}}_{01}, \ldots, \tilde{\mathbf{q}}_{06}].
$$

(3.2)

As the six equilibrium modes are independent they can be used in a linear combination to represent a general equilibrium state. The weighting factors attributed to each equilibrium mode can be introduced in a column vector $\mathbf{s} = [s_1, \ldots, s_6]^T$, whereby the nodal displacements, nodal forces, and mid-section internal forces associated with the general equilibrium state follow from a linear combination of the form

$$
\mathbf{v} = \sum_{j=1}^{6} \tilde{\mathbf{v}}_j s_j = \mathbf{V} \mathbf{s}, \quad \mathbf{p} = \sum_{j=1}^{6} \tilde{\mathbf{p}}_j s_j = \mathbf{P} \mathbf{s}, \quad \mathbf{q}_0 = \sum_{j=1}^{6} \tilde{\mathbf{q}}_{0j} s_j = \mathbf{R} \mathbf{s}.
$$

(3.3)

The elastic energy of the beam can now be expressed alternatively in terms of the flexibility matrix $\mathbf{H}$ either by use of (2.4), or as the product of the nodal forces $\mathbf{p}$ and displacements $\mathbf{v}$,

$$
W_e = \frac{1}{2} \mathbf{q}_0^T \mathbf{H} \mathbf{q}_0 = \frac{1}{2} \mathbf{v}^T \mathbf{p}.
$$

(3.4)

When the nodal displacements $\mathbf{v}$, nodal forces $\mathbf{p}$, and section forces at the middle of the beam $\mathbf{q}_0$ are given by the representations in (3.3), the elastic energy of the beam can be written in the form

$$
W_e = \frac{1}{2} \mathbf{s}^T \mathbf{R}^T \mathbf{H} \mathbf{R} \mathbf{s} = \frac{1}{2} \mathbf{s}^T \mathbf{V}^T \mathbf{P} \mathbf{s}.
$$

(3.5)

This equation must be satisfied for any choice of weighting factors $\mathbf{s}$ and noting that the matrix $\mathbf{R}$ is nonsingular, the flexibility matrix $\mathbf{H}$ follows as

$$
\mathbf{H} = \mathbf{R}^{-T} (\mathbf{V}^T \mathbf{P}) \mathbf{R}^{-1}.
$$

(3.6)

This procedure determines the 36 elements of the flexibility matrix $\mathbf{H}$ from the six equilibrium load cases solved by the Finite Element analysis of the 3D beam slice.
Note that the procedure does not require advanced beam kinematic theories. Moreover, the recovery of the cross-section stiffness matrix $D$ from the six deformation modes does not involve any approximations. Hereby the accuracy of the solution is only limited by the discretization of the cross-section. Users therefore only require knowledge of Finite Element modelling to be able to properly use this cross-section stiffness analysis method.

### 3.2. Numerical examples

Several examples have been used to demonstrate the accuracy of the full six by six set of stiffness coefficients obtained from the presented cross-section analysis method. The applications considered covered solid and thin-walled sections as well as isotropic and general anisotropic materials. In [C1] an isotropic square cross-section was studied. In [C2] an isotropic rectangular cross-section and a wind turbine blade-like section studied by Chen et al. [8] was analysed. In [C3] a thin-walled composite ellipse representing the cross-section of a 6 m beam experimentally tested as well a box section exhibiting bend-twist coupling via the use off-axis fibers from Chen et al. [8] was studied. In [P1] an isotropic circular section, a composite solid rectangular section with off-axis fibers, and a wind turbine blade section was studied. In what follows the results of the analysis of the isotropic circular section and of the wind turbine blade section from [P1] are presented.

**Isotropic circular section**

The first example is associated with the cross-section analysis of the isotropic circular section with radius $r$ illustrated in Fig. 3.3(a). The example is particularly useful as a benchmark problem as it allows comparison with known analytical solutions, such as Renton [41]. The cross-section is discretized using $n$ layers in the radial direction and $4n$ segments in the circumferential direction for a total of $4n^2$ solid elements with quadratic interpolation in the cross-section plane. The case with two layers of elements $n = 2$ is shown in Fig. 3.3(b). The use of a lengthwise cubic

![Figure 3.3: (a) Schematic of a circular section, (b) $n = 2$ Finite Element discretization. From [P1].](image)
Hermitian interpolation results in the nodes to be concentrated on the front and back faces of the beam, as shown in Fig. 3.3(b). Note that a thickness of the beam slice comparable to the in-plane element dimensions is used to avoid ill-conditioned elements. The relative error of the stiffness coefficients with respect to the analytical solution obtained using different mesh sizes are plotted in Fig. 3.4. It is shown that all parameters have a cubic convergence towards the analytical solution. The 3D deformation of the six deformation modes are presented graphically in Fig. 3.5. It can be seen that the displacement in the axial direction is modelled with the use of a single layer of elements. Furthermore, the extension, bending and twist deformation modes have a uniform transverse deformation along the axial direction.

Figure 3.4: Relative error in stiffness coefficients with respect to the mesh refinement parameter, where $\alpha = 1, 2$. From [P1].

Figure 3.5: Deformation modes: (a) Extension, (d) twist, (b,e) bending, (c,f) shear. From [P1].
### Wind turbine blade section

This example considers the cross-section of a wind turbine blade shown in Fig. 3.6 that exhibits bend-twist coupling via the use of off-axis fibers in the spar caps. The section is constructed using a single web design with the shell and spar cap made of fiberglass-epoxy, while the sandwich core present in the skin and web are made of balsa. This example serves to illustrate the use of the presented formulation to gain insight into the potential coupling and limitations associated with using off-axis fibers in an industrial type rotor blade cross-section.

Results of the bending stiffness about the $x_1$ axis and the bend-twist coupling with respect to the theoretical maximum coupling are shown in Fig. 3.7 as function of the thickness and material orientation of the spar cap $\theta$. It can be seen in Fig. 3.7(b) that the coupling is insensitive to the spar cap thickness whereas Fig. 3.7(a) demonstrates the dependence of the bending stiffness on both the thickness and material.

---

**Figure 3.6:** Schematic of a wind turbine blade section. From [P1].

**Figure 3.7:** Effect of varying spar cap thickness and spar cap fiber orientation: (a) Bending stiffness $EI_1$, (b) bend-twist coupling. From [P1].
orientation. The results indicate the trade-off from increasing the material offset between an increase in the bend-twist coupling and a reduction in the bending stiffness. The lower sensitivity of the bending stiffness to small spar cap material angles however points towards a potential range of angles to obtain bend-twist coupling, while limiting the increase in spar cap thickness needed to maintain a constant bending stiffness. Similar results were obtained in Wetzel [46], which found an optimal angle for coupling of 7° based on a full blade analysis.
Chapter 4

Internally layered solid elements

The continual introduction of new blade designs requires the cross-section stiffness analysis to provide accurate predictions, while being able to easily accommodate geometry and material updates from previous designs, e.g. changing the number of webs and introducing carbon fibers. Two important factors associated with the flexibility of the analysis are the computational and mesh generation effort. These two factors as well as the accuracy of the full six by six set of stiffness coefficients using Finite Element based theories as the one presented in Chapter 3 largely depend on the discretization approach.

The conventional Finite Element meshing approach is to model each lamina in the blade wall using one or more elements through the thickness [8, 20, 28]. With this approach, the number of elements depends on the number of layers which for a typical blade cross-section requires significant meshing effort and limits the design flexibility. This approach however enables to recover the stresses in each layer directly from the constitutive relation. An alternative method consists of representing the thin-walled parts using a single element in the wall thickness and its material properties are taken as the thickness weighted average of the lamina properties. This approach has been used to model wind turbine cross-sections in [P1] and in Høgsberg and Krenk [24]. This discretization provides a large reduction in the number of elements needed, which in turn reduces the mesh generation effort. The averaging of the properties however prevents the calculation of interlaminar stresses and limits its use to thin walled parts.

To circumvent these limitations, internally layered solid elements previously used to model 3D composite structures [7, 26, 39, 45] were extended in [P2] to Finite Element modelling of composite cross-sections where thin to thick laminates are modelled using a single element through the wall thickness. The stiffness is obtained using Gaussian quadrature through each layer, whereby the layup sequence effects are captured. A postprocessing scheme was also developed to recover interlaminar stresses via equilibrium equations of 3D elasticity. The analysis of several composite sections highlight that this modelling approach can significantly reduce the mesh generation and computational effort, while maintaining accuracy and the stress recovery capability for thin to thick-walled sections with general anisotropic materials. The main points of the theory are summarized in the following.
Chapter 4. Internally layered solid elements

4.1. Internally layered element

The theory for cross-section analysis presented in Chapter 3 can accommodate any displacement based elements with any shape, any interpolation function, and any anisotropic material distribution in the cross-section plane. Using this flexibility of the analysis method, one can use the concept of numerical integration through the laminate used in plate and shell theories [23] to develop a meshing approach which requires very few elements.

In an internally layered element one element of thickness $t$ contains $n$ layers of different material. The element is described in terms of the intrinsic coordinates $[\xi, \eta, \zeta]$ which cover the range $-1 \leq \xi, \eta, \zeta \leq 1$. Note that in the current formulation, the coordinate $\zeta$ is defined as being collinear with the global axial coordinate $x_3$ and $\eta$ is chosen to be perpendicular to the layer surface. Each layer is described by a separate coordinate system defined by the layer intrinsic coordinates $[\xi, \eta_k, \zeta]$ where $\eta_k$ ranges from -1 to 1 in each layer. The material constitutive matrix and thickness of the $k$'th layer is defined as $E_k$ and $h_k$, respectively. The stiffness matrix of each element $K_e$ is described by the following volume integral performed over each layer

$$K_e = \sum_{k=1}^{n} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T E_k B J \frac{h_k}{t} d\xi d\eta_k d\zeta,$$  \hspace{1cm} (4.1)

where $B$ is the strain-displacement matrix, and $J$ is the determinant of the Jacobian matrix. This format enables a simple Gaussian quadrature where the same Gauss point and weight factors can be used within each layer irrespective of the layer thickness distribution. The Gauss point and weight factors for various polynomial orders can be found in standard Finite Element textbook, e.g. [1, 10]. Note that if the element contains only one material layer $n = 1$, (4.1) reduces to the standard 3D solid element stiffness volume integral.

Figure 4.1 shows an illustrative comparison between the discretization of a wall section using the present internally layered element approach (labelled as Layered Element), and two other approaches namely, the use of a single element through the wall with properties based on a thickness weighted average of the lamina properties (labelled as Average Properties), and a conventional very detailed mesh (labelled as Solid Elements). It can be seen that the very detailed mesh requires several elements through the wall thickness. The reduced number of nodes associated with using a single element over the wall thickness yields a smaller global stiffness matrix which in turn provides performance gains. The Gauss points located in each layer illustrate how the laminated element approach integrates the stiffness of the individual laminas thereby retaining the effect of the stacking sequence, e.g. stiff laminates closer to the outer surface of the blade shell should increase the bending stiffness. This information is lost when the material properties of the element are taken as the thickness weighted average of the lamina properties. The layered element has also the advantage of having its nodal position defined by the outer
4.2 Stress recovery

Cross-section stiffness analysis procedures based on discretizing the section using 3D Finite Elements allow the recovery of the strain and resultant stress field. When using conventional solid elements, no special kinematic behaviour of the laminate is assumed and the behaviour of individual layers is explicitly solved. Stresses in each layer can therefore be recovered directly from the constitutive relation. Internally layered elements are able to capture the in-plane strain distribution and consequently can also recover the in-plane stress distribution from the constitutive relations. However, internally layered elements are unable to capture the inter-laminar strain discontinuity, thus preventing calculation of the interlaminar stress distribution directly from the constitutive relations. Interlaminar stresses govern delamination of laminates and therefore can provide valuable information in the design of the cross-section.

A direct way to recover the transverse stresses was developed in [P2] by considering a differential element inside the laminate with a local laminate coordinate system \([y_1, y_2, y_3]\), as shown in Fig. 4.2. The coordinate \(y_3\) is collinear with the slice longitudinal axis \(x_3\), the coordinate \(y_2\) is in the laminate thickness direction, and the \(y_1\) axis follows the local laminate curvature defined by the radius of curvature \(R\) in the cross-section plane. The equilibrium equations of 3D elasticity of the differential element follows from adding the surface forces from the three opposing

and inner geometry of the section, thereby making the meshing independent of the material layup. This enables the same nodal positions to be used for different material layup which can reduce the pre-processing work between designs.
Chapter 4. Internally layered solid elements

sets of sides,

\[
\frac{\partial \tau_{13}}{\partial y_1} + \frac{\partial \tau_{23}}{\partial y_2} + \frac{\partial \sigma_3}{\partial y_3} - \frac{\tau_{23}}{R} = 0, \quad (4.2)
\]

\[
\frac{\partial \sigma_1}{\partial y_1} + \frac{\partial \tau_{12}}{\partial y_2} + \frac{\partial \tau_{13}}{\partial y_3} - \frac{2\tau_{12}}{R} = 0, \quad (4.3)
\]

\[
\frac{\partial \tau_{12}}{\partial y_1} + \frac{\partial \sigma_2}{\partial y_2} + \frac{\partial \tau_{23}}{\partial y_3} + \frac{\sigma_1 - \sigma_2}{R} = 0. \quad (4.4)
\]

These equations must be satisfied by any static stress field, irrespective of the material behaviour. The interlaminar stresses \(\tau_{23}, \tau_{12},\) and \(\sigma_2\) inside the laminated element can therefore be obtained by integrating Eqs. (4.2), (4.3) and (4.4) over the thickness of the element. With this approach only the in-plane stresses need to be computed from constitutive relations based on the solution from the Finite Element analysis.

The in-plane stress gradients needed for integrating Eqs. (4.2), (4.3) and (4.4) could be obtained by transforming the stress gradients from the global coordinate system to the laminate coordinate system. Alternatively, a more direct approach is achieved by working on the stresses in the laminate coordinate system \([y_1, y_2, y_3]\). The stress field in each \(y_1-y_3\) plane containing Gauss points is described by interpolation between the stresses at the \(n\) Gauss points of the plane which are contained in the column vector \(\phi = [\phi_1^T, ..., \phi_n^T]^T\). The three in-plane stresses at each Gauss point are defined as \(\phi_i = [\sigma_1, \sigma_3, \tau_{13}]^T\). The in-plane stress vector field \(\sigma(y_1, y_3) = [\sigma_1, \sigma_3, \tau_{13}]^T\) in each \(y_1-y_3\) plane in terms of the stresses at the Gauss points has the form

\[
\sigma(y_1, y_3) = N(\xi, \zeta) \phi, \quad (4.5)
\]

where \(N(\xi, \zeta)\) are the interpolation functions corresponding to the Gauss point stresses. An illustration of the stress mapping between the intrinsic coordinate system and the plane in the laminate coordinate system is shown in Fig. 4.3

![Figure 4.2: Differential element of the laminate showing the laminate coordinate system. From [P2].](image-url)
4.3 Numerical examples

The in-plane stress gradients with respect to \( y_1 \) and \( y_3 \) follow directly from differentiation of Eq. (4.5). Using a piecewise interpolation of the in-plane stress gradients in each \( k' \)th layer in the thickness direction \( y_2 \), the interlaminar stresses are obtained by integrating Eqs. (4.2), (4.3) and (4.4) by assuming traction free surface on the bottom surface \( y_2 = -t/2 \). This integration through the element thickness guaranties interlaminar stress continuity between the layers and at least one stress free surface.

Note that the accuracy of the interlaminar stress distributions recovered is governed by the accuracy of the in-plane stress distribution obtained from the constitutive relations. It was shown in Høgsberg and Krenk [24] and in [P1] that the use of an element with a cubic interpolation function in \( y_1 \) and a linear interpolation in \( y_2 \) can effectively capture the in-plane stress distributions of extensive flanges and parts of thin-walled structures. This element can accurately represent shear stress variations and curved geometries which permits a further reduction in the number of elements needed to discretize the cross-section.

4.3. Numerical examples

The effectiveness of using internally layered elements to calculate cross-section stiffness properties and stress distributions has been illustrated by several examples in [P2]. Here, the main results are summarized based on two selected examples. In particular, the capability of representing multi-layered composites with few elements and the accurate recovery of interlaminar stresses using the presented postprocessing scheme are considered.

Multi-layer composite pipe

The first example concerns the analysis of a multi-layer composite pipe that was numerically investigated by Hodges [20] and Chen et al. [8]. Figure 4.4 illustrates the two layer construction of the pipe where the layup sequence for the top and
bottom straight walls are \((0^\circ, 90^\circ)\), and \((\theta, -\theta)\) for the left and right semi-circle walls. Comparison between different discretization approaches was presented in [P2] by analysing the cross-section using four different meshes. Model I, shown in Fig. 4.5(a), represents the conventional meshing approach used in Hodges [20] and Chen et al. [8], which uses a total of 2800 solid elements with quadratic interpolation. Model II, shown in Fig. 4.5(b), uses two layers of isoparametric elements with cubic-linear interpolation for a total of 36 elements. Model III and IV, shown in Fig. 4.5(c) and Fig. 4.5(d), respectively, use one layer of internally layered solid elements with a total of 18 elements for Model III and 34 elements for the more refined mesh of Model IV.

The full six by six set of stiffness coefficients obtained using each of the four models shows very good agreement with the results obtained by Hodges [20] and Chen et al. [8]. Moreover, an error of less than 2.3% on the stiffness coefficients between the linear-cubic internally layered element models (Model III and IV) and the highly

Figure 4.4: Schematic of multi-layer composite pipe originally from Hodges [20]. Current representation from [P2].

Figure 4.5: (a) Model I: 2800 solid quadratic elements (b) Model II: 36 solid linear-cubic elements (c) Model III: 18 solid internally layered linear-cubic elements (d) Model IV: 34 solid internally layered linear-cubic elements. From [P2].
4.3 Numerical examples

Figure 4.6: Composite pipe: (a) Constitutive (C) in-plane shear stresses $\tau_{13}$, (b) constitutive (C) and equilibrium (E) transverse shear stresses $\tau_{23}$. From [P2].

discretized model (Model I) indicates that the through thickness material property variations are well captured by the layered formulation. Distributions of the in-plane stress component $\tau_{13}$ and the interlaminar stress component $\tau_{23}$ through the curved wall section under a shear force $Q_1$ obtained using Model I, III, and IV are illustrated in Fig. 4.6. Very good agreement has been found for the in-plane stress distribution between all three models. Furthermore, close agreement in the interlaminar results between Model I and Model IV indicate the effectiveness of the postprocessing scheme. The larger discrepancy in the recovered interlaminar stresses using Model III shows that the curved section of interest must be well discretized in order to capture the complex interlaminar stress distribution that arises from the sudden material property change between the straight and curved segments.

**Wind turbine blade section**

This example illustrates the application of the internally layered formulation to model a two-cell cross-section of a Siemens Wind Power A/S wind turbine blade shown in Fig. 4.7. The cross-section is modelled using a single layer of elements, whereby a total of only 45 elements are used. From the cross-section stiffness analysis the full six by six set of stiffness coefficients can be obtained as well as the location of the elastic center and the shear center, which are shown in Fig. 4.7(b) using a circle and a cross, respectively. Distributions of the axial stress $\sigma_3$ and the

Figure 4.7: (a) Schematic of rotor blade cross-section (b) discretization using 45 internally layered elements. From [P2].
transverse interlaminar stress $\sigma_2$ through the section cut indicated by the dashed line in Fig. 4.7(b) under an axial force $Q_3$ is shown in Fig. 4.8. The distribution of the axial load between the various layers through the wall thickness is shown in Fig. 4.8(a), where it can be seen that the inner layer is the most loaded. Recovered transverse stresses in Fig. 4.8(b) are present from varying in-plane contraction from dissimilar Poisson’s ratio between the curved layers. Equation (4.4) indicates that the magnitude of these transverse stresses are proportional with the wall curvature. The results show that the lower part of the wall experiences transverse compression, while most of the upper part of the wall is under transverse tension. It can also be seen that the recovery process captures interlaminar stress continuity between the layers and the stress free surfaces.
Chapter 5

Cross-section deformation and instability

The introduction of larger wind turbine rotors results in an increase of the bending and in-plane cross-section deformation flexibility in relation to the operational loads. The curvature of a blade from operational aerodynamic loads shown in Fig. 5.1 gives an appreciation of the bending flexibility of modern large rotors. The limiting capacity in bending of thin-walled structures, such as wind turbine blades, is defined by two types of large geometrically non-linear deformations. The first type is governed by a progressive homogeneous flattening deformation of the cross-sections, a behaviour commonly referred to as the Brazier effect after a paper by Brazier [3]. The flattening is a result of the transverse component of the membrane stresses which act to compress the cross section. The reduction in height of the section leads to a reduction of the beam’s moment of inertia which creates a limit-point in the moment-curvature curve. The second type of non-linear deformation is local bifurcation buckling which typically has a local character with a lengthwise variation such as the formation of a kink/wrinkle on the compressive side of the beam. Note that the cross-section deformation from the Brazier effect can influence the critical moment at which local buckling occurs.

Non-linear analysis of wind turbine blades is normally performed using 3D Finite Element models with shell elements [12,25,38]. This approach allows the modelling

Figure 5.1: Blade bending during operation. Photo: Courtesy of Siemens.
of the deformations from the Brazier effect as well as bifurcation instabilities in
the entire structure. The structural details needed and the preprocessing as well
as computational time associated with 3D shell models however prohibits its use
in early design stages. The cost of not considering buckling early in the design
however can lead to unplanned redesign and an increase of the blade mass. Several
attempts have been made to develop a satisfactory asymptotic series solution for
the Brazier effect of circular tubes which would provide fast analysis. The variety of
solutions found however demonstrates the difficulties associated with an approach
which truncates terms [4,11,13,18,40]. More recently, Houliara and Karamanos [22,
studied the Brazier and local buckling problems of circular composite shells using
special-purpose finite elements.

In [P3] an efficient two step method is presented to solve the homogeneous large-
deformation bending problem and the buckling problem for prismatic thin-walled
cross-sections of arbitrary geometry with isotropic and orthotropic materials. In
an effort to avoid the complications associated with asymptotic analysis, the ho-
mogeneous cross-section deformation from the Brazier effect is modelled using two
dimensional non-linear co-rotating beam elements, developed in Krenk [33]. The
problem corresponds to solving the non-linear deformation of a planar frame shaped
like the cross-section acted upon by forces corresponding to the transverse com-
ponents of the membrane stresses. The local buckling is modelled using the Finite
Strip Method using the formulation of Li and Schafer [35]. The buckling analy-

sis can be performed on the deformed structure obtained from the beam model,
whereby the interaction between the Brazier deformation and local buckling can be
studied. The analysis requires a simple two-dimensional line mesh which provides
substantial savings in pre-processing and computational effort, making the analysis
suitable at early stages of the design. The main points of the two step method are
summarized in the following.

5.1. Numerical modelling of the Brazier effect

Similar to the beam theory presented in Chapter 2, the thin-walled beam is located
in the \([x_1, x_2, x_3]\) coordinate system, as shown in Fig. 5.2(a). The origin of the
coordinate system for this analysis is however aligned with the principal axes of
bending to uncouple the extension and bending problems. The walls of the beam
are made of laminated fiber-reinforced composites. Figure 5.2(a) shows the \([x, y, z]\)
axis defining the local mid-plane wall coordinate system, where the \(x\) axis is in the
longitudinal direction, while the \(z\) axis defines the thickness direction.

The beam is subjected to a state of homogeneous bending about the \(x_1\) axis with
a curvature \(\kappa_1\), as illustrated in Fig. 5.2(b) for the case of a circular tube. Under a
state of homogeneous bending, each cross-section deforms identically. The in-plane
displacement field along the longitudinal \(x_3\) axis of the beam can therefore by defined
by the in-plane deformation of any cross-section. The displacement of a point in an undeformed cross-section represented by the position vector $x_0 = [x_0^1, x_0^2]^T$ due to an in-plane cross-section deformation denoted by $u = [u_1, u_2]^T$ is defined as

$$x = x_0 + u.$$  

(5.1)

The stress-strain behaviour of the wall can be described using the first-order shear deformation theory. Following this kinematic representation, the static state and the corresponding deformations of a fully coupled laminate are related by eight coupled equations [27]. In the work presented in [P3], the laminates are limited to symmetric balanced layups and the longitudinal curvature and Poisson’s ratio effects at the lamina level are neglected. This allows the simplification of the governing equations to four independent constitutive relations, defining the normal in-plane force resultants $N_x$ and $N_y$, the bending moment about the $x$ axis $M_y$, and the transverse shear force $Q_{yz}$,

$$N_x = E_x h \varepsilon_x^0, \quad N_y = E_y h \varepsilon_y^0, \quad M_y = D_{22} \kappa_y, \quad Q_{yz} = \bar{A}_{22} \gamma_{yz}^0,$$  

(5.2)

where $h$ is the wall thickness of the laminate. The deformations associated with the forces and moment are described in terms of the mid-surface strains $\varepsilon_x^0$ and $\varepsilon_y^0$, the mid-surface bending curvature $\kappa_y$, and the transverse shear strain $\gamma_{yz}^0$. The term $D_{22}$ is the bending stiffness, while $E_x$ and $E_y$ are the effective laminate in-plane moduli. The transverse shear stiffness $\bar{A}_{22}$ permits the modelling of moderately thick and sandwich type walls which can exhibit in-plane shear deformation.

Under a state of homogeneous bending the part of the beam above the bending axis will be in compression and the lower part in tension. Both have a component directed towards the center line which tends to flatten the section, an effect referred to as the Brazier effect. From (5.1) and (5.2), the component of the resultant pressure that acts along the $x_2$ axis can be expressed in terms of the effective...
modulus $E_x$ and the distance $h$ to the bending axis as

$$p = N_x \kappa_1 = E_x h \left( x_o^2 + u_2 \right) \kappa_1^2 .$$  

(5.3)

It is seen that the resultant pressure $p$ increases with the square of the curvature and linearly with the distance $h$ from the neutral axis. Note that the latter is defined in terms of the deformed distance from the bending axis.

Following [P3], an efficient numerical solution method to calculate the large in-plane deformations $u_2$ associated with the Brazier effect can be obtained by discretizing the cross-section using two-dimensional non-linear co-rotating beam elements developed in Krenk [33]. The beam elements include shear deformability and geometrical stiffness expressed in terms of the normal force. The analysis procedure first consists of creating a planar frame by positioning beam elements on the centerline of the walls of the cross-section of the beam. The distributed load $p$ from the Brazier effect in (5.3) associated with a given curvature $\kappa_1$ is then converted into an equivalent reduced nodal load vector. Finally, a Newton-Raphson method using a total Lagrangian formulation is used to solve the non-linear deformation of the cross-section subject to the flattening pressure. No a priori knowledge of the deformation shape is needed with this approach. This enables the analysis of general thin-walled structures with isotropic and composite wall materials.

With the displacement $u_2$ obtained from the resolved 2D Finite Element beam problem and the constitutive relation (5.2) for $N_x$, the non-linear relation between the curvature of the beam $\kappa_1$ and the applied bending moment over the cross-section of the structure $M_1$ can be found from the contour integral

$$M_1 = \oint_{\Gamma_0} N_x x_2 \, dy = \kappa_1 \oint_{\Gamma_0} E_x h \left( x_o^2 + u_2 \right)^2 \, dy .$$  

(5.4)

Note that the resultant force $N_x$ is associated with the initial cross-section area and as such the integration is carried out over the shape of the initial undeformed cross-section $\Gamma_0$. The non-linear moment-curvature curve is obtained by repeating the analysis using different curvatures $\kappa_1$. Eventually, the moment will reach a limit point $dM_1/d\kappa_1 = 0$ indicating that the applied moment exceeds the load-carrying capability of the cross-section.

### 5.2. Local buckling

Many structures will collapse from local bifurcation buckling prior to reaching the limit point from ovalization of the cross-section. However, it has been shown for circular tubes made of isotropic and orthotropic materials that the cross-section deformation from the Brazier effect influences the critical moment at which local buckling occurs [4,11,22,44]. In [P3], a linear eigenvalue analysis is performed with
the cross-section shape and axial pre-stress determined by the non-linear homogeneous finite bending solution. The critical curvature for local buckling is found when the bending moment $M_1$ calculated from (5.4) becomes larger than the critical moment. The intersection of the moment-curvature and critical moment-curvature curves in connection with an example of a composite circular section is shown in Fig. 5.4(a).

The buckling analysis is based on the Finite Strip theory [9], using the implementation of Li and Schafer in the tool CUFSM 4.05 [35]. In the Finite Strip theory, the in-plane displacements of the cross-section are defined by polynomial functions, whereas the displacements in the longitudinal direction are defined by trigonometric functions corresponding to pre-set boundary conditions. Contrary to the Finite Element method, all the degrees of freedom when using the Finite Strip theory are located on a single cross-section plane. This is illustrated in Fig. 5.3 for the case of an I beam. The planar mesh can be defined directly by the nodal positions of the non-linear beam model. The reduced number of nodes and simpler mesh needed for the Finite Strip method provides substantial savings in pre-processing and computational effort compared with the Finite Element method. Note that the constitutive relation of CUFSM was modified to enable modelling of thin-walled structures made of composite materials. The kinematic description of CUFSM however remains based on Kirchhoff plate theory, which limits the analysis to thin-walled structures where shear deformation is not important.

5.3. Numerical examples

The capacity of the proposed method to analyse the homogeneous large deformation bending problem and associated flattening instability from the Brazier effect as well as the bifurcation instability from longitudinal stresses has been illustrated by several examples in [P3]. Here, the main results of a composite circular tube example and of a comparison between a Box, a C, and an I profile are used to demonstrate the
capability of the method to analyse thin-walled beams with arbitrary cross-section geometry and isotropic and orthotropic material distribution.

**Composite tube**

The first example considers the non-linear deformation associated with the bending of a circular tube made of AS3501 graphite-epoxy material with simply-supported conditions at the two opposite ends of the beam. This example has previously been studied by Corona and Rodrigues [11] and later by Houliara and Karamanos [22], using non-linear Finite Element models developed for analysing composite circular tubes. The cross-section is discretized using 100 elements placed on the centerline of the tube. The moment and critical moment for local buckling with respect to the curvature obtained using the non-linear model is shown in Fig. 5.4(a). The moment and curvature are expressed in dimensionless form as $m_o$ and $\alpha_o$, respectively, based on the normalization proposed by Houliara and Karamanos [22]. It can be seen that the curvature $\alpha_o$ at which the critical moment curve intersects the bending moment curve supersedes the limit point in the bending moment curve. This indicates that the tube will collapse from local buckling before flattening instability occurs. Although the tube will collapse from local buckling, the ovalization of the section plays a significant role in the mechanism of local buckling. This can be seen in Fig. 5.4(a) where the critical moment at local buckling is 35% smaller than the critical moment of the undeformed cross-section $\alpha_o = 0$. The reduction in critical moment is associated with change in in-plane local curvature, where a section with higher curvature will tend to be able to support higher stresses before exhibiting local buckling. It is shown in [P3] that the dimensionless curvature and moment at the onset of local buckling and flattening instability obtained with the proposed method agree very well with the results of Corona and Rodrigues [11] and Houliara and Karamanos [22].

![Figure 5.4: Composite tube: (a) Dimensionless bending moment and critical moment for local buckling with respect to curvature, (b) circumferential wall bending moment distribution. From [P3].](image-url)
as function of the circumferential coordinate $y$ starting at the $x_1$ axis is shown in Fig. 5.4(b). It can be seen that maximum moments occur at the top and bottom of the section and at the intersection with the bending axis. Furthermore, the curve associated with the larger bending curvature exhibits larger moments at the bending axis.

**Box, C, and I profiles**

This example is associated with the buckling analysis of a Box, a C, and an I profile with the same moment of inertia. The undeformed geometry of the three profiles are illustrated in Fig. 5.5 using solid lines. This example was used to study the influence of the web position on the non-linear deformation associated with bending of typical structural beam profiles. The sections are discretized using 20 elements in each vertical segment and 10 elements in each horizontal segment. The relation between the bending moment and critical moment with respect to the curvature obtained using the non-linear method and Finite Strip method, respectively, are shown in Fig. 5.6. The moment and curvature are expressed in dimensionless form as $m_b$ and $\alpha_b$, respectively, following the normalization presented in [P3]. It can be seen that the C section reaches its limiting moment first, followed by the Box section and then the I profile. The C and I profiles fail due to local buckling prior to reaching the limit point from ovalization of the cross-section. Conversely, the Box profile reaches the limit point prior to the formation of local wrinkles. Plots of the buckling mode of the three profiles shown in Fig. 5.5 provide insight into these behaviours. The deformation of the Box profile illustrated in Fig. 5.5(a) is characterized by an ovalization and flattening of the cross-section, which results in a reduction of the moment of inertia, and explains the formation of a limit point. The deformation of the C profile illustrated in Fig. 5.5(b) is associated with the formation of a kink in the top flange under compression. The flange is only supported at one end which results in a low critical stress before buckling and hence a low critical

Figure 5.5: Cross-section deformation at buckling: (a) Box profile, (b) C profile, (c) I profile. From [P3].
curvature. The deformation of the I profile illustrated in Fig. 5.5(c) is characterized by an undeformed web, which explains why the profile does not exhibit non-linear bending-curvature relation in the range shown in Fig. 5.6. In this configuration, the center position of the web results in the moment of each flange segment, which tends to bend the web to cancel.
Chapter 6

Conclusions

The main topic of the present thesis is the development of methods for structural analysis of composite beams with special emphasis on modelling wind turbine blades. A common aim in all the theoretical developments in this thesis is to obtain accurate predictions of the beam behaviour with as little computational and preprocessing effort as possible. Hereby enabling complex geometries and coupled beam properties to be considered even at an early design iteration.

It was demonstrated in [P4] that a two-node beam element with stiffness matrix and representation of distributed loads based on a complementary energy formulation can be used to effectively model complex 3D composite structures. The approach avoids the shortcomings of classic kinematics based beam theories, which require calculating shape functions corresponding to the stiffness properties of the element. Instead, the approach hinges on the distribution of internal forces and moments, which are obtained directly from statics. This enables a simple and direct calculation of the stiffness matrix accounting for variations in the stiffness properties inside the structure. It was also shown in [P4] how complex distributed loads on a beam structure can be represented in the form of equivalent nodal forces via integrals of the internal equilibrium force distributions. An important point of the method is the ability to recover the exact distribution of internal forces and moments. This removes the limitation of classical beam theories of recovering the section-forces only at nodal positions. The approach is limited to static analysis, however it could be extended to include inertial loads associated with dynamic problems.

The accuracy of a beam element is limited by the calculation of the general cross-section stiffness properties expressed in the form of a cross-section stiffness or flexibility matrix. In [P1] it was shown how the cross-section flexibility matrix can be extracted directly from an equivalent prismatic beam using standard 3D finite elements by imposing a set of six representative displacement modes. The advantage of the present 3D slice approach is that it enables a direct representation of both material discontinuities and general anisotropy via the representation in 3D elements. A single layer of high-order Hermitian interpolation is used to exactly capture the displacement field with respect to the axial direction associated with prismatic beams. It was shown that the accuracy of the solution is only limited by the discretization of the cross-section. The method builds on the concept of
complementary elastic energy presented in [P4], whereby the cross-section flexibility matrix is obtained from the elastic energy of the slice, calculated in terms of the generalized internal force components and elastic energy associated with the six displacement modes. This approach allows the calculation of the full six by six cross-section stiffness matrix of elastic beams with arbitrary cross-section geometry and material distribution.

The cross-section analysis procedure presented in [P1] can accommodate displacement based elements with any shape and any interpolation function in the cross-section plane. This flexibility was used in [P2] to present an efficient Finite Element modelling approach for thin and thick-walled cross-sections. The approach makes use of internally layered elements where a single element is used through the wall thickness, whereby the element’s stiffness is obtained using Gaussian quadrature through each layer. It was demonstrated in [P2] that this modelling approach can significantly reduce the number of elements required, thereby reducing the computational effort while maintaining accuracy up to moderately thick walls. It was also shown in [P2] how the interlaminar stresses in internally layered elements can be recovered by postprocessing the in-plane stress gradients via equilibrium equations of 3D elasticity derived in the laminate coordinate system. The ease of meshing associated with few elements and the independence of the nodal positions with the material layup would facilitate the parametrization of a full rotor blade, making this approach well suited for conducting design space exploration.

Modelling large geometrically non-linear deformations associated with bending of thin-walled beams was considered in [P3]. It was demonstrated how a procedure suitable for preliminary instability studies can be achieved by describing the deformation of the beam cross-section using a 2D centerline approach. Two types of instabilities were studied namely, flattening instability from the Brazier effect, and bifurcation instability from longitudinal stresses. Both were analysed using models which build on two well established Finite Element analysis techniques. The cross-section deformation from the Brazier effect was modelled using two dimensional non-linear co-rotating beam elements, whereas the local buckling was modelled using the Finite Strip method where the longitudinal deformation is modelled using trigonometric functions. The Finite Strip local buckling analysis could be extended to include the effect of transverse shear deformation which would allow modelling of sandwich composite type structures. Furthermore, the importance of coupling terms neglected from this 2D approach, e.g. Poisson’s ratio effects, could be investigate by formulating the analysis using non-linear 3D isoparametric elements.
References


[35] Z. Li, B.W. Schafer, Buckling analysis of cold-formed steel members with general boundary conditions using CUFSM: Conventional and Constrained Finite Strip Methods, Proceedings of the 20th Intl. Spec. Conf. on Cold-Formed Steel Structures, November, St. Louis, MO, 2010.


Beam section stiffness properties using a single layer of 3D solid elements

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*Computers and Structures,*
Beam section stiffness properties using a single layer of 3D solid elements

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Abstract

A method is presented for analysis of the properties of general cross-sections with arbitrary geometry and material distribution. The full six by six cross-section stiffness matrix is evaluated from a single element thickness slice represented by 3D solid elements with lengthwise Hermitian interpolation with six independent imposed deformation modes corresponding to extension, torsion, bending and shear. The flexibility matrix of the slice is obtained from complementary elastic energy, and the stiffness matrix is obtained by extracting and inverting the cross-section flexibility. Three examples illustrate the accuracy of the method for solid and thin-walled sections with isotropic and general anisotropic materials.

1. Introduction

Rotor blades of wind turbines have increased considerably in size in recent years making them more flexible in relation to the operational loads. To meet deflection, fatigue and performance requirements, they must be designed with complex cross-section geometries made up of composite fiber materials. For most of the design process, full three-dimensional Finite Element analysis is computationally expensive and does not lend itself well to design space exploration and the analysis of numerous load scenarios. However, because of the slender nature of rotor blades their behavior can accurately and effectively be predicted using beam models. In a beam approximation, the three-dimensional elasticity problem is reduced to a one-dimensional structural problem. Irrespective of the beam theory used to model the blade, the cross-section properties constitute an essential part of the beam model.

Rotor blade cross-sectional analysis methodology can generally be grouped in two categories based on whether a centerline approach or a Finite Element discretization is used [1–3]. Methods based on a centerline approach have been shown to be very efficient and provide satisfactory results for many thin-walled cross-sections. However, the centerline approach is sensitive to relative wall curvature, shear-flow representation at corners in the cross-section layout, and limitations regarding transverse displacement effects. Because of these limitations, approaches which model beams with complex geometries and general anisotropic materials rely on Finite Element discretization. The most widely recognized of these approaches in the field of rotor blade modeling is the Variational Asymptotic Beam Sectional Analysis (VABS) based on the theory developed by Hodges and Yu [4,5]. In this procedure the beam is considered as a three dimensional body described by a lengthwise variation of properties associated with cross-sections. It results in a split of the elasticity problem into a two-dimensional linear cross-sectional analysis and a one-dimensional beam analysis. In the cross-section problem the displacements are represented by isoparametric 2D elements.

Another finite element based approach, which has been shown to accurately extract general cross-sectional properties of many beam structures, was developed by Giavotto et al. [6,7]. The theory is based on defining two types of solutions of a virtual work formulation of which the non-decaying solution is used to recover the cross-section stiffness properties. The theory was implemented in two computer codes, namely Anisotropic Beam Analysis (ANBA) developed by the original authors and more recently in Beam Cross section Analysis Software (BECAS) developed by Blasques [8]. Both the VABS and the ANBA theories reduce the inherent three dimensional nature of the problem to a two dimensional form.

This paper builds on the concept of six equilibrium states of a beam previously used by Krenk and Jøppesten [9], Høgsberg and Krenk [10] and Jonnalagadda and Whitcomb [11] for the analysis of cross-sections made of orthotropic materials. The theories developed in these papers are restricted to orthotropic material properties and include restrictive assumptions regarding the
detailed distribution of deformations and stresses over the cross-section. In the present method, the cross-section stiffness matrix is calculated based on the analysis of a slice of the beam in the form of a single layer of elements with cubic shape-function variation in the length-wise direction and with arbitrary anisotropic properties. Six independent equilibrium deformation modes corresponding to extension, torsion, homogeneous bending and homogeneous shear are generated by imposing suitable displacement increments across the slice via Lagrange multipliers. Elastic energy equivalence of the 3D slice and the complementary elastic energy calculated in terms of the internal force/moment distribution is then used to define the full six by six flexibility matrix of the cross-section. Finally, the corresponding cross-section stiffness matrix follows by inversion of this flexibility matrix.

The advantages of the present finite-thickness slice method are that it avoids the development of any special 2D theory for the stress and strain distributions over the cross-section and enables a simple and direct representation of material discontinuities and general anisotropy via their well-established representation in 3D elements. Elements with cubic Hermite length-wise interpolation in combination with linear, quadratic and cubic in-plane interpolation have been implemented. Combination of these interpolation types enables accurate modelling of thin-walled parts, lamina built-up parts, as well as more massive parts.

2. Beam statics description

Consider a beam of length $l$ with longitudinal coordinate $x_1$ and cross-section coordinates $x_2$ and $x_3$ as shown in Fig. 1(a). The origin is located at the center cross-section plane of the beam, whereby the front and back of the beam are located at $x_1 = 1/2$ and $x_1 = -1/2$, respectively. The beam supports the equilibrium states of tension, torsion, bending, and shear.

The static states of a beam are described by three forces and three moments at each cross-section along $x_1$ which are statically equivalent to the exact distribution of the in-plane stresses. These six forces and moments are grouped together in the generalized force vector $\mathbf{q}(x_1) = [Q_1(x_1), Q_2(x_1), Q_3(x_1), M_1(x_1), M_2(x_1), M_3(x_1)]^T$. The components $Q_1(x_1)$ and $Q_2(x_1)$ are two shear forces, and $Q_3(x_1)$ is the axial force. The components $M_1(x_1)$ and $M_2(x_1)$ are bending moments, and $M_3(x_1)$ is the torsion moment component with respect to the origin of the reference coordinate system.

A compact notation is achieved when representing the two in-plane directions using Greek subscripts $\alpha, \beta = 1, 2$ which allows summation over repeated Greek subscripts. This notation includes use of the two-dimensional permutation symbol $\varepsilon_{\alpha\beta}$ which takes the following values based on the indices $\varepsilon_{12} = 1, \varepsilon_{21} = -1,$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. Using the Greek subscripts and the permutation symbol, internal forces and moments are defined in terms of stresses on a cross-section as

$$
\begin{align*}
Q_1(x_1) &= \int_A \sigma_{22} \, dA_1, \\
Q_2(x_1) &= \int_A \sigma_{22} \, dA_1, \\
Q_3(x_1) &= \int_A \sigma_3 \, dA_1, \\
M_1(x_1) &= \int_A x_2 \varepsilon_{21} \sigma_1 \, dA_1, \\
M_2(x_1) &= \int_A x_2 \varepsilon_{21} \sigma_1 \, dA_1, \\
M_3(x_1) &= \int_A x_2 \varepsilon_{21} \sigma_1 \, dA_1.
\end{align*}
$$

The internal force and moment components are illustrated in Fig. 1(b).

2.1. Equilibrium states

The statics of a beam is described by six equilibrium states, namely the homogeneous states of tension, torsion, bending, and shear. The case of homogeneous tension is illustrated in Fig. 2(a). Opposing axial forces of magnitude $Q_1^T$ are acting at the ends of the beam. Similarly the case of homogeneous torsion illustrated in Fig. 2(d) is characterized by opposing torsion moments of magnitude $M_3^T$ acting at the ends of the beam. The homogeneous bending states are illustrated in Fig. 2(b) and (e). Opposing bending moments of magnitude $M_2^T$ act at the ends of the beam. Finally, the states of homogeneous shear are illustrated in Fig. 2(c) and (f). Here, opposing shear forces of magnitude $Q_3^T$ are applied to the ends of the beam. This results in a total external moment that is counteracted by identical bending moments at the end-sections equal to

$$
M_s \left( \frac{1}{2} l^2 \right) = \frac{1}{2} k_{\text{w}} Q_3^T.
$$

From equilibrium considerations a beam without external loads will exhibit constant internal normal force, shear forces and torsion moment, while the bending moments will vary linearly with the shear force as gradient. The six equilibrium states are therefore fully defined by the internal force vector $\mathbf{q}_0 = [Q_1^T, Q_2^T, Q_3^T, M_1^T, M_2^T, M_3^T]^T$ at the center section of the beam. The equilibrium distribution of internal forces in the beam in terms of $\mathbf{q}_0$ follows as

$$
\begin{bmatrix}
Q_1(x_1) \\
Q_2(x_1) \\
Q_3(x_1) \\
M_1(x_1) \\
M_2(x_1) \\
M_3(x_1)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & Q_1^T \\
0 & 1 & 0 & 0 & 0 & Q_2^T \\
0 & 0 & 1 & 0 & 0 & Q_3^T \\
0 & x_1 & 0 & 1 & 0 & M_1^T \\
0 & 0 & 0 & 0 & 1 & M_2^T \\
0 & 0 & 0 & 0 & 0 & M_3^T
\end{bmatrix}
$$

The distribution of internal forces written in a more compact notation becomes

$$
\mathbf{q}(x_1) = \mathbf{T}(x_1) \mathbf{q}_0,
$$

where the interpolation matrix $\mathbf{T}(x_1)$ is defined in (3).

2.2. Flexibility matrix

Following the general formulation of equilibrium based beam elements in [12], the deformation associated with the internal forces and moments $\mathbf{q}(x_1)$ is described in terms of six strains defined by the strain vector $\gamma(x_1) = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \kappa_1, \kappa_2, \kappa_3]^T$. The components $\varepsilon_1$ and $\varepsilon_2$ are generalized shear strains, and $\kappa_3$ is the axial strain. Similarly $\kappa_1$ and $\kappa_2$ are the components of bending curvature, while $\kappa_3$ is the rate of twist. The generalized strain vector

![Fig. 1. (a) Coordinate system. (b) Section forces and moments.](image-url)
(\gamma(x_i)) is defined such that it is conjugate to the internal force vector \(Q(x_i)\) with respect to energy. Thus, the specific elastic energy associated with a cross-section is given as

\[ W_i(x_i) = \frac{1}{2} \gamma(x_i)^T \mathbf{C} \gamma(x_i) \]  

(5)

For linear elastic beams there is a linear relation between the internal forces and the conjugate strains. This relation can be written either in flexibility or stiffness format,

\[ \gamma(x_i) = \mathbf{C} \mathbf{q}(x_i), \quad \mathbf{q}(x_i) = \mathbf{D} \gamma(x_i), \]  

(6)

where \(\mathbf{C}\) and \(\mathbf{D} = \mathbf{C}^{-1}\) are the cross-section flexibility and stiffness matrix, respectively. Both are six by six symmetric matrices and as such can contain up to twenty-one independent entries in the case of a fully general anisotropic cross-section without geometric or material symmetries. In this case, for example, the principal axes of shear and bending may be different, and torsion may couple to extension and bending. Considering a beam with constant cross-section properties \(\mathbf{C}\) in the longitudinal direction and eliminating \(\gamma\) in (5) using (6), the following representation of the energy per unit length at \(x_i\) expressed in terms of the cross-section flexibility matrix is obtained

\[ W_i(x_i) = \frac{1}{2} \mathbf{q}(x_i)^T \mathbf{C} \mathbf{q}(x_i). \]  

(7)

The flexibility matrix of the equilibrium states of the beam follows from integration of the cross-section flexibility relation (7) over the beam length

\[ W_e = \int_{-l/2}^{l/2} W_i(x) dx = \int_{-l/2}^{l/2} \frac{1}{2} \mathbf{q}(x)^T \mathbf{C} \mathbf{q}(x) dx, \]  

(8)

The energy \(W_e\) can be represented in terms of the internal forces and moments by the center-section values \(\mathbf{q}_e\) by (4) as

\[ W_e = \frac{1}{2} \mathbf{q}_e^T \mathbf{H} \mathbf{q}_e, \]  

(9)

where the beam flexibility matrix \(\mathbf{H}\) corresponding to the six equilibrium states is defined by the integral

\[ \mathbf{H} = \int_{-l/2}^{l/2} \mathbf{T}(x_i)^T \mathbf{C} \mathbf{T}(x_i) dx_i, \]  

(10)

Carrying out the \(x_i\)-integration in explicit form with \(\mathbf{T}(x_i)\) given by the matrix in (3) provides the relation between the beam flexibility matrix \(\mathbf{H}\) and the cross-section flexibility matrix \(\mathbf{C}\).

It is seen how the bending flexibility parameters \(C_{0i}, C_{4i}, C_{6i}\), and \(C_{3i}\) of the cross-section enter the constant shear modes of the beam due to their linearly varying bending moment.

### 2.3. Energy equivalence

The beam displacement field representation with respect to \(x_i\) varies at most as a third degree polynomial. In what follows the solution to this displacement field is defined using a Hermitian representation in terms of the displacement and displacement derivative fields on the front (+) and back (−) faces of the beam as shown in Fig. 2. The implementation of the Hermitian interpolation is presented in Appendix A. The displacement vector field \(\mathbf{u}(x) = [u_1, u_3, u_5]^T\) is described in terms of the coordinates \(x = [x_1, x_2, x_3]^T\) by the representation

\[ \mathbf{u}(x) = \mathbf{N}(x) \mathbf{v}, \]  

(12)

The matrix \(\mathbf{N}(x)\) contains the shape functions corresponding to the generalized nodal displacements contained in the column vector \(\mathbf{v} = [v_1, \ldots, v_m]^T\) where \(m\) is the number of nodes on each side of the slice. The six nodal degrees of freedom are defined as \(\mathbf{v}_f = [\mathbf{v}_1, 0, \mathbf{v}_6, 0]^T\), with the nodal forces defined as \(\mathbf{f}_f = [\mathbf{f}_1, 0, \mathbf{f}_6, 0]^T\). The components \(f_j\) and \(f_j^f\) represent the three force components, while the components \(f_j^e\) and \(f_j^{f^e}\) represent three moment components conjugate to the displacement derivatives \(u_j\) and \(u_j^e\), respectively.

The procedure consists in solving the finite element problem corresponding to six independent equilibrium states, e.g. those illustrated in Fig. 2. The generalized nodal displacements \(\mathbf{v}_f\) and conjugate generalized nodal forces \(\mathbf{f}_f\), for \(j = 1, \ldots, 6\) are arranged as columns in the following two matrices

\[ \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 \mathbf{v}_3 \mathbf{v}_5 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{f}_1 \mathbf{f}_3 \mathbf{f}_5 \end{bmatrix}. \]  

(13)

The solution \(\mathbf{v}_f, \mathbf{f}_f\) for each of the states are calculated by imposing appropriate displacements on the end-sections of the beam slice, as discussed in Section 3. Once a solution has been calculated, the section-force component vector \(\mathbf{q}_e\) at the center section is evaluated. These six center section-force vectors are arranged in the matrix

\[ \mathbf{u}(x_i) - \mathbf{f}(x_i). \]  

(11)

Fig. 3. Degrees of freedom and static components at the front (+) and the back (−).
\[ \mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{21} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} \end{bmatrix}. \]  
(14)

When the six equilibrium states are linearly independent the matrix \( \mathbf{R} \) is regular and can be inverted.

A general equilibrium state can now be represented as a linear combination of the six equilibrium states introduced above. The components of this representation are denoted \( \mathbf{s} = [s_1, \ldots, s_6]^T \), whereby the nodal displacements and nodal forces take the form

\[ \mathbf{v} = \sum_{j=1}^{6} v_j \mathbf{s}_j = \mathbf{V} \mathbf{s}, \quad \mathbf{p} = \sum_{j=1}^{6} p_j \mathbf{s}_j = \mathbf{P} \mathbf{s}, \]  
(15)

The section-forces of the center section are expressed similarly as

\[ \mathbf{q}_b = \sum_{j=1}^{6} q_j \mathbf{s}_j = \mathbf{R} \mathbf{q}_b, \]  
(16)

The elastic energy of the slice can now be expressed alternatively in terms of the flexibility matrix \( \mathbf{H} \) by use of (9) or as the product of the nodal forces and displacements as given by the representations in (15),

\[ W_e = \frac{1}{2} \mathbf{s}^T \mathbf{R}^T \mathbf{H} \mathbf{R} \mathbf{s} = \frac{1}{2} \mathbf{v}^T \mathbf{V} \mathbf{H} \mathbf{V} \mathbf{p} \]  
(17)

The matrix \( \mathbf{R} \) is regular due to the assumed independence of the imposed displacement conditions, and thus the energy can be expressed in terms of the internal force components \( \mathbf{q}_b \) at the center cross-section. Elimination of the parameters \( \mathbf{s} \) by use of (16) gives

\[ W_e = -\frac{1}{2} \mathbf{q}_b^T \mathbf{H} \mathbf{q}_b = \frac{1}{2} \mathbf{q}_b^T \mathbf{R}^{-T} \mathbf{V}^T \mathbf{P} \mathbf{R}^{-1} \mathbf{q}_b \]  
(18)

It follows from this result that the columns of the matrix \( \mathbf{VR}^{-1} \) contain the nodal displacement vectors of the six equilibrium states corresponding to a unit value of the corresponding internal force \( \mathbf{q}_b \) at the center cross-section, while the nodal forces of these six equilibrium states are contained in the columns of the matrix \( \mathbf{PR}^{-1} \).

By Betti’s result of equality of the work of one equilibrium state of stresses through a different set of equilibrium displacements with the work obtained when interchanging the roles of the two states it follows that the matrix \( \mathbf{V}^T \mathbf{P} \) is symmetric, and thus the flexibility matrix \( \mathbf{H} \) is obtained in the form

\[ \mathbf{H} = \mathbf{R}^{-T} \mathbf{V}^T \mathbf{P} \mathbf{R}^{-1}. \]  
(19)

In particular, for orthotropic beams with symmetric cross-section the basic deformation cases are conveniently chosen in accordance with this symmetry, leading to a diagonal matrix \( \mathbf{R} \) corresponding to a simple normalization of the imposed deformation states.

The stiffness matrix of a beam element, formulated in terms of the displacements and rotations at the beam ends, is easily obtained explicitly in terms of the inverse \( \mathbf{H}^{-1} \) as described in [12].

### 3. Imposed deformation modes

The six independent equilibrium states are chosen as the deformation modes corresponding to extension, twist, bending, and shear. These deformation modes are illustrated in Fig. 4 for the case of a square orthotropic cross-section where the geometry before deformation is sketched using dotted lines. The equilibrium states are defined in terms of differences in forces and moments at the front and back face of the beam. It is therefore convenient to group the nodal displacements at the front (++) and back (–) faces as

\[ \mathbf{v}_k = \mathbf{v} \left( \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \end{array} \right) \]  
(20)

from which the nodal displacements can be split into an increment and a mean value across the length of the beam,

\[ \Delta \mathbf{v} = \mathbf{v}_k - \mathbf{v}, \quad 2 \mathbf{v} = \mathbf{v}_k + \mathbf{v}. \]  
(21)

The six displacement modes are generated by imposing suitable displacement increments defined in terms of \( \Delta \mathbf{v} \), while rigid body motion is constrained by using the mean displacements \( \mathbf{v} \).

It is convenient to describe the nodal displacement increment \( \Delta \mathbf{v} \) of the six deformation modes in terms of six deformation components grouped in the vector \( \mathbf{\zeta} = [\zeta_1, \ldots, \zeta_6]^T \). The components \( \zeta_1 \) and \( \zeta_2 \) are associated with the displacement increments of the two shear modes, and \( \zeta_3 \) with the displacement increments of the extension mode. Similarly \( \eta_1 \) and \( \eta_2 \) are associated with the displacement increments of the two bending modes, and \( \eta_3 \) with the displacement increments of the twist mode. Each of the deformation modes corresponds to activating one of the components of \( \mathbf{\zeta} \) while setting the remaining five to zero.

#### 3.1. Extension mode

The extension deformation mode illustrated in Fig. 4(a) is described by an elongation of the beam equal to \( \zeta_3 \). No internal generalized shear forces \( Q_0 \) occur in this mode, which yields constant forces and moments along the longitudinal axis, \( \mathbf{q}_b (x) = \mathbf{q}_b \). This leads to a uniform transverse contraction along the beam and thereby the in-plane displacement increment \( \Delta \mathbf{u}_k (x) \) for each node pair is zero. The displacement increments defining the extension deformation mode therefore follow as

\[ \Delta \mathbf{u}_k (x) = \mathbf{0}, \quad \Delta \mathbf{u}_k (x) = \zeta_3, \quad \text{Node pairs } i = 1, \ldots, m \]  
(22)

#### 3.2. Twist mode

The twist deformation mode illustrated in Fig. 4(b) is defined by a constant rate of twist \( \eta_3 \) about the \( x_1 \)-axis. The assumption of constant rate of twist corresponds to assuming homogeneous St. Venant torsion with identical cross-sectional warping along the

---

**Fig. 4. Deformation modes:** (a) Extension. (b) Twist. (c) Bending. (d) Shear.
beam. As in the extension mode, no internal generalized shear forces \( Q_s \) occur which leads to zero in-plane displacement increment \( \Delta u_s(x_i) \) for each node pair. The displacement increments defining the twist deformation mode then follow as
\[
\Delta u_s(x_i) = -e_{ij}X_i \eta_j, \quad \Delta u_t(x_i) = 0, \quad \text{Node pairs } i = 1, \ldots, m \quad (23)
\]

3.3. Bending modes

The two bending deformation modes illustrated in Fig. 4(c) are characterized by a constant bending curvature about the in-plane coordinate axes \( x_i \) equal to \( \hat{\eta}_i \). As no internal generalized shear forces \( Q_s \) occur in this mode, the in-plane displacement increment \( \Delta u_s(x_i) \) for each node pair is zero. The displacement increments defining the two bending deformation modes then follow as
\[
\Delta u_s(x_i) = 0, \quad \Delta u_t(x_i) = \hat{\eta}_i e_{ij}X_i \quad \text{Node pairs } i = 1, \ldots, m \quad (24)
\]

3.4. Shear modes

The two shear deformation modes illustrated in Fig. 4(d) are characterized by the combination of a transverse shearing displacement and an antisymmetric deformation associated with the presence of linearly varying bending moments. The latter contribution can be calculated from the nodal displacements of the extension, twist, and bending deformation modes. The nodal displacements of these four deformation modes \( \mathbf{v}_i \) where \( j = 3, 4, 5, 6 \) and their corresponding mid section internal force vectors without the null shear force components defined as
\[
q_{ij} = \begin{bmatrix} \end{bmatrix}
\]

are arranged as columns in the following two matrices
\[
\mathbf{V}_i = \begin{bmatrix} \end{bmatrix}, \quad \mathbf{R}_i = \begin{bmatrix} \end{bmatrix}. \quad (25)
\]

A matrix containing the nodal displacements corresponding to unit equilibrium loads can then be obtained by post-multiplication by the matrix \( \mathbf{R}_i^{-1} \),
\[
\mathbf{V}_i = \mathbf{V}_i \mathbf{R}_i^{-1}, \quad (26)
\]

where the column vectors of the matrix \( \mathbf{V}_i \) follow as
\[
\mathbf{v}_i = \begin{bmatrix} \end{bmatrix}. \quad (27)
\]

The column vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{v}_4 \) are the nodal displacements associated with the equilibrium states of tension, bending, and torsion for imposed unit load components.

The deformation associated with linearly varying bending moments along the beam can now be represented as a linear combination of the mid plane nodal displacements of the bending equilibrium states with unit equilibrium loads \( \mathbf{v}_j, \mathbf{v}_j \).
\[
u_t^j(x_i) = \hat{\eta}_i M_t(x_i) + \hat{\eta}_j M_b(x_i), \quad \text{Node pairs } i = 1, \ldots, m. \quad (28)
\]

Substituting the representation of the moments in terms of the shear forces according to (2) yields
\[
u_t^j(x_i) = \hat{\eta}_i x_i \begin{bmatrix} \end{bmatrix}, \quad \text{Node pairs } i = 1, \ldots, m \quad (29)
\]

Taking the difference in displacement across the beam, the nodal displacement increment from the presence of the linearly varying bending moments follows as
\[
\Delta u_t(x_i) = \begin{bmatrix} \end{bmatrix}, \quad \text{Node pairs } i = 1, \ldots, m \quad (30)
\]

The displacement increments which define the two shear deformation modes now follow as the summation of the contribution from the antisymmetric bending moment \( \Delta u_b(x_i) \) and the transverse shearing displacement defined by \( \hat{\eta}_i \),
\[
\Delta u_s(x_i) = \hat{\eta}_i + \Delta u_b(x_i), \quad \Delta u_t(x_i) = \Delta u_t(x_i), \quad \text{Node pairs } i = 1, \ldots, m. \quad (31)
\]

3.5. Rigid body motion

Rigid body displacements must be constrained to completely define the kinematics associated with the six deformation modes. Since the beam is located in a 3D space, three rigid body rotations and three rigid body translations must be constrained. To ensure no generalized shear forces are present in the extension, twist, and bending deformation modes, constraints are imposed as to allow shearing of the beam. Shear deformation is tied with transverse deformation and inclination of the end faces. Constraining the increments in transverse displacements \( \Delta u_s(x_i) = 0 \) for each node pair and not imposing constraints on the rotation about the \( x \)-axis allows the beam to shear, while preventing rigid body rotation. The remaining rigid body rotation about the \( x \)-axis and the three rigid body translations can be constrained in terms of averages of the mean displacements
\[
\sum_{i=1}^{m} u_{di} = 0, \quad \sum_{i=1}^{m} u_{ei} = 0, \quad \sum_{i=1}^{m} e_{ij} e_{ij} = 0, \quad (32)
\]

where \( x_d \) and \( x_e \) are the in-plane nodal coordinates and the three mean displacements of node pair \( i \), respectively.

For the two shear deformation modes, rigid body motion is constrained by ensuring that the deformation is orthogonal to the deformation in the other four modes. The orthogonality is expressed in terms of energy such that the displacements \( \mathbf{v}_1, \mathbf{v}_2 \) of the shear modes do not produce work with respect to the force distributions of the extension, twist and bending deformation modes,
\[
\mathbf{v}_1^T \mathbf{p}_s - \mathbf{v}_2^T \mathbf{p}_s = 0, \quad x = 1, 2; \quad j = 3, 4, 5, 6. \quad (33)
\]

The extension, twist and bending deformation modes have constant generalized internal forces, whereby the forces at the front and back face are of equal magnitude but with opposite sign. \( \mathbf{p}_s = -\mathbf{p}_s \). Using this relation the orthogonality conditions simplify to
\[
\mathbf{v}_1^T \mathbf{p}_s = 0, \quad x = 1, 2; \quad j = 3, 4, 5, 6 \quad (34)
\]

These relations clearly indicate that the orthogonality constraints act only on the mean displacements. From the nature of the deformation modes, the orthogonality conditions prevent all three rigid body rotations and rigid body translation in the \( x_i \) direction. The remaining two rigid body translations in the \( x_e \) direction can be constrained in terms of averages of the mean displacements by use of the second formula in (32),
\[
\sum_{i=1}^{m} u_{ei} = 0. \quad (35)
\]

4. Deformation modes using finite elements

The properties of the cross-section are evaluated based on the global stiffness matrix of the slice obtained using the discretization method presented in Appendix A. The stiffness equations of the slice in terms of the nodal displacement at the front (‘+’) and back (‘−’) faces \( \mathbf{v}_k \) take the following block matrix format
\[
\begin{bmatrix} \end{bmatrix} = \begin{bmatrix} \end{bmatrix}. \quad (36)
\]
The stiffness equations in terms of the increment and mean of forces and displacements across the length of the beam follow as
\[
\begin{bmatrix}
\mathbf{A}_1 & \mathbf{J}_1 \\
\vdots & \vdots \\
\mathbf{A}_n & \mathbf{J}_n
\end{bmatrix}
= \begin{bmatrix}
\mathbf{B}_1 \\
\vdots \\
\mathbf{B}_m
\end{bmatrix},
\tag{37}
\]
where the block matrices are defined as
\[
\mathbf{K}_{11} = \frac{1}{2}(\mathbf{K}_{++} - \mathbf{K}_{+-} - \mathbf{K}_{-+} + \mathbf{K}_{--}),
\]
\[
\mathbf{K}_{12} = \frac{1}{2}(\mathbf{K}_{++} - \mathbf{K}_{+-} - \mathbf{K}_{-+} + \mathbf{K}_{--}),
\]
\[
\mathbf{K}_{21} = \frac{1}{2}(\mathbf{K}_{++} - \mathbf{K}_{+-} - \mathbf{K}_{-+} + \mathbf{K}_{--}),
\]
\[
\mathbf{K}_{22} = \frac{1}{2}(\mathbf{K}_{++} + \mathbf{K}_{+-} + \mathbf{K}_{-+} + \mathbf{K}_{--}).
\tag{38}
\]

The six deformation modes are imposed by representing the translation increments \(\Delta \mathbf{w}_i\) for all node pairs in terms of the six component deformation mode vector \(\zeta\) as described in Section 3. The corresponding axial derivatives \(\Delta \mathbf{w}^t\) are kept as part of the finite element problem to be solved. This is accomplished by introducing the representation
\[
\Delta \mathbf{v} = \Delta \mathbf{w} + \Delta \mathbf{b}_h,
\tag{39}
\]
in which the six deformation components and the derivative increments are contained in the vector \(\mathbf{w} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n \end{bmatrix}^T\), while the second term adds the displacement increments associated with an antisymmetric bending moment defined in (30). Note, that the second term is proportional to the length \(l\). The transformation matrix \(\mathbf{A}\) and the antisymmetric bending moment contribution matrix \(\mathbf{B}\) are defined by the block matrix format
\[
\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{J}_1 \\
\vdots & \vdots \\
\mathbf{A}_n & \mathbf{J}_n
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\
\vdots \\
\mathbf{B}_m
\end{bmatrix},
\tag{40}
\]
where the matrix \(\mathbf{A}\), defining the displacement increments and the matrix \(\mathbf{J}\), retaining the derivative increments are given as
\[
\mathbf{A} =
\begin{bmatrix}
0 & 0 & 0 & 0 & -x_2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & x_1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \mathbf{J} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\tag{41}
\]

Based on (30) the antisymmetric moment contribution matrix \(\mathbf{B}\) is defined in terms of the beam length \(l\) and the mean nodal displacements from the two unit bending equilibrium states \(\mathbf{p}^a\) and \(\mathbf{p}^b\),
\[
\mathbf{B}_i = l
\begin{bmatrix}
-\mathbf{p}^{a}_{15} & \mathbf{p}^{a}_{14} & 0 & 0 & 0 \\
-\mathbf{p}^{a}_{25} & \mathbf{p}^{a}_{24} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\tag{42}
\]
Elimination of \(\Delta \mathbf{v}\) in (37) by (39) gives
\[
\begin{bmatrix}
\mathbf{A}_1 & \mathbf{J}_1 \\
\vdots & \vdots \\
\mathbf{A}_n & \mathbf{J}_n
\end{bmatrix}
= \begin{bmatrix}
\mathbf{B}_1 \\
\vdots \\
\mathbf{B}_m
\end{bmatrix},
\tag{43}
\]
In this form, the forces associated with each of the six deformation components \(\zeta\) contained in the vector \(\mathbf{w}\) are equal to minus twice the mid-section generalized forces \(\mathbf{q}_0\).

### 4.1. Enforcing deformation modes using constraints

Activating each of the six deformation modes is accomplished by the addition of constraints which enforce relationships among the degrees of freedom. Two types of constraints are needed, namely constraints to fix the value of the six deformation components \(\zeta\) and constraints which prevent rigid body motion of the slice. The constraints that define the value of the six deformation components \(\zeta\) provide the handle needed to set one degree of freedom to unity and all others to zero. This corresponds to activating one deformation mode, while setting the remaining five to zero. These constraints are defined as
\[
\zeta = \mathbf{g}_i,
\tag{44}
\]
where the vector \(\mathbf{g}_i\) is used to select which degree of freedom to activate while setting the remaining five to zero, where the index \(j = 1, \ldots, 6\) specifies which deformation is activated, e.g. \(\mathbf{g}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\) for the extension case.

Two different sets of constraints to prevent rigid body motion are needed in the analysis, namely one for the extension, twist, and bending deformation modes and the other one for the two shear deformation modes. These constraints act on the mean displacements and can be expressed as
\[
\mathbf{g}_i \mathbf{b}_h = \mathbf{0},
\tag{45}
\]
where \(\mathbf{G}_i\) is the rigid body constraint matrix where the index \(n\) ranging from 1 to 2 specifies which of the two sets of constraints are enforced. From (32) the constraint matrix using average displacements when solving the extension, twist, and bending deformation modes takes the form
\[
\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, m.
\tag{46}
\]
where the sub-matrices \(\mathbf{G}_{ii}\) containing the in-plane nodal coordinates for node \(i\) are defined as
\[
\mathbf{G}_{ii} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, m.
\tag{47}
\]
The constraints of the shear deformation problems make use of the combined orthogonality and mean value conditions (34) and (35). These are combined in the constraint matrix
\[
\mathbf{G}_2 = \begin{bmatrix} \end{bmatrix},
\tag{48}
\]
in which the matrix \(\mathbf{G}_2\) enforcing the orthogonality conditions is defined as
\[
\mathbf{G}_2 = \begin{bmatrix} \end{bmatrix}^T.
\tag{49}
\]
The matrix \(\mathbf{G}_2\) enforcing rigid body constraints using mean displacements is defined as
\[
\mathbf{G}_2 = \begin{bmatrix} \end{bmatrix},
\tag{50}
\]
where the sub-matrices \(\mathbf{G}_{ii}\) take the form
\[
\mathbf{G}_{ii} = \begin{bmatrix} \end{bmatrix}, \quad i = 1, \ldots, m.
\tag{51}
\]
The constraints are added to the system of linear equations using the method of Lagrange multipliers where each constraint is enforced by solving for the associated Lagrange multiplier, acting as the force needed to impose the constraint. [13,14]. When no external forces are applied to the slice in addition to constraint
forces, the Lagrange multipliers associated with the degrees of freedom of the deformation modes are the constraint forces, here in the form of minus twice the mid section generalized forces. This enables the antisymmetric moment contribution to be moved inside the stiffness matrix. Incorporating the constraints used to enforce the deformation modes, the constraints that prevent rigid body motion, and their associated Lagrange multipliers \( \lambda_i \) and \( \lambda_j \), respectively, the system of equations in (43) takes the following form:

\[
\begin{align*}
A'K_{11}A'K_{12} & = -\frac{1}{2}A'K_{11}B \\
K_{22} & = 0
\end{align*}
\]

\[
\begin{bmatrix}
L & 0 & 0 \\
0 & G_0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The matrix \( L \) used to select the deformation components \( \xi \) takes the form:

\[
L = \begin{bmatrix}
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

The form of the rigid body constraint matrix \( G_0 \) depends on the analysis step, with \( G_0 \) for the extension, torsion and bending modes and \( G_0 \) for the two shear modes.

It is convenient to present (52) in a compact notation where the displacement and force terms are grouped together

\[
U_j = \left[ \begin{array}{c}
\xi_i \\
\eta_i \\
\lambda_i
\end{array} \right], \quad F_j = \left[ \begin{array}{c}
g_1 \\
g_2 \\
0
\end{array} \right].
\]

In this notation, the system of equations takes the form

\[
K_j U_j = F_j,
\]

in which the stiffness matrix supplemented by the constraint matrices is contained in the matrix \( K_j \).  

5. Cross-section stiffness matrix analysis procedure

The cross-section stiffness analysis procedure is summarized in Table 1 in pseudo-code format. The first part of the analysis consists of assembling both the stiffness matrix of the slice in block format and the transformation matrix \( A \) based on (40).

The next step is to analyze the six deformation modes. First the extension, torsion and bending deformation modes are analyzed using the rigid body constraint matrix \( G_0 \) defined in (46). The nodal displacements \( \psi_j \) from these four deformation modes are used to populate the antisymmetric moment contribution matrix \( B \) based on (40), while the corresponding nodal force distributions \( p_i \) are used to create the constraint matrix \( G_0 \) from (48). The two shear deformation modes are then calculated.

The nodal displacements, conjugate nodal forces and mid section force vectors of the six deformation modes are then arranged as columns in the matrices \( V, P \) and \( R \), respectively. The beam flexibility matrix \( H \) can then be determined from (19). The cross-section flexibility matrix \( C \) is then calculated using (11) and finally the corresponding cross-section stiffness matrix \( D \) is obtained by inversion of this flexibility matrix.

6. Examples

This section contains an assessment of the capacity of the present method for calculating the full six by six set of stiffness coefficients of general cross-sections. Three examples are used to cover solid and thin-walled sections as well as isotropic and general anisotropic materials, namely an isotropic circular section, a rectangular section with antisymmetric composite layup, and a realistic wind turbine blade cross-section with off-axis fibers in the spar cap. It is noted, that for thin-walled, single-layer, orthotropic cross-sections the isoparametric representation used in [10] comes out as a special case of the present formulation, although set up in a different way as a two-dimensional theory for the cross-section.

6.1. Isotropic circular section

The first example concerns the circular cross-section geometry shown in Fig. 5(a), where the circle has a radius of \( r = 1 \) and is made of an isotropic material with Poisson ratio \( \nu = 0.3 \). The cross-section is discretized using \( n \) layers in the radial direction and \( 4n \) segments in the circumferential direction using elements with quadratic interpolation in the cross-section plane and Hermite cubic interpolation in the axial direction. The case \( n = 2 \) is shown in Fig. 5(b). The reference axis being at the center, only the diagonal terms in the stiffness matrix are non-zero. The diagonal terms are \( G_{x1} = G_{x2} = E_1 = E_2 \), and \( G_j \) representing the shear stiffness about both in-plane axes, the extension stiffness, the bending stiffness about both in-plane axes, and the torsion stiffness, respectively. From symmetry of the section \( G_{x1} = G_{x2} \) and \( E_1 = E_2 \). Results for isotropic cross-sections obtained using different mesh sizes and an analytical solution from Renton [15] are listed in Table 2. The relative error of the stiffness coefficients

![Fig. 5. (a) Schematic of a circular section. (b) n = 2 finite element discretization.](image-url)

<table>
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<td>(1) Initialize ( g_1, B ) and ( G_0 ).</td>
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<td>(2) Calculate matrices ( K_{11}, K_{12}, K_{11}, K_{22}, ) and ( A ).</td>
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<td>Run MODES with ( G_0 ) for extension, bending and twist modes,</td>
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<td>( j = 1, 4, 5, 6 ).</td>
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<td>Run MODES with ( G_0 ) for shear modes ( j = 1, 2 ).</td>
<td></td>
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<tr>
<td>(5) Flexibility and stiffness matrices:</td>
<td></td>
</tr>
<tr>
<td>( V = [ \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \xi_6 ] )</td>
<td></td>
</tr>
<tr>
<td>( H = \begin{bmatrix} V^T \end{bmatrix} P C^T )</td>
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<td>( C ) from (11).</td>
<td></td>
</tr>
<tr>
<td>( D = C^{-1} ).</td>
<td></td>
</tr>
</tbody>
</table>

| Table 2 | Normalized cross-section stiffness properties for an isotropic circle. |
|---|---|---|
| Mesh refinement parameter \( n \) | Analytical |
| 1 | 2 | 4 | 8 |
| \( A_{x1}/\pi r^2 \) | 0.9882 | 0.9992 | 0.9999 | 1.0000 | 1.0000 |
| \( A_{x2}/\pi r^2 \) | 0.2498 | 0.2500 | 0.2500 | 0.2500 | 0.2500 |
| \( A_{x1}/\pi r^2 \) | 0.4883 | 0.4992 | 0.5000 | 0.5000 | 0.5000 |
| \( A_{x2}/\pi r^2 \) | 0.8604 | 0.8504 | 0.8507 | 0.8507 | 0.8507 |
with respect to the analytical solution are plotted in Fig. 6, illustrating cubic convergence for all parameters towards the analytical solution. Two elements in the radial direction are required to obtain convergence of all the stiffness coefficients to within 1% relative error. This is due to the inability of four quadratic triangular elements to capture exactly the geometry of the circle as well as the occurrence of cubic terms in the cross-section coordinates in the transverse shear problem.

The 3D deformation of the six deformation modes are presented graphically in Fig. 7 using two layers of elements and a slice thickness of 2 to clearly illustrate the deformation modes. The quadratic curvature associated with the bending deformations is seen in Fig. 7(b) and (e). Furthermore in the two shear cases, the cubic displacement in the thickness direction is modelled with the use of a single element in the thickness direction via the Hermitian interpolation. It can be seen in Fig. 7(d) that the twist mode does not develop an axial warping due to the circular cross-section. The extension, bending and twist deformation modes exhibit constant in-plane contractions along the axial direction, whereas the shear modes have varying in-plane displacements because of the presence of linearly varying bending moments.

6.2. Composite rectangular section

The second example considers the solid rectangular section made of layered orthotropic material as shown in Fig. 8(a) with width of \( 2a = 0.05 \) m and height of \( 2b = 0.1 \) m. The upper and lower halves of the section are made of composite material with properties taken from [3], namely \( E_x = 141,963 \) GPa, \( E_y = E_z = 9,79056 \) GPa, \( G_{xy} = G_{xz} = G_{yz} = 59,9844 \) GPa and \( v_{xy} = v_{xz} = v_{yz} = 0.42 \), where the \( ij \) system constitutes the principal material directions in the plane of the lamina, as shown in Fig. 8(b). Moreover, the \( x_4 \)-plane is in the plane of the lamina and parallel to the \( x_4 \)-plane while the \( x_1 \) axis is along the fiber direction. The fiber orientation angle \( \theta \) is defined as the angle between the \( x_1 \) axis and the \( x_1 \) axis.

The cross-section parameters of the composite rectangular section with \( \theta = 45^\circ \) using a regular mesh of \( 10 \times 10 \) elements with quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction are presented in Table 3. The results are compared to the values obtained using the commercial code VABS using 744 quadratic elements. It is observed that for this section, results match well those calculated using VABS, with the maximum percentage difference (0.25%) occurring.

![Figure 6](image6.png) Fig. 6. Relative error in stiffness coefficients with respect to the mesh refinement parameter.

![Figure 7](image7.png) Fig. 7. Deformation modes: (a) Extension. (d) Twist. (b) and (e) Bending. (c) and (f) Shear.

![Figure 8](image8.png) Fig. 8. (a) Schematic of a composite rectangular section. (b) Definition of principal material axes.
for the shear stiffness in the \( x_1 \) direction. This configuration exhibits strong shear-bending coupling and extension–torsion coupling, and the torsional rigidity is an order of magnitude less than the coupling. A measure of the importance of the off-diagonal terms can be obtained by normalizing them with respect to their associated diagonal terms,

\[
\gamma_i = \frac{D_{ij}}{\sqrt{D_{ii}D_{jj}}}.
\]  

The non-dimensional coupling parameters \( \gamma_i \) range from \(-1 < \gamma_i < 1\) where the extreme values indicate maximum possible coupling. Fig. 9 shows the value of the coupling parameters with respect to the ply angle. It is seen that the maximum value for each coupling does not occur for the same fiber orientation angle, with the maximum for the shear-bending coupling \( \gamma_{14} \) occurring at 43° for the shear-bending coupling \( \gamma_{25} \) at 75°, and for the extension–torsion coupling \( \gamma_{36} \) at 50°.

The deformation modes are shown graphically in Fig. 10. It is seen that the extension, bending, and torsion modes have a uniform transverse deformation along the \( x_1 \) axis. Furthermore, shearing occurs in these four modes because of the presence of off-axis fibers.

6.3. Wind turbine blade section

A concept developed in the late 1980s for reducing loads seen by wind turbines is to promote blade twisting to decrease the angle of attack when subject to a wind gust with the use of biased layups in the blade spar cap [16,17]. This final example of the present paper concerns the analysis of the two-cell cross-section of a wind turbine blade shown in Fig. 11 that exhibits bend–twist coupling via the use of off-axis fibers in the spar cap. The section has a 1 m chord and the outer geometry of a S825 airfoil which is

---

**Table 3**

Cross-section stiffness properties for the composite rectangular section with \( \theta = 45\)°.

<table>
<thead>
<tr>
<th></th>
<th>Units</th>
<th>VABS</th>
<th>Mesh 10 ( \times ) 10</th>
<th>% Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{A1} )</td>
<td>[N]</td>
<td>5.395E+07</td>
<td>5.409E+07</td>
<td>0.25</td>
</tr>
<tr>
<td>( C_{A2} )</td>
<td>[N]</td>
<td>9.660E+07</td>
<td>9.659E+07</td>
<td>0.01</td>
</tr>
<tr>
<td>( E_1 )</td>
<td>[N]</td>
<td>2.955E+08</td>
<td>2.957E+08</td>
<td>0.05</td>
</tr>
<tr>
<td>( H_1 )</td>
<td>[Nm°]</td>
<td>2.434E+05</td>
<td>2.435E+05</td>
<td>0.06</td>
</tr>
<tr>
<td>( H_2 )</td>
<td>[Nm°]</td>
<td>4.945E+04</td>
<td>4.955E+04</td>
<td>0.19</td>
</tr>
<tr>
<td>( G_f )</td>
<td>[Nm°]</td>
<td>1.320E+05</td>
<td>1.320E+05</td>
<td>0.01</td>
</tr>
<tr>
<td>( D_{44} )</td>
<td>[Nm]</td>
<td>1.812E+06</td>
<td>1.816E+06</td>
<td>0.21</td>
</tr>
<tr>
<td>( D_{33} )</td>
<td>[Nm]</td>
<td>5.198E+05</td>
<td>5.197E+05</td>
<td>0.03</td>
</tr>
<tr>
<td>( D_{36} )</td>
<td>[Nm]</td>
<td>-2.960E+06</td>
<td>-2.960E+06</td>
<td>0.00</td>
</tr>
</tbody>
</table>

---

**Fig. 9.** Coupling parameters as function of the fiber orientation angle \( \theta \).

**Fig. 10.** Deformation modes of the composite rectangular section: (a) Extension. (d) Twist. (b and e) Bending. (c and f) Shear.

**Fig. 11.** Schematic of a wind turbine blade section.
normally used at 75% of the blade radius. The origin of the coordinate system is placed at the leading edge of the section and the $x_1$ axis is co-linear with the airfoil chord line. The spar cap extends from 0.13 m to 0.47 m with the shear web positioned vertically at the center of the spar cap at 0.30 m. The material properties are taken from Griffin [18]. The two different material layups used in the cross section are listed in Table 4. Note, that the thickness of the triaxial fabric has been modified for ease of modelling. Each section is made of three laminas having the triaxial fabric on the outer and inner surfaces and a balsa core for the skin and web sections and a spar cap mixture core for the spar cap sections. The spar cap mixture is made of alternate layers of triaxial fabric and uniaxial fabric. The stiffness properties for each layup are listed in Table 5. Because the shear stiffness $G_{12}$ was not provided in the reference, it was calculated assuming transversely isotropic composites.

Previous work by Hegsgaard and Krenk [10] has shown that extensive flanges and parts of thin-walled structures can be effectively modelled using elements with cubic-linear interpolations in the cross-section plane. Fig. 12 shows the mesh of the wind turbine cross-section used in the current analysis obtained using cubic-linear elements. The skin, web and spar cap are modelled using thirteen 16-node elements with cubic-linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. The trailing edge and transitions between sections with different thicknesses are modelled using eight 8-node elements with linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. The discretization of the slice contains a total of 184 nodes.

The effect of the thickness and material orientation of the spar cap on the bending stiffness about the $x_1$ axis $E_{11}$ and the bend–twist coupling parameter $\gamma_{40}$ is shown in Fig. 13. The spar cap thickness is varied from 20 mm to 35 mm and the material orientation from $0^\circ$ to $-45^\circ$ where the material orientation follows from Fig. 8(b). A maximum bend–twist coupling parameter value of $\gamma_{40} = 0.068$ is obtained with a material orientation of $\theta = -30^\circ$. It can be seen in Fig. 13(b) that the coupling parameter is rather insensitive to the spar cap thickness. Conversely, from Fig. 13(a) it is seen that both the spar cap thickness and fiber orientation have an effect on the bending stiffness, where a smaller material angle and thicker spar cap yield larger bending stiffness. The bending stiffness varies almost linearly with the thickness for the range studied. The bending stiffness is less sensitive to changes in material orientation close to $\theta = 0^\circ$. This is expected from stress and strain coordinate transformation, where the axial stiffness component varies with the cosine of the angle while the shear-extension stiffness component varies with the sine of the angle. Small material angles can therefore be used to obtain bend–twist coupling while limiting the increase in spar cap thickness needed to maintain a constant bending stiffness. For example, a $-3^\circ$ material orientation would yield a coupling parameter of $\gamma_{40} = 0.027$ and would only require a 4% increase in the spar cap thickness to maintain the same bending stiffness as a configuration with no off-axis fibers (i.e. $0^\circ$) and a 20 mm spar cap. Doubling the value of the coupling parameter would require a $-16^\circ$ material orientation and an increase in thickness of 22% to maintain the same bending stiffness.

7. Conclusions

A method for analyzing the cross-section stiffness properties of elastic beams with arbitrary cross-section geometry and material distribution has been presented. The method is based on analysis of a slice of the beam on which six independent deformation modes corresponding to extension, torsion, bending and shear are prescribed by imposing displacement increments across the slice via Lagrange multipliers. The six by six cross-section flexibility matrix is obtained using complementary elastic energy

---

**Table 4**

Cross-section layup definition.

<table>
<thead>
<tr>
<th>Section</th>
<th>Material</th>
<th>Thickness [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skin and Web</td>
<td>Triaxial fabric</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>Balsa</td>
<td>7.0</td>
</tr>
<tr>
<td>Spar cap</td>
<td>Triaxial fabric</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>Spar cap mixture</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>Triaxial fabric</td>
<td>3.5</td>
</tr>
</tbody>
</table>

**Table 5**

Material properties.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{11}$ [GPa]</th>
<th>$E_{22} = E_{13}$ [GPa]</th>
<th>$G_{12} = G_{13}$ [GPa]</th>
<th>$G_{23}$ [GPa]</th>
<th>$v_{12} = v_{13} = v_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balsa</td>
<td>2.07</td>
<td>2.07</td>
<td>0.14</td>
<td>0.863</td>
<td>0.22</td>
</tr>
<tr>
<td>Spar cap mixture</td>
<td>27.1</td>
<td>8.35</td>
<td>4.70</td>
<td>3.05</td>
<td>0.37</td>
</tr>
<tr>
<td>Triaxial fabric</td>
<td>24.2</td>
<td>8.97</td>
<td>4.97</td>
<td>3.23</td>
<td>0.39</td>
</tr>
</tbody>
</table>

**Fig. 12.** Finite element discretization of blade section.

**Fig. 13.** Effect of varying spar cap thickness and spar cap fiber orientation: (a) Bending stiffness $E_{11}$, (b) Bend–twist coupling $\gamma_{40}$. 

\[ \text{Fig. 12. Finite element discretization of blade section.} \]

\[ \text{Fig. 13. Effect of varying spar cap thickness and spar cap fiber orientation: (a) Bending stiffness} \; E_{11}, \; \text{(b) Bend–twist coupling} \; \gamma_{40}. \]
calculated in terms of the internal force/moment components and then inverted to give the cross-section stiffness matrix. The slice is modelled using a single layer of 3D finite elements with standard discretization in the cross-section plane and Hermitian cubic interpolation in the length-wise direction, whereby the degrees of freedom are concentrated on the front and back faces of the slice.

Three examples of increasing complexity have been used to demonstrate the accuracy of the procedure for a number of cases including both solid and thin-walled sections and isotropic and general anisotropic materials. Results show excellent agreement with analytical solutions and values predicted by the commercial code VABS. Finally, the method has been used to illustrate the compromise between bend–twist coupling and bending stiffness of a realistic cross-section of a wind turbine blade with off-axis fibers in its spar-cap.

Acknowledgment

This work has been supported by Siemens Wind Power A/S.

Appendix A. Finite element representation of a slice

In the present analysis the beam is discretized using three-dimensional isoparametric finite elements with a Hermitian interpolation in the longitudinal direction $x_s$. The Hermitian interpolation provides the required third degree polynomial representation of the displacement field as well as places all the nodes on the front and back face of the beam slice, simplifying the definition of the deformation modes. To avoid ill-conditioned elements, a slice of thickness comparable to the in-plane element dimensions is used.

The theory of the finite element method is well established, see e.g. [13,14], and the focus here is on the modifications in the interpolation method needed for Hermitian interpolation in the axial direction. The elements are described in terms of the intrinsic coordinates $\xi = [\xi, \eta]$ where the coordinate $\xi$ is collinear with the global axial coordinate $x_s$. For hexahedron elements, the intrinsic coordinates cover the range $-1 \leq \xi, \eta \leq 1$. An 8-node element in the intrinsic coordinate system is shown in Fig. A.1. For this element, nodes 1–8 represent the corner nodes.

The displacement vector field $\mathbf{u}(x) = [u_1, u_2, u_3]^T$ is described using the representation

$$\mathbf{u}(\mathbf{x}) = N(\xi, \eta) \mathbf{v} = \sum_{i=1}^{2n} N_i(\xi, \eta) \mathbf{v}_i,$$

where the shape functions associated with the nodal degrees of freedom $\mathbf{v}_i$ are contained in the block matrix $N_i(\xi, \eta)$.

$$N_i(\xi, \eta) = \begin{bmatrix} N_i(\xi) & 0 & 0 & N_i(\xi) \\ 0 & N_i(\xi) & 0 & 0 \\ 0 & 0 & N_i(\xi) & 0 \\ 0 & 0 & 0 & N_i(\xi) \end{bmatrix},$$

in which the shape functions associated with the vector components of the displacement field are labelled as $N$ and the shape functions associated with the displacement gradients are labelled as $N_v$.

The shape functions for an 8-node element with Hermitian interpolation are given in Table A.1. They are constructed using Lagrange product formulas. Shape functions associated with the corner node 7 are shown in Fig. A.2. The element is characterized by a linear variation in the $\xi, \eta$ plane and a cubic variation in $\zeta$. It should be noted that shape functions $N_i$ associated with the gradients of the displacement field are made proportional to the thickness of the slice, so, in order to have the gradients defined in the global coordinate system. The bending deformation of such an element is shown in Fig. A.3. The Hermitian interpolation enables capturing the quadratic displacement without intermediate nodes, and similarly for the cubic variation in the shear problem.

References


Wind turbine cross-sectional stiffness analysis using internally layered solid elements

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AIAA Journal, (accepted).
Wind Turbine Cross-Sectional Stiffness Analysis using Internally Layered Solid Elements

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An efficient finite element modelling approach is presented for analyzing the general cross-section stiffness properties and stress distribution of thin and thick-walled sections with isotropic and general anisotropic materials. The procedure is based on discretizing the walls of the section using a single layer of displacement based elements whereby the element’s stiffness is obtained using Gaussian quadrature through each layer. The interlaminar stresses are recovered at points of interest via a 3D equilibrium-based postprocessing scheme that utilises the distribution of in-plane stress gradients derived in the local laminate coordinate system. The theory is illustrated by application to three composite sections of various shapes and material layups.

I. Introduction

Rotor blade design in current multi-megawatt wind turbines must adapt to a continually changing market which is driven by the increasing use worldwide of wind energy as a way to diversify the energy supply. To meet the requirements associated with specific installation sites such as power rating and wind conditions, rotor blades are designed using complex cross-section geometries made of composite fiber materials. The modelling approach used must provide accurate predictions of the blade behaviour while being able to easily accommodate drastic geometry and material changes from previous designs.

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2 Professor, Department of Mechanical Engineering.
Given that cross-section dimensions of rotor blades are much smaller than their overall length and that the cross-sections retain their integrity, beam models can be used to precisely and effectively predict their behaviour. Using beam elements, the complex 3D behaviour of each blade can be modelled using only $10^2 - 10^3$ degrees of freedom (DOF) compared to $10^5 - 10^6$ DOF when using shell elements. Cross-section properties constitute an essential part of any beam model. As such the cross-section program should be able to accurately model thin-walled parts, lamina built-up parts, and more massive parts, while maintaining design flexibility.

Three theories which rely on Finite Element discretization of the cross-section have been shown to provide accurate generalized stiffness properties for wind turbine rotors. The first two approaches are based on advanced kinematic analysis of beams namely the theories developed by Giavotto et al. [1] and Ghirengelli and Mantegazza [2], called nonhomogeneous anisotropic beam section analysis (ANBA), and that of Hodges et al. [3–5], called variational asymptotic beam sectional analysis (VABS). More recently, Couturier et al. [6] developed a cross-sectional analysis procedure which avoids the use of any special 2D theory by analyzing a thin slice of the beam using a single layer of 3D finite elements with cubic lengthwise displacement interpolation on which six independent deformation modes corresponding to extension, torsion, homogeneous bending and homogeneous shear are prescribed by imposing suitable displacement increments across the slice. These theories are based on the full equations of elasticity theory as represented in the six equilibrium states of a prismatic beam. These modes can in principle be extended to an additional infinite set of decaying deformation modes, see e.g. [7, 8], but for cellular sections the attenuation length is small and the effect of these modes usually negligible.

The accuracy of the generalized stiffness properties as well as the flexibility of modelling using one of these Finite Element based theories largely depend on the discretization approach. The conventional finite element meshing approach is to model each layer in the blade walls using one or more elements through the thickness [4, 9, 10]. No special kinematic behaviour of the laminate is assumed and the behaviour of individual layers is explicitly taken into considerations. Stresses in each layer can therefore be recovered directly from the constitutive relation. The number of elements will depend on the number of layers which for a typical wind turbine cross-section may be
around 20, leading to around $10^3$-$10^5$ elements, see e.g. [9]. This large number of elements requires a significant meshing effort and limits the flexibility in geometry and material layup of an automatic meshing procedure.

An alternative discretization approach is to model thin-walled parts of the blades using elements with cubic-linear interpolation in the cross-section plane. Only one element is used in the wall thickness and its material properties are taken as the thickness weighted average of the lamina properties [6, 11]. This enables to model a wind turbine cross-section using less than fifty elements. The averaging of material properties does not capture the effect of the position of each lamina, thus preventing the calculation of interlaminar stresses and limits its use to thin walled parts.

A preliminary investigation of the use of partial hybrid elements to model cross-sections has been done by Márquez [12]. The method requires the development of new elements which use independent approximations for the in-plane displacements and transverse stress components.

In the present paper, internally layered solid elements previously used to model 3D composite structures [13–16] are extended to finite-element modelling of thin to thick composite cross-sections. Laminates are modelled using a single element through the wall thickness whereby the element’s stiffness is obtained from integration of the stiffness of the individual lamina across the thickness. In-plane stresses are determined directly via the constitutive relation and transverse stresses are recovered via equilibrium equations of 3D elasticity derived in the laminate coordinate system. The internally layered element formulation has been implemented in the cross-sectional analysis program from Couturier et al [6], and this program has been used to provide realistic examples of two composite tubular sections and a wind turbine blade cross-section.

The advantage of the present internally layered element modelling approach is that it provides an easy and efficient way of modelling thin and thick walled composite cross-sections using few elements which are based on well-established 3D displacement Finite Element theories. The stress recovery method allows to calculate all stress components in the wall of the structure regardless of the curvature and material layup.
II. Summary of Cross-Sectional Stiffness Calculation

This section summarises a method for analysis of the stiffness properties of general cross-sections with arbitrary geometry and material distribution that is the basis for the developments of this paper. The reader is referred to Couturier et al. [6] for an exhaustive presentation of the formulation.

The method is based on the analysis of a cross-section of a beam which is extruded into a 3D straight prismatic slice of finite thickness \( l \). The slice has a coordinate system with its origin located in the center cross-section plane and with longitudinal coordinate \( x_3 \) and cross-section coordinates \( x_1 \) and \( x_2 \), as shown in Fig. 1(a). The six generalized section forces and moments associated with each cross-section plane along \( x_3 \) are grouped together in the force vector \( \mathbf{q}(x_3) = [Q_1(x_3) \ Q_2(x_3) \ Q_3(x_3) \ M_1(x_3) \ M_2(x_3) \ M_3(x_3)]^T \). The components \( Q_1(x_3) \) and \( Q_2(x_3) \) are two shear forces, and \( Q_3(x_3) \) is the axial force. The components \( M_1(x_3) \) and \( M_2(x_3) \) are bending moments, and \( M_3(x_3) \) is the torsion moment component with respect to the origin of the reference coordinate system. The internal force and moment components are illustrated in Fig. 1(b).

In the present paper, the procedure to obtain the cross-section stiffness properties is summarized in the following two subsections. In the first, the six equilibrium states of the slice are used to obtain a relation between the cross-section stiffness matrix and the slice flexibility matrix. In the second, a displacement based Finite Element approach is used to obtain the slice flexibility matrix.

A. Cross-Section Flexibility

The statics of the slice can be described by six equilibrium modes, namely the homogeneous states of extension, torsion, bending, and shear. From equilibrium considerations the slice without external loads will exhibit constant internal normal force, shear forces and torsion moment, while the bending moments will vary linearly with the shear force as gradient. The six equilibrium modes
can therefore be defined by the internal force $\mathbf{q}_0 = [Q_1^0 \ Q_2^0 \ Q_3^0 \ M_1^0 \ M_2^0 \ M_3^0]^T$ at the center of the slice. Since the constant shear forces $Q_1^0$ and $Q_2^0$ lead to linear anti-symmetric moment variation, the distribution of internal forces in the slice in terms of their values $\mathbf{q}_0$ at the center cross-section of the slice follows as

$$
\begin{bmatrix}
Q_1(x_3) \\
Q_2(x_3) \\
Q_3(x_3) \\
M_1(x_3) \\
M_2(x_3) \\
M_3(x_3)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & x_3 & 0 & 1 & 0 & 0 \\
-x_3 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
Q_1^0 \\
Q_2^0 \\
Q_3^0 \\
M_1^0 \\
M_2^0 \\
M_3^0
\end{bmatrix},
$$

(1)

or in a more compact notation

$$
\mathbf{q}(x_3) = \mathbf{T}(x_3) \mathbf{q}_0,
$$

(2)

where the interpolation matrix $\mathbf{T}(x_3)$ is defined in Eq. (1).

Following the general formulation of equilibrium based beam elements in Krenk [17], the deformation associated with the internal forces and moments $\mathbf{q}$ is described in terms of six generalized strains, defined by the strain vector $\mathbf{\gamma} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \kappa_1 \ \kappa_2 \ \kappa_3]^T$. The components $\varepsilon_1$ and $\varepsilon_2$ are generalized shear strains, and $\varepsilon_3$ is the axial strain. Similarly $\kappa_1$ and $\kappa_2$ are the components of bending curvature, while $\kappa_3$ is the rate of twist. The generalized strain $\mathbf{\gamma}$ is defined such that it is conjugate to the internal force vector $\mathbf{q}$ with respect to energy. Thus, the specific elastic energy associated with a cross-section can be given either as a product of the generalized strain $\mathbf{\gamma}$ and internal force vector $\mathbf{q}$ or using the cross-section flexibility matrix $\mathbf{C}$ of the linear elastic slice

$$
W_s(x_3) = \frac{1}{2} \mathbf{\gamma}(x_3)^T \mathbf{q}(x_3) = \frac{1}{2} \mathbf{q}(x_3)^T \mathbf{C} \mathbf{q}(x_3),
$$

(3)

The inverse of the flexibility matrix $\mathbf{C}$ is the six by six cross-section stiffness matrix $\mathbf{D}$. Both matrices can contain up to 21 independent entries where the off-diagonal entries represent geometrical or material induced couplings. From the constant properties in the longitudinal direction $x_3$, the flexibility matrix of the slice’s equilibrium modes follows from integration of the cross-section flexibility.
relation Eq. (3) over the slice length

$$W_e = \int_{-l/2}^{l/2} W_s(x_3) \, dx_3 = \frac{i}{2} \int_{-l/2}^{l/2} q(x_3)^T \mathbf{C} q(x_3) \, dx_3.$$  

(4)

Representing the internal forces and moments via the mid-point values $q_0$ by Eq. (2) yields

$$W_e = \frac{1}{2} q_0^T \mathbf{H} q_0.$$  

(5)

where the slice flexibility matrix $\mathbf{H}$ corresponding to the six equilibrium modes is defined by the integral

$$\mathbf{H} = \int_{-l/2}^{l/2} \mathbf{T}(x_3)^T \mathbf{C} \mathbf{T}(x_3) \, dx_3.$$  

(6)

Carrying out the integration in explicit form provides the relation between the slice flexibility matrix $\mathbf{H}$ and the cross-section flexibility matrix $\mathbf{C}$

$$\mathbf{H} = l \begin{bmatrix}
C_{11} + \frac{\rho^2}{12} C_{55} & C_{12} - \frac{\rho^2}{14} C_{54} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} - \frac{\rho^2}{12} C_{45} & C_{22} + \frac{\rho^2}{14} C_{44} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}.$$  

(7)

It is seen how the slice flexibility matrix is a function of the cross-section properties as well as the thickness $l$ of the slice. It is observed that for a finite-thickness slice the bending flexibility parameters $C_{55}$, $C_{44}$, $C_{54}$, and $C_{54}$ of the cross-section contribute to the constant-shear modes of the beam due to their linearly varying bending moment. If the flexibility matrix of a slice with arbitrary length is known, the relation in Eq. (7) can be used to recover the representative cross-section stiffness properties.

**B. Finite Element Representation**

In the present methodology, the slice flexibility matrix $\mathbf{H}$ is obtained using the Finite Element Method whereby the slice is discretized using 3D isoparametric finite elements. The use of 3D finite elements enables a simple and direct representation of material discontinuities and general
anisotropy. The model needs to capture the displacement field in the axial direction $x_3$ associated with prismatic beams. This displacement field is captured using a single layer of elements in the thickness direction with lengthwise cubic Hermite interpolation. The displacement field in vector components $\mathbf{u}(x_j) = [u_j]^T$ where the $j$ index ranges from one to three are described using finite elements as

$$\mathbf{u}(x_j) = \mathbf{N}(x_j)\mathbf{v},$$

where $\mathbf{N}(x_j)$ are the shape functions corresponding to the nodal displacement which are contained in the column vector $\mathbf{v} = [v_1, \ldots, v_m]^T$ where $m$ is the total number of nodes. The nodal degrees of freedom are defined as $\mathbf{v}_i = [u_\alpha, u_3, u'_\alpha, u'_3]^T$. The components $u_\alpha$ and $u_3$ represent the in-plane and axial displacements, respectively, while the components $u'_\alpha$ and $u'_3$ represent the corresponding derivatives with respect to the axial coordinate $x_3$. The static component vector conjugate to the displacement vector $\mathbf{v}_i$ is the force vector $\mathbf{p}_i = [f_\alpha, f_3, f'_\alpha, f'_3]^T$. The components $f_\alpha$ and $f_3$ represent the three force components, while the components $f'_\alpha$ and $f'_3$ represent three moment components conjugate to the displacement derivatives $u'_\alpha$ and $u'_3$, respectively.

The procedure consists in solving the finite element problem corresponding to six independent equilibrium states. The six independent equilibrium states are chosen as the deformation modes corresponding to extension, twist, bending, and shear. Each of the states are obtained by imposing appropriate displacements on the end-sections of the beam slice. The deformation modes are illustrated in Fig. 2 for the case of a square orthotropic cross-section where the undeformed slice is sketched using dotted lines. The generalized nodal displacements, conjugate generalized nodal

![Fig. 2 Deformation modes: (a) Extension, (b) Twist, (c) Bending, (d) Shear.](image-url)
forces, and mid-section-force component vector of the six equilibrium states \( j = 1, \ldots, 6 \) are contained in the column vectors \( \tilde{v}_j, \tilde{p}_j \) and \( \tilde{q}_{0j} \). The vectors are grouped in the following three matrices

\[
V = [\tilde{v}_1, \ldots, \tilde{v}_6], \quad P = [\tilde{p}_1, \ldots, \tilde{p}_6], \quad R = [\tilde{q}_{01}, \ldots, \tilde{q}_{06}].
\]  

(9)

A general equilibrium state can now be represented as a linear combination of the six basic equilibrium states introduced above. The components of this representation are denoted \( s = [s_1, \cdots, s_6]^T \), whereby the nodal displacements, nodal forces and section-forces of the center section take the form

\[
v = \sum_{j=1}^{6} \tilde{v}_j s_j = Vs, \quad p = \sum_{j=1}^{6} \tilde{p}_j s_j = Ps, \quad q_0 = \sum_{j=1}^{6} \tilde{q}_{0j} s_j = Rs.
\]  

(10)

The elastic energy of the slice can now be expressed alternatively in terms of the flexibility matrix \( H \) either by use of Eq. (5), or as the product of the nodal forces \( P \) and displacements \( V \) as given by the representations Eq. (10),

\[
W_e = \frac{1}{2} s^T R^T H R s = \frac{1}{2} s^T V^T P s.
\]  

(11)

This is an identity valid for arbitrary component vectors \( s \) and noting that the matrix \( R \) is regular, the flexibility matrix \( H \) follows as

\[
H = R^{-T} (V^T P) R^{-1}.
\]  

(12)

This procedure determines the 36 elements of the flexibility matrix \( H \) from the six load-cases solved by the finite element analysis of the 3D slice. The cross-section flexibility matrix \( C \) can then be calculated using the finite-length flexibility relation Eq. (7), and the corresponding cross-section stiffness matrix \( D \) is obtained by inversion of the flexibility matrix, \( D = C^{-1} \). The present cross-section analysis method has been implemented and validated in a computer program called CrossFlex (Cross-section Flexibility) \([6, 18, 19]\).

### III. Internally Layered Element

The discretization of the slice needed for the finite element analysis presented in the previous section is general in the sense that it can accommodate displacement based elements with any shape, any interpolation function, and any anisotropic material distribution in the cross-section plane. Using this flexibility of the method, a meshing approach which requires very few elements
can be developed using elements which can each capture the behaviour of several different material layers.

Consider a hexahedron finite element described in terms of the intrinsic coordinates \(\xi = [\zeta, \eta, \zeta]^T\) where the coordinate \(\zeta\) is collinear with the global axial coordinate. The intrinsic coordinates cover the range \(-1 \leq \xi, \eta, \zeta \leq 1\). The element stiffness matrix is then given by \([20, 21]\)

\[
K_e = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T E B J \, d\xi d\eta d\zeta.
\]  

(13)

where \(E\) is the material constitutive matrix, \(B\) is the strain-displacement matrix, and \(J\) is the determinant of the Jacobian matrix. The equation Eq. (13) is evaluated using Gaussian quadrature.

Consider now that the element of thickness \(t\) contains \(n\) layers of different material as shown in Fig. 3. Note that in the current formulation, \(\eta\) is chosen to be perpendicular to the layer plane. Care must be taken to ensure proper orientation of the elements in the mesh. The material constitutive matrix and thickness of the \(k^{th}\) layer is defined as \(E_k\) and \(h_k\), respectively. In order to use the same Gauss point and weight factor values, the limits of integration should remain from -1 to 1. This is achieved using the following coordinate transformation \([14]\) where \(\eta\) is replaced by \(\eta_k\) which ranges from -1 to 1 in each layer

\[
\eta = -1 + \frac{1}{t} \left[-h_k(1 - \eta_k) + 2 \sum_{j=1}^{k} h_j\right].
\]  

(14)

from which it follows that

\[
d\eta = \frac{h_k}{t} d\eta_k.
\]  

(15)

Fig. 3 Internally layered element intrinsic \(\{\xi, \eta, \zeta\}\) coordinate system and layer \(\{\xi, \eta_k, \zeta\}\) coordinate system.
Carrying out the numerical integration over each layer by substitution of Eqs. (14) and (15) in Eq. (13) yields

\[ K_e = \sum_{k=1}^{n} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T E_k B J \frac{h_k}{t} d\xi d\eta d\zeta. \]  

(16)

It is clear from this expression that the number of Gauss integration points is proportional to the number of layers. Moreover, if the element contains only one material layer the expression for a 3D solid element is recovered. This numerical integration through the laminate is also used in plate and shell theories [22]. Although out of the scope of this paper, the concept of internally layered elements can also be used to determine the general cross-section mass properties. The required analysis is simpler than the one to obtain the stiffness properties as only the density of each layer needs to be integrated over the element.

### IV. Stress Recovery

Layered composites have continuous in-plane strain components across lamina interfaces and depending on the lamina material properties may have discontinuous interlaminar strain components across lamina interfaces. From the displacement continuity inside the element, internally layered elements with appropriate interpolation functions are able to capture the in-plane strain distribution and consequently the in-plane stress distribution can be obtained from the constitutive relations. Internally layered elements are however unable to capture the interlaminar strain discontinuity, thus preventing calculation of the interlaminar stress distribution directly from the constitutive relations.

The importance of the representation of the interlaminar stresses depends on the goal of the analysis. For example, the cross-sectional stiffness analysis is governed by the in-plane stress distribution where the axial stresses dominate the extension and bending stiffnesses and the in-plane shear stresses dominate the torsion and shear stiffnesses. Conversely, delamination of laminates is governed by the interlaminar stress distribution.

A direct way to obtain the transverse shear stress distribution is to consider a differential element inside the laminate with a local laminate coordinate system \( \{y_1, y_2, y_3\} \) as shown in Fig. 4 where the coordinate \( y_3 \) is collinear with the slice longitudinal axis \( x_3 \), the coordinate \( y_2 \) is in the laminate thickness direction, and the \( y_1 \) axis follows the local laminate curvature defined by the radius of
curvature $R$ in the cross-section plane. Adding the surface forces in the three coordinate directions gives the following three equilibrium equations in terms of the stress components in the laminate coordinate system

\[ \frac{\partial \tau_{13}}{\partial y_1} + \frac{\partial \tau_{23}}{\partial y_2} + \frac{\partial \sigma_3}{\partial y_3} - \frac{\tau_{23}}{R} = 0, \]  

\[ \frac{\partial \sigma_1}{\partial y_1} + \frac{\partial \tau_{12}}{\partial y_2} + \frac{\partial \tau_{13}}{\partial y_3} - \frac{2\tau_{12}}{R} = 0, \]  

\[ \frac{\partial \tau_{12}}{\partial y_1} + \frac{\partial \tau_{23}}{\partial y_2} + \frac{\sigma_1 - \sigma_2}{R} = 0. \]

Note that when the curvature of the laminate approaches zero, i.e. the radius of curvature $R$ approaches infinity, Eqs. (17)–(19) reduce to the standard equilibrium equations of 3D elasticity in Cartesian coordinate. The interlaminar stresses $\tau_{23}$, $\tau_{12}$, and $\sigma_2$ can be obtained by integrating Eqs. (17), (18) and (19) over the thickness of the element. With this approach only the in-plane stresses need to be computed from constitutive relations based on the solution from the finite element analysis.

The distribution of in-plane stress gradients in an element needed in Eqs. (17)–(19) is obtained by interpolating the in-plane stress gradients in each $y_1$-$y_3$ plane containing Gauss points by defining the in-plane stress vector field $\mathbf{\sigma}(y_1, y_3) = [\sigma_1, \sigma_3, \tau_{13}]^T$ in each of these $y_1$-$y_3$ planes using the representation

\[ \mathbf{\sigma}(y_1, y_3) = \mathbf{N}(\xi, \zeta) \phi. \]  

The matrix $\mathbf{N}(\xi, \zeta)$ contains the interpolation functions corresponding to the Gauss point stresses contained in the column vector $\phi = [\phi_1, ..., \phi_n]^T$ where $n$ is the number of Gauss points. The three

Fig. 4 Differential element of the laminate showing the laminate coordinate system.
in-plane stresses at each Gauss point are defined as $\phi_i = [\sigma_1, \sigma_3, \tau_{13}]^T$. An illustration of the stress mapping between the intrinsic coordinate system and the plane in the laminate coordinate system is shown in Fig. 5.

The in-plane stress gradients with respect to $y_1$ and $y_3$ follow directly from differentiation of Eq. (20) by ensuring that the $y_1$ and $y_3$ axis are orthogonal

$$\sigma_{y_1}(y_1, y_3) = \frac{d\xi}{dy_1} N_\xi(\xi, \zeta) \phi = \frac{2}{\Delta s_1} N_\xi(\xi, \zeta) \phi,$$

$$\sigma_{y_3}(y_1, y_3) = \frac{d\zeta}{dy_3} N_\zeta(\xi, \zeta) \phi = \frac{2}{l} N_\zeta(\xi, \zeta) \phi,$$

where a comma denotes partial differentiation. Note that the scale factors mapping $\xi$ to $y_1$ and $\zeta$ to $y_3$ have been replaced by the ratio of the plane dimensions in the $\xi - \zeta$ space being two by two to its dimensions in the $y_1 - y_3$ space being $\Delta s_1$ by $l$ as shown in Fig. 5. The arc length $\Delta s_1$ can be obtained by fitting a polynomial curve through the Gauss points. Obtaining the in-plane stress gradients in this way avoids the need to transform stress gradients from the global coordinate system to the laminate coordinate system.

Using a piecewise interpolation of the in-plane stress gradients in each $k$’th layer in the thickness direction $y_2$, the interlaminar stresses are obtained by integrating Eqs. (17)–(19) by assuming traction free surface on the bottom surface $y_2 = -t/2$. This integration through the element thickness guaranties interlaminar stress continuity between the layers and at least one stress free surface.
V. Applications

The present section contains an assessment of the effectiveness of using internally layered elements to calculate cross-section stiffness properties and stress distribution of thin and thick walled composite structures. Three applications are considered, namely a composite tube section with soft and stiff core, a multi-layer composite pipe and a realistic wind turbine cross-section.

A. Composite Circular Tube Cross-Section

This example concerns the circular section with outer radius of $R = 0.5 \text{ m}$ shown in Fig. 6. The cross-section is build-up as a symmetric sandwich structure where the core occupies 75% of the total wall thickness and the remaining 25% is divided equally between the two faces. The faces are made of fiberglass-epoxy with the main fiber direction collinear with the beam axis. Two core materials are considered, namely fiberglass-epoxy with the main fiber direction in the circumferential direction and balsa. The material properties are taken from Griffin [23] and Couturier et al. [6] and are listed for reference in Table 1.

The cross-section is analyzed using four models each with a different discretization and material representation combination. Model A, shown in Fig. 7(a), represents the conventional highly discretized meshing approach where in the thickness direction each fiberglass-epoxy faces are discretized using two elements and the core is discretized using six elements. In the circumferential direction 16 segments are used for a total of 160 solid elements and 8064 DOF. The elements have quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. Model B, shown in Fig. 7(b), uses a single internally layered element with quadratic

![Fig. 6 Schematic of circular section with $t/R = 0.3$.](image-url)
shape functions over the thickness and 16 segments in the circumferential direction for a total of 16 elements and 1152 DOF. Model C, shown in Fig. 7(c), uses internally layered elements with linear-cubic interpolation with a single element in the thickness direction and eight segments in the circumferential direction for a total of eight elements and 576 DOF. Model D also uses eight elements with linear-cubic interpolation, however it uses a thickness-weighted average constitutive relations.

The effect of the wall thickness on the cross-section stiffness coefficients calculated using the four models for the case of the fiberglass-epoxy core is shown in Fig. 8. The wall thickness was varied from $t/R = 1/10$ to $t/R = 3/5$. The reference axis being at the center, only the diagonal terms in the stiffness matrix are non-zero. The diagonal terms are $G_{A1}$, $G_{A2}$, $EA$, $EI_1$, $EI_2$, and $GJ$ representing the shear stiffness about both in-plane axes, the extension stiffness, the bending stiffness about both in-plane axes, and the torsion stiffness, respectively. From symmetry of the section $G_{A1} = G_{A2}$ and $EI_1 = EI_2$.

Stiffness properties obtained using internally layered elements with either linear-cubic (Model C) or quadratic interpolation (Model B) are in good agreement with those obtained using the detailed

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{11}$ [GPa]</th>
<th>$E_{22} = E_{33}$ [GPa]</th>
<th>$G_{12} = G_{13}$ [GPa]</th>
<th>$G_{23}$ [GPa]</th>
<th>$\nu_{12} = \nu_{13} = \nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triaxial fabric</td>
<td>24.2</td>
<td>8.97</td>
<td>4.97</td>
<td>3.23</td>
<td>0.39</td>
</tr>
<tr>
<td>Balsa</td>
<td>2.07</td>
<td>2.07</td>
<td>0.14</td>
<td>0.863</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Fig. 7 (a) Model A: 160 solid quadratic elements (b) Model B: 16 solid internally layered quadratic elements (c) Model C: 8 solid internally layered linear-cubic elements.
Fig. 8 Stiffness properties for fiberglass-epoxy core section normalized with the value obtained using 160 solid elements (Model A) at $t/R = 0.1$.

model (Model A) with a maximum error in bending and shear at $t/R = 0.6$ of 1.6% and 2.4%, respectively. Although comparable results are obtained, models using internally layered elements (Model B and C) required substantially less meshing and computational effort given they need an order of magnitude less elements and only require information about the structure’s geometry for node positioning.

It can be seen in Fig. 8 that for $t/R \leq 0.3$ results obtained using the averaged constitutive relations (Model D) agree well with those obtained with the other methods. As the thickness is further increased it underpredicts the bending and torsional stiffnesses with a maximum error in bending and torsion for the balsa core at $t/R = 0.6$ of 12% and 15%, respectively. This error stems from the nature of the averaging process where the effect of the position of the stiff faces with respect to the neutral axis and the center of twist is lost.

The effect of the wall thickness on the cross-section stiffness coefficients calculated using the four models for the case of the balsa core is shown in Fig. 9. Sections modelled using internally
layered elements with either linear-cubic (Model C) or quadratic interpolation (Model B) are in good agreement with those obtained using the detailed model (Model A) for the extension, bending and torsion stiffnesses for the range of thicknesses studied with a maximum error in bending at $t/R = 0.6$ of 1.8%. Shear stiffness obtained using internally layered elements showed a deviation larger than 5% for wall thicknesses larger than $t/R = 0.3$. The reason for this discrepancy is that as the soft balsa core becomes thicker the structure should behave more like two independent concentric tubes. The kinematics of this effect is not captured with the use of single element through the thickness whereby the structure behaves more as two rigidly connected tubes which has a larger stiffness [24]. One should therefore be aware of parts of structures that become increasingly independent not to use a model with a single internally layered element if shear stiffness is important.

The stress distribution through the wall section at 45 degrees from the $x_1$ axis under a shear load $Q_1 = -1$ is shown for the $t/R = 0.2$ tube section with the fiberglass-epoxy core and the balsa core in Fig. 10 and Fig. 11, respectively. The stresses are obtained using the highly discretized 

Fig. 9 Stiffness properties for balsa core section normalized with the value obtained using 160 solid elements (Model A) at $t/R = 0.1$. 

[Diagram showing stiffness properties for balsa core section]
model (Model A) and the linear-cubic internally layered element model (Model C). Stresses are expressed in the laminate coordinate system. Stresses obtained using the highly discretized model are calculated using the constitutive relations and are shown at the Gauss point locations. For the internally layered element model, the in-plane and interlaminar stress distributions are obtained using the constitutive relations and the equilibrium equations, respectively.

It can be seen from Fig. 10(a) and Fig. 11(a) that the in-plane shear stress distribution $\tau_{13}$ obtained using the internally layered elements (Model C) are in good agreement with the Gauss point stresses of the detailed mesh (Model A) with a discrepancy of the maximum stress for the fiberglass-epoxy and balsa core of 0.2% and 1.6%, respectively. It is seen in Fig. 10(b) and Fig. 11(b) that both models predict that the maximum interlaminar stress $\tau_{23}$ for the fiberglass-epoxy core section occurs at the outer bonded-joint while the maximum for the balsa core section occurs at the
inner bonded-joint. The discrepancy of the maximum stress for the fiberglass-epoxy and balsa core section is 7.5% and 12%, respectively. The continuous interlaminar stresses and stress free surfaces is well captured by the equilibrium approach.

B. Multi-Layer Composite Pipe

This second example concerns the analysis of the multi-layer composite pipe that exhibits extension-twist coupling via the use of-axis fibers shown in Fig. 12. The section has been used in Hodges [25] and Chen et al. [9] to validate cross-sectional analysis theories. The walls of the section are made up of two laminas with $E_1 = 141.963$ GPa (20.59E+06 psi), $E_2 = E_3 = 9.79056$ GPa (1.42E+06 psi), $G_{12} = G_{13} = G_{23} = 5.99844$ GPa (8.7E+05 psi) and $\nu_{12} = \nu_{13} = \nu_{23} = 0.42$. The layup sequence for the top and bottom straight walls are $(0, 90)$, and $(\theta, -\theta)$ for the left and right curved walls. The layup sequence is defined from the outside surface towards the inside of the section.

![Fig. 12 Schematic of multi-layer composite pipe.](image)

In the present example, the cross-section is discretized using the four meshes illustrated in Fig. 13. Model I, shown in Fig. 13(a), represents the very detailed meshing from Hodges [25] and Chen et al. [9], which here corresponds to 20 elements in the thickness direction and 140 segments in the circumferential direction for a total of 2800 solid elements. The elements have quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. Model II, shown in Fig. 13(b), uses two layers of homogeneous isoparametric elements with linear-cubic interpolation with 18 segments in the circumferential direction for a total of 36 elements. The present simple cross-section with only two layers makes this model a plausible candidate, while
more realistic composite cross-sections typically have considerably more layers. Model III, shown in Fig. 13(c), corresponds to a similar discretization in the circumferential direction in terms of 18 segments, but now with the layers represented within elements, reducing the number of elements to 18. In model IV, shown in Fig. 13(d), the number of elements of Model III has been increased to 34, by increasing the number of elements on the curved sides of the cross-section, while maintaining a single element with embedded layers in the thickness direction.

The particular cross-section with $\theta = 45^\circ$ has been analysed using VABS and ANABA with 143 segments and 20 layers, amounting to 2860 elements by Hodges [25] and Chen et al. [9]. These results are presented in Table 2 together with results obtained by the present method using the four meshes shown in Fig. 13. It may be noted that the very fine meshes used in the references and in Model I correspond to a very large number of elements in order to serve as a reference, and that this does not necessarily imply a need for this number of elements in a practical calculation.

The $\theta = 45^\circ$ configuration exhibits extension-twist coupling, represented by the cross-section stiffness matrix element $D_{36}$. A measure of the importance of the off-diagonal terms can be obtained by normalizing them with respect to their associated diagonal terms,

$$\gamma_{ij} = \frac{D_{ij}}{\sqrt{D_{ii}D_{jj}}}.$$  \hspace{1cm} (23)

This normalized measure of coupling, referred to as the coupling parameter, ranges from $-1 < \gamma_{ij} < 1$, where the extreme values indicate maximum possible coupling. For this cross-section,
the diagonal stiffness terms and coupling parameters obtained using the detailed mesh (Model I) match well those calculated using VABS and ANBA. Very good agreement is also obtained with the internally layered element model (Model III) with two orders of magnitude less elements with the maximum percentage difference when comparing the dominant terms of 2.3% occurring for the shear stiffness in the $x_2$ direction. The extension twist coupling parameter $\gamma_{36}$ has a 1.1% difference with the detailed models. It is noted that the differences on all parameters are smaller than the variability of the properties of real fiber reinforced cross-sections. One can observe little variations in the stiffness properties between the present internally layered element model (Model III) and the model with two layers of homogeneous isoparametric elements (Model II) which indicates that the through thickness material property variations are well captured by the layered formulation.

Figure 14 shows the value of the extension-twist coupling parameter with respect to the ply angle obtained using the internally layered element model with 18 elements (Model III) and the detailed mesh (Model I). It can be seen that both models agree well for all fiber orientations. The maximum coupling occurs at a fiber orientation $\theta = 22^\circ$ with a coupling value of $\gamma_{36} = 7.14E^{-02}$.

The stress distribution through the right curved wall section at 45 degrees from the $x_1$ axis under the shear load $Q_1 = -1$ is shown for $\theta = 45^\circ$ in Fig. 15. The stresses are obtained using the highly discretized model (Model I) and the linear-cubic internally layered element models (Model II and Model III).

<table>
<thead>
<tr>
<th>Units</th>
<th>VABS</th>
<th>ANBA</th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
<th>Model IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GA_1$ [N]</td>
<td>3.489E+06</td>
<td>3.493E+06</td>
<td>3.493E+06</td>
<td>3.500E+06</td>
<td>3.500E+06</td>
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</tr>
<tr>
<td>$GA_2$ [N]</td>
<td>1.463E+06</td>
<td>1.465E+06</td>
<td>1.464E+06</td>
<td>1.496E+06</td>
<td>1.496E+06</td>
<td>1.496E+06</td>
</tr>
<tr>
<td>$EI_1$ [Nm$^2$]</td>
<td>5.402E+03</td>
<td>5.402E+03</td>
<td>5.402E+03</td>
<td>5.424E+03</td>
<td>5.424E+03</td>
<td>5.424E+03</td>
</tr>
<tr>
<td>$EI_2$ [Nm$^2$]</td>
<td>1.547E+04</td>
<td>1.547E+04</td>
<td>1.547E+04</td>
<td>1.553E+04</td>
<td>1.553E+04</td>
<td>1.552E+04</td>
</tr>
<tr>
<td>$GJ$  [Nm$^2$]</td>
<td>1.971E+03</td>
<td>1.972E+03</td>
<td>1.972E+03</td>
<td>1.988E+03</td>
<td>1.988E+03</td>
<td>1.988E+03</td>
</tr>
<tr>
<td>$\gamma_{14}$</td>
<td>-6.738E-03</td>
<td>-6.759E-03</td>
<td>-6.771E-03</td>
<td>-6.598E-03</td>
<td>-6.597E-03</td>
<td>-6.641E-03</td>
</tr>
<tr>
<td>$\gamma_{25}$</td>
<td>-3.895E-02</td>
<td>-3.911E-02</td>
<td>-3.912E-02</td>
<td>-3.725E-02</td>
<td>-3.725E-02</td>
<td>-3.726E-02</td>
</tr>
<tr>
<td>$\gamma_{36}$</td>
<td>3.681E-02</td>
<td>3.685E-02</td>
<td>3.687E-02</td>
<td>3.621E-02</td>
<td>3.620E-02</td>
<td>3.613E-02</td>
</tr>
</tbody>
</table>
III and IV). Stresses are expressed in the laminate coordinate system. Stresses obtained using the highly discretized model are calculated using the constitutive relations and are shown at the Gauss point locations. For the internally layered element models, the in-plane and interlaminar stress distributions are obtained using the constitutive relations and the equilibrium equations, respectively.

It can be seen from Fig. 15(a) that the in-plane shear stress distribution $\tau_{13}$ obtained using the internally layered element models (Model III and IV) agree well with the Gauss point stresses of the detailed mesh (Model I) with a discrepancy of the maximum stress of 5%. The identical in-plane stress distribution of the models using internally layered elements agrees with their identical cross-section stiffness properties listed in Table 2 as the stiffness properties are governed by the stress components on the cross-section plane. It is seen in Fig. 15(b) that all models predict that the maximum interlaminar stress $\tau_{23}$ occurs in the outer layer. The discrepancy of the maximum interlaminar stress for the model with 18 and 34 laminate elements is 21.4% and 5.5%, respectively. This shows that the curved section of interest must be well discretized in order to capture the complex interlaminar stress distribution that arises from the sudden material property change between the straight and curved segments. However, Model IV with near-doubling of the elements along the curved parts illustrates that rather few additional elements are needed to obtain a good prediction of interlaminar stresses.
Fig. 15 (a) Constitutive (C) in-plane shear stresses $\tau_{13}$, (b) Constitutive (C) and equilibrium (E) transverse shear stresses $\tau_{23}$.

C. Wind Turbine Blade Cross-Section

This final example of the present paper concerns the analysis of the two-cell cross-section of a Siemens Wind Power A/S wind turbine blade shown in Fig. 16(a). The cross-section’s shell and spar cap are made of fiberglass-epoxy, while the sandwich core present in the trailing edge walls and tail are made of balsa and foam, respectively. This section is located near the tip of the blade which is indicated by the thick spar cap. Figure 16(b) shows the mesh of the wind turbine cross-section obtained using a single layer of internally layered elements. The walls, web and spar cap are modelled using 28 16-node internally layered elements with cubic-linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. The trailing core and transitions junctions are modelled using 17 8-node internally layered elements with linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. The discretization contains a total of 45 elements. The position of the nodes depends only on the geometry of the section and is therefore independent of the material layup.

The location of the elastic center and the shear center are shown in Fig. 16(b) using a circle

Fig. 16 (a) Schematic of rotor blade cross-section (b) Discretization using 45 internally layered elements.
Table 3 Stiffness properties of the blade section.

<table>
<thead>
<tr>
<th>EA [N]</th>
<th>EI_1 [Nm^2]</th>
<th>EI_2 [Nm^2]</th>
<th>GJ [Nm^2]</th>
<th>GA_1 [N]</th>
<th>GA_2 [N]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.22E+09</td>
<td>2.28E+07</td>
<td>9.40E+07</td>
<td>3.24E+06</td>
<td>1.65E+08</td>
<td>2.50E+07</td>
</tr>
</tbody>
</table>

and a cross, respectively. The principal axes of bending and shear shown at the elastic center and shear center, respectively, are practically parallel. Furthermore, the axis associated with edgewise bending is almost aligned with the chord line of the section. The extension and bending stiffnesses about the principal axes of bending as well the torsion and shear stiffnesses about the principal axes of shear for the blade cross-section are presented in Table 3. The stiffness properties indicate that the bending stiffness in the edgewise direction is about four times as large as the bending stiffness in the flapwise direction. The shear stiffness in the edgewise direction is about seven times as large as the shear stiffness in the flapwise direction which agrees with the results in Høgsberg and Krenk [11] for a similar section where a detailed stress distribution shows the dominating effect of the spar cap in the edgewise shear stiffness. The shear center is almost located at the quarter-chord point which means that little twist will be induced from the lift distribution at a cross-sectional level.

Figure 17 shows the stress distribution at mid-height of the web for two pure shear load cases with $Q_1 = 1$ and $Q_2 = 1$, respectively, and one pure torsion load case with $M_3 = 1$. The loads are applied in the principal axes of shear. The thickness coordinate $\eta$ is positive towards the leading edge. The in-plane stress distribution $\tau_{13}$ shown in Fig. 17(a) is obtained using the constitutive relation, while the transverse interlaminar stress distribution $\tau_{23}$ shown in Fig. 17(b) is obtained using the stress recovery method. The stresses are normalized with respect to the largest shear stress of the pure shear load case with $Q_2 = 1$.

The sandwich construction of the web is apparent from the in-plane stress distribution $\tau_{13}$ shown in Fig. 17(a) where it can be seen that the faces carry most of the shear flow, while the soft core has little contribution in all three load cases. The largest in-plane stress $\tau_{13}$ occurs in the pure shear load case with $Q_2 = 1$ which confirms engineering intuition that the web has the largest influence on the flapwise shear stiffness $GA_2$. The largest in-plane stress in the other shear load case and torsion load case are 28% and 1% of the largest stress of the load case with $Q_2 = 1$, respectively.
Fig. 17 (a) Normalized constitutive in-plane shear stresses $\tau_{13}$, (b) Normalized equilibrium transverse shear stresses $\tau_{23}$.

Fig. 18 (a) Normalized constitutive in-plane axial stresses $\sigma_3$, (b) Normalized equilibrium transverse interlaminar stresses $\sigma_2$.

The transverse shear stresses $\tau_{23}$ shown in Fig. 17(b) are three orders of magnitude smaller than the in-plane stresses $\tau_{13}$. The pure shear load case with $Q_1 = 1$ has interlaminar stresses $\tau_{23}$ with larger magnitude and opposite direction than those of the other shear load case with $Q_2 = 1$ even though the former’s in-plane stresses are smaller and in the same direction. From Eq. (17) this indicates that for the shear load case with $Q_1 = 1$ the stress gradients $\partial\tau_{13}/\partial y_1$ and $\partial\sigma_3/\partial y_3$ in the web are larger and with opposite directions compared to the shear load case with $Q_2 = 1$.

The stress distribution through the leading edge wall section under an axial force $Q_3$ applied at the elastic center is shown in Fig. 18. The location of the section is indicated by the dashed line in Fig. 16(b). The axial stress distribution $\sigma_3$ shown in Fig. 18(a) is obtained using the constitutive relation, while the transverse interlaminar stress distribution $\sigma_2$ shown in Fig. 18(b) is obtained using the stress recovery method. Stresses are expressed in the laminate coordinate system and normalized with respect to the maximum axial stress $\sigma_3$. 
The variation in axial stress distribution $\sigma_3$ shown in Fig. 18(a) comes from the different fiber orientation of each layer. The inner layer carries most of the axial load since it had the largest proportion of longitudinal fibers. Although only loaded axially, transverse stresses are present from varying in-plane contraction from dissimilar Poisson's ratio between the curved layers. This interlaminar stress is dominated by the curvature in the wall which corresponds to the last term in Eq. (19) indicating that the magnitude of the transverse stresses will grow proportional with the wall curvature. The stress recovery indicates that the lower part of the wall experiences transverse compression, while the largest tensile transverse stress occurs at the bonded joint between the middle and inner laminas. The largest transverse stress $\sigma_2$ is three orders of magnitude smaller than the maximum axial stress $\sigma_3$. It is noted that transverse tensile strength is typically two orders of magnitude smaller than the longitudinal tensile strength. The stress recovery method captures the interlaminar stress continuity between the layers and the stress free surfaces.

VI. Conclusion

A theory for cross-sectional analysis developed previously has been extended using finite elements which can accommodate several laminas in one element and is supplemented by a technique to accurately determine the interlaminar stress distribution. In the approach, the in-plane stress distribution is obtained using the constitutive relation, while the transverse stress distribution is recovered by postprocessing the in-plane stress gradients via equilibrium equations of 3D elasticity derived in the laminate coordinate system. The method shows good agreement with a conventional highly discretized model, illustrated by a composite tube cross-section with soft and stiff core and a multi-layer composite pipe. The results highlight that the use of internally layered elements for cross-section analysis can significantly reduce mesh generation and computational effort while maintaining accuracy up to moderately thick walls. Postprocessing of stresses of a wind turbine cross-section showed the presence of transverse interlaminar stresses from in-plane contraction associated with varying Poisson's ratio between layers. Although this paper deals only specifically with the stiffness properties, the present approach can be easily extended to provide the general cross-section mass properties.
Acknowledgments

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References


Non-linear collapse of general thin-walled cross-sections under pure bending

P.J. Couturier and S. Krenk

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Thin-walled beams exhibit a non-linear response to bending moments due to the progressive flattening of the cross-section, a behaviour commonly referred to as the Brazier effect. Most approaches to model this effect are limited to either circular cross-sections or to cross-sections made of isotropic materials. This article proposes an efficient two-step method of predicting the non-linear collapse of thin-walled cross-sections of arbitrary geometry with isotropic and orthotropic materials. The procedure relies on representing the cross-section by two-dimensional non-linear co-rotating beam elements with imposed in-plane loads proportional to the curvature, combined with a finite strip buckling analysis based on the deformed cross-section. By comparison with existing analytical and numerical modelling approaches, it is demonstrated that the present method can capture the cross-section flattening and critical moment for buckling of thin-walled structures commonly found in the industry.

I. Introduction

When thin-walled prismatic structures are bent the induced bending curvature leads to a transverse distributed load along the wall. This effect has been known and studied for about a century and is commonly denoted the ‘Brazier effect’ after a paper by Brazier[1] from 1927. In the typical Brazier problem a beam with thin-walled section of area $A$ is subjected to a constant bending mo-
ment $M$. In the basic form it is assumed that the deformation is identical for all cross-sections, and the elastic energy per unit length therefore consists of the energy $U_L$ due to the longitudinal strain $\varepsilon = \kappa(y + v)$ plus the energy $U_C$ associated with in-plane deformation,

$$U = U_L + U_C = \frac{1}{2}\kappa^2 \int_A E_L (y + v)^2 \, dA + \frac{1}{2} \int_A \sigma_{\alpha\beta} \varepsilon_{\alpha\beta} \, dA.$$  \hspace{1cm} (1)

In this expression $y$ is the initial distance from the neutral axis of bending, and $v$ is the displacement in the $y$-direction, while $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are the stress and strain components in the cross-section plane.

In most of the publications on the various forms of the Brazier problem the structure is assumed to be a thin-walled tube, described by the centerline and its curvature. Brazier used this assumption together with an integration of single-term representation of the in-plane deformation pattern of a circular cylinder to obtain the first correction term to the bending stiffness and the first term of the cross-section flattening. This analysis was extended by Reissner\cite{2, 3} to an asymptotic series expansion for a cylinder of general cross-section, including initial curvature along the bending axis and internal pressure, with details given for the circular cylinder. Fabian\cite{4} extended this analysis, also including axial loads. A synthesis of these solutions was given by Calladine\cite{5}. Later works of Libai and Bert\cite{6} and Harursampath and Hodges\cite{7} based on variational principles, essentially using (1) with a large-deformation representation of the last term, led to solutions with different asymptotic form – suggesting dependence on the implied truncations. A recent numerical integration by Levyakov\cite{8} of the differential equation used by Reissner supports the accuracy of the asymptotic series expansion beyond the point of maximum moment.

The limiting capacity in bending of thin-walled members is reached either in the form of a limit point, formed by the maximum of the moment-curvature curve of the homogeneous Brazier type solution of the bending problem, or by a bifurcation point that may be located before the limit point. The bifurcation deformation typically has a local character with a lengthwise variation, and therefore lies outside the assumptions of homogeneous bending. This problem can be circumvented by a procedure introduced by Weingarten and Seide \cite{9} and Hutchinson \cite{10}. The procedure is based on an analogy between the local curvature in bending just before bifurcation, and the radius of an equivalent cylinder which buckles by bifurcation at this particular load level. Thus, increasing local flattening of the part of the cross-section in compression reduces the bifurcation load level. The
The bifurcation problem of cylindrical tubes in bending has been analyzed by three-dimensional numerical methods e.g. by Corona and Rodrigues[11], Tatting et al.[12] and Houliara and Karamanos[13].

It was demonstrated by Houliara and Karamanos[13] that the approximate explicit bifurcation solution based on an ‘equivalent cylinder’ is rather accurate for isotropic cylinders, but deteriorates with increasing degree of anisotropy.

In the present paper the homogeneous large-deformation bending problem is solved for arbitrary thin-walled cross-sections by using the energy expression (1) in a form where the in-plane deformation problem is represented via large-displacement two-dimensional co-rotating beam elements, developed in Krenk[14]. The co-rotating elements are given in closed form based on complementary energy including shear deformability, and also includes the effect of the normal force on the bending stiffness. Essentially, the problem corresponds to that of a planar frame of shape like the cross-section, acted upon by forces proportional to the distance from the neutral bending axis. Thus, the displacement based part of these forces, $\kappa E_L v$, acts as an elastic spring support along the section. The homogeneous bending analysis is supplemented by a buckling analysis based on finite strip theory[15], using the formulation of Li and Schafer[16] with cross-section shape and axial pre-stress as determined by the nonlinear homogeneous finite bending solution. This combined solution procedure applies for general cross-sections and is implemented using standard finite element and strip solution procedures. It is demonstrated to be highly accurate by comparisons with published results for the circular cylinder, and its use to non-circular profiles is illustrated by three different thin walled sections.

II. Numerical Modelling of the Brazier Effect

Consider a thin-walled beam with longitudinal coordinate $x_3$ and transverse coordinates of the cross-section $x_1$ and $x_2$, as illustrated in Fig. 1(a) for the case of a circular tube. The origin is positioned at the elastic center and the coordinates are aligned with the principal axes of bending as to uncouple the extension and bending problems. The cross-section coordinates of a point in the undeformed beam is represented by the position vector $x_0 = [x_1^0, x_2^0]^T$. The displacement of the point $x_0$ due to in-plane cross-section deformation is denoted $u = [u_1, u_2]^T$, and the total position
of a point in the deformed state is

\[ \mathbf{x} = \mathbf{x}_0 + \mathbf{u} . \]  

(2)

The walls of the beam are of made of laminated fiber-reinforced composites. The \( x-y-z \) axis define the local mid-plane wall coordinate system where the \( x \) axis is in the longitudinal direction and the \( z \) axis defines the thickness direction, as shown in Fig. 1(b). Based on the first-order shear deformation theory, the static states of a segment of the wall are described by three in-plane forces, three moments, and two transverse forces [17]. These forces and moments are grouped together in the in-plane force vector \( \mathbf{N} = [N_x \ N_y \ N_{xy}]^T \), the bending moment vector \( \mathbf{M} = [M_x \ M_y \ M_{xy}]^T \), and the transverse shear force vector \( \mathbf{Q} = [Q_{xz} \ Q_{yz}]^T \). The components \( N_x \) and \( N_y \) are normal force resultants in the \( x \) and \( y \) directions, respectively, and \( N_{xy} \) is the in-plane shear force resultant. The components \( M_x \) and \( M_y \) are bending moments about the \( y \) and \( x \) axis, respectively, and \( N_{xy} \) is the twisting moment resultant. The components \( Q_{xz} \) and \( Q_{yz} \) define transverse shear forces.

The deformation associated with the forces and moments \( \mathbf{N} \), \( \mathbf{M} \), and \( \mathbf{Q} \) is described in terms of the mid-surface strains \( \mathbf{\varepsilon}^o = [\varepsilon^o_x \ \varepsilon^o_y \ \gamma^o_{xy}]^T \), the mid-surface curvatures \( \mathbf{\kappa} = [\kappa_x \ \kappa_y \ \kappa_{xy}]^T \), and the shear strains \( \mathbf{\gamma}^o = [\gamma^o_{xz} \ \gamma^o_{yz}]^T \), respectively. The components \( \varepsilon^o_x \) and \( \varepsilon^o_y \) are mid-surface normal strains, and \( \varepsilon^o_{xy} \) is the mid-surface shear strain. The components \( \kappa_x \) and \( \kappa_y \) are bending curvatures, and \( \kappa_{xy} \) is the twisting curvature. The components \( \gamma^o_{xz} \) and \( \gamma^o_{yz} \) are transverse shear strains.

Fig. 1 Circular tube: (a) Beam coordinate system 1-2-3 and wall coordinate system \( x-y-z \), (b) State of homogeneous bending.
For linear elastic laminates there is a linear relation between the forces $N$, $M$, and $Q$ and the strains $\varepsilon^o$, $\kappa$, and $\gamma^o$. The relation can be written in stiffness format as

$$
\begin{bmatrix}
N \\
M \\
Q
\end{bmatrix} =
\begin{bmatrix}
A & B & 0 \\
B & D & 0 \\
0 & 0 & \tilde{A}
\end{bmatrix}
\begin{bmatrix}
\varepsilon^o \\
\kappa \\
\gamma^o
\end{bmatrix},
$$

(3)

where the matrices $A$, $B$, $D$, and $\tilde{A}$ group the extension stiffness components, the bending-extension coupling components, the bending stiffness components, and the transverse shear stiffness components, respectively. In the present analysis, laminates are limited to symmetric balanced layups whereby there is no extension-bending coupling $B = 0$ and no shear-extension coupling $A_{13} = A_{23} = 0$. It is assumed that bend-twist coupling terms $D_{13}$ and $D_{23}$ are negligible which is true for cross-ply laminates and a good approximation for many structures.

A. Homogeneous Bending

The beam is subjected to a state of homogeneous bending about the $x_1$ axis with a curvature $\kappa_1$, as illustrated in Fig. 1(b) for the case of a circular tube. Under a state of pure bending, no shear stresses in the longitudinal direction develop $N_{xy} = Q_{xz} = 0$ which results in uniform transverse deformation along the beam whereby the twisting curvature $\kappa_{xy}$ must be zero. Under these conditions and for the composite material properties defined in the previous section Eq. 3 reduces to

$$
\begin{bmatrix}
N_x \\
N_y \\
M_x \\
M_y \\
Q_{yz}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 \\
A_{12} & A_{22} & 0 & 0 & 0 \\
0 & 0 & D_{11} & D_{12} & 0 \\
0 & 0 & D_{12} & D_{22} & 0 \\
0 & 0 & 0 & 0 & \tilde{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\varepsilon^o_x \\
\varepsilon^o_y \\
\kappa_x \\
\kappa_y \\
\gamma^o_{yz}
\end{bmatrix}.
$$

(4)

Two simplifying assumptions often used when studying the Brazier effect [2, 3, 12] are introduced at this point. First, the longitudinal curvature $\kappa_x$ effect at the lamina level is ignored as it is assumed much smaller than the wall curvature caused by the cross-section deformation $\kappa_y$. Second, the Poisson’s ratio effect is neglected whereby the two normal force resultants $N_x$ and $N_y$ decouple.
These two assumptions allow to further simplify Eq. 4 into four independent constitutive relations

\[ N_x = E_x h \varepsilon_x^0, \quad N_y = E_y h \varepsilon_y^0, \quad M_y = D_{22} \kappa_y, \quad Q_{yz} = \bar{A}_{22} \gamma_{yz}^0, \tag{5} \]

where \( h \) is the wall thickness and \( E_x \) and \( E_y \) are the effective moduli defined in terms of uniaxial extension [18]

\[ E_x = \frac{A_{11} A_{22} - A_{12}^2}{A_{22} h}, \quad E_y = \frac{A_{11} A_{22} - A_{12}^2}{A_{11} h}. \tag{6} \]

For an isotropic material with Young’s Modulus \( E \) and Poisson’s ratio \( \nu \), the stiffness components take the following value \( E_x = E_y = E, \quad D_{22} = E h^3/12 (1 - \nu^2) \), and \( \bar{A}_{22} = k_{yz} E / 2 (1 + \nu) \) where the shear correction factor \( k_{yz} \) takes the value 5/6.

From geometric consideration, the relation between the beam longitudinal curvature \( \kappa_1 \) and the mid-surface normal strain \( \varepsilon_x^0 \) follows as

\[ \varepsilon_x^0 = \kappa_1 x_2. \tag{7} \]

Note that the distance from the bending axis is defined terms of the deformed distance from the bending axis. With the constitutive relation Eq. 5 for \( N_x \) and the kinematic relation Eq. 7 for \( \varepsilon_x^0 \), the non-linear relation between the curvature of the beam \( \kappa_1 \) and applied bending moment over the cross-section of the structure \( M_1 \) can be found from

\[ M_1 = \oint_{\Gamma_0} N_x x_2 \, dy = \kappa_1 \oint_{\Gamma_0} E_x h (x_2^0 + u_2)^2 \, dy. \tag{8} \]

Note that the resultant force \( N_x \) is associated with the initial cross-sectional area and as such the integration is carried over the shape of the initial undeformed cross-section \( \Gamma_0 \).

From elementary mechanics under a state of homogeneous bending, the section of the beam above the bending axis will be in compression and the lower part in tension. Both have a component directed towards the center line which tends to flatten the section. The component of the resultant pressure that acts along the \( x_2 \) axis is given by

\[ p = N_x \kappa_1. \tag{9} \]

Using Eq. 5 and Eq. 7, the resultant pressure can be expressed in terms of the effective modulus \( E_x \) and the distance away from the bending axis

\[ p = E_x h x_2 \kappa_1^2. \tag{10} \]
It can be seen that the resultant pressure \( p \) increases with the square of the curvature. Furthermore, for a section with uniform properties the crushing pressure varies linearly with the distance from the neutral axis. One recovers the expression for the vertical pressure described by Brazier [1] if the material is made isotropic and the cross-section deformation is neglected when defining the distance away from the bending axis.

**B. Cross-sectional Deformation**

Under a state of homogeneous bending, each cross-section deforms identically. The in-plane displacement field along the longitudinal axis of the beam can therefore be defined by the in-plane deformation of any cross-section. An efficient numerical solution method to model large in-plane deformations associated with the Brazier effect can be obtained by discretizing the cross-section using two dimensional non-linear co-rotating beam elements. No a priori knowledge of the deformation shape is needed with this approach. This enables the analysis of general thin-walled structures with isotropic and composite wall materials. The formulation of the finite element beam model used in the analysis is presented in Appendix A. For a detailed derivation of the co-rotating beam theory, the reader is referred to Krenk [14].

The structure of the non-linear model algorithm is presented in Table 1. The algorithm solves the deformation of the cross-section subject to the flattening pressure using the Newton-Raphson method using a total Lagrangian formulation. The first part of the analysis consists of calculating the nodal load stiffness matrix \( K_p \) which accounts for the linear relation between the in-plane load associated with the Brazier effect and the nodal displacements. The total nodal load \( p \) is then obtained from the product of the nodal load stiffness matrix and the nodal position vector \( x_0 \). The solution then involves \( n_{max} \) load steps \( f_1, \ldots, f_{n_{max}} \) where in each load step the residual force \( r \) is evaluated as the difference between the external forces \( f_n \) and the internal forces \( g \) plus a correction from the change in load based on the deformation. The displacement iteration increment \( \delta u \) is computed by use of the tangent stiffness matrix. The latter is composed of three parts: one from the constitutive and geometrical stiffness associated with the deformation of the element in terms of the deformation modes by \( K_d \), the second from the co-rotation of the element’s frame of reference.
Table 1 Algorithm using Newton-Raphson method

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial state:</td>
<td>$x_0$, $u_0 = f_0 = \Delta f = 0$</td>
</tr>
<tr>
<td>Calculate:</td>
<td>$K_p$, $p = K_p x_0$, $\Delta f = p / n_{max}$</td>
</tr>
<tr>
<td>Load steps</td>
<td>$n = 1, 2, ..., n_{max}$</td>
</tr>
</tbody>
</table>

- $f_n = f_{n-1} + \Delta f + K_p \Delta u$
- $\Delta u = 0$
- Iterations $i = 1, 2, ..., i_{max}$
- $r = f_n - g(u_{n-1} + \Delta u) + K_p \Delta u$
- Stop iteration when $\|r\| < \epsilon \|\Delta f\|$ |

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$K = K_d + K_r - K_p$</td>
</tr>
<tr>
<td>$\delta u$</td>
<td>$\delta u = K^{-1} r$</td>
</tr>
<tr>
<td>$\Delta u$</td>
<td>$\Delta u = \Delta u + \delta u$</td>
</tr>
<tr>
<td>$u_n$</td>
<td>$u_n = u_{n-1} + \Delta u$</td>
</tr>
<tr>
<td>End of load step</td>
<td></td>
</tr>
</tbody>
</table>

by $K_r$, and the third from the nodal load stiffness matrix $K_p$ which accounts for the reduction in the load as the cross-section flattens. The iterations end when the residual force is within a specified tolerance limit of the applied load or the maximum number of iterations has been reached $i_{max}$.

### III. Local Buckling

It has been observed that many structures will collapse from local bifurcation buckling prior to reaching the flattening instability at the limit point. The cross-section deformation from the Brazier effect can however influence the critical moment at which local buckling will occur. In the current analysis, a linear eigenvalue analysis is performed on the deformed structure obtained from the non-linear beam model. By repeating the analysis for various curvatures, the critical curvature for local buckling can be found when the bending moment associated with the curvature from Eq. (8) becomes larger than the critical moment.
The linear eigenvalue analysis is performed using the buckling analysis tool CUFSM 4.05 [16] developed at Johns Hopkins University which is based on the finite strip method proposed by Cheung [15] for thin-walled structures with uniform geometrical and material properties in the longitudinal direction. The kinematics in CUFSM’s implementation follows Kirchhoff plate theory. As with the Finite Element Method, the Finite Strip Method uses shape functions to define the displacement field. However, unlike the finite element method which only uses polynomial displacement functions, the Finite Strip Method uses polynomial displacement functions in the transverse directions and trigonometric functions in the longitudinal direction corresponding to pre-set boundary conditions.

Fig. 2 I beam discretization: (a) Finite Element Method, (b) Finite Strip Method.

An illustrative comparison between the discretization of the Finite Element Method and the Finite Strip Method is shown in Fig. 2 for the case of an I beam. It can be seen that the Finite Element Method requires several nodes along the lengthwise direction whereas the Finite Strip Method has all of its degrees of freedom located at a single cross-section plane. The reduced number of nodes and simpler mesh needed for the Finite Strip Method provides substantial pre-processing and computational effort savings compared with the Finite Element Method.

For the present analysis, CUFSM was modified to enable modelling thin-walled structures made of composite materials. As such, the original isotropic constitutive relation was replaced by the extension stiffness matrix $\mathbf{A}$ and the bending stiffness matrix $\mathbf{D}$ for composite materials presented in Eq. 3. The bending-extension components $\mathbf{B}$ were not included as only symmetric balanced layups are considered.
IV. Applications

The present section contains an assessment of the capacity of the present method for calculating the limit point instability and bifurcation point of composite thin-walled cross-sections. Three studies are used to cover several common thin-walled sections, namely a study of an isotropic circular tube, a composite circular tube and a comparison between a Box, C, and I profile.

A. Isotropic Tube

The first example concerns a circular tube with an outer radius \( r \) and a wall thickness \( t \) made of an isotropic material with Young’s modulus \( E \) and Poisson’s ratio \( \nu \). The ends of the tube are free to deform. The section is discretized using 100 elements placed on the perimeter of the tube. The relation between the bending moment and flattening of the tube with respect to the curvature obtained using the non-linear method are shown in Fig. 3. The flattening is measured as the radial displacement of the point on the section furthest away from the axis of bending. The bending moment \( M \), curvature \( \kappa \), and flattening \( v \) for a tube can be expressed in dimensionless form by using the parameters, see e.g. Reissner [2, 3],

\[
\begin{align*}
    m_t &= \frac{M}{\pi r \sqrt{AD}}, \\
    \alpha_t &= \kappa r^2 \sqrt{\frac{A}{D}}, \\
    \nu_t &= \frac{u_2}{r},
\end{align*}
\]

where \( D = Et^3/12(1 - \nu^2) \) is the wall bending stiffness and \( A = Et \) is the axial stiffness factor of the tube.

Fig. 3 Thin-walled tube: (a) Dimensionless moment curvature relation, (b) Dimensionless flattening curvature relation.
Figure 3 also shows the analytical solutions obtained by Brazier [1] and Reissner [2, 3] as well as that of Harursampath and Hodges [7]. It can be seen that the bending moment and flattening predicted by the present non-linear method agree well with the results of Reissner [3] and Fabian [4] up to the limit point. The small discrepancy after the limit point stems from the omission of higher order terms in the truncated series expansion solution of Reissner [2] and Fabian [4]. These results also agree with those obtained by Tatting et al. [12] and Houlara and Karamanos [13] who used a non-linear shell theory. Brazier’s results show good agreement of the load path curve but under-predicts the flattening. Calladine [5] offers an explanation for this discrepancy. He shows that the omitted higher order terms in Brazier’s analysis nearly cancel at the limit-point thereby providing a good estimate of the collapse moment for circular sections.

The normalized moment, curvature, and flattening at the limit point predicted by Brazier [1], Reissner [2, 3] and Fabian [4] as well as by Harursampath and Hodges [7] are presented in Table 2. Insight into the kinematic assumptions taken by the different authors can be obtained by analysing which modifications to the algorithm shown in Table 1 need to be made to recreate their results. The flattening predicted by Brazier can be obtained by using a linear analysis whereby no updating of the resultant pressure and geometry is performed. This is accomplished by setting \( n = 1 \) and \( i = 1 \) and \( K_p = 0 \) in the algorithm. The normalized bending moment and flattening obtained using this linear version of the algorithm at the critical curvature predicted by Brazier’s method are shown in bracket in Table 2. It can be seen that the flattening agrees very well. The discrepancy in the bending moment stems from the omission in Brazier’s analysis of the second order term of the bending stiffness expression. Similarly, the solution obtained by Harursampath and Hodges can be obtained by performing the analysis without updating the geometry and using the resultant

<table>
<thead>
<tr>
<th>Method</th>
<th>( \alpha_t )</th>
<th>( m_t )</th>
<th>( v_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brazier [Algorithm: linear]</td>
<td>1.63 [1.63]</td>
<td>1.09 [1.14]</td>
<td>0.222 [0.222]</td>
</tr>
<tr>
<td>Harursampath and Hodges [Algorithm: linear + correction]</td>
<td>2.53 [2.53]</td>
<td>1.34 [1.34]</td>
<td>0.370 [0.368]</td>
</tr>
<tr>
<td>Reissner-Fabian [Algorithm: non-linear]</td>
<td>1.68 [1.68]</td>
<td>1.06 [1.06]</td>
<td>0.268 [0.266]</td>
</tr>
</tbody>
</table>
The critical moment curvature relation calculated using the Finite Strip Method is shown in Fig. 4(a). The curve represents the smallest normalized moment at which local buckling is observed for a straight beam with a cross-section geometry equal to the deformed cross-section calculated at each beam curvature. Simply-supported condition at the two opposite ends of the beam were used in the analysis. It can be seen that the bending moment curve intersects the critical moment curve at \( \alpha_t=1.45 \) indicating that the tube will collapse from local buckling before the limit point is reached. A plot of the buckling mode of the undeformed circular cross-section and of the cross-section prior to local buckling at \( \alpha_t=1.45 \) is shown in Fig. 4(b). It can be seen that the buckling modes are associated with the creation of a kink/wrinkle on the compressive side of the tube. Although the tube will collapse from local buckling, the ovalization of the section plays a significant role in the mechanism of local buckling. This can be seen in Fig. 4(a) where the critical moment at local buckling is 50% of the critical moment of the undeformed geometry. The reduction in critical moment is associated with change in in-plane local curvature where a section with higher curvature will tend to have a higher critical stress [19]. The crushing pressure tends to flatten the region of the circular tube with maximum compressive stress, as shown in Fig. 4(b), which reduces the critical stress and hence
reduces the critical moment.

An analytical criterion for local buckling of isotropic tubes under homogeneous bending was
proposed by Weingarten and Seide [9] and Hutchinson [10]. Their combined work showed that local
buckling under pure bending, occurs when the point of maximum compressive stress reaches the
buckling stress of the tube under uniform axial compression. For isotropic material, this critical
stress $\sigma_{cr}$ is

$$\sigma_{cr} = \frac{Et}{\rho \sqrt{3(1-\nu)}}$$

where $\rho$ is the local radius of curvature at the point where compressive stress is the largest. The
effect of a flattening of the section is evident from Eq. 12 where a flattening of a section will
reduce the critical stress. Using this criterion of local buckling, one obtains a normalized critical
moment for the undeformed section and at the point of local buckling of $m_t=2.00$ and $m_t=1.01$,
respectively, which is 4% and 3% smaller than what is obtained with the Finite Strip Method as
shown in Fig. 4(a).

B. Composite Tube

circular tube made of AS3501 graphite-epoxy material with 0° layers and a radius to thickness
ratio $r/t = 50$. The material properties of the composite are $E_{11} = 138.0$ GPa (20.02 Mpsi),
$E_{22} = 9.0$ GPa (1.3 Mpsi), $G_{12} = 7.10$ GPa (1.03 Mpsi), and $\nu_{12} = 0.3$. Following Houliara and
Karamanos [13], the moment and curvature are normalized by the following quantities

$$m_o = \frac{M_0 q \sqrt{1-\nu_{12}\nu_{21}}}{E_{22} r t^2}, \quad \alpha_o = \frac{\kappa r^2 q \sqrt{1-\nu_{12}\nu_{21}}}{t}$$

where $q = \sqrt{E_{22}/E_{11}}$ defines the level of anisotropy. For an isotropic material, the relation between
the moment and curvature normalization in Eq. 11 and Eq. 13 are $m_o = m_t \pi/\sqrt{12}$ and $\alpha_o = \alpha_t/\sqrt{12}$, respectively.

The dimensionless moment and critical moment for local buckling with respect to curvature
obtained using the present non-linear model using 100 elements is shown in Fig. 5(a). It can be seen
that the response of the composite tube resembles that of the isotropic tube in Fig. 4(a). As such,
the curves shows a linear response for small curvatures and a non-linear moment-curvature relation for large curvatures resulting from the decrease of the moment of inertia from the flattening of the section. It can also be seen that local buckling supersed flattening instability. The moment at which a straight beam with a cross-section equal to that of the undeformed tube would exhibit local buckling was calculated to be \( m_o = 1.376 \). As can be seen in Fig. 5(a), the critical moment curve intersects the bending moment curve at \( m_o = 0.889 \). This 35\% reduction in the critical moment is, similarly to the isotropic tube, caused by the flattening of the section of the tube undergoing maximum compressive stress as curvature increases.

The dimensionless curvature and moment at the onset of local buckling and flattening instability obtained with the present method and by Corona and Rodrigues [11] and Houliara and Karamanos [13] are listed in Table 3. It is observed that for this cross-section, results match well those

**Table 3 Comparison of flattening and bifurcation instabilities.**

<table>
<thead>
<tr>
<th>Method</th>
<th>( \alpha_o )</th>
<th>( m_o )</th>
<th>( \alpha_o )</th>
<th>( m_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>0.478</td>
<td>0.959</td>
<td>0.355</td>
<td>0.889</td>
</tr>
<tr>
<td>Houliara and Karamanos</td>
<td>0.480</td>
<td>0.954</td>
<td>0.354</td>
<td>0.882</td>
</tr>
<tr>
<td>Corona and Rodrigues</td>
<td>0.49</td>
<td>0.98</td>
<td>0.37</td>
<td>0.92</td>
</tr>
</tbody>
</table>
previously published. The moment at the onset of flattening instability predicted by the present method differs by 0.5% and 2.2% compared to the moments obtained by Corona and Rodrigues [11] and Houliara and Karamanos [13], respectively. This shows close prediction in the non-linear flattening of the cross-section. Similarly, there is a difference of 0.8% and 3.5% in the moment at the onset of local buckling compared to the moments obtained by Corona and Rodrigues [11] and Houliara and Karamanos [13], respectively. This shows close prediction of the non-linear flattening as well as of the onset of local-buckling. The differences on all parameters between the three methods for this cross-section are all smaller than the variability of the properties of real fiber reinforced cross-sections.

Figure 5(b) shows the wall bending moment about the $x$ direction as function of the circumferential coordinate $y$ starting at the $x_1$ axis. The moment distribution is shown for the curvature at which local buckling occurs $\alpha_o = 0.355$ as well as for a curvature which is two orders of magnitude smaller, well within the linear range of the moment-curvature relation shown in Fig. 5(a). The wall bending moment is expressed in dimensionless form using

$$m_w = \frac{4M_x}{E_x t r^3 \kappa^3}.$$  \hspace{1cm} (14)

It can be seen that the largest positive bending moment occurs at the maximum distance from the bending axis whereas the largest negative bending moment occurs at the bending axis. For the small curvature where the tube is undeformed, the magnitude of the maximum positive and negative bending moments are equal. Conversely, the flattening of the tube at $\alpha_o = 0.355$ yields a 10% larger magnitude of the moment at the bending axis compared to at the top and bottom of the section.

C. Box, C, and I Profiles

Three commonly used thin-walled beam profiles for structural applications are the Box, C, and I profiles. Figure 6 shows the cross-section of three such beams with a height $h$, a width $w = h/2$, and a flange wall thickness $t_f = h/10$. The C and I profiles have a web wall thickness $t_w$ equal to the flange wall thickness whereas the two webs of the Box profile have a wall thickness of half the flange thickness. The three cross-sections are homogeneous with Young’s modulus $E$ and Poisson’s ratio.
\[ \nu = 0.3 \]. The sections are discretized using 20 elements in each vertical segment and 10 elements in each horizontal segment.

The relation between the bending moment and critical moment with respect to the curvature obtained using the non-linear method and Finite Strip Method, respectively, are shown in Fig. 7. The bending \( M \) and curvature \( \kappa \) are normalized with respect to the critical moment for local buckling \( M_{cr} \) and associated curvature \( \kappa_{cr} \) for Euler column buckling of the web of the I profile,

\[
\alpha_b = \frac{\kappa}{\kappa_{cr}} = \kappa \sqrt{\frac{6 \, w \, h^3 \, (1 - \nu^2)}{t_f^2 \, \pi^4}}, \quad m_b = \frac{M}{M_{cr}} = \frac{M}{EI_0} \sqrt{\frac{6 \, w \, h^3 \, (1 - \nu^2)}{t_f^2 \, \pi^4}},
\]

where \( EI_0 \) is the cross-section bending stiffness of the undeformed cross-section, which is identical for all three cross-sections of the present example.

As seen in Fig. 7 (15) all three profiles have the same tangent bending stiffness for small curvature. As the curvature becomes larger, the C profile is the first to show a non-linearity in the bending-curvature relation. For sufficiently large curvature the C profile reaches a limit point. Local buckling which occurs at \( \alpha_b = 0.37 \) however supersedes the large in-plane deformation which leads to the flattening instability. A plot of the buckling mode and the deformed cross-section at \( \alpha_b = 0.37 \) of the C profile is shown in Fig. 8(b). It can be seen that the buckling mode is associated with the deformation of the flange under compression. The almost constant critical moment from \( \alpha_b = 0 \) to \( \alpha_b = 0.37 \) indicates that the flattening of the section has little influence on the critical moment for this section. This is due to the fact that the flange associated with local buckling remains straight during the flattening of the section as is shown in Fig. 8(b).

The Box profile is the only section which reached the limit point prior to the formation of local
wrinkles. The limit point is reached at a normalized curvature of $\alpha_b = 0.79$ which is double the critical curvature of the C profile. The deformed cross-section at the limit point of the Box section is shown in Fig. 8(a). It can be seen that the section ovalizes and flattens under the resultant pressure. The increase in critical moment shown in Fig. 7 prior to reaching the limit point is due to the large increase in curvature of the wall which, similar to the circular section, leads to an increase in the critical stress.

It can be seen in Fig. 7 that the I section does not exhibit a non-linear bending-curvature relation in the curvature range studied. The insensitivity of the I profile to the Brazier effect comes from the center position of the web whereby the resultant moment of each flange segment which tends to bend the web cancels. The I profile also exhibits the highest critical bending moment which occurs at a normalized curvature of $\alpha_b = 1.26$. The higher critical buckling comes from the shorter flange segments. The normalized critical curvature is however larger than unity which indicates that the web would reach the Euler column buckling load from the in-plane compressive forces prior to reaching the limit point.
Fig. 8 Cross-section deformation at buckling: (a) Box profile, (b) C profile, (c) I profile.

V. Conclusion

This article proposes a flexible method for analyzing the buckling under pure bending of thin-walled beams with arbitrary cross-section geometry and isotropic and orthotropic material distribution. Two types of instability are studied namely, flattening instability from the Brazier effect, and bifurcation instability from longitudinal stresses. Both are analyzed using models which build on two well established Finite Element Analysis techniques. The cross-section deformation from the Brazier effect is modelled using two dimensional non-linear co-rotating beam elements whereas the local buckling is modelled using the Finite Strip Method where the longitudinal deformation is modelled using trigonometric functions. The analysis requires a simple two-dimensional line mesh which provides substantial pre-processing and computational effort savings making the analysis suitable even at early stages of design. Through various cross-section shapes studied, it was demonstrated that the sensitivity of the critical moment for local buckling to the cross-section deformation from the Brazier effect depends strongly on the initial curvature of the part of the section undergoing maximum longitudinal compressive stress. It was also shown that the position of a web in a cross-section can determine if the cross-section will buckle first from flattening instability, local bifurcation, or Euler column buckling of the web itself.
Appendix A. Co-Rotating Beams in Two Dimensions

Following the finite element formulation using co-rotating beams in two dimensions in Krenk [14], the displacement field is separated in two components: the displacement of the element local coordinate system in the global $x_1 - x_2$ plane, and the deformation of the element in its own local coordinate system based on three deformation modes namely extension, symmetric bending, and anti-symmetric bending.

The beam elements considered in the present analysis consist of a straight beam connecting two end-nodes defined as A and B as illustrated in Fig. 9(a). The initial position of the beam in the fixed global frame of reference is defined in terms of the nodal coordinates contained in the columns vector $x_0 = [x^A_0, x^B_0]^T$. The three nodal coordinates are defined as $x^i_0 = [x^i_1, x^i_2, \varphi_i]$, where the components $x_1$ and $x_2$ represent the in-plane coordinates, while the component $\varphi_i$ is the angle at the nodes which corresponds for straight beams to the inclination of the element. The nodal displacements are contained in the column vector $du = [du_A, du_B]^T$. The three nodal degrees of freedom are defined as $du_i = [du_1, du_2, d\varphi]$, where the components $du_1$ and $du_2$ represent the in-plane displacements, while the component $d\varphi$ is the nodal rotation. The nodal coordinates of the element in a deformed state contained in the column vector $x = [x^A, x^B]^T$ is obtained from

$$x = x_0 + u .$$

The displacement vector $du$ is conjugate to the force vector $dq = [dq_A, dq_A]^T$, with the nodal forces defined as $dq_i = [df^i_1, df^i_2, dm^i]$. The components $df_1$ and $df_2$ represent the two in-plane force components, while the component $dm$ is the moment conjugate with the rotation $d\varphi$.

The static states of a beam element are described by a normal force $N$, a constant moment $M_s$ and an anti-symmetric moment $M_a$ which can be grouped in the internal force vector $t = [N, M_s, M_a]^T$. The three static states are illustrated in Fig. 9(b). Note that the anti-symmetric bending mode must be complemented by shear forces $Q = -2M_a/l$. The deformation associated with the internal force and moments $t$ is described in terms of three displacements defined by the internal incremental displacement vector $dv = [du, d\varphi_s, d\varphi_a]^T$, where the component $du$ represent axial elongation, $d\varphi_s$ represent symmetric rotation of the end-points, and $d\varphi_a$ represent counter-clockwise rotation of both end points. The element generalized force components $q_e$ which are
expressed in a coordinate system aligned with the element is related to the internal force vector \( t \) via

\[
q_c = S t , \tag{17}
\]

where the transformation matrix \( S \) in terms of the length of the element \( l \) takes the form

\[
S = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 2/l \\
0 & -1 & 1 \\
1 & 0 & 0 \\
0 & 0 & -2/l \\
0 & 1 & 1
\end{bmatrix} \tag{18}
\]

In the present analysis, the element tangent stiffness matrix is composed of the three parts: the deformation stiffness matrix \( K_d \), the rotation stiffness matrix \( K_d \), as well as the nodal load stiffness matrix \( K_p \). The final form of these three stiffness matrices is presented next.

Element Deformation Stiffness

The element stiffness matrix associated with the three deformation mode \( K_d \) is composed of two parts: the constitutive stiffness \( K_d^c \), and the geometrical stiffness \( K_d^g \),

\[
K_d^c = S (K_d^c + K_d^g) S^T. \tag{19}
\]
The constitutive stiffness matrix is expressed in terms of the constitutive relations for laminated composites presented in Eq. 5 while the geometrical stiffness is expressed terms of the normal force, \[ K_e^c = \frac{1}{l} \begin{bmatrix} E_y h & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & 3\psi_a D_{22} \end{bmatrix}, \quad K_g^e = l \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} N & 0 \\ 0 & 0 & \frac{1}{24} N \end{bmatrix}. \] (20)

The shear flexibility parameter \( \psi_a \) is defined as
\[ \psi_a = \frac{1}{(1 + \Phi)} \] (21)

Note that for a compressive normal force \( N \), the components of the geometrical stiffness matrix \( K_g \) take negative values which accounts for the effect of increased flexibility.

**Element Rotation Stiffness Matrix**

The combined effect of the co-rotating frame of reference and the change of the shear force due to changes in the beam length \( l \) is given by the rotation stiffness matrix \( K_r^e \),

\[ K_r^e = \begin{bmatrix} K_{11} & -K_{11}^r \\ -K_{11}^r & K_{11} \end{bmatrix}, \] (22)

in terms of the block matrix

\[ K_{11}^r = \frac{1}{l} \begin{bmatrix} 0 & -Q & 0 \\ -Q & N & 0 \\ 0 & 0 & 0 \end{bmatrix}. \] (23)

**Nodal Load Stiffness Matrix**

The distributed load \( p \) from the Brazier effect in Eq 10 that tends to flatten the section is illustrated in Fig. 10(a) for one element. It can be seen that the load varies linearly with the distance from the neutral axis. The reduced nodal load vector associated with \( p \) can be expressed in terms of the length of the undeformed element \( l_0 \) and the nodal position in the deformed state in the fixed global frame of reference \( x_A^2 \) and \( x_B^2 \) as

\[ f_A^2 = -h l_0 E_x \kappa^2 \left( \frac{1}{6} x_A^2 + \frac{1}{6} x_B^2 \right), \quad f_B^2 = -h l_0 E_x \kappa^2 \left( \frac{1}{6} x_A^2 + \frac{1}{6} x_B^2 \right). \] (24)
Fig. 10 (a) Distributed load $p$ applied to a beam element, (b) Reduced nodal loads.

In the calculation of the reduced nodal load, the rotational degrees of freedom are ignored. As such, the moment terms in the nodal load are omitted which leads to the reduced nodal loading shown in Fig. (b). The nodal load can be expressed in matrix notation as

$$
\begin{bmatrix}
q_A \\
q_B
\end{bmatrix} =
\begin{bmatrix}
K_{11}^p & K_{12}^p \\
K_{21}^p & K_{22}^p
\end{bmatrix}
\begin{bmatrix}
x_A \\
x_B
\end{bmatrix}
$$

(25)

with

$$
K_{11}^p = -hl_0 E_x \kappa^2
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix},
K_{12}^p = -hl_0 E_x \kappa^2
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

(26)

It is convenient to present Eq. 25 in a compact notation where the nodal load and nodal position terms are grouped together $\mathbf{q} = [q_A, q_A]^T$ and $\mathbf{x} = [x_A, x_A]^T$. In this notation, the reduced load takes the form

$$
\mathbf{q} = \mathbf{K}_p \mathbf{x},
$$

(27)

in which the linear relation between the nodal load and the nodal position is contained in the nodal load stiffness matrix $\mathbf{K}_p$. Following Eq. 16, the total nodal position can be split into the nodal position of the undeformed cross-section $\mathbf{x}_0 = [x_0^A, x_0^B]^T$ and the total nodal displacements $\mathbf{u} = [u_A, u_B]^T$,

$$
\mathbf{q} = \mathbf{K}_p \mathbf{x}_0 + \mathbf{K}_p \mathbf{u}.
$$

(28)

In this form, the first term represents the nodal load $\mathbf{p}$ associated with the nodal position of the
undeformed cross-section while the second term represents a correction for the change in the load as the section deforms. The nodal load $p$ remains unchanged during the analysis and is therefore treated as an external load whereas the correction term depends on the displacement and is therefore added to the tangent stiffness matrix. Although out of the scope of this paper, the load stiffness matrix could be extended to include the effect of internal pressure.

**Element Tangent Stiffness in Global Frame**

The element tangent stiffness matrix is defined as the sum of the stiffness associated with deformation mode $K_d$, the rotation stiffness matrix $K_r$, and the negative of the nodal load stiffness matrix $K_p$. When summing the matrices care must be taken to transform the deformation mode and rotation stiffness matrices from the local element coordinate system into the global frame of reference,

$$K_e = R_e (K_d^e + K_r^e) R_e^T - K_p^e.$$  \hspace{1cm} (29)

The element rotation matrix $R_e$ is defined as

$$R_e = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix},$$  \hspace{1cm} (30)

with $R$ representing the standard two dimensional rotation matrix

$$R = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$  \hspace{1cm} (31)

where $\varphi$ is the element angle, shown in Fig. 9(a). The element stiffness matrix $K_e$ expressed in the global frame of reference can be assembled into the global stiffness matrix of the structure following standard finite element assembling procedure.

**Acknowledgments**

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References


[16] Li, Z., and Schafer, B. W., “Buckling Analysis of Cold-Formed Steel Members with General Boundary Conditions using CUFSM: Conventional and Constrained Finite Strip Methods,” *Proceedings of the..."


Equilibrium based non-homogeneous anisotropic beam element

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(submitted)
Equilibrium Based Non-homogeneous Anisotropic Beam Element

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Abstract

Complementary energy is used to obtain the stiffness matrix of a non-homogeneous anisotropic elastic beam element and the nodal forces associated with loads distributed over the element. The element flexibility matrix is obtained by integrating the complementary energy density corresponding to six beam equilibrium states. Distributed element loads are represented via corresponding sets of internal forces in equilibrium with the loads. The element can in principle include any variation of cross-section properties and load variation. The ability to represent variable cross-section properties, coupling from anisotropic materials, and distributed element loads are illustrated by numerical examples.

Keywords: Elastic beam element, Beam stiffness matrix, Beam flexibility matrix, Equilibrium modes, Anisotropic beam, Distributed element load.

1. Introduction

Beam elements constitute an essential part of many forms of engineering analysis, e.g. for representing beams and columns of civil engineering and aerospace structures, offshore steel structures, and recently large scale composite wind turbine blades. In each case it is desirable to use beam elements that represent a suitable part of the structure, and thus beam elements that permit varying and fully coupled beam properties as well as distributed loads are of considerable interest. The classical formulation of beam elements, still in extensive use, is based on displacement shape functions, and the corresponding internal forces are then obtained by multiplication with the appropriate stiffness parameters. Apart from simple beams with constant properties along the beam axis this typically leads to complications, e.g. in relation to coupled modes, variable cross-section properties, shear deformation etc.

Additionally, variable stiffness along a beam element will influence the representation of distributed loads in the form of equivalent nodal loads.

Within linear elastic beam theory most of the indicated restrictions for displacement based formulations can be alleviated by changing to a complementary energy formulation in terms of suitable sets of internal forces. Within the general framework of finite elements the use of complementary energy formulations is limited by the need for a suitable set of internal equilibrium force distributions, see e.g. [1]. For two- and three-dimensional elements this complicates the formulation, and mixed energy principles have been developed. However, for beams suitable internal force equilibrium distributions are readily available, and very compact and versatile formulations can be obtained. A direct and simple solution for a beam in plane bending was obtained directly from elementary statics by Livesley [2]. A complementary energy formulation was obtained for the Bernoulli beam element with variable cross section and the similar Timoshenko by Friedman and Kosmatka [3, 4], and a more general formulation combining shear flexibility, element curvature and distributed loads was presented by Krenk [5]. This formulation was developed for arches with variable cross section in [6] and for distributed loads in [7]. The flexibility formulation has also been used in a large deformation context as part of a co-rotational beam formulation, see e.g. [8].

Common to the papers just mentioned is the assumption of symmetry, leading to one or two-dimensional problems. A central point of the present paper is the development of a beam element with varying non-symmetric cross-sections and general coupling of the various deformation modes via anisotropic material properties. This extension requires representation of general cross-section properties in the form of a cross-section stiffness or flexibility matrix. This can be obtained in a simple way by a two-dimensional representation of the classic beam deformation modes including torsion and shear warping in terms of isoparametric elements [9]. A more detailed description, which in principle condenses the three-dimensional behavior into the cross-section plane, is the so-called Variational Asymptotic Sectional Beam Analysis (VABS) proposed by Hodges and co-workers in [10], and described in more detail in [11] with further developments in [12]. An alternative procedure was developed by Giavotto et al. [13, 14], in which a beam with constant cross-section representing a cross-section of the original beam is analyzed by a 2D or 3D eigenvalue technique, where the non-decaying
deformation modes are extracted and associated with bending, shear, extension and torsion. The technique goes by the name of Anisotropic Beam Analysis (ANBA). Further developments of the associated eigenmode technique have been presented in [15] and [16]. Closer associated with the present paper is a related method, in which the cross-section flexibility properties associated with the non-decaying beam modes are extracted directly from an equivalent prismatic beam by imposing a set of six representative displacement modes [17, 18].

The present paper develops a beam element for variable cross-sections with general anisotropy, and derives specific formulas for the representation of internally distributed loads by equivalent nodal forces. The basic notation and definition of the section properties are introduced in section 2. The following theory is divided into two parts. First, the element stiffness matrix is derived from the elementary concept of complementary energy for a beam without distributed loads in section 3. The absence of external distributed loads makes this part quite elementary. In section 4 the theory is extended to include distributed loads via a corresponding distribution of the internal forces. A proper complementary energy functional is introduced, and elimination of the parameters of the homogeneous internal force distributions then identify the equivalent nodal forces of the element. Finally, section 5 gives examples illustrating the effect of anisotropic coupling of the displacement modes and the effect of stiffness variation on the nodal forces, and furthermore combines the effects in the analysis of a realistic wind turbine blade.

2. Definition of General Section Properties

Consider a straight beam, and introduce a coordinate system with the $z$-axis acting as beam axis, while the cross-sections are parallel with the $xy$-plane. The internal force vector $\mathbf{Q}(z) = [Q_x(z), Q_y(z), Q_z(z)]^T$ at a cross-section defined by $z$ is defined in terms of the stresses on the cross-section $[\sigma_{xx}, \sigma_{zy}, \sigma_{zz}]$ as

$$Q_x = \int_A \sigma_{xx} \, dA \quad , \quad Q_y = \int_A \sigma_{zy} \, dA \quad , \quad Q_z = \int_A \sigma_{zz} \, dA$$

(1)

In theories for homogeneous prismatic isotropic beams the combined problems of bending and extension are usually referred to the elastic center, while the torsion and shear problems
are referred to the center of twist. The resulting relations can then be transformed into a common point of reference. In the present formulation the deformation modes can be coupled, and no assumptions about particular characteristic points in the cross-section are introduced. Assuming a common point of reference $A$ with coordinates $(x_A, y_A)$ in the cross-section the moment vector $\mathbf{M}(z) = [M_x(z), M_y(z), M_z(z)]^T$ at a cross-section at $x$ is defined by the two bending moments

$$M_x = \int_A (y - y_A) \sigma_{xx} \, dA, \quad M_y = -\int_A (x - x_A) \sigma_{xx} \, dA$$

and the torsion moment

$$M_z = \int_A [(x - x_A) \sigma_{zy} - (y - y_A) \sigma_{zx}] \, dA$$

The different sign on the bending moment components is due to the vector format of the bending moments. The internal force and moment components are illustrated in Fig. 1b. By these definitions zero moment corresponds to the force acting at the reference point $A$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{a) Generalized strains $\gamma(z)$ and curvatures $\kappa(z)$, b) Internal forces $\mathbf{Q}(z)$ and moments $\mathbf{M}(z)$.}
\end{figure}

It is convenient to collect the six generalized internal force components in the vector $\mathbf{q}(z) = [Q_x(z), Q_y(z), Q_z(z), M_x(z), M_y(z), M_z(z)]^T$. In elastic beam theory the deformations associated with the generalized internal forces $\mathbf{q}(z)$ are described in terms of a generalized strain vector $\mathbf{\gamma}(z) = [\gamma_x(z), \gamma_y(z), \gamma_z(z), \kappa_x(z), \kappa_y(z), \kappa_z(z)]^T$. The components $\gamma_x$ and $\gamma_y$ are generalized shear strains, while $\gamma_z$ is the axial strain of the beam. Similarly, $\kappa_x$ and $\kappa_y$ are the components of bending curvature, while $\kappa_z$ is the rate of twist. The generalized strain and curvature components are defined such that they are conjugate to the internal force and moment components with respect to energy. Thus, the elastic energy per unit length is given as

$$W_s = \frac{1}{2} \mathbf{\gamma}^T \mathbf{q}$$
For linear elastic beams there is a linear relation between the generalized internal forces contained in the vector $\mathbf{q}$ and the conjugate generalized strain-curvature vector $\mathbf{\gamma}$. This relation can be written either in stiffness format as

$$\mathbf{q} = \mathbf{D} \mathbf{\gamma}$$  \hspace{1cm} (5)

or in the inverse flexibility, or compliance, format

$$\mathbf{\gamma} = \mathbf{C} \mathbf{q}$$  \hspace{1cm} (6)

In these relations $\mathbf{D}$ is the cross-section stiffness matrix, and $\mathbf{C} = \mathbf{D}^{-1}$ is the cross-section flexibility matrix. In general they are both six by six symmetric non-negative definite matrices and may depend on the axial coordinate $z$. Thus, a general formulation, permitting e.g. anisotropic and inhomogeneous materials, may require up to 21 stiffness of flexibility parameters to describe the deformation properties of a thin slice of the beam.

In classic beam theory without shear flexibility, the shear strain components $\gamma_x$ and $\gamma_y$ vanish. This implies that the second and third row and column of the cross-section flexibility matrix $\mathbf{C}$ vanish, and the cross-section stiffness matrix $\mathbf{D}$ becomes singular. In this case a reduced format for the cross section properties can be used. However, in the present formulation the cross-section flexibility matrix $\mathbf{C}$ is used as basis, and no modification is necessary.

The energy per unit length of the beam can be expressed either in terms of the cross-section stiffness matrix $\mathbf{D}$ or the cross-section flexibility matrix $\mathbf{C}$ as

$$W_s = \frac{1}{2} \mathbf{\gamma}^T \mathbf{D} \mathbf{\gamma} = \frac{1}{2} \mathbf{q}^T \mathbf{C} \mathbf{q}$$  \hspace{1cm} (7)

The stiffness matrix of a beam element can be developed from either of these forms. In the first case the distribution of the strains and curvatures $\mathbf{\gamma}(z)$ along the beam is required, while in the second case it is the distribution of the internal forces and moments $\mathbf{q}(z)$ that is required. While the distribution of strain and curvature along the element depend on the stiffness properties of the element, the distribution of the internal forces and moments is determined directly by statics. This leads to a simple and general procedure for the stiffness matrix of fairly general beam elements based on flexibility, [5]. The following derivation is based on the flexibility approach. First, the stiffness matrix of the element is derived, and then the appropriate formulae for representing distributed loads are obtained.
3. Beam Element Stiffness Matrix

The stiffness matrix of a beam element accommodating linear bending and constant extension and torsion has six degrees of freedom at each of its two end nodes as shown in Fig. 2a. The nodal displacements are conveniently organized in a 12 component generalized displacement vector \( \mathbf{u}^T = [\mathbf{u}_1^T, \varphi_1^T, \mathbf{u}_2^T, \varphi_2^T] \), where \( \mathbf{u}_j = [u_x, u_y, u_z]^T \) and \( \varphi_j = [\varphi_x, \varphi_y, \varphi_z]^T \) denote the displacement and rotation, respectively, at node \( j = 1, 2 \). The corresponding generalized nodal forces are shown in Fig. 2b and collected in the 12 component generalized force vector \( \mathbf{f}^T = [\mathbf{f}_1^T, \mathbf{m}_1^T, \mathbf{f}_2^T, \mathbf{m}_2^T] \) illustrated in Fig. 2b, with nodal forces and moments represented by \( \mathbf{f}_j = [f_x, f_y, f_z]^T \) and \( \mathbf{m}_j = [m_x, m_y, m_z]^T \), respectively.

![Figure 2: a) Element displacements \( \mathbf{u}_j \) and rotations \( \varphi_j \), b) Element forces \( \mathbf{f}_j \) and moments \( \mathbf{m}_j \).](image)

3.1. Local deformation modes

The beam element shown in Fig. 2 has 12 degrees of freedom. Of these 6 describe rigid body displacement, while the remaining 6 describe deformation modes of the element. Only the deformation modes contribute to the stiffness of the element, and thus the stiffness matrix of the element can be obtained from the stiffness of these 6 modes, when combined with suitable variable transformations. It is convenient to define the local deformation modes such that they each correspond to a simple set of end loads in equilibrium. There are three modes corresponding to a constant internal force \( \mathbf{Q}_0 \) in the element, and three modes corresponding to a constant moment \( \mathbf{M}_0 \) in the element. It is noted that constant shear forces \( Q_{0x} \) and \( Q_{0y} \) lead to linear anti-symmetric moment variation. Thus, \( \mathbf{Q}_0 \) is the constant internal force in the beam, while \( \mathbf{M}_0 \) is the internal moment in the mid-section of the beam.
Let the axial position in the beam element be described by a normalized coordinate $\xi$ on the interval $[-1, 1]$ and let the length of the element be $2a$. The distribution of internal forces is given by the relation

$$q(\xi) = T(\xi) q_0$$

in terms of the value of the generalized internal force at the center of the beam $q_0$, and the distribution matrix

$$T(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a\xi & 0 & 1 & 0 & 0 \\ -a\xi & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(9)

It is seen that the distribution matrix is linear in the normalized length coordinate, and that the value of the matrix at the two beam nodes $T(\pm 1)$ correspond to a unit matrix supplemented by two off-diagonal terms $\pm a$.

3.2. Flexibility matrix of the equilibrium modes

The flexibility matrix of the equilibrium modes of the element follows from integration of the cross-section flexibility relation (7b) over the element length,

$$W_e = \int_0^l W_s(s) \, ds = a \int_{-1}^1 \frac{1}{2} q(\xi)^T C(\xi) q(\xi) \, d\xi$$

(10)

The generalized internal forces $q(\xi)$ are represented via the mid-point values $q$ by (8). The energy of the beam element then takes the form

$$W_e = \frac{1}{2} q_0^T H q_0$$

(11)

where the element flexibility matrix $H$ corresponding to the six equilibrium deformation modes is given by the integral

$$H = a \int_{-1}^1 T(\xi)^T C(\xi) T(\xi) \, d\xi$$

(12)

For a beam of variable cross-section, e.g. a tapered beam or a beam with initial twist, the section flexibility matrix $C$ is a function of the axial coordinate $\xi$. It will then be most
convenient to evaluate the integral in (12) numerically, either by Gauss quadrature or a combination of mid- and end-point values.

For beam elements with constant section flexibility matrix $C$ it follows from symmetry that the $\xi$-terms in $T(\xi)$ only contribute to the integral via the quadratic terms. There are very few of these, and it is then convenient to carry out the integration in explicit form. The result is the element equilibrium mode flexibility matrix

$$H = 2a \begin{bmatrix}
C_{11} + \frac{1}{3}a^2C_{55} & C_{12} - \frac{1}{3}a^2C_{54} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} - \frac{1}{3}a^2C_{45} & C_{22} + \frac{1}{3}a^2C_{44} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix} \quad (13)$$

It is seen, how the bending parameters $C_{44}, C_{45}, C_{54}, C_{55}$ of the cross-section enter the constant shear modes of the beam element due to their linearly varying bending moment. In the case of beams without shear deformations the first and the second rows and columns in the section flexibility matrix $C$ vanish, and the bending flexibility terms constitute the only contribution to these rows and columns in the element flexibility matrix. However, the presence of these terms ensures that the element equilibrium mode flexibility matrix $H$ can be inverted to give the equilibrium mode stiffness matrix $H^{-1}$. A similar effect occurs in the general expression (12), but is less directly visible.

The equilibrium mode flexibility and stiffness matrices define a set of generalized element strains $\gamma_0$ by flexibility and stiffness relations similar to (5) and (6) for the cross-section,

$$\gamma_0 = H q_0, \quad q_0 = H^{-1} \gamma_0 \quad (14)$$

In contrast to the cross-section relations both the equilibrium mode relations are non-singular also for beams without shear flexibility.

3.3. From mode flexibility to element stiffness

The relation (8) gives the internal forces in terms of the mid-point values $q_0$, defining the equilibrium modes. The nodal forces are the internal forces at the nodes with appropriately
chosen signs. Therefore the nodal force vector \( f^T = [f_1^T, m_1^T, f_2^T, m_2^T] \) is expressed in terms of the mid-point value of the internal force vector by a relation of the form

\[
f = G q_0
\]

where the 12 by 6 matrix \( G \) is expressed by the end point values of the internal force distribution matrix as

\[
G = \begin{bmatrix}
-T(-1) \\
T(1)
\end{bmatrix}
\]

with \( T(\pm 1) \) given by (9).

The relation between the generalized nodal forces \( f \) and the corresponding generalized nodal displacements \( u^T = [u_1^T, \varphi_1^T, u_2^T, \varphi_2^T] \) follows from use of the principle of virtual work. The virtual work can be expressed both in the 12 component element format and in the 6 component deformation mode format.

\[
\delta V = \delta u^T f = \delta \gamma_0^T q_0
\]

where \( \gamma_0 \) denotes the the generalized strains corresponding to the mid-point generalized force vector \( q_0 \). When the generalized force components in the 12 component format are represented in terms of their 6 component counterpart by the transformation (16), the following representation of the 6 component deformation measures is obtained,

\[
\gamma_0 = G^T u
\]

The transformation matrix in the present kinematic strain-displacement relation is the transpose of the transformation matrix in the static external-internal force relation (15).

The stiffness matrix in the 12 component format follows from expressing the energy of the element - first in the 6 component deformation mode format, and then in the 12 component displacement format by use of the transformation (18). In the 6 component format the energy is

\[
W = \frac{1}{2} \gamma_0^T H^{-1} \gamma_0
\]

where \( H^{-1} \) is the stiffness matrix of the equilibrium mode representation. Substitution of the generalized equilibrium mode strain vector by use of (18) gives

\[
W = \frac{1}{2} u^T K u
\]
where the element stiffness matrix $K$ is

$$K = G H^{-1} G^T$$

(21)

This transformation generates the 12 by 12 element stiffness matrix $K$ from the inverse of the 6 by 6 equilibrium mode flexibility matrix $H$. As already mentioned, the additional terms in the upper left 2 by 2 block of the flexibility matrix $H$ in (13) ensures the existence of the inverse $H^{-1}$, also in the case of vanishing shear flexibility.

4. Representation of distributed load

While the previous section concentrated on the beam element stiffness matrix it is often of interest to provide a correct representation of distributed loads. The detailed distribution of the equivalent nodal loads, used to represent the distributed load, depends on the properties of the element. As the load distribution depends on the element properties along the beam, the equivalent loads constitute a natural part of the theoretical basis of the complementary energy based element.

4.1. Basic theory

The first step in the formulation of an equilibrium element with distributed load is to identify a set of generalized internal forces that are in equilibrium with the distributed load. These are collected in the six-component vector $\tilde{q}(z) = [\tilde{Q}_x(z), \tilde{Q}_y(z), \tilde{Q}_z(z), \tilde{M}_x(z), \tilde{M}_y(z), \tilde{M}_z(z)]^T$, and the full distribution of the internal forces then has the form

$$q(\xi) = \tilde{q}(\xi) + T(\xi) q_0$$

(22)

It is noted that the selection of the set of internal forces $\tilde{q}(\xi)$ that keep equilibrium with the external load is not unique, as they may contain any linear combination of the generalized internal force distributions from the homogeneous solutions. This gives a considerable freedom in the specific choice of the distributions in $\tilde{q}(\xi)$, and it is convenient to choose these as corresponding to simple boundary conditions. The generalized section forces contained in the vector $q_0$ in (22) correspond to the component values of the additional homogeneous part at the center of the element.
The specific complementary elastic energy corresponding to a unit length of the beam is now given as

\[ W_s = \frac{1}{2} q^T C q = \frac{1}{2} q^T C q + \bar{q}^T C \bar{T} q_0 + \frac{1}{2} q_0^T T^T C T q_0 \]  

(23)

This corresponds to the total complementary energy

\[ W_c = \int_0^l W_s(s) \, ds - u^T f \]  

(24)

The vector \( f \) contains the generalized reaction forces at the ends of the beam, here given as

\[ f = g + G q_0 \]  

(25)

with the 12-component vector \( g \) containing the end point values corresponding to the section forces \( \bar{q}(\xi) \),

\[ g = \begin{bmatrix} -\bar{q}(-1) \\ \bar{q}(1) \end{bmatrix} \]  

(26)

and the 12 by 6 matrix \( G \) defined in (16).

When carrying out the integration, the complementary energy (24) can be expressed in the form

\[ W_c = \frac{1}{2} q_0^T H q_0 + h^T q_0 + h - u^T (G q_0 + g) \]  

(27)

where the matrix \( H \) is given by (12), the vector \( h \) is defined by

\[ h = a \int_{-1}^1 T(\xi)^T C(\xi) \bar{q}(\xi) \, d\xi \]  

(28)

and the scalar \( h \) is without importance for the element properties.

The static variables \( q_0 \), defining the homogeneous part of the internal forces (22), are determined from the stationarity condition

\[ \frac{\partial W_c}{\partial q_0} = H q_0 + h - G^T u = 0 \]  

(29)

whereby

\[ q_0 = H^{-1}(G^T u - h) \]  

(30)

Upon substitution of this value, the complementary energy (27) takes the form

\[ W_c = -\frac{1}{2} u^T K u + u^T r + \text{const} \]  

(31)
where $\mathbf{K}$ is the element stiffness matrix (21), while the equivalent nodal forces on the element are given by the vector

$$
\mathbf{r} = \mathbf{G} \mathbf{H}^{-1} \mathbf{h} - \mathbf{g}
$$

(32)

In this expression the second term is the nodal forces corresponding to the non-homogeneous part of the internal force distribution $\tilde{\mathbf{q}}$, given explicitly by (26), while the first term represents the contribution from the homogeneous part of the internal force distribution activated to satisfy the kinematic boundary conditions of the element.

### 4.2. Simple internal loads

The equivalent nodal forces $\mathbf{r}$ are determined by (32) via the corresponding distribution of the internal forces $\tilde{\mathbf{q}}(\xi)$, in part by using the end point values to define the vector $\mathbf{g}$, given by (26), and in part by using the internal force distribution to evaluate the integral $\mathbf{h}$, given by (28). As already mentioned, the internal force distributions $\tilde{\mathbf{q}}(\xi)$ must represent equilibrium with the distributed load on the element, but this leaves considerable freedom in the construction of these distributions corresponding to different support conditions at the nodes. The two obvious candidates are a simply supported beam, in which the load components are distributed to the supports in the same way as the transverse force components, and a cantilever beam with all reaction forces at one of the nodes, while the other is free. In the following, the non-homogeneous internal force distribution is obtained for a cantilever beam supported at node 1, and it is presented in terms of the non-dimensional length coordinate $\xi$ also used in the internal distribution matrix $\mathbf{T}(\xi)$.

The first load case consists of a concentrated external force $\mathbf{P}_* = [P^*_x, P^*_y, P^*_z]^T$ and a concentrated moment $\mathbf{M}_* = [M^*_x, M^*_y, M^*_z]^T$ acting at the internal point defined by the non-dimensional coordinate $1 < \xi_* < 1$ as illustrated in Fig. 3a, where the load components are conveniently collected in the six-dimensional generalized load vector $\mathbf{p}_*^T = [\mathbf{P}_*, \mathbf{M}_*]^T$. The internal force distributions contain a discontinuity at $\xi = \xi_*$, suggesting the use of equilibrium distributions corresponding to a beam with a free right end and a constrained left end. Each load component in $\mathbf{p}_*$ directly generates a corresponding constant internal force component in $\tilde{\mathbf{q}}(\xi)$ to the left of the point of application of the load, i.e. for $\xi < \xi_*$. It
follows from elementary statics that the internal force distributions are

\[ \tilde{q}_s(\xi) = p_\ast - (\xi_s - \xi) a \left[ 0, 0, 0, P_x^0, -P_y^0, 0 \right]^T, \text{ for } \xi \leq \xi_s \]  

while \( \tilde{q}_s(\xi) \equiv 0 \) for \( \xi > \xi_s \). The constant internal force represented by the first term is shown in Fig. 3b, while the linear moment distribution represented by the second term is illustrated in Fig. 3c. These internal force distributions are used to generate the vector \( h \) via the integral (28). The corresponding direct reaction forces at the two element nodes then follow from (26) in the form

\[ g_s^T = -\left[ p_s^T, 0_{1 \times 6} \right] + (1 + \xi_s) a \left[ 0, 0, 0, P_x^0, -P_y^0, 0 \right]_{1 \times 6} \]  

It is seen, how the concentrated load components appear directly with opposite sign in the reactions at node 1, and that the transverse forces generate moment reactions as well.

![Figure 3](image)

**Figure 3:** a) Element with concentrated load, b) Direct constant internal forces, c) Linear moments from transverse forces.

The second load case consists of a uniformly distributed load. The force and moment intensities are contained in the generalized load vector \( p_0 = [p_x^0, p_y^0, p_z^0, m_x^0, m_y^0, m_z^0]^T \). Also in this case the equilibrium distribution of the generalized internal forces \( \tilde{q}(\xi) \) is determined with reference to a cantilever beam, constrained at node 1. The internal forces can be determined directly from elementary statics, but in the present case it is more direct to consider the solution (33) for a concentrated load \( p_0 d \xi_s \), and to integrate the solution over the interval \( \xi < \xi_s < 1 \). This gives the load distribution

\[ \tilde{q}_0(\xi) = (1 - \xi) a p_0 - \frac{1}{2}(1 - \xi)^2 a^2 \left[ 0, 0, 0, P_y^0, -P_x^0, 0 \right]^T \]  

13
The common factor \((1 - \xi)\) eliminates contributions at the right end, and thus the reaction vector (26) takes the simple form

\[
\mathbf{g}_0^T = -2a \left[ \mathbf{p}_0^T ; \mathbf{0}_{1\times 6} \right] + 2a^2 \left[ 0, 0, 0, p_0^y, -p_0^y, 0 ; \mathbf{0}_{1\times 6} \right]
\]  

(36)

The reaction moments in the last term are recognized as the total force, e.g. \(2ap_0^y\), acting at the distance \(a\) from the support.

In order to represent loads with variation along the beam element it is convenient to have the basic forms of triangular variation. A triangular load distribution with intensity \(\mathbf{p}_1 = [p_1^x, p_1^y, m_1^x, m_1^y, m_1^z]^T\) at node 1 tapering linearly towards zero at node 2, can be represented as a distribution of individual loads of the form \(\frac{1}{l}(1 - \xi_*)\mathbf{p}_1 d\xi_*\). Integration of the concentrated load solution (33) with this intensity over the interval \(\xi < \xi_* < 1\) gives the internal force distributions

\[
\tilde{\mathbf{q}}_1(\xi) = \frac{1}{4}(1 - \xi)^2 a \mathbf{p}_1 - \frac{1}{12}(1 - \xi)^3 a^2 \left[ 0, 0, 0, p_0^y, -p_0^y, 0 \right]^T
\]  

(37)

The corresponding reaction vector follows from (26) as

\[
\mathbf{g}_1^T = -a \left[ \mathbf{p}_1^T ; \mathbf{0}_{1\times 6} \right] + 2a^2 \left[ 0, 0, 0, p_0^y, -p_0^y, 0 ; \mathbf{0}_{1\times 6} \right]
\]  

(38)

Again, the second term contains easily identifiable moment contributions from the transverse loads. The internal force distributions and the reactions from a triangular load distribution growing linearly from node 1 towards node two follows immediately by a linear combination, in which the present linear load distribution is subtracted from a constant load distribution.

4.3. Power function load distribution

The procedure used above permits a simple generalization to polynomial load distributions. The key result is the solution for a load vector distributed as a power of the relative distance from one of the ends, here taken as the right end at node 2. For the element length \(l = 2a\) the relative distance takes the form \((l-s)/l = (1 - \xi)/2\). Thus, a load distribution of power \(n\) over a beam element can be expressed in the form

\[
\tilde{\mathbf{P}}_n(\xi) = \mathbf{p}_n \left( \frac{l-s}{l} \right)^n = \mathbf{p}_n \left( \frac{1 - \xi}{2} \right)^n
\]  

(39)
where the vector $\mathbf{p}_n = [p^n_x, p^n_y, p^n_z, m^n_x, m^n_y, m^n_z]^T$ contains the values at node 1 of the load components with power $n$ variation. The load intensity $\mathbf{p}_n(\xi^*)$ is substituted into the generalized internal force distribution (33) for a concentrated load, and integration over the interval $\xi < \xi^* < 1$ then gives

$$\tilde{\mathbf{q}}_n(\xi) = \frac{2}{n+1} \left(\frac{1-\xi}{2}\right)^{n+1} a \mathbf{p}_n - \frac{2}{n+1} \frac{2}{n+2} \left(\frac{1-\xi}{2}\right)^{n+2} a^2 \left[0, 0, 0, p^n_y, -p^n_x, 0\right]^T \quad (40)$$

The corresponding reaction vector follows from (26) in the form

$$\mathbf{g}_n^T = -\frac{2}{n+1} a \left[\mathbf{p}_n^T; \mathbf{0}_{1 \times 6}\right] + \frac{2}{n+1} \frac{2}{n+2} a^2 \left[0, 0, 0, p^n_y, -p^n_x, 0; \mathbf{0}_{1 \times 6}\right] \quad (41)$$

It is observed that the present notation is fully compatible with that of the previous simple solutions for $n = 0, 1$, and that these appear as special cases. Clearly, the present general solution for power function distribution of the load densities permits representation of arbitrary polynomial load distributions by simple linear combination.

5. Examples

The equilibrium formulation and its capacity to include anisotropy and distributed loads for non-homogeneous beam elements are illustrated by three examples: a homogeneous box beam with bend-twist coupling introduced via composite wall properties, a linearly tapered beam with solid circular cross-section, and finally the analysis of a realistic modern wind turbine blade.

5.1. Composite box beam with end load

This example concerns the analysis of a composite box beam, shown in Fig. 4, that exhibits bend-twist coupling via the use of fibers forming an angle with the beam axis. The particular beam properties were introduced by Stample and Lee [21] and investigated experimentally by Chandra et al. [19] for three different fiber orientations. The beam has a length of $l = 762$ mm (30 in.), a width of $w = 24.2$ mm (0.953 in.), a height of $h = 13.6$ mm (0.537 in.) and a uniform wall thickness of $t = 0.76$ mm (0.030 in.). The walls of the section are made up of six laminas with $E_i = 142.0$ GPa (20.59E+06 psi), $E_j = E_k = 9.79$ GPa (1.42E+06 psi), $G_{ij} = G_{ik} = 6.00$ GPa (8.7E+05 psi), $G_{jk} = 4.80$ GPa (6.96E+05 psi) and
\[ \nu_{ij} = \nu_{ik} = 0.42, \nu_{jk} = 0.02 \]  where \( i \) denotes the fiber direction, \( j \) the transverse direction, and \( k \) the direction normal to the plane of the lamina. The fiber orientation angle \( \alpha \) is defined as the angle between the longitudinal axis \( z \) and the fiber direction. The layup sequence for the top and bottom walls are \((−\alpha)_{6}\) and \((\alpha)_{6}\), respectively, and for left and right walls are \((−\alpha, \alpha)_{3}\) and \((\alpha, −\alpha)_{3}\), respectively. The layup sequence is defined from the innermost to the outermost layers. The \(−\alpha \) fiber angle of the top wall can be seen in Fig. 4.

The cross-section stiffness parameters for the three different fiber orientations \( \alpha = 15^\circ \), \( 30^\circ \), and \( 45^\circ \) were obtained by the computer code CrossFlex based on the representation of

<table>
<thead>
<tr>
<th>Units</th>
<th>( \alpha = 15^\circ )</th>
<th>( \alpha = 30^\circ )</th>
<th>( \alpha = 45^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( GA_1 )</td>
<td>3.94E+05</td>
<td>5.37E+05</td>
<td>4.12E+05</td>
</tr>
<tr>
<td>( GA_2 )</td>
<td>1.76E+05</td>
<td>3.02E+05</td>
<td>3.08E+05</td>
</tr>
<tr>
<td>( EA )</td>
<td>6.11E+06</td>
<td>2.80E+06</td>
<td>1.14E+06</td>
</tr>
<tr>
<td>( EI_1 )</td>
<td>1.75E+02</td>
<td>8.20E+01</td>
<td>3.53E+01</td>
</tr>
<tr>
<td>( EI_2 )</td>
<td>4.10E+02</td>
<td>1.83E+02</td>
<td>8.09E+01</td>
</tr>
<tr>
<td>( GJ )</td>
<td>4.98E+01</td>
<td>7.53E+01</td>
<td>6.18E+01</td>
</tr>
<tr>
<td>( \gamma_{12} )</td>
<td>1.42E−03</td>
<td>1.98E−03</td>
<td>1.48E−03</td>
</tr>
<tr>
<td>( \gamma_{13} )</td>
<td>−5.28E−01</td>
<td>−5.61E−01</td>
<td>−4.19E−01</td>
</tr>
<tr>
<td>( \gamma_{23} )</td>
<td>−6.62E−04</td>
<td>−7.94E−04</td>
<td>−3.21E−05</td>
</tr>
<tr>
<td>( \gamma_{45} )</td>
<td>−4.02E−03</td>
<td>−4.19E−03</td>
<td>−2.07E−03</td>
</tr>
<tr>
<td>( \gamma_{46} )</td>
<td>5.55E−01</td>
<td>6.14E−01</td>
<td>4.62E−01</td>
</tr>
<tr>
<td>( \gamma_{56} )</td>
<td>−7.15E−03</td>
<td>−7.22E−03</td>
<td>−5.05E−03</td>
</tr>
</tbody>
</table>
the cross-section as a slice with the thickness of a single element with cubic displacement interpolation, [18]. Each of the six lamina were represented by a layer of elements and 50 segments were used in the circumferential direction for a total of 300 solid elements. A modified cross-section analysis in terms of layered elements would lead to a substantially reduced model size, but this aspect is outside the scope of the present paper. The result of the cross-section analysis is the full six by six symmetric cross-section stiffness matrix. The parameters in the stiffness matrix are listed in Table 1, in which the off-diagonal elements are given in terms of the non-dimensional coefficients

$$\gamma_{ij} = \frac{D_{ij}}{\sqrt{D_{ii} D_{jj}}}.$$  (42)

representing coupling in the normalized form $-1 < \gamma_{ij} < 1$, with $\pm 1$ representing the

![Figure 5: Twist at mid span of box beam under tip torque of 0.113 Nm [1 lb in.].](image)

![Figure 6: Bending slope at mid span of box beam under tip torque of 0.113 Nm [1 lb in.].](image)
maximum possible coupling. All three configurations exhibit a large bend-twist coupling \( \gamma_{13} \) and extension-shear coupling \( \gamma_{46} \), where the beam with \( \alpha = 30^\circ \) exhibits the largest coupling.

The static behaviour of the composite box beam when clamped at one end and loaded with a transverse force at the free end has been investigated experimentally by Chandra et al. [19] for the three different fiber orientations. The measured twist and the bending slope at the middle of the beam due to a tip torque are shown in Fig. 5 and Fig. 6, respectively, together with results obtained from the present model using a single beam element with the cross-section stiffness parameters given in Table 1. These results are compared with results obtained by Smith and Chopra [20] using an analytical model, with results from a finite element approach developed by Stample and Lee [21], as well as results obtained by Ghiringhelli [22] using a finite element beam model and a 3D finite element model.

For all three fiber configurations the results obtained using the present beam model agree well with the 3D finite element model, the finite element beam model developed by Ghiringhelli, as well as with the beam model developed by Stample and Lee. Good agreement is also obtained with the experimental results with the exception of the bending slope for the beam with \( \alpha = 30^\circ \). Ghiringhelli [22] offers an explanation for this discrepancy by pointing out that the experimental result at \( \alpha = 30^\circ \) deviates from the regular curve found by evaluating the bending slope at every 5\(^\circ\) fiber angle.

5.2. Composite box beam with distributed load

Consider the cantilever box beam with \( \alpha = 15^\circ \) subjected to a uniformly distributed torque with intensity \( m_z = 1/l \text{N} \). A distributed torque may arise from pitching moments on airfoils and from distributed loads with an offset from the shear center of the beam. The element nodal load vector for such a prismatic beam with bend-twist coupling subject to a distributed torque can be obtained from (32) as

\[
\mathbf{r} = m_z \left[ -\psi_1, \psi_2, 0, -a\psi_2, a, \psi_1, -\psi_2, 0, -a\psi_2, -a\psi_1, a \right]^T,
\]

where the contributions from the coupling parameters are represented via the terms \( \psi_1 \) and \( \psi_2 \),

\[
\psi_1 = \frac{\psi a^2 C_{56}}{a^2 C_{55} + 3 C_{11}}, \quad \psi_2 = \frac{\psi a^2 C_{46}}{a^2 C_{44} + 3 C_{22}}.
\]
The non-dimensional parameter $\psi$ is used to represent the effect of neglecting the contribution from bend-twist coupling in the nodal load vector where setting $\psi = 1$ gives the consistent load vector whereas $\psi = 0$ gives the reduced load vector which neglects fixed-end moments.

The internal torque distribution recovered using the present approach with $\psi = 1$ is shown in Fig. 7. Results are compared with the exact linear distribution as well as the stress calculated by conventional finite element analysis directly from nodal displacements using $\psi = 0$. It can be seen that the present beam modelling approach recovers the exact internal torque distribution with the use of a single element without the need for ad hoc assumptions or averaging. The well known staircase torque distribution obtained from conventional finite element analysis approaches the exact distribution as the number of elements are increased. For the latter elements, it can be seen that the internal forces in this example are accurate at the element centers.

The tip displacements of the beam from the distributed torque using the consistent and the reduced load vectors are presented in Table 2. It can be seen that the tip displacements $u_x$ and $u_y$ calculated using the reduced load vector converge to the correct values, obtained using a single element with the consistent nodal loads. Moreover, the models with one and six elements using the reduced load vector under-predict the in-plane displacements by 25% and 0.7%, respectively. The tip rotation in this example is identical between all models as it is unaffected by the type of load vector used. Omission of the coupling terms with
coefficients $\psi_1$ and $\psi_2$ in (43) does not prevent convergence towards the correct results as the mesh is refined since these coefficients tend to zero more rapidly than the torque terms.

The symbolic solution for the in-plane displacement and rotation at the tip of a prismatic beam with bend-twist coupling subject to a distributed torque modelled using a single beam element can be obtained by using the stiffness matrix (21) and the load vector in (43),

\[
\begin{align*}
    u_x &= \frac{2}{3} m_z a^3 C_{56} (3 + \psi) , \\
    u_y &= -\frac{2}{3} m_z a^3 C_{46} (3 + \psi) , \\
    \phi_z &= 2 m_z a^2 C_{66} .
\end{align*}
\] (45)

It is seen that the tip rotation $\phi_z$ is independent of the non-dimensional parameter $\psi$ and hence of the type of load vector used, as was observed from the numerical results listed in Table 2. It can also be seen that using the reduced nodal load $\psi = 0$ yields 25% smaller in-plane deflections independent of the geometrical and material properties of the beam.

5.3. Tapered beam with circular solid cross-section

Consider a cantilever beam with solid circular cross-section shown in Fig. 8. The cross-section tapers from radius $r_1$ to $r_2$ over the length $l = 2a$. Let the ratio between the end radius be defined by $\beta = r_2/r_1$. The area has a quadratic variation and the moment of inertia has a quartic variation. The analytical solution for the tip deflection $u_x,y(l)$ from a uniformly distributed transverse load $p_{x,y}$ can be obtained using the principle of virtual work with $\beta < 1$,

\[
\begin{align*}
    u_{x,y}(l) &= \frac{p_{x,y} l^2}{\pi} \left[ \frac{\beta^2 (6 \ln(1/\beta) + 2\beta^3 - 9\beta^2 + 18 \beta - 11)}{3E r_1^4 (1 - \beta)^4} + \frac{\beta - 1 + \ln(1/\beta)}{kG r_1^2 (1 - \beta)^2} \right] ,
\end{align*}
\] (46)

where $E$, $G$, and $k$ are the modulus of elasticity, the shear modulus, and the shear correction factor, respectively. It is seen that the tip displacement consists of two additive contributions, a contribution from bending flexibility, and a contribution from shear deformation.
The consistent nodal loads for a tapered circular beam element without including the effect of shear flexibility are obtained by integrating (28),

\[
\mathbf{r} = \begin{bmatrix} p_x f_1, p_y f_1, 0, -p_y m_1, p_x m_1, 0, p_x f_2, p_y f_2, 0, p_y m_2, -p_x m_2, 0 \end{bmatrix}^T. \tag{47}
\]

Note, that consistent nodal loads from transverse loading from both directions \( p_x \) and \( p_y \) are determined identically apart from changes of sign of the end moments. The parameters \( f_1 \) and \( f_2 \) defining the distribution of the transverse shear forces between the two end nodes are determined as

\[
f_1 = 2a \frac{1 - \beta + \beta \ln(\beta)}{(1 - \beta)^2}, \quad f_2 = 2a \beta \frac{-1 + \beta - \ln(\beta)}{(1 - \beta)^2}. \tag{48}
\]

The parameters \( m_1 \) and \( m_2 \) defining the corresponding fixed-end moments are expressed as

\[
m_1 = 2a^2 \frac{1 + \beta - 2 \beta^2 + \beta (\beta + 2) \ln(\beta)}{(\beta^2 + \beta + 1) (1 - \beta)^2}, \quad m_2 = 2a^2 \beta^2 \frac{-2 + \beta + \beta^2 - (2 \beta + 1) \ln(\beta)}{(\beta^2 + \beta + 1) (1 - \beta)^2}. \tag{49}
\]

The values of the load distribution parameters in (48) and (49) are shown in Fig. 9 as function of the taper ratio \( \beta \). It is seen that for a cylindrical beam, \( \beta = 1 \), the shear force and end moment parameters are equal at both nodes with \( f_1 = f_2 = a \) and \( m_1 = m_2 = a^2/3 \). As the beam approaches a perfect cone with \( \beta = 0 \), all the equivalent nodal loads are shifted to the node with the larger radius with \( f_1 = 2a, f_2 = 0 \) and \( m_1 = 2a^2, m_2 = 0 \). The graph also illustrates that the redistribution becomes more pronounced as the tapering is increased, i.e. for a smaller \( \beta \).
Consider a specific tapered beam with solid circular cross-section with length $l = 64\,\text{m}$, root diameter $r_1 = 1.2\,\text{m}$ and tip diameter $r_2 = 0.12\,\text{m}$, corresponding to $\beta = 0.1$. The beam is made of an orthotropic material with axial modulus of elasticity $E = 10.0\,\text{GPa}$ and shear modulus $G = 2.0\,\text{GPa}$. The cross-section dimensions and elasticity parameters correspond roughly to the flapwise bending of a wind turbine rotor blade with a bending stiffness ratio of $\beta^4 = 10^{-4}$. The beam is loaded by a uniformly distributed transverse load $p_y = 1.0\,\text{N/m}$.

The tip deflection of the beam when loaded by a distributed load with constant intensity follows from either the full element formulation with stiffness matrix $K$ from (21) and nodal forces from (32), or the analytical expression (46). The result for a solid cross-section including the effect of shear flexibility with shear correction factor $k = 0.85$ is presented in the first line of Table 3. Any subdivision of the beam using the nodal loads for the conical beam element as given by (32) will recover this result. This result is compared with the deflection calculated by use of the equilibrium element, with the equivalent nodal loads (47), in which the shear flexibility effect has been neglected. It is seen that while a single element is sufficient to obtain the correct tip deflection if fully integrated, four elements are needed for reducing the error to $1.2\%$ when the load is distributed according to (47) with load distribution coefficients (48) and (49) that do not account for the shear flexibility. Thus, in principle it is necessary for obtaining the full accuracy of the equilibrium element in connection with distributed loads to evaluate the equivalent nodal loads by detailed integration. However, in practice it may be convenient to use a small number of elements.
Table 3: Tapered beam tip deflection from a uniformly distributed load $p_y = 1.0 \text{ N/m}$.

<table>
<thead>
<tr>
<th>Element</th>
<th>Load Vector</th>
<th>$N_{\text{elem}}$</th>
<th>$u_y$ [m]</th>
<th>% diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tapered</td>
<td>Yes</td>
<td>1</td>
<td>5.94E−04</td>
<td>0.0</td>
</tr>
<tr>
<td>Tapered</td>
<td>No</td>
<td>1</td>
<td>4.96E−04</td>
<td>16.4</td>
</tr>
<tr>
<td>Tapered</td>
<td>No</td>
<td>2</td>
<td>5.61E−04</td>
<td>5.5</td>
</tr>
<tr>
<td>Tapered</td>
<td>No</td>
<td>4</td>
<td>5.86E−04</td>
<td>1.2</td>
</tr>
<tr>
<td>Straight</td>
<td>Yes</td>
<td>1</td>
<td>14.08E−04</td>
<td>137.3</td>
</tr>
<tr>
<td>Straight</td>
<td>Yes</td>
<td>2</td>
<td>10.57E−04</td>
<td>78.1</td>
</tr>
<tr>
<td>Straight</td>
<td>Yes</td>
<td>4</td>
<td>7.65E−04</td>
<td>28.9</td>
</tr>
<tr>
<td>Straight</td>
<td>Yes</td>
<td>16</td>
<td>6.04E−04</td>
<td>1.8</td>
</tr>
</tbody>
</table>

To obtain a simple representation of the resulting displacements and section-forces directly at the nodes.

The lower part of Table 3 shows the effect of representing the beam in terms of cylindrical elements with radius determined as the radius at the center cross-section. This is a method often used in practice. It is seen that for this kind of approximation 16 elements are needed to reduce the tip deflection error to 1.8%, demonstrating the considerable gain in accuracy obtained by use of equilibrium based elements.

The internal shear and moment distributions recovered using the present approach are shown in Fig. 10. The results are compared with the exact linear shear distribution and the...
quadratic moment distribution as well as the stress calculated directly from nodal displacements of the cylindrical elements. As expected from the equilibrium basis of the present method the internal force is recovered exactly, even when using only a single element. The piecewise constant shear force distribution obtained from conventional finite element analysis approaches the exact distribution as the number of elements is increased. The displacement based approach with equivalent nodal loads determined by static equivalence can not capture the quadratic distribution of the bending moment within a single element, and four elements are required to have less than one percent error in the maximum bending moment.

5.4. Wind Turbine Blade

This third and final example concerns the analysis of a 75 m long wind turbine blade currently manufactured by Siemens Wind Power A/S and illustrated in Fig. 11. The blade is constructed using a single web design with the shell and spar cap made of fiberglass-epoxy, while a sandwich core present in the trailing edge walls and tail is made of balsa and foam. The distribution along the blade length of bending stiffness about each of the principal axes of bending normalized with respect to the bending stiffness of the circular root section are shown in Fig. 12a. This gives an illustration of the large cross-section property variations that must be captured when modelling wind turbine blades. It can be seen that for the first half of the blade the bending stiffness in the edgewise direction is typically twice as large as the stiffness in the flapwise direction. The increase in the edgewise bending stiffness near the root is associated with the transition from a circular to an airfoil cross-section.

In the current analysis, the blade is discretized using five different beam element meshes, each with a different number of elements and location of the nodes as shown in Fig. 11. The nodes are positioned along the elastic axis. A fine mesh with 75 elements of equal length.
has been omitted from the figure for clarity. The node positions for the mesh with two, four and eight elements are optimized to minimize the error of the first four natural frequencies. As expected, the nodes are skewed towards the more compliant outward part of the blade, as shown in Fig. 11.

The blade is loaded by the distributed force $p_y$, acting normal to the local secant direction. The lengthwise distribution of the wind load $p_y$ is shown in Fig. 12b. The linear increasing lift for the inner two-thirds of the blade that can be observed is a result of the blade’s pre-twist. Furthermore, the reduction in aerodynamic load at the blade tip is attributed to tip losses. The deflection associated with this component of the wind pressure must be determined with high accuracy as it tends to bend the blade back against the tower. The relative error of the in-plane tip displacement obtained using the models with two, four, height, and sixteen elements relative to a reference deflection calculated using 75 elements under the wind pressure are shown in Fig. 13a. The elastic axis of the blade is not straight, and therefore four elements are required to determine both in-plane displacement components to within 1% relative error. If the blade was straight, the deflection under a tip load could be calculated exactly using a single element.

The distributions of the internal shear force $Q_2$ and the moment $M_1$ recovered using the present equilibrium method are shown in Fig. 13b for 2 and 75 elements. It is seen that both the shear and the moment distributions obtained using only two elements are in excellent
agreement with the distributions obtained using the refined mesh, with a discrepancy on the maximum moment and shear force at the root of less than 0.1%. The continuous internal forces and moments and stress-free blade tip are well captured by the current approach. Note, that the magnitude of the maximum shear force is approximately 50 times smaller than that of the maximum bending moment.

6. Conclusions

A complementary energy formulation has been presented for a two-node straight beam element including the stiffness matrix and the representation of distributed loads in terms of equivalent nodal forces. The three main features are that the formulation permits: i) arbitrary lengthwise variation of the cross-section properties by integrating local cross-section flexibility weighted with simple and known internal force distributions, ii) representation of arbitrary cross-sections with coupled properties, e.g. from material anisotropy, represented by a full six by six local flexibility matrix, and iii) an exact formula for the equivalent nodal loads for arbitrary distributed loads, represented via equilibrium internal forces. The formulation includes the effect of shear flexibility automatically, and is non-singular, even in the limiting case of vanishing shear flexibility. The element stiffness properties determine the distribution of internal element loads to the equivalent concentrated loads at the element nodes, and it has been demonstrated that the correct evaluation of the equivalent nodal

Figure 13: Distributed lift force: a) Static tip deflection, b) Moment and shear force.
loads is essential for retaining the accuracy, when considering beams with large stiffness variations.

In principle the theory is exact, and the accuracy is only limited by approximations that may be involved in the evaluation of integrals of the cross-section flexibility matrix, weighted by the equilibrium internal force distributions. The use of equilibrium internal force distributions limits the full accuracy of the formulation to static problems. However, in most dynamics problems the local displacements due to element deformation are quite limited and the convected motion of the element dominates. This permits to retain a considerable part of the accuracy of the equilibrium based element stiffness formulation, also in dynamics problems.

Acknowledgment

The present paper is an extension of an unpublished internal note from 2006 describing the element stiffness matrix of a fully coupled nonhomogeneous beam without distributed loads. The present work has been supported by Siemens Wind Power A/S.


Beam section stiffness properties using 3D finite elements

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Beam section stiffness properties using 3D finite elements

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Summary. The cross-section properties of a beam is characterized by a six by six stiffness matrix, relating the six generalized strains to the conjugate section forces. The problem is formulated as a single-layer finite element model of a slice of the beam, on which the six deformation modes are imposed via Lagrange multipliers. The Lagrange multipliers represent the constraining forces, and thus combine to form the cross-section stiffness matrix. The theory is illustrated by a simple isotropic cross-section.

Key words: cross-section analysis, coupled beam deformation, anisotropic beam

Introduction

With recent advances in manufacturing capabilities, beams with more complex geometries and materials with general anisotropy are being used in wind turbine blades. While the global response of the blades can be represented by a beam model, the accuracy depends on the use of appropriate description of the cross-section properties, including coupling from e.g. pretwist and material anisotropy. Several theories have been developed to calculate the cross-section properties of beams. Many are based on simplifications which limit their use to simple geometries or isotropic materials [1]. Two methodologies have been found to provide the correct stiffness matrix for most engineering structures which are based on advanced kinematic analysis of beams namely the theories developed by Giavotto et al. [2] and Hodges and Yu [3], respectively.

This paper presents a method to calculate the cross-section stiffness matrix of the deformation modes of classic beam theory. The method is based on the analysis of a thin slice of the beam, on which the six modes of deformation corresponding to the equilibrium modes are imposed by use of Lagrange multipliers. Each deformation mode corresponds to activating one kinematic degree-of-freedom, while setting the remaining five to zero. Thus, each kinematic load case generates six Lagrange multipliers, representing the section forces needed to impose that particular kinematic mode. Together the six sets of Lagrange multipliers, each with six components, form the cross-section stiffness matrix. The analysis is carried out by a three-dimensional finite element model of a thin slice of the beam. This format enables correct representation of effects like transverse contraction and coupling due to anisotropy.

The slice approach

The properties of a cross-section of a beam can be assessed by considering a thin slice of the beam as shown in Fig. 1. The slice is given a unit thickness for numerical simplification. The slice is characterised by six equilibrium states, namely extension, two homogeneous shear components, torsion, and two homogeneous bending components. The stiffness matrix linking the equilibrium states and their conjugate six deformation modes can be determined by imposing a displacement of the end cross-section planes of the slice and calculating the required forces. The degrees of
freedom of the slice are defined in terms of the displacements at the front (+) and back (−) faces of the slice as \( u_{\pm} = [u_{1}^T, u_{2}^T, \cdots, u_{n}^T]^T \), where \( n \) is the number of nodes and \( u_{i} \) defines the 3D displacements at the node \( i \).

**The six deformation load cases**

The properties of the slice are analysed using the finite element method. Within linear elasticity the stiffness equations of the slice take the following block matrix format

\[
\begin{bmatrix}
K_{++} & K_{+-} \\
K_{-+} & K_{--}
\end{bmatrix}
\begin{bmatrix}
u_{+} \\
u_{-}
\end{bmatrix}
= \begin{bmatrix}
f_{+} \\
f_{-}
\end{bmatrix},
\]

(1)

The deformation modes are defined in terms of differences in displacement at the two sides of the slice and it is therefore convenient to rewrite the stiffness equations in terms of increments and mean values

\[ \Delta u = u_{+} - u_{-}, \quad 2\bar{u} = u_{+} + u_{-}. \]

(2)

Substitution into (1) gives

\[
\begin{bmatrix}
(K_{++} - K_{+-} + K_{-+} - K_{--}) & (K_{++} + K_{+-} - K_{-+} - K_{--}) \\
(K_{++} - K_{+-} + K_{-+} - K_{--}) & (K_{++} + K_{+-} + K_{-+} + K_{--})
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
2\bar{u}
\end{bmatrix} =
\begin{bmatrix}
f_{+} - f_{-} \\
f_{+} + f_{-}
\end{bmatrix}.
\]

(3)

In order to define the deformation of the slice explicitly in terms of the deformation modes, the displacement vector is further transformed as to include the six generalized strains \( \gamma = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \kappa_x, \kappa_y, \kappa_z]^T \). The components \( \varepsilon_x, \varepsilon_y \) and \( \varepsilon_z \) represent the axial strain and both generalized shear strains, respectively. Similarly, the components \( \kappa_x, \kappa_y \) and \( \kappa_z \) represent the rate of twist and both bending curvatures, respectively. If one uses elements with Hermitian interpolation of the transverse displacements in the axial direction and nodal degrees of freedom defined as \( u_{i} = [u, v, w, u', v']^T \), the transformation is done by defining the difference in displacement, \( \Delta u \), as

\[ \Delta u = \Phi u_{\gamma}, \]

(4)

where \( u_{\gamma} = [\gamma^T, \Delta u_{1}', \Delta u_{2}', \cdots, \Delta u_{n}', \Delta v_{n}']^T \). The transformation matrix \( \Phi \) takes the form

\[
\Phi = \begin{bmatrix}
\Gamma_{1} & \Theta_{1} \\
\vdots & \ddots \\
\Gamma_{n} & \Theta_{n}
\end{bmatrix}.
\]

(5)

The matrix \( \Gamma_{i} \) defining the displacement increments and the matrix \( \Theta_{i} \) storing the rotation
increments are defined as

\[ \Gamma_i = \begin{bmatrix} 1 & 0 & 0 & z_i & -y_i \\ 0 & 1 & 0 & -z_i & 0 \\ 0 & 0 & 1 & y_i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Theta_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \]

(6)

where \( x_i, y_i, z_i \) are the global Cartesian coordinates of node \( i \).

Elimination of \( \Delta u \) in (3) by (4) gives

\[ \begin{bmatrix} \Phi^T K_{11} \Phi & \Phi^T K_{12} \\ K_{21} \Phi & K_{22} \end{bmatrix} \begin{bmatrix} u_\gamma \\ 2\bar{u} \end{bmatrix} = 2 \begin{bmatrix} \Phi^T (f_+ - f_-) \\ (f_+ + f_-) \end{bmatrix}, \]

(7)

where \( K_{ij} \) are the block components of the stiffness matrix in (3).

In order to impose the six deformation modes independently, the values of \( \gamma \) are defined via constraints in the form of

\[ [C_\gamma, C_f] \begin{bmatrix} u_\gamma \\ 2\bar{u} \end{bmatrix} = q_j. \]

(8)

The vector \( q_j \) is used to activate one kinematic degree-of-freedom, while setting the remaining five to zero, e.g. for the extension case \( q_1 = [1, 0, 0, 0, 0, 0]^T \). The constraints are added to the system of linear equations using the method of Lagrange multipliers where each constraint is enforced by solving for the associated Lagrange multiplier which acts as the force needed to impose the constraint [4]. As such, if no external forces are applied to the slice, the Lagrange multipliers associated with the generalized strains come out as the generalized forces. Incorporating the constraints and Lagrange multipliers, \( \lambda \), to be solved and setting the external forces to zero the system of equations takes the form

\[ \begin{bmatrix} \Phi^T K_{11} \Phi & \Phi^T K_{12} & C_\gamma^T \\ K_{21} \Phi & K_{22} & C_f^T \\ C_\gamma & C_f & 0 \end{bmatrix} \begin{bmatrix} u_\gamma \\ 2\bar{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

(9)

Using this formulation, the cross-section stiffness matrix can be populated by imposing one displacement mode at a time and solving for the generalized section forces.

It is to be noted that in the case of both shear modes and torsion mode, additional constraints need to be added to enforce that the work is orthogonal to the work done in extension, and both bending modes. Since the internal work equals the external work done by the forces on the nodes, the orthogonality conditions can be expressed as

\[ f_{\pm \alpha}^T u_{\pm \beta} = 0, \]

(10)

where the indices define the displacement modes based on the order set in \( \gamma \), i.e. \( \alpha = 1, 5, 6 \) and \( \beta = 2, 3, 4 \).

**Simple example**

This section presents the analysis of a square cross-section using an implementation of the methodology described in the previous section. Eight-node elements are used with Hermitian shape functions in the thickness direction. The square has a width of \( b = 2 \) with a Young’s modulus of \( E = 1 \) and Poisson’s ratio of \( \nu = 0.3 \). The reference axis being at the center, only diagonal terms in the stiffness matrix are non-zero. Furthermore, the diagonal terms come out as \( EA, GA_y, GA_z, GJ, EI_y, \) and \( EI_z \) which are the extensional stiffness, shear stiffness about both in-plane axes, the torsional stiffness and bending stiffness about both in-plane axes, respectively.
Table 1. Normalized cross-section stiffness properties for a square

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>1x1</th>
<th>4x4</th>
<th>9x9</th>
<th>19x19</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A/b^2$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$A_y/b^2 = A_z/b^2$</td>
<td>1.000</td>
<td>0.8788</td>
<td>0.8424</td>
<td>0.8353</td>
<td>0.8333</td>
</tr>
<tr>
<td>$J/b^4$</td>
<td>0.1667</td>
<td>0.1479</td>
<td>0.1421</td>
<td>0.1409</td>
<td>0.1408</td>
</tr>
<tr>
<td>$I_y/b^4 = I_z/b^4$</td>
<td>0.09619</td>
<td>0.08416</td>
<td>0.08350</td>
<td>0.08338</td>
<td>0.08333</td>
</tr>
</tbody>
</table>

Results obtained using different mesh sizes and using analytical solutions for isotropic cross-sections are listed in Table 1, illustrating convergence for all parameters towards the analytical solution.

The associated 3D deformation of the six modes are represented graphically in Fig. 2. It can be seen that the cubic displacement associated with shear is captured. Furthermore in the two bending cases, the quadratic curvature in the thickness direction is modelled with the use of a single element via the Hermitian interpolation. Contraction from Poisson’s ratio can also be observed.

![Graphical representation of elastic beam deformation modes](image)

Figure 2. Elastic beam deformation modes for a square cross-section

References


C2

General beam cross-section analysis using a 3D finite element slice

P. Couturier and S. Krenk

ABSTRACT
A formulation for analysis of general cross-section properties has been developed. This formulation is based on the stress-strain states in the classic six equilibrium modes of a beam by considering a finite thickness slice modelled by a single layer of 3D finite elements. The displacement variation in the lengthwise direction is in the form of a cubic polynomial, which is here represented by Hermite interpolation, whereby the degrees of freedom are concentrated on the front and back faces of the slice. The theory is illustrated by application to a simple cross-section for which an analytical solution is available. The paper also shows an application to wind turbine blade cross-sections and discusses the effect of the finite element discretization on the cross-section properties such as stiffness parameters and the location of the elastic and shear centers.

INTRODUCTION
Wind turbine rotors are typically composed of three blades, and these are becoming increasingly slender with the introduction of larger rotors. This places greater emphasis on the deformation characteristics of the blades. Irrespective of the beam theory used to model the blade, the cross-section properties constitute an essential part of the beam model.

Several theories have been developed to calculate the cross-section properties of beams. Many are based on simplifications which limit their use to simple geometries or isotropic materials [1]. Two methodologies have been found to provide the correct stiffness matrix for most engineering structures which are based on advanced kinematic analysis of beams namely the theories developed by Giavotto et al. [2, 3] and Hodges and Yu [4].

This paper presents a method to calculate the cross-section stiffness matrix of the classic six homogeneous equilibrium load cases. The method is based on the analysis of a slice of the beam made up of linear elastic material, on which six independent deformation modes corresponding to extension, torsion, homogeneous bending and homogeneous shear are prescribed by imposing suitable displacement increments across the slice via Lagrange multipliers. Each of the six imposed set of displacements generates six resulting internal force/moment components identified by the Lagrange multipliers, representing the internal force components needed to impose that particular kinematic mode. The complementary elastic energy calculated in terms of the internal force/moment components is then used to define the full six by six flexibility matrix of the cross-section. The corresponding cross-section stiffness matrix then follows by inversion of this flexibility matrix.

The advantages of the present finite-thickness slice method are that it avoids the development of any special 2D theory for the stress and strain distributions over the cross-section and enables a simple and direct representation of thin-walled parts and more massive parts via their well-established representation in 3D elements.

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BEAM STATICS DESCRIPTION

Consider a beam of length \( l \) with longitudinal coordinate \( x_3 \) and cross-section coordinates \( x_1 \) and \( x_2 \) as shown in figure 1(a). The origin is located at the center cross-section plane of the beam, whereby the front and back of the beam are located at \( x_1 = l/2 \) and \( x_1 = -l/2 \), respectively. The beam supports the equilibrium states of tension, torsion, bending, and shear.

The exact distribution of the surface stresses at a cross-section defined by \( x_3 \) can be replaced by three forces and three moments which are statically equivalent that is, which have the same force and the same moment. These six forces and moments are grouped together in the force vector \( \mathbf{q}(x_3) = [Q_1(x_3) \ Q_2(x_3) \ Q_3(x_3) \ M_1(x_3) \ M_2(x_3) \ M_3(x_3)]^T \). The components \( Q_1(x_3) \) and \( Q_2(x_3) \) are two shear forces, and \( Q_3(x_3) \) is the axial force. The components \( M_1(x_3) \) and \( M_2(x_3) \) are bending moments, and \( M_3(x_3) \) is the torsion moment component with respect to the origin of the reference coordinate system. A compact notation is achieved when representing the two in-plane directions using Greek subscripts \( \alpha, \beta = 1,2 \) which allows summation over repeated Greek subscripts. This notation includes use of the two-dimensional permutation symbol \( \epsilon_{\alpha \beta} \) which takes the following values based on the indices \( e_{12} = 1 \), \( e_{21} = -1 \), and \( e_{11} = e_{22} = 0 \). Using the Greek subscripts and the permutation symbol, internal forces and moments in terms of stresses on a cross-section are defined as

\[
\begin{align*}
Q_\alpha(x_3) &= \int_A \sigma_{\alpha \gamma} dA, \\
M_\alpha(x_3) &= \int_A \epsilon_{\alpha \gamma} x_\gamma \sigma_{\gamma \delta} dA,
\end{align*}
\]

The internal forces and moments components are illustrated in figure 1(b).

Equilibrium modes

The beam can be described by six equilibrium modes, namely the homogeneous states of extension, torsion, bending, and shear. The case of homogeneous tension is illustrated in figure 2(a). Opposing axial forces of magnitude \( M_0^\alpha \) are acting at the ends of the beam. Similarly the case of homogeneous torsion illustrated in figure 2(d) is characterized by opposing torsion moment of magnitude \( M_0^\alpha \) acting at the ends of the beam. The homogeneous bending modes are illustrated in figure 2(b) and (e). Opposing bending moments of magnitude \( M_0^\beta \) are acting at the ends of the beam. Finally the cases of homogeneous shear are illustrated in figure 2(c) and (f). Here, shear forces \( Q_0^\alpha \) are applied to the end of the beam. This results in a total external moment that is counteracted by identical bending moments \( M_0(\pm \tfrac{l}{2}) = -\tfrac{1}{2} \epsilon_{\alpha \beta} l Q_0^\alpha \).

From equilibrium consideration a beam without external loads will exhibit constant internal normal force, shear forces and torsion moment, while the bending moments will vary linearly with the shear force as gradient. The six equilibrium modes can therefore be defined by the internal force \( \mathbf{q}_0 = [Q_0^1 \ Q_0^2 \ Q_0^3 \ M_0^1 \ M_0^2 \ M_0^3]^T \) at the center of the beam. Since the constant shear forces \( Q_0^\alpha \) lead to linear anti-symmetric moment variation, the force vector \( \mathbf{q}_0 \) is the internal force and moment at the center cross-section of the beam. The distribution of internal forces in the beam in terms of \( \mathbf{q}_0 \) follows as

\[
\begin{bmatrix}
Q_1(x_3) \\
Q_2(x_3) \\
Q_3(x_3) \\
M_1(x_3) \\
M_2(x_3) \\
M_3(x_3)
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & x_3 & 0 & 1 & 0 & 0 \\
-\frac{1}{l} & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
Q_0^1 \\
Q_0^2 \\
Q_0^3 \\
M_0^1 \\
M_0^2 \\
M_0^3
\end{bmatrix},
\]

The distribution of internal forces written in a more compact notation becomes

\[
\mathbf{q}(x_3) = \mathbf{T}(x_3) \ \mathbf{q}_0 ,
\]

where the interpolation matrix \( \mathbf{T}(x_3) \) is defined in (2).

Flexibility matrix

Following the general formulation of equilibrium based beam elements in [5], the deformation associated with the internal forces and moments \( \mathbf{q} \) is described in terms of six strains defined by the strain vector \( \mathbf{\gamma} = [\epsilon_1 \ \epsilon_2 \ \kappa_1 \ \kappa_2 \ \kappa_3]^T \). The components \( \epsilon_1 \) and \( \epsilon_2 \) are generalized shear strains, and \( \epsilon_3 \) is the axial strain. Similarly \( \kappa_1 \) and \( \kappa_2 \) are the components of bending curvature, while \( \kappa_3 \) is the rate of twist. The generalized strain \( \mathbf{\gamma} \) is defined such that it is conjugate to the internal force vector \( \mathbf{q} \) with respect to energy. Thus, the specific elastic energy associated with a cross-section is given as

\[
W_s = \frac{1}{2} \mathbf{\gamma}^T \mathbf{q}.
\]

For linear elastic beams there is a linear relation between the internal forces and the conjugate strains. This relation can be
written either in flexibility or stiffness format,
\[ \mathbf{\gamma} = \mathbf{C} \mathbf{q}, \quad \mathbf{q} = \mathbf{D} \mathbf{\gamma}, \]  
(5)

where \( \mathbf{C} \) and \( \mathbf{D} = \mathbf{C}^{-1} \) are the cross-section flexibility and stiffness matrix, respectively. Both are six by six symmetric matrices and as such can contain up to twenty-one independent entries in the case of a fully general anisotropic cross-section. Eliminating \( \mathbf{\gamma} \) in (4) using (5), the following representation of the energy per unit length at \( x_3 \) expressed in terms of the cross-section flexibility matrix is obtained

\[ W_c(x_3) = \frac{1}{4} \mathbf{q}(x_3)^T \mathbf{C} \mathbf{q}(x_3). \]  
(6)

Consider the beam shown in figure 1(a) as having a constant material distribution in the longitudinal direction, that is a cross-sectional flexibility matrix \( \mathbf{C} \) independent of \( x_3 \). The flexibility matrix of the equilibrium modes of the beam follows from integration of the cross-section flexibility relation (6) over the beam length

\[ W_c = \int_{-l/2}^{l/2} W_c(x_3) dx_3 = \frac{1}{4} \int_{-l/2}^{l/2} \mathbf{q}(x_3)^T \mathbf{C} \mathbf{q}(x_3) dx_3. \]  
(7)

Representing the internal forces and moments via the mid-point values \( \mathbf{q}_0 \) by (2) yields

\[ W_c = \frac{1}{4} \mathbf{q}_0^T \mathbf{H} \mathbf{q}_0. \]  
(8)

where the beam flexibility matrix \( \mathbf{H} \) corresponding to the six equilibrium modes is defined by the integral

\[ \mathbf{H} = \int_{-l/2}^{l/2} \mathbf{T}^T(x_3) \mathbf{C} \mathbf{T}(x_3) dx_3. \]  
(9)

Carrying out the integration in explicit form provides the relation between the beam flexibility matrix and the cross-section flexibility matrix

\[
\mathbf{H} = \mathbf{I} \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}.
\]
(10)

It is seen how the bending parameters \( C_{33}, C_{44}, C_{55}, \) and \( C_{34} \) of the cross-section enter the constant shear modes of the beam due to its linearly varying bending moment.

**Energy equivalence**

The displacement field representation with respect to \( x_3 \) when no distributed loads are present is at most cubic. Thus, the solution to this displacement field is defined by four parameters. In what follows the parameters are chosen as the displacement and displacement derivative fields on the front and back faces of the beam as shown in figure 3. The degrees of freedom are grouped in the displacement vector \( \mathbf{u}(x_3) = [u(x_3)]^T, \) where the degrees of freedom at the front \( (+) \) and back \( (−) \) faces are \( \mathbf{u}(x_3) = [u(x_3), \frac{d}{dx_3} u(x_3)]^T \) with the \( j \) index ranging from one to three. The components \( u_j \) represent the three displacement components, while the components \( u'_j \) represent the corresponding derivatives with respect to the axial coordinate \( x_3. \)

The static component conjugate to the displacement vector \( \mathbf{u}(x_3) \) is the force vector \( \mathbf{f}(x_3) = [f(x_3)]^T, \) where the components at the front and back faces are \( \mathbf{f}(x_3) = [f_1(x_3), \pm f_2] \) for \( j = 1 \). The components \( f_j \) represent the three force components conjugate to the displacement derivatives \( u'_j. \)

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The elastic energy of the beam can be expressed in terms of the displacement vector \( \mathbf{u}(x_a) \) and the force vector \( \mathbf{f}(x_a) \) as

\[
W_e = \frac{1}{2} \int_A \mathbf{u}^T \mathbf{f} dA.
\] (11)

The energy of the 3D beam must equal the energy defined in terms of the beam flexibility matrix in (8) which leads to the following equality

\[
W_e = \frac{1}{2} \mathbf{q}_0^T \mathbf{H} \mathbf{q}_0 = \frac{1}{2} \int_A \mathbf{u}^T \mathbf{f} dA.
\] (12)

The elements of the six by six flexibility matrix can be obtained from a linear combination of the displacements and forces associated six orthogonal modes. The solution for these elements takes a simple form if the six modes are chosen as the six equilibrium modes. The displacement and forces of the six equilibrium modes is not available directly but can be constructed from deformation modes described in the next section. Let the displacement vectors describing the deformation of the six equilibrium modes take unit components in \( \mathbf{q}_0 \) in sequence be grouped in the matrix

\[ \mathbf{U}(x_a) = \left[ \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3 \mathbf{u}_4 \mathbf{u}_5 \mathbf{u}_6 \right]^T. \]

The components \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are the displacement vectors containing the degrees of freedom of two shear equilibrium modes and \( \mathbf{u}_3 \) is the displacement vector associated with the torsion equilibrium mode. The static component conjugate to the equilibrium mode displacement matrix \( \mathbf{U}(x_a) \) is the force matrix \( \mathbf{F}(x_a) = \left[ \mathbf{f}_1 \mathbf{f}_2 \mathbf{f}_3 \mathbf{f}_4 \mathbf{f}_5 \mathbf{f}_6 \right]^T. \) From (12) using the displacement and force vectors of the six equilibrium modes, the elements of the six by six flexibility matrix take the following simple form

\[
H_{ij} = \frac{1}{2} \int_A \left( \mathbf{u}_i^T \mathbf{f}_j + \mathbf{u}_j^T \mathbf{f}_i \right) dA, \quad i, j = 1, \ldots, 6
\] (13)

From this methodology the 36 elements of the flexibility matrix can be determined. The cross-section flexibility matrix \( \mathbf{C} \) can then be calculated using (10) and the corresponding cross-section stiffness matrix \( \mathbf{D} \) is obtained by inversion of this flexibility matrix.

\[ \mathbf{U}(x_a) = \mathbf{C}(\mathbf{F}(x_a) - \mathbf{f}(x_a)), \]

\[ 2 \mathbf{u}(x_a) = \mathbf{u}(x_a, \mathbf{f}_1) + \mathbf{u}(x_a, -\mathbf{f}_1). \] (14)

Only the increment in displacement \( \Delta \mathbf{u} \) will generate work and thereby contribute to the flexibility matrix of the equilibrium modes of the beam. The increment in displacement therefore needs to be defined in terms of the six degrees of freedom of the
deformation modes $\tilde{v}$. The mean value of the displacement $\bar{u}$ is associated with rigid body motion of the beam.

Extension deformation mode illustrated in figure 4(a) is described by an elongation of the beam equal to $\tilde{v}_3$. No internal shear forces $Q_3$ occur in this mode which yields identical forces and moments at each cross-section along the longitudinal axis $x_3$. This leads to a uniform transverse contraction along the beam and thus increment displacement derivatives $\Delta u_3'$ and $\Delta u'_3$ of zero.

Twist deformation mode illustrated in figure 4(b) is defined by a constant rate of twist about the axial coordinate $x_3$ equal to $\tilde{v}_3$. The assumption of constant rate of twist corresponds to assuming homogeneous St. Venant torsion with identical cross-section warping along the beam. As in the extension mode, no internal shear forces occur which leads to increment displacement derivatives $\Delta u_3'$ and $\Delta u'_3$ of zero.

The two bending deformation modes illustrated in figure 4(c) are characterized by a constant bending curvature about the in-plane coordinates $x_a$ equals to $\tilde{v}_a$. No internal shear forces occur in this mode and the axial increment displacement derivative $u'_3$ is zero and the in-plane increment displacement derivative $u'_a$ is defined by the bending curvature.

The two shear deformation modes illustrated in figure 4(d) are characterized by a transverse increment in the direction of the in-plane coordinates $x_a$ defined by $\tilde{v}_a$. These two modes will have internal shear forces and from equilibrium consideration will also have linearly varying bending moments with end moments $M_{a}(\alpha \frac{1}{2})=-\frac{1}{2} \epsilon_{abg} Q_{bg}^a$. This variation of internal moments leads to in-plane contractions that vary with $x_3$ contrary to the other four deformation modes. The displacement increment for the shear deformation modes is then defined by the superposition of the in-plane translation displacement $\tilde{v}_a$ and the deformation associated with the presence of anti-symmetric bending.

**Rigid body motion**

Rigid body displacements must be constrained to completely define the kinematics associated with the six deformation modes. Since the beam is located in a 3D space, three rigid body rotations and three rigid body displacements must be constrained. The constraints are expressed in terms of orthogonality conditions between the mean displacement $\bar{u}$ and the internal force distribution at the center of the beam associated with the deformation modes. For the two shear deformation modes, orthogonality between the six deformation modes is imposed,

$$\int_A \Delta \bar{u}^T \bar{u} \, dA = 0, \quad i = 1, \ldots, 6 \quad (15)$$

The increment value of the force over the thickness of the beam associated with the six deformation modes $\Delta \bar{f}(x)$ is defined as

$$\Delta \bar{f}(x_a) = \bar{f}(x_a, \frac{1}{2}) - \bar{f}(x_a, \frac{1}{2}). \quad (16)$$

To ensure no generalized shear forces are present in the extension, twist, and bending deformation modes, the orthogonality condition with the force distribution of the bending modes is not enforced, thereby allowing shearing of the beam. For these modes the constraints on rigid body motion therefore take the following form

$$\int_A \Delta \bar{f}^T \bar{u} \, dA = 0, \quad i = 1, 2, 3, 6 \quad (17)$$

It is to be noted that the force distributions $\bar{f}(x_a)$ needed for the orthogonality constraints are not known prior to solving the deformation modes. However, the orthogonality conditions are only dependent on the shape of the force distributions but not their magnitude. This allows the use of an intermediate prediction step to calculate the force distributions where constraints on average displacements are used to prevent rigid body displacements.

**FINITE ELEMENT REPRESENTATION OF SLICE**

The kinematic conditions used to impose the deformation modes presented in the previous sections are independent of the solution method selected to solve the full displacement field. However, since the cross-section flexibility matrix is obtained by imposing incremental displacements on a beam, it is convenient to solve the total deformation using three-dimensional isoparametric finite elements. The same discretization can be used for all load cases as long as care is given to select element inter- polation of sufficient order to capture the main effects. Since the displacement field in the longitudinal direction varies at maximum as a third degree polynomial, an efficient discretization in that direction can be done using either four nodes or a Hermite interpolation. The latter option was selected as it follows the description of the deformation modes in the previous section. Using Hermite interpolation in the $x_3$-direction, the degrees of freedom of the slice are defined in terms of the displacements at the front (+) and back (−) faces as $u_{\Delta} = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n]^T$, where $n$ is the number of nodes on one face and the nodal degrees of freedom are defined as $\mu_j = [u_j, u'_j]^T$ where the $j$ index ranges from one to three. The components $u_{a\Delta}$ and $u_{3\Delta}$ represent the in-plane and axial displacements, while the components $u_{a\Delta}'$ and $u_{3\Delta}'$ represent the corresponding derivatives with respect to the axial coordinate $x_3$. Large distortion of the elements is prevented by using a slice of thickness comparable to the in-plane element dimensions.
TABLE 1. Shape functions of 8-node element with Hermitian interpolation.

<table>
<thead>
<tr>
<th>Corner nodes: $\xi_i = \pm 1, \eta_i = \pm 1, \zeta_i = \pm 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = \frac{1}{16}(1 + \xi \xi)(2 - \xi \xi)(1 + \eta \eta)(1 + \zeta \zeta)$</td>
</tr>
<tr>
<td>Corner nodes: $\xi_i = \pm 1, \eta_i = \pm 1, \zeta_i = \pm 1$</td>
</tr>
<tr>
<td>$N^h = \frac{1}{32}(\xi + \xi_i)^2(\xi - \xi_i)(1 + \eta \eta)(1 + \zeta \zeta)$</td>
</tr>
</tbody>
</table>

The theory of the finite element method is well established, see e.g. [6, 7], as such our interest here are the modifications in the interpolation method that are needed to have a Hermitian interpolation in the axial direction. The elements are described in terms of the intrinsic coordinates $\xi_i = \{x, y, z\}$ where the coordinate $\xi$ is collinear with the global axial coordinate $x$. For hexahedral elements, the intrinsic coordinates cover the range $-1 \leq \{\xi, \eta, \zeta\} \leq 1$. An 8-node element in the intrinsic coordinate system is shown in figure 5.

The displacement field in vector components $u(x) = [u]^T$ are described using finite elements as

$$u(x) = N(\xi)\hat{u},$$

where the nodal values of the displacement are contained in the column vector $\hat{u} = [u_1, u_2, \ldots, u_m]^T$. The matrix containing the associated shape functions is defined as $N(\xi) = [N_1, N_2, \ldots, N_m, N_m]$, where the shape functions associate with the vector components of the displacements are labelled as $N$ and the shape functions associated with the gradients of the displacement field are labelled as $N^h$.

The shape functions for an 8-node element with Hermitian interpolation are given in table 1. They are constructed using Lagrange product formulas. Shape functions associated with the corner node 4 are shown in figure 5. The element is characterized by a linear variation in the $\eta, \zeta$ plane and a cubic variation in $\xi$. It is to be noted that shape functions $N^h$ associated with the gradients of the displacement field are made proportional to the thickness of the slice, $t$, in order to have the gradients defined in the global coordinate system.

The stiffness equations in terms of the sum and difference of forces and displacements then take the form

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_x - u_\eta \\ u_x + u_\eta \end{bmatrix} = \begin{bmatrix} f_x - f_\eta \\ f_x + f_\eta \end{bmatrix},$$

where the block matrices are defined by

$$K_{11} = \frac{1}{2}(K_{++} - K_{-+} - K_{+-} + K_{--}),$$
$$K_{12} = \frac{1}{2}(K_{++} + K_{-+} - K_{+-} - K_{--}),$$
$$K_{21} = \frac{1}{2}(K_{++} - K_{-+} + K_{+-} - K_{--}),$$
$$K_{22} = \frac{1}{2}(K_{++} + K_{-+} + K_{+-} + K_{--}).$$

In this system of equations, the displacements are split into an increment over the thickness of the slice and a mean value

$$\Delta u = u_x - u_\eta, \quad 2\bar{u} = u_x + u_\eta.$$ (22)

The nodal displacement increments over the thickness of the slice for node pair $k$ can be defined in terms of the degrees of freedom representing the deformation modes $\tilde{\nu}$ and an antisymmetric bending contribution,

$$\Delta u_k = A_k \tilde{\nu} + B_k q_0,$$ (23)

where the transformation matrix $A_k$ is given as

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 & -x_2^2 \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & x_2 - x_1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ (24)
The anti-symmetric bending contribution matrix \( B_k \) is defined as
\[
B_k = I \left( \hat{u}_4 \right), \quad \left( \hat{u}_4, \hat{u}_5 \right), \quad 0, 0, 0, 0, \right)_k,
\] (25)
where the nodal displacement vector associated with the two bending equilibrium modes are defined as \( \hat{u}_4 \) and \( \hat{u}_5 \). One can rewrite the stiffness equations in terms of the deformation mode where all degrees of freedom \( \Delta u \) are expressed in terms of the degrees of freedom representing the deformation modes \( \hat{v} \). When using elements with Hermitian interpolation in the axial direction, the transformation is done by defining the displacement increments \( \Delta u \) as
\[
\Delta u = A \hat{v} + B q_0,
\] (26)
The transformation matrix \( A \) and the anti-symmetric contribution matrix \( B \) take the form
\[
A = \begin{bmatrix}
A_1 \\
\vdots \\
A_n
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1 \\
\vdots \\
B_n
\end{bmatrix},
\] (27)
where \( A_k \) and \( B_k \) are defined in (24) and (25). Elimination of \( \Delta u \) in (20) by (26) gives
\[
\begin{bmatrix}
A^T K_{11} A & A^T K_{12} \\
K_{21} A & K_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{v} \\
\Delta u
\end{bmatrix} = \begin{bmatrix}
A^T (f_e - f_r - K_{11} B q_0) \\
(f_e + f_r - K_{21} B q_0)
\end{bmatrix}.
\] (28)
In this form, the six forces associated with the six degrees of freedom of the deformation modes correspond to the generalized forces and moments.

**Enforcing deformation modes using constraints**

Activating each of the six deformation modes is accomplished by the addition of constraints which enforces relationships among the degrees of freedom. Two types of constraints are needed, namely constraints to fix the value of the six degrees of freedom of the deformation modes \( \hat{v} \), and constraints to prevent rigid body motion of the slice.

The constraints that define the value of the six degrees of freedom of the deformation modes provide the handle needed to set one degree of freedom to unity and all others to zero. This corresponds to activating one deformation mode, while setting the remaining five to zero. These constraints are defined as
\[
I \hat{v} = g_j,
\] (29)
where \( I \) is a six by six identity matrix. The vector \( g_j \) is used to select which degree of freedom to activate while setting the remaining five to zero where the index \( j \) ranging from 1 to 6 specifies which deformation is activated, e.g. for the extension case \( g_j = \{0, 0, 1, 0, 0, 0\}^T \).

Two different sets of constraints to prevent rigid body motion are needed in the analysis. These constraints act on the mean displacements and can be expressed as
\[
G_m \bar{u} = 0,
\] (30)
where \( G_m \) is the rigid body constraint matrix where the index \( m \) ranging from 1 to 2 specifies which of the two sets of constraints is enforced. From (17) the constraint matrix using orthogonality conditions when solving the extension, twist, and bending deformation modes takes the form
\[
G_1 = [\Delta \hat{f}_1, \Delta \hat{f}_2, \Delta \hat{f}_3, \Delta \hat{f}_4]^T,
\] (31)
Similarly, from (15) the constraint matrix to enforce the six work orthogonality conditions when solving the two shear deformation modes takes the form
\[
G_2 = [\Delta \hat{f}_1, \Delta \hat{f}_2, \Delta \hat{f}_3, \Delta \hat{f}_4]^T.
\] (32)
The constraints are added to the system of linear equations using the method of Lagrange multipliers where each constraint is enforced by solving for the associated Lagrange multiplier which acts as the force needed to impose the constraint [6]. If no external forces are applied to the slice, the Lagrange multipliers associated with the degrees of freedom of the deformation modes come out as the negative of the generalized forces. Incorporating the constraints and Lagrange multipliers \( \lambda \) to be solved and setting the external forces to zero the system of equations of (28) takes the following form
\[
\begin{bmatrix}
A^T K_{11} A & A^T K_{12} & 0 & 0 \\
K_{21} A & K_{22} & -K_{21} B & G_m^T \\
1 & 0 & 0 & 0 \\
0 & G_m & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{v} \\
\Delta u \\
\lambda
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
g_j
\end{bmatrix},
\] (33)
The anti-symmetric bending contribution has been moved inside the stiffness matrix since the Lagrange multipliers \( \lambda \) associated with the degrees of freedom of the deformation modes come out as the negative of the internal forces \( q_0 \). The form of the rigid body constraint matrix \( G_m \) depends on which deformation mode is analysed.
EXAMPLES

This section contains an assessment of the ability of the present methodology to calculate cross-section stiffness coefficients. Two examples are used to cover solid and thin-walled sections, namely an isotropic rectangular section and an isotropic wind turbine blade-like section.

Isotropic rectangular section

The first example concerns the rectangular cross-section geometry shown in figure 6(a), where the rectangle has a width of \(2a = 0.05\), a height of \(2b = 0.1\), and is made of an isotropic material. As shown in figure 6(b) the section is discretized by \(n \times n\) elements with quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. Reference axis being at the center, only the diagonal terms in the stiffness matrix are non-zero. The stiffness matrix therefore takes the following form

\[
\begin{bmatrix}
Q_{11} & GA_1 & 0 & 0 & 0 & 0 \\
Q_{21} & 0 & GA_2 & 0 & 0 & 0 \\
Q_{31} & 0 & 0 & EA & 0 & 0 \\
Q_{41} & 0 & 0 & 0 & EI_1 & 0 \\
Q_{51} & 0 & 0 & 0 & 0 & EI_2 \\
Q_{61} & 0 & 0 & 0 & 0 & GJ \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix} = (34)
\]

where the diagonal terms \(GA_1, GA_2, EA, EI_1, EI_2\), and \(GJ\) are the shear stiffness about both in-plane axes, the extensional stiffness, the bending stiffness about both in-plane axes, and the torsion stiffness, respectively.

Results for isotropic cross-sections obtained using different mesh sizes and an analytical solution from Renton [8] are listed in table 2. The error of the shear and torsion stiffness coefficients with respect to the analytical solution are plotted in figure 7, illustrating convergence for all parameters towards the analytical solution. Both shear stiffness and torsion stiffness require more than one element to converge below 1\% relative error. This is due to the inability of the quadratic interpolation to exactly represent the cubic deformation associated with shear.

The 3D deformation of the six deformation modes are presented graphically in figure 8 using a \(4 \times 4\) mesh and a slice thickness of 0.2 to clearly illustrate the deformation modes. The cubic displacement associated the shear deformations is seen in figures 8(c) and (f). Furthermore in the two bending cases, the quadratic curvature in the thickness direction is modelled with the use of a single element in the thickness direction via the Hermitian interpolation. The shear modes exhibit in-plane contractions that vary along the axial direction as a result of the linearly varying bending moments. It is seen that the rectangular section behaves more like a plate when subject to shear deformation in the \(x_1\) direction by having a larger anticlastic curvature as compared to when sheared in the \(x_2\) direction. This explains the lower shear stiffness coefficient in the \(x_1\) direction listed in table 2.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>1x1</th>
<th>2x2</th>
<th>4x4</th>
<th>8x8</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1/4Eab)</td>
<td>0.9113</td>
<td>0.7878</td>
<td>0.7844</td>
<td>0.7844</td>
<td>0.7844</td>
</tr>
<tr>
<td>(A_2/4Eab)</td>
<td>0.9967</td>
<td>0.8416</td>
<td>0.8335</td>
<td>0.8330</td>
<td>0.8329</td>
</tr>
<tr>
<td>(A/4Eab)</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(I_1/Eb^2a)</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
</tr>
<tr>
<td>(I_1/Eb^2b)</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
<td>1.3333</td>
</tr>
<tr>
<td>(J_1^{16}a^2b)</td>
<td>0.2667</td>
<td>0.2332</td>
<td>0.2291</td>
<td>0.2287</td>
<td>0.2287</td>
</tr>
</tbody>
</table>

FIGURE 6. (a) Schematic of a rectangular section, (b) \(3 \times 3\) finite element discretization.

FIGURE 7. Relative error in shear and torsion stiffness with respect to number of elements per side.
Isotropic blade-like section

The second example concerns the wind turbine blade-like section shown in figure 9(a) which has been used in Chen et al. [1] to assess the performance of blade modelling tools. The isotropic section is made of material with properties $E = 206.843$ GPa and $v = 0.49$. In the current analysis, the cross-section is discretized using elements with quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction with $n$ layers of elements in the thickness direction and 10$\pi$ segments in the circumferential direction. The mesh with a single layer of elements is shown in figure 6(b).

Results obtained using three different mesh sizes and the results reported in [1] obtained using VABS which is based on a variational asymptotic method with a 1200 element mesh are presented in table 3. The absolute relative difference of the stiffness coefficients with respect to the values obtained using VABS are plotted in figure 10, illustrating convergence for all parameters towards the solution obtained using VABS. It can be seen that an difference of less than 2% is obtained with the use of a single layer of elements $n = 1$ for all stiffness coefficients with the exception of the shear-torsion coupling $D_{26}$ which exhibits a difference of 30%. Four layers of elements are required to bring the shear-torsion coupling difference below 2%. Engineering intuition of the importance of the discretization of the tail section because of the sharp inner corner was confirmed from local mesh refinement. The coordinate of the shear center $a_2$ and the elastic center $c_2$ calculated from the stiffness matrix as explained in [9] are listed in table 3. From the symmetry of the section $a_2 = c_2 = 0$. As expected, the convergence of the shear center coordinate follows that of the shear-torsion coupling coefficient.

CONCLUSIONS

A method for analysing the cross-section stiffness properties of beams such as wind turbine blades has been presented. The methodology is based on the analysis of a thin slice of the beam, on which six independent deformation modes corresponding to extension, torsion, bending and shear are described by imposing displacement increments across the slice via Lagrange multipliers. The six by six stiffness matrix is defined using complementary elastic energy calculated in terms of the internal force/moment components. The slice is modelled using

![Figure 8](image_url)  
**FIGURE 8.** Deformation modes: (a) Extension, (d) Twist, (b,e) Bending, (c,f) Shear.

![Figure 9](image_url)  
**FIGURE 9.** (a) Schematic of isotropic blade-like section, (b) $n = 1$ finite element discretization.
a single layer of 3D finite elements with standard discretization in the cross-section plane and Hermitian cubic interpolation in the length-wise direction, whereby the degrees of freedom are concentrated on the front and back faces of slice. It is illustrated that the method can effectively calculate cross-section stiffness coefficients for solid and thin-walled structures.

ACKNOWLEDGMENT
This work has been supported by Siemens Wind Power A/S.

REFERENCES


TABLE 3 Cross-section properties for the blade-like section.

<table>
<thead>
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<th>Mesh refinement parameter $n$</th>
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Composite beam cross-section analysis
by a single high-order element

P.J. Couturier and S. Krenk

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Composite Beam Cross-Section Analysis
by a Single High-Order Element Layer

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An analysis procedure of general cross-section properties is presented. The formulation is based on the stress-strain states in the classic six equilibrium modes of a beam by considering a finite thickness slice modelled by a single layer of 3D finite elements. The theory is illustrated by application to composite sections of various shapes and material layups.

I. Introduction

Cross-section properties constitute an essential part of any beam model. As such the cross-section theory should be able to accurately model thin-walled parts, lamina built-up parts, and more massive parts. Two methodologies have been shown to provide accurate stiffness matrix for most engineering structures which are based on advanced kinematic analysis of beams namely the theories developed by Giavotto et al.\textsuperscript{1,2} called nonhomogeneous anisotropic beam section analysis (NABSA), and that of Cesnik and Hodges\textsuperscript{3,4} called variational asymptotic beam sectional analysis (VABS).

This paper builds on the stress states associated with the six equilibrium modes of a beam. These states have previously been used by Krenk and Jeppesen\textsuperscript{5} and Høgsberg and Krenk\textsuperscript{6} in connection with a center-line and 2D representation of the cross-section for the analysis of cross-sections made of orthotropic materials. In the present methodology, the cross-section stiffness matrix is calculated based on the analysis of a 3D slice of the beam in the form of a single layer of high-order elements, on which six independent deformation modes corresponding to extension, torsion, homogeneous bending and homogeneous shear are prescribed by imposing suitable displacement increments across the slice. The advantages of the present finite-thickness slice method are that it avoids the development of any special 2D theory for the stress and strain distributions over the cross-section and enables a simple and direct representation of material discontinuities and general anisotropy via their well-established representation in 3D elements. The present formulation appeals to implementation in standard finite element codes.

II. Beam Static Description

Consider a beam of length $l$ with longitudinal coordinate $x_3$ and cross-section coordinates $x_1$ and $x_2$ with the origin located in the center cross-section plane of the beam as shown in figure 1(a). The exact distribution of the surface stresses on a cross-section defined by $x_3$ can be replaced by three forces and three moments which are statically equivalent. These six forces and moments are grouped together in the force vector $\mathbf{q}(x_3) = [Q_1(x_3) \ Q_2(x_3) \ Q_3(x_3) \ M_1(x_3) \ M_2(x_3) \ M_3(x_3)]^T$. The components $Q_1(x_3)$ and $Q_2(x_3)$ are two shear forces, and $Q_3(x_3)$ is the axial force. The components $M_1(x_3)$ and $M_2(x_3)$ are bending moments, and $M_3(x_3)$ is the torsion moment component with respect to the origin of the reference coordinate system. The internal forces and moments components are illustrated in figure 1(b). The internal forces and moments

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are defined in terms of stresses on a cross-section as

\[ Q_\alpha(x_3) = \int_A \sigma_{3\alpha} \, dA, \quad M_\alpha(x_3) = \int_A e_{\alpha\beta} \, x_\beta \, \sigma_3 \, dA, \]

where Greek subscripts \( \alpha, \beta = 1, 2 \) represent the two in-plane directions. Summation over repeated Greek subscripts is implied. The \( e_{\alpha\beta} \) is the two-dimensional permutation symbol.

III. Equilibrium Modes

The statics of the beam can be described by six equilibrium modes, namely the homogeneous states of extension, torsion, bending, and shear. From equilibrium consideration a beam without external loads will exhibit constant internal normal force, shear forces and torsion moment, while the bending moments will vary linearly with the shear force as gradient. The six equilibrium modes can therefore be defined by the internal force \( q_0 = [Q_1^0 \ Q_2^0 \ Q_3^0 \ M_1^0 \ M_2^0 \ M_3^0]^T \) at the center of the beam. Since the constant shear forces \( Q_0^0 \) lead to linear anti-symmetric moment variation, the distribution of internal forces in the beam in terms of their values \( q_0 \) at the center cross-section of the beam follows as

\[
\begin{bmatrix}
Q_1(x_3) \\
Q_2(x_3) \\
Q_3(x_3) \\
M_1(x_3) \\
M_2(x_3) \\
M_3(x_3)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & x_3 & 0 & 1 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
Q_1^0 \\
Q_2^0 \\
Q_3^0 \\
M_1^0 \\
M_2^0 \\
M_3^0
\end{bmatrix},
\]

or in a more compact notation

\[ q(x_3) = T(x_3) \ q_0, \]

where the interpolation matrix \( T(x_3) \) is defined in (2).

IV. Flexibility Matrix

Following the general formulation of equilibrium based beam elements in Krenk, the deformation associated with the internal forces and moments \( q \) is described in terms of six generalized strains, defined by the strain vector \( \gamma = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \kappa_1 \ \kappa_2 \ \kappa_3]^T \). The components \( \varepsilon_1 \) and \( \varepsilon_2 \) are generalized shear strains, and \( \varepsilon_3 \) is the axial strain. Similarly \( \kappa_1 \) and \( \kappa_2 \) are the components of bending curvature, while \( \kappa_3 \) is the rate of twist. The generalized strain \( \gamma \) is defined such that it is conjugate to the internal force vector \( q \) with respect to energy. Thus, the specific elastic energy associated with a cross-section is given as

\[ W_s(x_3) = \frac{1}{2} \gamma(x_3)^T \ q(x_3) = \frac{1}{2} \ q(x_3)^T \ C \ q(x_3), \]

where \( C \) is the symmetric cross-section flexibility matrix for the linear elastic beam. Consider the beam shown in figure 1(a) as having a constant material distribution in the longitudinal direction. The flexibility matrix of the equilibrium modes of the beam follows from integration of the cross-section flexibility relation.
(4) over the beam length
\[ W_e = \int_{l/2}^{l/2} W_a(x_3) \, dx_3 = \frac{1}{2} \int_{l/2}^{l/2} q(x_3)^T C q(x_3) \, dx_3. \]  

(5)

Representing the internal forces and moments via the mid-point values by (3) yields
\[ W_e = \frac{1}{2} q_0^T H q_0. \]

(6)

where the beam flexibility matrix \( H \) corresponding to the six equilibrium modes is defined by the integral
\[ H = \int_{l/2}^{l/2} T(x_3)^T C T(x_3) \, dx_3. \]

(7)

Carrying out the integration in explicit form provides the relation between the beam flexibility matrix and the cross-section flexibility matrix
\[
H = l \begin{bmatrix}
C_{11} + \frac{i^2}{12} C_{55} & C_{12} - \frac{i^2}{12} C_{54} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} - \frac{i^2}{12} C_{45} & C_{22} + \frac{i^2}{12} C_{44} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}.
\]

(8)

It is seen how the beam flexibility matrix is a function of the cross-section properties as well as the length \( l \) of the beam. It is observed that for a finite-length beam the bending flexibility parameters \( C_{55}, C_{44}, C_{54}, \) and \( C_{54} \) of the cross-section contribute to the constant-shear modes of the beam due to their linearly varying bending moment.

V. Energy Equivalence and Finite Elements

The displacement field representation with respect to \( x_3 \) varies at most as a third degree polynomial. In what follows the solution to this displacement field is defined using a Hermitian representation in terms of the displacement and displacement derivative fields on the front (+) and back (−) faces of the beam. The displacement field in vector components \( u_j(x_j) = [u_j]^T \) where the \( j \) index ranges from one to three are described using finite elements as
\[ u(x_j) = N(x_j) v, \]

(9)

where \( N(x_j) \) are the shape functions corresponding to the nodal displacement which are contained in the column vector \( v = [v_1, \ldots, v_m]^T \) where \( m \) is the total number of nodes. The nodal degrees of freedom are defined as \( v_i = [u_{\alpha}, u_3, u_{\alpha}', u_3']^T \). The components \( u_\alpha \) and \( u_3 \) represent the in-plane and axial displacements, respectively, while the components \( u_\alpha' \) and \( u_3' \) represent the corresponding derivatives with respect to the axial coordinate \( x_3 \). The static component vector conjugate to the displacement vector \( v_j \) is the force vector \( p_i = [f_\alpha, f_3, f_\alpha', f_3']^T \). The components \( f_\alpha \) and \( f_3 \) represent the three force components, while the components \( f_\alpha' \) and \( f_3' \) represent three moments components conjugate to the displacement derivatives \( u_\alpha' \) and \( u_3' \), respectively.

The procedure consists in solving the finite element problem corresponding to six independent equilibrium states. The generalized nodal displacements, conjugate generalized nodal forces, and mid-section-force component vector of the six states are contained in the column vectors \( \tilde{v}_j, \tilde{p}_j \) and \( \tilde{q}_{0j} \), respectively, where \( j = 1, \ldots, 6 \). The vectors can be grouped in the following three matrices
\[ V = [\tilde{v}_1, \ldots, \tilde{v}_6], \quad P = [\tilde{p}_1, \ldots, \tilde{p}_6], \quad R = [\tilde{q}_{01}, \ldots, \tilde{q}_{06}]. \]

(10)

A general equilibrium state can now be represented as a linear combination of the six basic states introduced above. The components of this representation are denoted \( s = [s_1, \ldots, s_6]^T \), whereby the nodal displacements, nodal forces and section-forces of the center section take the form
\[ v = \sum_{j=1}^{6} \tilde{v}_j s_j = V s, \quad p = \sum_{j=1}^{6} \tilde{p}_j s_j = P s, \quad q_0 = \sum_{j=1}^{6} \tilde{q}_{0j} s_j = R s. \]

(11)
The elastic energy of the slice can now be expressed alternatively in terms of the flexibility matrix $H$ by use of (6) or as the produce of the nodal forces and displacements as given by the representations (11),

$$W_e = \frac{1}{2} s^T R^T H R s = \frac{1}{2} s^T V^T P s.$$  \hspace{1cm} (12)

Considering arbitrary component vectors $s$ and noting that the matrix $R$ is regular, the flexibility matrix $H$ follows as

$$H = R^{-T} (V^T P) R^{-1}. \hspace{1cm} (13)$$

This procedure determines the 36 elements of the flexibility matrix $H$ from the six load-cases solved by the finite element analysis of the 3D slice. The cross-section flexibility matrix $C$ can then be calculated using the finite-length flexibility relation (8), and the corresponding cross-section stiffness matrix $D$ is obtained by inversion of this flexibility matrix.

**VI. Deformation Modes**

The six independent equilibrium states are chosen as the deformation modes corresponding to extension, twist, bending, and shear. Each of the states are obtained by imposing appropriate displacements on the end-sections of the beam slice. The deformation modes are illustrated in figure 2 for the case of a square orthotropic cross-section where the undeformed beam is sketched using dotted lines.

Since the equilibrium modes are defined in terms of differences in forces and moments at the two sides of the beam, it is convenient to split the displacement into an increment and mean value over the thickness of the beam,

$$\Delta v(x_\alpha) = v(x_\alpha, \frac{1}{2}l) - v(x_\alpha, -\frac{1}{2}l),$$
$$2 \bar{v}(x_\alpha) = v(x_\alpha, \frac{1}{2}l) + v(x_\alpha, -\frac{1}{2}l). \hspace{1cm} (14)$$

Only the increment in displacement $\Delta v$ will generate work and thereby contribute to the flexibility matrix of the equilibrium modes of the beam. The mean value of the displacement $\bar{v}$ is associated with rigid body motion of the beam.

The extension deformation mode illustrated in figure 2(a) is described by an elongation of the beam. No internal shear forces $Q_\alpha$ occur in this mode which yields identical forces and moments at each cross-section along the longitudinal axis $x_3$. This leads to a uniform transverse contraction along the beam and thus increment displacement derivatives $\Delta u'_\alpha$ and $\Delta u'_3$ of zero.

The twist deformation mode illustrated in figure 2(b) is defined by a constant rate of twist about the axial coordinate $x_3$. The assumption of constant rate of twist corresponds to assuming homogeneous St. Venant torsion with identical cross-section warping along the beam. As in the extension mode, no internal shear forces occur which leads to increment displacement derivatives $\Delta u'_\alpha$ and $\Delta u'_3$ of zero.

The two bending deformation modes illustrated in figure 2(c) are characterized by a constant bending curvature about the in-plane coordinates $x_\alpha$. No internal shear forces occur in this mode and the axial

![Figure 2. Deformation modes: (a) Extension, (b) Twist, (c) Bending, (d) Shear.](image)
increment displacement derivative \( u'_3 \) is zero, while the in-plane increment displacement derivative \( u'_\alpha \) is defined by the bending curvature.

The two shear deformation modes illustrated in figure 2(d) are characterized by a transverse displacement increment in the direction of the in-plane coordinates \( x_\alpha \). These two modes will have internal shear forces and from equilibrium consideration will also have linearly varying bending moments with end-moments \( M_\alpha(\pm \frac{l}{2}) = -\frac{1}{2} e_{\alpha\beta} l Q_0^\beta \). This variation of internal moments leads to in-plane contractions that vary with \( x_3 \) contrary to the other four deformation modes. The displacement increment for the shear deformation modes is then defined by the superposition of the in-plane translation displacement and the deformation associated with the presence of anti-symmetric bending.

VII. Examples

This section illustrates the application of the present approach to calculate cross-section stiffness coefficients of composite sections. Two examples are used to cover different cross-section shapes and lamina layups, namely a composite ellipse cross-section and a box beam with symmetric fiber layup.

A. Composite Ellipse Cross-Section

This example concerns the composite section geometry shown in figure 3(a). It represents the cross section of a 6 m beam tested in torsion and flap bending at the Advanced Structures and Composites Center at The University Maine for Siemens Wind Power A/S. A schematic of the bending and torsion test set-up is shown in figure 4, illustrating the vertical set-up with one end of the beam clamped and the other attached to two winches. The cross-section of the beam was build-up as a symmetric sandwich structure with a balsa core and fiberglass-epoxy faces. In-plane properties have been determined from coupons taken from three different span-wise locations which provided a mean equivalent elastic modulus \( E_{11} = 8.46 \) GPa with 4\% coefficient of variation (COV) and mean shear modulus \( G_{12} = 2.31 \) GPa with 15\% COV. In-plane property variations were caused by the presence of ply overlaps.

![Figure 3. (a) Schematic of composite ellipse section, (b) \( n = 1 \) finite element discretization.](image)

In the current analysis, the cross-section is discretized using elements with quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction with \( n \) layers of elements in the thickness direction and 10\( n \) segments in the circumferential direction. The mesh with a single layer of element is shown in figure 3(b).

The mean value and coefficient of variation of the bending and torsion stiffnesses taking into account the variation of the in-plane properties using two different mesh sizes are presented in Table 1. The table also includes results calculated using standard beam theory and the results obtained from the experimental test using winch load data and digital image correlation of the center 2 m section of the beam. Variation in experimental data was mainly attributed to small spanwise material variation. It can be observed from Table 1 that the model results show an excellent agreement with the analytical solution. Moreover, a difference of less than 1\% with the analytical solution is obtained with the use of a single layer of elements \( n = 1 \) for both stiffness coefficients. The stiffnesses and 68\% confidence intervals normalized with respect to the experimental mean stiffness values are plotted in figure 5, illustrating that the experimental bending and torsional stiffness lies within the model confidence interval. This shows agreement in the results when the variation in elastic modulus are accounted for.
B. Box Beam

Aeroelastic tailoring can be achieved with the use of composite materials where the orthotropic axis of the laminate are off-axis to the structural axis. Research examples of this type of structural tailoring are load alleviation in wind turbine blades through bend-twist coupling and the X-29 Forward-Swept Wing Flight Demonstrator by Grumman where off-axis fibers were used to increase wing divergence speed. This second and final example of the present paper concerns the analysis of the box section shown in figure 6(a) that exhibits bend-twist coupling via the use off-axis fibers. The section has been used in Yu et al. and Smith and Chopra to validate cross-sectional analysis theories.

The box has a width of \( w = 24.2 \, \text{mm} \) (0.953 in.), a height of \( h = 13.5 \, \text{mm} \) (0.530 in.) and a uniform wall thickness of \( t = 0.76 \, \text{mm} \) (0.030 in.). The walls of the section are made up of six laminas with \( E_i = 142.0 \, \text{GPa} \) (20.59E+06 psi), \( E_j = E_k = 9.79 \, \text{GPa} \) (1.42E+06 psi), \( G_{ij} = 6.00 \, \text{GPa} \) (8.7E+05 psi), \( G_{ik} = G_{jk} = 4.80 \, \text{GPa} \) (6.96E+05 psi) and \( \nu_{ij} = \nu_{ik} = \nu_{jk} = 0.42 \) where the \( ijk \) system constitutes the principal material directions.
in the plane of the lamina, as shown in figure 6(b). The fiber orientation angle $\alpha$ is defined as the angle between the longitudinal axis $x_3$ and the axis along the fiber length $x_i$. The lamina orientation angle $\beta$ is defined as the angle between the $x_1$ axis and the $y_1$ axis in the plane of the lamina. The layup sequence for the top and bottom walls are $(-\alpha)_6$ and $(\alpha)_6$, respectively, and for left and right walls are $(-\alpha, \alpha)_3$ and $(\alpha, -\alpha)_3$, respectively. The layup sequence is in the direction of $y_2$ shown in figure 6(a).

In the current analysis, the cross-section is discretized using two meshes which are illustrated in figure 7. The first mesh, show in figure 7(a), contains 300 elements with quadratic interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction where 15 $\times$ 6 elements are used along the width and 10x6 elements along the height. Previous work by Høgsberg and Krenk has shown that extensive flanges and parts of thin-walled structures can be effectively modelled using elements with cubic-linear interpolations in the cross-section plane. Figure 7(b) shows the second mesh of the box section obtained using cubic-linear elements where the nodes are shown for clarity. The walls are modelled using four 16-node elements with cubic-linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. Each corner is modelled using two triangular elements with linear interpolation in the cross-section plane and Hermitian cubic interpolation in the axial direction. This discretization of the slice contains a total of 12 elements. For the second mesh, an average of the lamina constitutive matrix are used at the element level.

The cross-section parameters of the composite box section with both mesh and those published in Yu et al. obtained with VABS with 300 elements and NABSA with 336 elements are presented in Table 2. This configuration exhibits strong bend-twist coupling and extension-shear coupling. A measure of the importance of the off-diagonal terms can be obtained by normalizing them with respect to their associated diagonal terms,

$$\gamma_{ij} = \frac{D_{ij}}{\sqrt{D_{ii}D_{jj}}}.$$  (15)

This measure of coupling, from this point referred to as the coupling parameter, ranges from $-1 < \gamma_{ij} < 1$ where the extreme values indicate maximum possible coupling. It is observed that for this cross-section, the diagonal stiffness terms and coupling parameters obtained using the detailed mesh match well those
calculated using VABS and NABSA. Very good agreement is also obtained with the single-layer mesh with one order of magnitude less elements with the maximum percentage difference when comparing the dominant terms of 1.4% occurring for the shear stiffness in the $x_2$ direction. The differences on all parameters due to fewer elements are all smaller than the variability of the properties of real fiber reinforced cross-sections.

The 3D deformation of the deformation modes are represented graphically in figure 8 using the reduced mesh and a slice thickness of twice the width where the geometry before deformation is sketched using dotted lines. It is seen that the extension, bending, and torsion modes have a uniform transverse deformation along the $x_3$ axis. Furthermore, shearing is present in these four modes from the presence of off-axis fibers.

![Figure 8. Deformation modes of the box section: (a) Extension, (d) Twist, (b,e) Bending, (c,f) Shear.](image-url)
VIII. Conclusion

A method for obtaining the full six by six cross-section stiffness matrix of beams with arbitrary cross-section geometry and material distribution has been presented. The procedure avoids the use of any special 2D theory by analysing a thin slice of the beam using standard 3D finite element theory where a single layer of high-order Hermitian interpolation is used to capture axial displacements associated with prismatic beams. The cross-section flexibility matrix is obtained from the elastic energy of the finite-thickness slice, calculated in terms of the generalized internal force components and elastic energy associated with six independent deformation modes corresponding to extension, torsion, bending and shear. The cross-section stiffness matrix then follows from matrix inversion.

The method shows good agreement with analytical solution and experimental data for a thin-walled ellipse cross-section. Furthermore, the capability of modelling beams with general anisotropy was demonstrated using a box beam with fiber directions forming an angle with the beam axis. From a modelling point of view it is interesting to note that for both examples the use of a single element over the thickness provided excellent results. This suggests that the method can be used to efficiently modelled complex beams with little meshing effort.

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References


Efficient beam-type structural modeling of rotor blades

P.J. Couturier and S. Krenk

Proceedings of the EWEA 2015 Annual Event,
Efficient Beam-Type Structural Modeling of Rotor Blades

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Abstract
The present paper presents two recently developed numerical formulations which enable accurate representation of the static and dynamic behaviour of wind turbine rotor blades using little modeling and computational effort. The first development consists of an intuitive method to extract fully coupled six by six cross-section stiffness matrices with limited meshing effort. Secondly, an equilibrium based beam element accepting directly the stiffness matrices and accounting for large variations in geometry and material along the blade is presented. The novel design tools are illustrated by application to a composite section with bend-twist coupling and a real wind turbine blade.

1 Introduction
Wind turbine rotors are becoming increasingly slender with the introduction of larger rotors, inviting global beam-type analysis which puts focus on underlying beam theory and general cross-sectional stiffness properties. The blade modeling approach must provide accurate predictions of the blade behavior, e.g. bend-twist response while being able to easily accommodate geometry and material updates from previous designs.

This paper presents a developed methodology to facilitate the design of wind turbine blades using anisotropic materials and complex geometry to generate desired displacement characteristics. The work was done as part of a collaborative project between the Technical University of Denmark and Siemens Wind Power A/S. In the first part of the project a formulation was developed for analysis of the properties of general cross-sections with arbitrary geometry and material distribution [1]. The work was later extended by an efficient finite element modelling approach for thin and thick-walled sections which substantially reduced the meshing effort by discretizing the walls of the section using a single layer of displacement based elements with the layers represented within the elements [2]. The last part of the project aimed at showing how these general properties enter an efficient equilibrium based beam element which accounts for geometry and material variations along the blade span based on Krenk [3]. The beam element has been incorporated into an aeroelastic program at Siemens Wind Power A/S.

2 Approach
The basic process of structural modeling routinely used for slender structures, like wind turbine blades, to obtain fast and accurate predictions of the natural frequencies, deflections, and the overall dynamic behaviour is shown in Fig. 1. The goal of the process is to construct a reduced model of the 3D composite structure using beam elements. For an accurate representation, the beam model must reproduce the elastic energy of the 3D structure. This can only be achieved if one accounts for the variations in cross-section geometry and material properties along the blade as well as the governing kinematic behaviours, e.g. deformation mode coupling, transverse shear deformation and warping.

The first step to reduce the dimensionality of a blade is to calculate the mechanical properties associated with the individual beam cross-sections. As part of the project, a formulation for cross-section analysis providing the full six by six stiffness matrix for non-homogeneous and anisotropic

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sections with coupling between the deformation modes has been developed. The stiffness matrix is evaluated using a single layer of 3D finite elements with cubic lengthwise displacement interpolation on which six independent deformation modes corresponding to extension, torsion, homogeneous bending and homogeneous shear are prescribed by imposing suitable displacement increments across the slice. The advantages of the present finite-thickness slice method are that it avoids the development of any special 2D theory for the stress and strain distributions over the cross-section and enables a simple and direct representation of material discontinuities and general anisotropy via their well established representation in 3D elements. The present cross-section analysis method has been implemented and validated in a computer program called CrossFlex (Cross-section Flexibility) for thin-walled and massive parts made of general anisotropic materials [1, 4, 5].

The accuracy of the six by six stiffness matrix as well as the ease to accommodate drastic geometry and material changes from previous designs using Finite Element based cross-section analysis methods largely depend on the discretization approach. The conventional finite element meshing approach is to model each layer in the blade walls using one or more elements through the thickness. Using this approach, the number of elements will depend on the number of layers which requires significant meshing effort and limits design flexibility. To circumvent these limitations, a new cross-section meshing approach was developed where thin to thick laminates are modelled using a single element through the wall thickness [2]. The elements stiffness is obtained from integration of the stiffness of the individual lamina across the thickness. A complementary postprocessing scheme was also developed to recover interlaminar stresses via equilibrium equations of 3D elasticity.

An illustrative comparison between the discretization of a lamina using conventional solid element and the internally layered elements is shown in Fig. 2. It can be seen that the conventional meshing approach requires several elements through the wall thickness whereas only one layered element is required to model the section of the lamina. Furthermore, by integrating the stiffness of the individual laminates the effect of the staking sequence is retained, e.g. stiff laminates closer to the outer surface of the blade shell should increase the bending stiffness. This information would be lost if the material properties of the element were taken as the thickness weighted average of the lamina properties. The reduced number of nodes and simpler mesh needed when using layered elements provide substantial pre-processing and computational effort savings compared with conventional meshing using solid elements. For example, less than 50 internally layered elements are needed to model the complexity of a typical blade cross-section [2].

The second step to create a reduced model of the 3D blade is to obtain the beam elements stiffness matrices. In the present project, a beam element was extended in which the stiffness properties are obtained via flexibility from equilibrium considerations that do not use assumed shape functions [3]. This so called complementary energy approach immediately accepts the six
by six cross-section stiffness matrices and accounts for geometry and material variations along the blade span. The beam element has been incorporated into an aeroelastic program at Siemens Wind Power A/S. It has been shown that a blade with complex geometry and material layup can accurately and effectively be modeled using 6-8 elements.

3 Applications
This section illustrates the application of the developed method of evaluating general cross-sectional properties and beam element stiffness matrices to model composite beams. Two studies are considered, first the cross-section properties and kinematics of a box beam with bend-twist coupling are studied and compared with published results. Second, the beam element capability is further demonstrated with its application to a realistic wind turbine rotor blade.

3.1 Box Beam
This example concerns the analysis of a composite box beam that exhibits bend-twist coupling via the use of off-axis fibers. The cross-section properties of this beam have been studied in Yu et al. [6] and by the present authors in Couturier and Krenk [4] to validate cross-sectional analysis theories. The box has a width of $w = 24.2$ mm (0.953 in.), a height of $h = 13.5$ mm (0.530 in.) and a uniform wall thickness of $t = 0.76$ mm (0.030 in.). The walls of the section are made up of six laminas with $E_i = 142.0$ GPa (20.59E+06 psi), $E_j = E_k = 9.79$ GPa (1.42E+06 psi), $G_{ij} = 6.00$ GPa (8.7E+05 psi), $G_{ik} = G_{jk} = 4.80$ GPa (6.96E+05 psi) and $\nu_{ij} = \nu_{ik} = \nu_{jk} = 0.42$ where $i$ denotes the fiber direction, $j$ the transverse direction, and $k$ the direction normal to the plane of the lamina. The layup sequence for the top and bottom walls are $(-\alpha)_6$ and $(\alpha)_6$, respectively, and for left and right walls are $(-\alpha, \alpha)_3$ and $(\alpha, -\alpha)_3$, respectively. The layup sequence is defined from the innermost to the outermost layers.

In the current analysis, the cross-section is discretized using two meshes which are illustrated in Fig. 3. The first mesh, show in Fig. 3(a), represents the conventional highly discretized meshing approach with one element placed in each of the six layers in the thickness direction and 50 segments in the circumferential direction for a total of 300 solid elements. The elements have quadratic interpolation in the cross-section plane. The second mesh, show in Fig. 3(b), uses internally layered elements with a single element in the thickness direction. The walls are modelled using four 16-node elements with cubic-linear interpolation in the cross-section plane. Each corner is modelled using two triangular elements with linear interpolation in the cross-section plane. This discretization of the slice contains a total of 12 elements. Both meshes use lengthwise cubic Hermitian interpolation to capture the displacement field associated with prismatic beams. This results in the degrees of freedom to be concentrated on the front and back faces of the slice, as shown in Fig. 3. Note that a thickness of the slice comparable to the in-plane element dimensions
is used to avoid ill-conditioned elements.

The particular cross-section with \( \alpha = 15^\circ \) has been analysed using VABS with 300 6-noded quadrilateral elements and NABSA with 336 9-noded quadrilateral elements by Yu et al. [6]. These results are presented in Table 1 together with results obtained by the present method using the two meshes shown in Fig. 3. The \( \alpha = 15^\circ \) configuration exhibits bend-twist coupling and extension-shear coupling. A measure of the importance of the off-diagonal terms in the cross-section stiffness matrix \( D \) can be obtained by normalizing them with respect to their associated diagonal terms,

\[
\gamma_{ij} = \frac{D_{ij}}{\sqrt{D_{ii}D_{jj}}}.
\]

This normalized measure of coupling, referred to as the coupling parameter, ranges from \(-1 < \gamma_{ij} < 1\), where the extreme values indicate maximum possible coupling. All stiffness terms obtained using the present detailed mesh agree well with those calculated using VABS and NABSA. Good agreement is also obtained with the internally layered element model with one order of magnitude less elements with the maximum percentage difference when comparing the dominant terms of 1.5\% occurring for the shear stiffness in the \( x_2 \) direction. The bend-twist coupling parameter \( \gamma_{46} \) and the extension-shear coupling parameter \( \gamma_{13} \) have a 0.9\% and 0.5\% difference with the detailed models, respectively. It is noted that the differences on all parameters are smaller than the variability of the properties of fiber reinforced structures used in the industry.

Figure 4(a) shows the value of the bend-twist and extension-shear coupling parameter with respect to the ply angle obtained using the internally layered element model with 12 elements and the detailed mesh with 300 elements. It can be seen that both models agree well for all fiber orientations. The maximum bend-twist coupling occurs at a fiber orientation \( \alpha = 25^\circ \) with a

![Figure 3: (a) Mesh using 300 solid elements, (b) Mesh using 12 solid internally layered elements.](image-url)

Table 1: Cross-section stiffness properties for the box section with \( \alpha = 15^\circ \).

<table>
<thead>
<tr>
<th></th>
<th>NABSA 336 elements</th>
<th>VABS 300 elements</th>
<th>Present Full 300 elements</th>
<th>Present Lam. 12 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G.A_1 ) [N]</td>
<td>3.94E+05</td>
<td>3.93E+05</td>
<td>3.94E+05</td>
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<tr>
<td>( G.A_2 ) [N]</td>
<td>1.72E+05</td>
<td>1.73E+05</td>
<td>1.72E+05</td>
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<td>( E.A ) [N]</td>
<td>6.09E+06</td>
<td>6.09E+06</td>
<td>6.08E+06</td>
<td>6.09E+06</td>
</tr>
<tr>
<td>( E.I_1 ) [Nm²]</td>
<td>1.70E+02</td>
<td>1.70E+02</td>
<td>1.70E+02</td>
<td>1.70E+02</td>
</tr>
<tr>
<td>( E.I_2 ) [Nm²]</td>
<td>4.05E+02</td>
<td>4.05E+02</td>
<td>4.06E+02</td>
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<tr>
<td>( G.J ) [Nm²]</td>
<td>4.88E+01</td>
<td>4.88E+01</td>
<td>4.87E+01</td>
<td>4.90E+01</td>
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<td>( \gamma_{12} )</td>
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<td>1.23E–03</td>
<td>1.36E–03</td>
<td>1.38E–03</td>
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<tr>
<td>( \gamma_{13} )</td>
<td>−5.28E–01</td>
<td>−5.29E–01</td>
<td>−5.29E–01</td>
<td>−5.26E–01</td>
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<tr>
<td>( \gamma_{23} )</td>
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<td>−6.57E–04</td>
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<tr>
<td>( \gamma_{45} )</td>
<td>−4.06E–03</td>
<td>−4.06E–03</td>
<td>−4.05E–03</td>
<td>−2.07E–03</td>
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<tr>
<td>( \gamma_{46} )</td>
<td>5.55E–01</td>
<td>5.55E–01</td>
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<td>5.52E–01</td>
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<tr>
<td>( \gamma_{56} )</td>
<td>−7.13E–03</td>
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coupling value of $\gamma_{46} = 6.26 \times 10^{-1}$ while the maximum extension-shear coupling occurs at a fiber orientation $\alpha = 24^\circ$ with a coupling value of $\gamma_{46} = -5.78 \times 10^{-1}$. The variation of the bending stiffness about the $x_1$ axis and the torsion stiffness with respect to the ply angle is shown in Fig. 4(b). It can be seen that using a fiber orientation $\alpha = 25^\circ$ leads to a reduction of the bending stiffness of 56% compared to the configuration with no off-axis fibers (i.e. $0^\circ$).

The linear static behaviour of a 762 mm (30 in.) long cantilever beam with such a section with a height of $h = 13.6$ mm (0.537 in.) instead of $h = 13.5$ mm (0.530 in.) built with three different fiber orientations $\alpha = 15^\circ$, $30^\circ$, and $45^\circ$ has been investigated experimentally by Chandra et al. [7]. The twist and bending slope at the middle of the beam they measured due to a tip torque are shown in Fig. 5 and Fig. 6, respectively, together with results obtained with the present modeling approach using a single beam element and the six by six stiffness matrix obtained using the two meshes shown in Fig. 3. Results are compared with data obtained from Smith and Chopra [8] using an analytical model (Smith and Chopra) and a finite element approach developed by Stample and Lee [9] (Stample and Lee), as well as data obtained from Ghiringhelli [10] using a finite element beam model (Ghiringhelli) and a 3D finite element model (3D FEM).

For the three beams modelled, the results obtained using the present beam model with the detailed cross-section mesh agree well with the 3D finite element model (3D FEM), the finite element beam model developed by Ghiringhelli (Ghiringhelli), as well as the beam model developed by Stample and Lee (Stample and Lee). Good agreement is also obtained with the experimental results with the exception of the bending slope for the beam with $\alpha = 30^\circ$. Ghiringhelli [10] offers an explanation for this discrepancy. He shows by evaluating the bending slope at every 5° fiber angle interval that the experimental result at $\alpha = 30^\circ$ deviates from the regular analytical curve.

Figure 5: Twist at mid span of box beam under tip torque of 0.113 Nm [1 lb in.].
Figure 6: Bending slope at mid span of box beam under tip torque of 0.113 Nm [1 lb in.].

Very good agreement is also obtained when using the cross-section stiffness properties from the internally layered element model with the maximum percentage difference compared to using the stiffness properties of the detailed cross-section mesh of 2.6% occurring for the bending slope with $\alpha = 30$. 

### 3.2 Wind Turbine Blade

This second and final example of the present paper concerns the analysis of a 75 m long wind turbine blade manufactured by Siemens Wind Power A/S illustrated in Fig. 7. The blade is constructed using a single web design with the shell and spar cap made of fiberglass-epoxy, while the sandwich core present in the trailing edge walls and tail are made of balsa and foam. The distribution along the blade length of bending stiffnesses about the principal axes of bending normalized with respect to the bending stiffness of the circular root section are shown in Fig. 8. This gives an appreciation of the large cross-section property variations that must be captured when modeling wind turbine blades. It can be seen that the bending stiffness in the edgewise direction is on average for the first half of the blade twice as large as the stiffness in the flapwise direction. The increase in the edgewise bending stiffness near the root is associated with the transition from a circular to an airfoil cross-section.

In the current analysis, the blade is discretized using six beam element meshes each with a different number of elements. The location of the nodes for the different meshes used are shown in Fig. 7. The nodes are positioned along the neutral axis (elastic axis). Omitted from the figure for clarity are the mesh with 32 and 75 equally spaced elements. Note that the node positions for the mesh with four and eight elements are optimized to minimize the error of the first four natural frequencies. As expected, the nodes are skewed towards the more compliant outboard part of the blade, as shown in Fig. 7.

The relative error of the in-plane tip displacement obtained using the models with one, four, and sixteen elements relative to a reference deflection calculated using 75 elements under a concentrated tip load $p_1 = p_2 = c$ are shown in Fig. 9(a). Four elements are required to obtain convergence of both in-plane displacement components to within 1% relative error. This is due
to the inability of one element to capture the effect of the pre-bend in the blade. If the blade was straightened, the deflection under a tip load would be calculated exactly using a single element.

The relative error of the first four undamped natural frequencies of the blade obtained using the models with 4, 8, 16, and 32 elements relative to the frequencies calculated using 75 elements are shown in Fig. 9(b). The mass matrix is obtained from a finer integration of the classic shape functions. For this case, eight elements are required to obtain convergence of all four natural frequencies to within 1% relative error. The increase in the number of elements required to describe the dynamic behavior is expected from the large material property variations, where a sufficient number of elements is needed to properly capture the mode shapes associated with the natural frequencies.

4 Conclusion
A method of evaluating general cross-sectional properties and beam element stiffness matrices of a rotor blade which accounts for all the possible couplings between the deformation modes and of variations in geometry and material along the blade has been developed. The present methods avoid the need for advanced kinematic analysis of beams and instead hinge on the beam equilibrium modes.

In spite of the very small number of elements used, the method shows good agreement with numerical solutions and experimental data for a box beam that exhibits bend-twist coupling via the use of off-axis fibers. Example of an analysis of a Siemens rotor blade demonstrated that very few elements are needed to describe dynamic and static behavior of blades using the proposed modeling approach.

Figure 9: (a) Static tip deflection, (b) Undamped natural frequencies.
Acknowledgments
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References


